# Efficient Batch and Recursive Least Squares for Matrix Parameter Estimation with Application to Adaptive MPC 

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#### Abstract

Traditionally, batch least squares (BLS) and recursive least squares (RLS) are used for identification of a vector of parameters that form a linear model. In some situations, however, it is of interest to identify parameters in a matrix structure. In this case, a common approach is to transform the problem into standard vector form using the vectorization (vec) operator and the Kronecker product, known as vec-permutation. However, the use of the Kronecker product introduces extraneous zero terms in the regressor, resulting in unnecessary additional computational and space requirements. This work derives matrix BLS and RLS formulations which, under mild assumptions, minimize the same cost as the vecpermutation approach. This new approach requires less computational complexity and space complexity than vec-permutation in both BLS and RLS identification. It is also shown that persistent excitation guarantees convergence to the true matrix parameters. This method can used to improve computation time in the online identification of multiple-input, multiple-output systems for indirect adaptive model predictive control.


## I. Introduction

Least squares based identification methods are foundational to systems and control theory, particularly identification, signal processing, and adaptive control [1], [2]. Batch least squares (BLS) and recursive least squares (RLS) are traditionally used to identify a vector of parameters in a linear measurement process [2], [3]. However, it may be of interest to identify parameters in a matrix structure, for example, in adaptive control of multiple-input, multiple-output (MIMO) systems [3], [4]. One approach is to use vec-permutation [5], a method which rewrites the linear measurement process such that the columns of the parameters to be identified are stacked into a vector. This is accomplished using the the vectorization operator and Kronecker product, and is a straightforward solution for various situations [4], [6]-[12].

A significant drawback, however, is that the vecpermutation method increases the dimension of the linear measurement process by using the Kronecker product, introducing extraneous zero terms in the regressor (e.g. equation (15) of [4]). This results in increased computational cost and storage requirements. Other methods for identifying parameters in a matrix structure have been proposed including square root filtering [13], multiinnovations [14], [15], and gradient-based methods [16]. However, these methods do not address whether a least squares cost function is globally minimized or the relationship of the method to standard least squares methods.

[^0]This work develops a batch and recursive least squares algorithm for identification of matrix parameters which, under mild assumptions, minimizes the same cost function used in the vec-permutation approach. This method provides a $\mathcal{O}\left(m^{3}\right)$ times improvement in computational cost and a $\mathcal{O}\left(m^{2}\right)$ times improvement in storage requirements over vecpermutation, where $m \geq 1$ is the number of columns of the identified parameter matrix. We also show how persistent excitation guarantees convergence of the identified matrix parameters to true matrix parameters, which extends established results for identification of vector parameters [17]-[19]. Finally, we show how this method can be used to significantly reduce computation time spent on online identification in predictive cost adaptive control (PCAC) [4].

## II. The Vec-permutation Approach

Consider a measurement process of the form ${ }^{1}$

$$
\begin{equation*}
y_{k}=\phi_{k} \theta, \tag{1}
\end{equation*}
$$

where $k=0,1,2, \ldots$ is the time step, $y_{k} \in \mathbb{R}^{p \times m}$ is the measurement at step $k, \phi_{k} \in \mathbb{R}^{p \times n}$ is the regressor at step $k$, and $\theta \in \mathbb{R}^{n \times m}$ is a matrix of unknown parameters. Parameters $\theta$ can be identified by minimizing the least squares cost function $J_{k}: \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$, defined as

$$
\begin{align*}
J_{k}(\hat{\theta})= & \sum_{i=0}^{k} \operatorname{vec}\left(y_{i}-\phi_{i} \hat{\theta}\right)^{\mathrm{T}} \bar{\Gamma}_{i} \operatorname{vec}\left(y_{i}-\phi_{i} \hat{\theta}\right) \\
& +\operatorname{vec}\left(\hat{\theta}-\theta_{0}\right)^{\mathrm{T}} \bar{R} \operatorname{vec}\left(\hat{\theta}-\theta_{0}\right) \tag{2}
\end{align*}
$$

where $\operatorname{vec}(\cdot)$ is the column stacking operator, positivedefinite $\bar{R} \in \mathbb{R}^{m n \times m n}$ is the regularization matrix, $\theta_{0} \in$ $\mathbb{R}^{n \times m}$ is an initial estimate of $\theta$, and, for all $k \geq 0$, positivedefinite $\bar{\Gamma}_{k} \in \mathbb{R}^{m p \times m p}$ is the measurement weighting matrix.

Using vec-permutation [5], (1) can be rewritten as

$$
\begin{equation*}
\bar{y}_{k}=\bar{\phi}_{k} \bar{\theta} \tag{3}
\end{equation*}
$$

where $\bar{y}_{k} \in \mathbb{R}^{m p}, \bar{\phi}_{k} \in \mathbb{R}^{m p \times m n}$, and $\bar{\theta} \in \mathbb{R}^{m n}$ are defined

$$
\begin{align*}
& \bar{y}_{k} \triangleq \operatorname{vec}\left(y_{k}\right),  \tag{4}\\
& \bar{\phi}_{k} \triangleq\left(I_{m} \otimes \phi_{k}\right),  \tag{5}\\
& \bar{\theta} \triangleq \operatorname{vec}(\theta), \tag{6}
\end{align*}
$$

[^1]and where $\otimes$ is the Kronecker product. Note, for all $k \geq 0$ and $\hat{\theta} \in \mathbb{R}^{n \times m}$ that $\operatorname{vec}\left(y_{k}-\phi_{k} \hat{\theta}\right)=\bar{y}_{k}-\bar{\phi}_{k} \operatorname{vec}(\hat{\theta})$ and $\operatorname{vec}\left(\hat{\theta}-\theta_{0}\right)=\operatorname{vec}(\hat{\theta})-\bar{\theta}_{0}$, where $\bar{\theta}_{0} \in \mathbb{R}^{m n}$ is defined as
\[

$$
\begin{equation*}
\bar{\theta}_{0} \triangleq \operatorname{vec}\left(\theta_{0}\right) \tag{7}
\end{equation*}
$$

\]

It then follows that, for all $k \geq 0$, the cost function $J_{k}$, given in (2), can be rewritten as

$$
\begin{align*}
J_{k}(\hat{\theta})= & \sum_{i=0}^{k}\left(\bar{y}_{i}-\bar{\phi}_{i} \operatorname{vec}(\hat{\theta})\right)^{\mathrm{T}} \bar{\Gamma}_{i}\left(\bar{y}_{i}-\bar{\phi}_{i} \operatorname{vec}(\hat{\theta})\right) \\
& +\left(\operatorname{vec}(\hat{\theta})-\bar{\theta}_{0}\right)^{\mathrm{T}} \bar{R}\left(\operatorname{vec}(\hat{\theta})-\bar{\theta}_{0}\right) \tag{8}
\end{align*}
$$

Propositions 1 and 2 respectively give the batch and recursive least squares methods using vec-permutation to minimize cost function $J_{k}$.
Proposition 1. Let $N \geq 0$. For all $0 \leq k \leq N$, let $\phi_{k} \in$ $\mathbb{R}^{p \times n}$, let $y_{k} \in \mathbb{R}^{p \times m}$, and let $\bar{\Gamma}_{k} \in \mathbb{R}^{m p \times m p}$ be positive definite. Furthermore, let $\bar{\theta}_{0} \in \mathbb{R}^{n \times m}$ and let $\bar{R} \in \mathbb{R}^{m n \times m n}$ be positive definite. Then, $J_{N}: \mathbb{R}^{n} \rightarrow \mathbb{R}$, defined in (2), has a unique minimizer whose vectorization is given by

$$
\begin{equation*}
\operatorname{vec}\left(\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{N}(\hat{\theta})\right)=\bar{A}_{N}^{-1} \bar{b}_{N} \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
& \bar{A}_{N} \triangleq \bar{R}+\sum_{i=0}^{N} \bar{\phi}_{i}^{\mathrm{T}} \bar{\Gamma}_{i} \bar{\phi}_{i}  \tag{10}\\
& \bar{b}_{N} \triangleq \bar{R} \operatorname{vec}\left(\theta_{0}\right)+\sum_{i=0}^{N} \bar{\phi}_{i}^{\mathrm{T}} \bar{\Gamma}_{i} \bar{y}_{i} \tag{11}
\end{align*}
$$

and where, for all $0 \leq k \leq N, \bar{y}_{k} \in \mathbb{R}^{m p}$ and $\bar{\phi}_{k} \in \mathbb{R}^{m p \times m n}$ are defined in (4) and (5), respectively.
Proof. This result follows directly from Lemma A.3.
Proposition 2. For all $k \geq 0$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in \mathbb{R}^{p \times m}$, and let $\bar{\Gamma}_{i} \in \mathbb{R}^{m p \times m p}$ be positive definite. Furthermore, let $\bar{\theta}_{0} \in \mathbb{R}^{n \times m}$ and let $\bar{R} \in \mathbb{R}^{m n \times m n}$ be positive definite. Then, for all $k \geq 0, J_{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$, defined in (2), has a unique minimizer, whose vectorization is denoted as

$$
\begin{equation*}
\bar{\theta}_{k+1} \triangleq \operatorname{vec}\left(\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{k}(\hat{\theta})\right) \tag{12}
\end{equation*}
$$

Moreover, for all $k \geq 0, \bar{\theta}_{k+1}$ is given recursively by

$$
\begin{align*}
\bar{P}_{k+1}^{-1} & =\bar{P}_{k}^{-1}+\bar{\phi}_{k}^{\mathrm{T}} \bar{\Gamma}_{k} \bar{\phi}_{k}  \tag{13}\\
\bar{\theta}_{k+1} & =\bar{\theta}_{k}+P_{k+1} \bar{\phi}_{k}^{\mathrm{T}} \bar{\Gamma}_{k}\left(\bar{y}_{k}-\bar{\phi}_{k} \bar{\theta}_{k}\right) \tag{14}
\end{align*}
$$

where $\bar{y}_{k} \in \mathbb{R}^{m p}$ and $\bar{\phi}_{k} \in \mathbb{R}^{m p \times m n}$ are defined in (4) and (5), respectively, $\bar{P}_{0} \triangleq \bar{R}^{-1}$, and, for all $k \geq 0$, $\bar{P}_{k} \in \mathbb{R}^{m n \times m n}$ is positive definite, and hence nonsingular. Moreover, for all $k \geq 0, \bar{P}_{k}$ can be expressed recursively as

$$
\begin{equation*}
\bar{P}_{k+1}=\bar{P}_{k}-\bar{P}_{k} \bar{\phi}_{k}^{\mathrm{T}}\left(\bar{\Gamma}_{k}^{-1}+\bar{\phi}_{k} \bar{P}_{k} \bar{\phi}_{k}^{\mathrm{T}}\right)^{-1} \bar{\phi}_{k} \bar{P}_{k} \tag{15}
\end{equation*}
$$

Proof. This result follows directly from Lemma A.3.

An inefficiency with this method is that the Kronecker product in (5) introduces extraneous zero terms in $\bar{\phi}_{k}$ when $m>1$, resulting in a sparse and higher dimension regressor matrix. However, there is no way to simplify the results of Propositions 1 and 2 as the regularization matrix $\bar{R}$ and measurement weighting matrices $\bar{\Gamma}_{k}$ are not necessarily sparse. The computational complexities of BLS and RLS with vecpermutation are shown in Tables I and II respectively.

## III. Independent Column Weighting

Next, we assume there exist positive-definite $R_{1}, \ldots, R_{m} \in \mathbb{R}^{n \times n}$ such that $\bar{R}$ is block diagonal of the form

$$
\bar{R}=\left[\begin{array}{ccc}
R_{1} & \cdots & 0_{n \times n}  \tag{16}\\
\vdots & \ddots & \vdots \\
0_{n \times n} & \cdots & R_{m}
\end{array}\right]
$$

Furthermore, assume that, for all $k \geq 0$, there exist positivedefinite $\Gamma_{1, k}, \ldots, \Gamma_{m, k} \in \mathbb{R}^{p \times p}$ such that $\bar{\Gamma}_{k}$ is block diagonal of the form

$$
\bar{\Gamma}_{k}=\left[\begin{array}{ccc}
\Gamma_{1, k} & \cdots & 0_{p \times p}  \tag{17}\\
\vdots & \ddots & \vdots \\
0_{p \times p} & \cdots & \Gamma_{m, k}
\end{array}\right]
$$

This corresponds to independent weighting of the columns of $y_{i}-\phi_{i} \hat{\theta}, i=0, \ldots, k$, and $\hat{\theta}-\theta_{0}$ in (2). Then, for all $k \geq 0$ and $\hat{\theta} \in \mathbb{R}^{n \times m}$, (2) can be rewritten as

$$
\begin{equation*}
J_{k}(\hat{\theta})=\sum_{j=0}^{m} J_{j, k}\left(\hat{\theta}_{j}\right) \tag{18}
\end{equation*}
$$

where, for all $j=1, \ldots, m, J_{j, k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is defined as

$$
\begin{align*}
J_{j, k}\left(\hat{\theta}_{j}\right)= & \sum_{i=0}^{k}\left(y_{j, i}-\phi_{i} \hat{\theta}_{j}\right)^{\mathrm{T}} \Gamma_{j, i}\left(y_{j, i}-\phi_{i} \hat{\theta}_{j}\right) \\
& +\left(\hat{\theta}_{j}-\theta_{j, 0}\right)^{\mathrm{T}} R_{j}\left(\hat{\theta}_{j}-\theta_{j, 0}\right) \tag{19}
\end{align*}
$$

where the vectors $y_{1, k}, \ldots, y_{m, k} \in \mathbb{R}^{p}, \theta_{1,0}, \ldots, \theta_{m, 0} \in \mathbb{R}^{n}$, and $\hat{\theta}_{1}, \ldots, \hat{\theta}_{m} \in \mathbb{R}^{n}$ are the $m$ columns of $y_{k}, \theta_{0}$, and $\hat{\theta}$, respectively. In particular,

$$
\begin{align*}
& y_{k} \triangleq\left[\begin{array}{lll}
y_{1, k} & \cdots & y_{m, k}
\end{array}\right]  \tag{20}\\
& \theta_{0} \triangleq\left[\begin{array}{lll}
\theta_{1,0} & \cdots & \theta_{m, 0}
\end{array}\right]  \tag{21}\\
& \hat{\theta} \triangleq\left[\begin{array}{lll}
\hat{\theta}_{1} & \cdots & \hat{\theta}_{m}
\end{array}\right] \tag{22}
\end{align*}
$$

The following Lemma shows that minimizing the cost function $J_{k}$, given by (18), can be done by separately minimizing $J_{j, k}$ for all $j=1, \ldots, m$.
Lemma 1. For all $k \geq 0$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in$ $\mathbb{R}^{p \times m}$, and let $\Gamma_{1, k}, \ldots, \Gamma_{m, k} \in \mathbb{R}^{p \times p}$ be positive definite. Furthermore, let $\bar{\theta}_{0} \in \mathbb{R}^{n \times m}$ and let $R_{1}, \ldots, R_{m} \in \mathbb{R}^{n \times n}$ be positive definite. Then, for all $k \geq 0, J_{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$, defined in (18), has a unique minimizer given by
$\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{k}(\hat{\theta})=\left[\begin{array}{lll}\underset{\hat{\theta}_{1} \in \mathbb{R}^{n}}{\arg \min } J_{1, k}\left(\hat{\theta}_{1}\right) & \ldots & \underset{\hat{\theta}_{m} \in \mathbb{R}^{n}}{\arg \min } J_{m, k}\left(\hat{\theta}_{m}\right)\end{array}\right]$

Proof. Note that, for all $j=1, \ldots, m$, since $J_{j, k}$ is a function of only $\hat{\theta}_{j}$, it follows from Lemma A. 3 that $J_{j, k}$ has a unique minimizer. Then, since $J_{k}(\hat{\theta})=\sum_{j=0}^{m} J_{j, k}\left(\hat{\theta}_{j}\right)$, it follows from (22) that the Lemma holds.

Propositions 3 and 4 respectively give the independent column weighting batch and recursive least squares methods to minimize cost function $J_{k}$ given by (18).

Proposition 3. Let $N \geq 0$. For all $0 \leq k \leq N$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in \mathbb{R}^{p \times m}$, and let $\Gamma_{1, k}, \ldots, \Gamma_{m, k} \in \mathbb{R}^{p \times p}$ be positive definite. Furthermore, let $\bar{\theta}_{0} \in \mathbb{R}^{n \times m}$ and let $R_{1}, \ldots, R_{m} \in \mathbb{R}^{n \times n}$ be positive definite. Then, $J_{N}: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}$, defined in (18), has a unique minimizer given by

$$
\underset{\hat{\theta} \in \mathbb{R} n \times m}{\arg \min } J_{N}(\hat{\theta})=\left[\begin{array}{lll}
A_{1, N}^{-1} b_{1, N} & \cdots & A_{m, N}^{-1} b_{m, N} \tag{23}
\end{array}\right],
$$

where, for all $j=1, \ldots, m$,

$$
\begin{align*}
& A_{j, N} \triangleq R_{j}+\sum_{i=0}^{N} \phi_{i}^{\mathrm{T}} \Gamma_{j, i} \phi_{i}  \tag{24}\\
& b_{j, N} \triangleq R_{j} \theta_{j, 0}+\sum_{i=0}^{N} \phi_{i}^{\mathrm{T}} \Gamma_{j, i} y_{j, i}, \tag{25}
\end{align*}
$$

and where, for all $0 \leq k \leq N, y_{j, k} \in \mathbb{R}^{p}$ is defined in (20) and $\theta_{j, 0} \in \mathbb{R}^{n}$ is defined (21).

Proof. It follows from Lemma A. 3 that, for all $j=1, \ldots, m$, $\arg \min _{\hat{\theta}_{1} \in \mathbb{R}^{n}} J_{j, N}\left(\hat{\theta}_{j}\right)=A_{j, N}^{-1} b_{j, N}$. Hence, Lemma 1 implies (23).

Proposition 4. For all $k \geq 0$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in$ $\mathbb{R}^{p \times m}$, and let $\Gamma_{1, k}, \ldots, \Gamma_{m, k} \in \mathbb{R}^{p \times p}$ be positive definite. Furthermore, let $\bar{\theta}_{0} \in \mathbb{R}^{n \times m}$ and let $R_{1}, \ldots, R_{m} \in \mathbb{R}^{n \times n}$ be positive definite. Then, for all $k \geq 0, J_{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$, defined in (18), has a unique minimizer, denoted as

$$
\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{k}(\hat{\theta})=\left[\begin{array}{lll}
\theta_{1, k+1} & \cdots & \theta_{m, k+1} \tag{26}
\end{array}\right]
$$

where, for all $j=1, \ldots, m, \theta_{j, k+1} \in \mathbb{R}^{n}$ is given by

$$
\begin{align*}
P_{j, k+1}^{-1} & =P_{j, k}^{-1}+\phi_{k}^{\mathrm{T}} \Gamma_{j, k} \phi_{k},  \tag{27}\\
\theta_{j, k+1} & =\theta_{j, k}+P_{j, k+1} \phi_{k}^{\mathrm{T}} \Gamma_{j, k}\left(y_{j, k}-\phi_{k} \theta_{j, k}\right) \tag{28}
\end{align*}
$$

where $y_{j, k} \in \mathbb{R}^{p}$ is defined in (20), $\theta_{j, 0} \in \mathbb{R}^{n}$ is defined (21), $P_{j, 0} \triangleq R_{j}^{-1}$, and, for all $k \geq 0$ and $j=1, \ldots, m$, $P_{j, k} \in \mathbb{R}^{n \times n}$ is positive definite, and hence nonsingular. Moreover, for all $k \geq 0$ and $j=1, \ldots, m, P_{j, k}$ can be expressed recursively as

$$
\begin{equation*}
P_{j, k+1}=P_{j, k}-P_{j, k} \phi_{k}^{\mathrm{T}}\left(\Gamma_{j, k}^{-1}+\phi_{k} P_{j, k} \phi_{k}^{\mathrm{T}}\right)^{-1} \phi_{k} P_{j, k} \tag{29}
\end{equation*}
$$

Proof. For all $j=1, \ldots, m$, let $\theta_{j, k+1} \triangleq \arg \min _{\hat{\theta}_{1} \in \mathbb{R}^{n}} J_{j, k}$. It then follows from Lemma A. 3 that, for all $j=1, \ldots, m$, (27) and (28) hold. Finally, Lemma 1 implies (23).

An advantage of independent column weighting versus vec-permutation is that no Kronecker product is used, implying that no sparse matrices are introduced. The computational complexities of BLS and RLS with independent column
weighting are shown in Tables I and II respectively. Independent column weighting results in a $\mathcal{O}\left(\mathrm{m}^{2}\right)$ times improvement in computational complexity over vec-permutation for both BLS and RLS as well as a $\mathcal{O}(m)$ times improvement in space complexity.

## IV. Independent Identical Column Weighting

Finally, we now assume there exist positive-definite $R \in$ $\mathbb{R}^{n \times n}$ such that $\bar{R}$ is block diagonal of the form

$$
\bar{R}=\left[\begin{array}{ccc}
R & \cdots & 0_{n \times n}  \tag{30}\\
\vdots & \ddots & \vdots \\
0_{n \times n} & \cdots & R
\end{array}\right]=I_{m} \otimes R .
$$

Furthermore, assume that, for all $k \geq 0$, there exist positivedefinite $\Gamma_{k} \in \mathbb{R}^{p \times p}$ such that $\bar{\Gamma}_{k}$ is block diagonal of the form

$$
\bar{\Gamma}_{k}=\left[\begin{array}{ccc}
\Gamma_{k} & \cdots & 0_{p \times p}  \tag{31}\\
\vdots & \ddots & \vdots \\
0_{p \times p} & \cdots & \Gamma_{k}
\end{array}\right]=I_{m} \otimes \Gamma_{k} .
$$

This corresponds to independent and identical weighting of the columns of $y_{i}-\phi_{i} \hat{\theta}, i=0, \ldots, k$, and $\hat{\theta}-\theta_{0}$ in (2). Then, it follows from Lemma A. 2 that, for all $k \geq 0$, (2) can be rewritten as

$$
\begin{align*}
J_{k}(\hat{\theta})=\operatorname{tr}[ & \sum_{i=0}^{k}\left(y_{i}-\phi_{i} \hat{\theta}\right)^{\mathrm{T}} \Gamma_{i}\left(y_{i}-\phi_{i} \hat{\theta}\right) \\
& \left.+\left(\theta-\theta_{0}\right)^{\mathrm{T}} R\left(\theta-\theta_{0}\right)\right] \tag{32}
\end{align*}
$$

Propositions 5 and 6 respectively give the independent identical column weighting batch and recursive least squares methods to minimize cost function $J_{k}$ given by (32).

Proposition 5. Let $N \geq 0$. For all $0 \leq k \leq N$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in \mathbb{R}^{p \times m}$, and let $\Gamma_{k} \in \mathbb{R}^{p \times p}$ be positive definite. Furthermore, let $\theta_{0} \in \mathbb{R}^{n \times m}$ and let $R \in \mathbb{R}^{n \times n}$ be positive definite. Then, $J_{N}: \mathbb{R}^{n} \rightarrow \mathbb{R}$, defined in (32), has a unique minimizer given by

$$
\begin{equation*}
\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{N}(\hat{\theta})=A_{N}^{-1} b_{N}, \tag{33}
\end{equation*}
$$

where

$$
\begin{align*}
& A_{N} \triangleq R+\sum_{i=0}^{N} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}  \tag{34}\\
& b_{N} \triangleq R \theta_{0}+\sum_{i=0}^{N} \phi_{i}^{\mathrm{T}} \Gamma_{i} y_{i} . \tag{35}
\end{align*}
$$

Proof. Note that (32) can be rewritten as (18) where, for all $k \geq 0$ and $j=1, \ldots, m, \Gamma_{j, k}=\Gamma_{k}$ and $R_{j}=R$. It then follows from Proposition 3 that $\arg \min _{\hat{\theta} \in \mathbb{R}^{n \times m}} J_{N}(\hat{\theta})=$ $\left[\begin{array}{lll}A_{N}^{-1} b_{1, N} & \cdots & A_{N}^{-1} b_{m, N}\end{array}\right]=A_{N}^{-1}\left[\begin{array}{lll}b_{1, N} & \cdots & b_{m, N}\end{array}\right]$, where $A_{N} \in \mathbb{R}^{n \times n}$ is defined in (34) and $b_{j, N}$ is defined in (25). Finally, note that $b_{N}=\left[b_{1, N} \cdots b_{m, N}\right]$, thus yielding (33).

Proposition 6. For all $k \geq 0$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in$ $\mathbb{R}^{p \times m}$, and let $\Gamma_{k} \in \mathbb{R}^{p \times p}$ be positive definite. Furthermore, let $\theta_{0} \in \mathbb{R}^{n \times m}$ and let $R \in \mathbb{R}^{n \times n}$ be positive definite. Then, for all $k \geq 0, J_{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$, defined in (32), has a unique minimizer, denoted as

$$
\begin{equation*}
\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{k}(\hat{\theta})=\theta_{k+1} \tag{36}
\end{equation*}
$$

Moreover, for all $k \geq 0, \theta_{k+1} \in \mathbb{R}^{n \times m}$ is given by

$$
\begin{align*}
P_{k+1}^{-1} & =P_{k}^{-1}+\phi_{k}^{\mathrm{T}} \Gamma_{k} \phi_{k}  \tag{37}\\
\theta_{k+1} & =\theta_{k}+P_{k+1} \phi_{k}^{\mathrm{T}} \Gamma_{k}\left(y_{k}-\phi_{k} \theta_{k}\right) \tag{38}
\end{align*}
$$

where $P_{0} \triangleq R^{-1}$ and, for all $k \geq 0, P_{k} \in \mathbb{R}^{n \times n}$ is positive definite, hence nonsingular. Finally, for all $k \geq 0, P_{k+1}$ can be expressed recursively as

$$
\begin{equation*}
P_{k+1}=P_{k}-P_{k} \phi_{k}^{\mathrm{T}}\left(\Gamma_{k}^{-1}+\phi_{k} P_{k} \phi_{k}^{\mathrm{T}}\right)^{-1} \phi_{k} P_{k} \tag{39}
\end{equation*}
$$

Proof. Note that (32) can be rewritten as (18) where, for all $k \geq 0$ and $j=1, \ldots, m, \Gamma_{j, k}=\Gamma_{k}$ and $R_{j}=R$. It then follows from Proposition 4 that, for all $k \geq 0$,

$$
\underset{\hat{\theta} \in \mathbb{R}^{n \times m}}{\arg \min } J_{k}(\hat{\theta})=\left[\begin{array}{lll}
\theta_{1, k+1} & \cdots & \theta_{m, k+1}
\end{array}\right]
$$

where, for all $j=1, \ldots, m, \theta_{j, k+1} \in \mathbb{R}^{n}$ is given by

$$
\begin{aligned}
P_{k+1}^{-1} & =P_{k}^{-1}+\phi_{k}^{\mathrm{T}} \Gamma_{k} \phi_{k} \\
\theta_{j, k+1} & =\theta_{j, k}+P_{k+1} \phi_{k}^{\mathrm{T}} \Gamma_{k}\left(y_{j, k}-\phi_{k} \theta_{j, k}\right)
\end{aligned}
$$

Applying (20) and (22) then implies (37) and (38).
Note that in independent identical column weighting, the same covariance matrix is used for each column of $\theta_{k}$ versus independent column weighting which computes a separate covariance matrix for each column of parameters $\theta_{j, k}, j=$ $1, \ldots, m$. The computational complexities of BLS and RLS with independent identical column weighting are shown in Tables I and II respectively. Independent column weighting results in a $\mathcal{O}\left(m^{3}\right)$ times improvement in computational complexity over vec-permutation as well as a $\mathcal{O}\left(m^{2}\right)$ times improvement in space complexity over vec-permutation.

Figure 1 shows numerical testing of BLS and RLS with vec-permutation, independent column weighting, and independent identical column weighting. We consider the measurement process (1) with $p=10, n=50$, and $1 \leq m \leq 20$. For larger values of $m$, we see significantly faster computation time for independent identical column weighting over vec-permutation and independent column weighting.

## A. Convergence of Matrix RLS

It is well-known that in standard RLS, the parameter estimate vector converges to the vector of true parameters if the sequence of regressors $\left(\phi_{k}\right)_{k=0}^{\infty}$ is persistently exciting [17]-[19]. Theorem 1 extends this result to matrix RLS. The following definition is from page 64 of [1].


Fig. 1: Consider the measurement process (1) with $p=10, n=50$, and $1 \leq m \leq 20$. Batch least squares (top) shows computation time with $N=100$ data points, averaged over 10 trials. Recursive least squares (bottom) shows computation time per step, averaged over 100 trials. Error bars show the $95 \%$ confidence intervals.

Definition 1. $\left(\phi_{k}\right)_{k=0}^{\infty} \subset \mathbb{R}^{p \times n}$ with weight $\left(\Gamma_{k}\right)_{k=0}^{\infty}$ is persistently exciting (PE) if

$$
\begin{equation*}
C \triangleq \lim _{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i} \in \mathbb{R}^{n \times n} \tag{40}
\end{equation*}
$$

exists and is positive definite.
Theorem 1. Let $\theta, \theta_{0} \in \mathbb{R}^{n \times m}$ and let $R \in \mathbb{R}^{n \times n}$ be positive definite. For all $k \geq 0$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in \mathbb{R}^{p \times m}$ be given by (1), let $\Gamma_{k} \in \mathbb{R}^{p \times p}$ be positive definite, and let $P_{k} \in \mathbb{R}^{n \times n}$ and $\theta_{k} \in \mathbb{R}^{n \times m}$ be given by (37) and (38), respectively. Assume that $\left(\phi_{k}\right)_{k=0}^{\infty}$ with weight $\left(\Gamma_{k}\right)_{k=0}^{\infty}$ is $P E$, and define $C \in \mathbb{R}^{n \times n}$ by (40). Then,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} k\left(\theta_{k}-\theta\right)=C^{-1} R\left(\theta_{0}-\theta\right) \tag{41}
\end{equation*}
$$

Proof. Note that

$$
\begin{aligned}
\theta_{k} & =\left(R+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}\right)^{-1}\left(R \theta_{0}+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} y_{i}\right) \\
& =\left(R+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}\right)^{-1}\left(R \theta_{0}-R \theta+R \theta+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i} \theta\right) \\
& =\left(R+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}\right)^{-1}\left[R\left(\theta_{0}-\theta\right)+\left(R+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}\right) \theta\right] \\
& =\left(R+\sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}\right)^{-1} R\left(\theta_{0}-\theta\right)+\theta
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\lim _{k \rightarrow \infty} k\left(\theta_{k}-\theta\right) & =\lim _{k \rightarrow \infty}\left(\frac{1}{k} R+\frac{1}{k} \sum_{i=0}^{k-1} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}\right)^{-1} R\left(\theta_{0}-\theta\right) \\
& =C^{-1} R\left(\theta_{0}-\theta\right)
\end{aligned}
$$

TABLE I: Batch identification summary and computational complexities for $N$ measurements

|  | Algorithm | $\mathcal{O}$ Comp. Complexity (No Assumptions) | Comp. Complexity <br> $(N>n \geq p)$ | Comp. Complexity <br> $(N>p \geq n)$ |
| ---: | :--- | :--- | :--- | :--- |
| Vec-Permutation | $(9),(10),(11)$ | $\mathcal{O}\left(\max \left\{N p n m^{3} \max \{p, n\}, n^{3} m^{3}\right\}\right)$ | $\mathcal{O}\left(N p n^{2} m^{3}\right)$ | $\mathcal{O}\left(N p^{2} n m^{3}\right)$ |
| Indep. Column Weight | $(23),(24),(25)$ | $\mathcal{O}\left(\max \left\{N p n m \max \{p, n\}, n^{3} m\right\}\right)$ | $\mathcal{O}\left(N p n^{2} m\right)$ | $\mathcal{O}\left(N p^{2} n m\right)$ |
| Indep. Iden. Column Weight | $(33),(34),(35)$ | $\mathcal{O}\left(\max \left\{N p \max \{n, m\} \max \{p, n\}, n^{3}\right\}\right)$ | $\mathcal{O}(N p n \max \{n, m\})$ | $\mathcal{O}\left(N p^{2} \max \{n, m\}\right)$ |

TABLE II: Recursive identification summary and computational/space complexities per step

|  | Algorithm <br> $(n \gg p)$ | Comp. Complexity <br> $(n \gg)$ | Algorithm <br> $(p \geq n)$ | Comp. Complexity <br> $(p \geq n)$ | Number of Parameters <br> in Memory |
| ---: | :---: | :--- | :---: | :--- | :--- |
| Vec-Permutation | $(15),(14)$ | $\mathcal{O}\left(p n^{2} m^{3}\right)$ | $(13),(14)$ | $\mathcal{O}\left(p^{2} n m^{3}\right)$ | $n^{2} m^{2}+n m$ |
| Indep. Column Weight | $(29),(28)$ | $\mathcal{O}\left(p n^{2} m\right)$ | $(27),(28)$ | $\mathcal{O}\left(p^{2} n m\right)$ | $n^{2} m+n m$ |
| Indep. Iden. Column Weight | $(39),(38)$ | $\mathcal{O}(p n \max \{n, m\})$ | $(37),(38)$ | $\mathcal{O}\left(p^{2} \max \{n, m\}\right)$ | $n^{2}+n m$ |

## V. Application to Online Identification for

 Indirect Adaptive Model Predictive ControlConsider a MIMO input-output system of the form

$$
\begin{equation*}
y_{k}=-\sum_{i=1}^{\hat{n}} F_{i} y_{k-i}+\sum_{i=0}^{\hat{n}} G_{i} u_{k-i}, \tag{42}
\end{equation*}
$$

where $k \geq 0$ is the time step, $\hat{n}$ is the model order, $u_{k} \in \mathbb{R}^{m}$ is the control, $y_{k} \in \mathbb{R}^{p}$ is the measurement, and $F_{1}, \ldots, F_{\hat{n}} \in$ $\mathbb{R}^{p \times p}$ and $G_{0}, \ldots, G_{\hat{n}} \in \mathbb{R}^{p \times m}$ are the system coefficient matrices to be estimated. A model of the form (42) is identified online in the indirect adaptive model predictive control scheme: predictive cost adaptive control (PCAC) [4]. For all $k \geq 0$, the system coefficient matrices are estimated by minimizing the cost function $J_{k}: \mathbb{R}^{p \times \hat{n}(m+p)+m} \rightarrow \mathbb{R}$, defined as

$$
\begin{equation*}
J_{k}(\hat{\theta})=\sum_{i=0}^{k} z_{i}^{\mathrm{T}}(\hat{\theta}) z_{i}(\hat{\theta})+\operatorname{vec}\left(\hat{\theta}-\theta_{0}\right)^{\mathrm{T}} \bar{P}_{0}^{-1} \operatorname{vec}\left(\hat{\theta}-\theta_{0}\right), \tag{43}
\end{equation*}
$$

where $z_{k}: \mathbb{R}^{p \times \hat{n}(m+p)+m} \rightarrow \mathbb{R}^{p}$ is defined

$$
\begin{equation*}
z_{k}(\hat{\theta}) \triangleq y_{k}+\sum_{i=1}^{\hat{n}} \hat{F}_{i} y_{k-i}-\sum_{i=0}^{\hat{n}} \hat{G}_{i} u_{k-i}, \tag{44}
\end{equation*}
$$

$\hat{\theta} \in \mathbb{R}^{p \times \hat{n}(m+p)+m}$ are the coefficients to be estimated, defined

$$
\hat{\theta} \triangleq\left[\begin{array}{llllll}
\hat{F}_{1} & \cdots & \hat{F}_{\hat{n}} & \hat{G}_{0} & \cdots & \hat{G}_{\hat{n}} \tag{45}
\end{array}\right],
$$

and where $\theta_{0} \in \mathbb{R}^{p \times \hat{n}(m+p)+m}$ is an initial guess of the coefficients and $\bar{P}_{0} \in \mathbb{R}^{[\hat{n} p(m+p)+m p] \times[\hat{n} p(m+p)+m p]}$ is positive definite. Note that, for all $k \geq 0, z_{k}(\hat{\theta})$ can be written as

$$
\begin{equation*}
z_{k}(\hat{\theta})=y_{k}-\hat{\theta} \phi_{k}, \tag{46}
\end{equation*}
$$

where $\phi_{k} \in \mathbb{R}^{\hat{n}(m+p)+m}$ is defined as

$$
\phi_{k} \triangleq\left[\begin{array}{lllll}
-y_{k-1}^{\mathrm{T}} & \cdots-y_{k-\hat{n}}^{\mathrm{T}} & u_{k}^{\mathrm{T}} & \cdots & u_{k-\hat{n}}^{\mathrm{T}} \tag{47}
\end{array}\right]^{\mathrm{T}} .
$$

Further defining $\bar{\phi}_{k} \in \mathbb{R}^{p \times \hat{n} p(m+p)+m p}$ as

$$
\begin{equation*}
\bar{\phi}_{k} \triangleq \phi_{k}^{\mathrm{T}} \otimes I_{p}, \tag{48}
\end{equation*}
$$

it follows that $z_{k}(\hat{\theta})$ can be written as

$$
\begin{equation*}
z_{k}(\hat{\theta})=y_{k}-\bar{\phi}_{k} \operatorname{vec}(\hat{\theta}) \tag{49}
\end{equation*}
$$

where $\operatorname{vec}(\hat{\theta}) \in \mathbb{R}^{\hat{n} p(m+p)+m p}$ is the vectorization of $\hat{\theta}$. Using (49), we derive the identification algorithm used in [4].

Proposition 7. For all $k \geq 0$, let $u_{k} \in \mathbb{R}^{m}$, $y_{k} \in$ $\mathbb{R}^{p}$. Furthermore, let $\theta_{0} \in \mathbb{R}^{p \times \hat{n}(m+p)+m}$ and let $\bar{P}_{0} \in$ $\mathbb{R}^{[\hat{n} p(m+p)+m p] \times[\hat{n} p(m+p)+m p]}$ be positive definite. Then, for all $k \geq 0$, $J_{k}$, defined in (43), has a unique global minimizer, denoted

$$
\begin{equation*}
\theta_{k+1} \triangleq \underset{\hat{\theta} \in \mathbb{R}^{p \times \hat{n}(m+p)+m}}{\arg \min } J_{k}(\hat{\theta}), \tag{50}
\end{equation*}
$$

which is given by

$$
\begin{align*}
\bar{P}_{k+1} & =\bar{P}_{k}-\bar{P}_{k} \bar{\phi}_{k}^{\mathrm{T}}\left(I_{p}+\bar{\phi}_{k} \bar{P}_{k} \bar{\phi}_{k}^{\mathrm{T}}\right)^{-1} \bar{\phi}_{k} \bar{P}_{k},  \tag{51}\\
\operatorname{vec}\left(\theta_{k+1}\right) & =\operatorname{vec}\left(\theta_{k}\right)+\bar{P}_{k+1} \bar{\phi}_{k}^{\mathrm{T}}\left(y_{k}-\bar{\phi}_{k} \operatorname{vec}\left(\theta_{k}\right)\right) . \tag{52}
\end{align*}
$$

Proof. This result follows from Lemma A.3. For further details, see equations (8) through (20) of [4].

Next, we provide an alternate formulation using matrix RLS.

Proposition 8. Consider the notation and assumptions of Proposition 7. If there exists $P_{0} \in \mathbb{R}^{[\hat{n}(m+p)+m] \times[\hat{n}(m+p)+m]}$ such that $\bar{P}_{0}=P_{0} \otimes I_{p}$, then, for all $k \geq 0, \theta_{k+1} \in$ $\mathbb{R}^{p \times \hat{n}(m+p)+m}$ is given by

$$
\begin{align*}
P_{k+1} & =P_{k}-\frac{P_{k} \phi_{k} \phi_{k}^{\mathrm{T}} P_{k}}{1+\phi_{k}^{\mathrm{T}} P_{k} \phi_{k}},  \tag{53}\\
\theta_{k+1} & =\theta_{k}+\left(y_{k}-\theta_{k} \phi_{k}\right) \phi_{k}^{\mathrm{T}} P_{k+1} . \tag{54}
\end{align*}
$$

Proof. Note that, for all $k \geq 0, z_{k}(\hat{\theta})^{\mathrm{T}} z_{k}(\hat{\theta})=$ $\operatorname{tr}\left(z_{k}(\hat{\theta})^{\mathrm{T}} z_{k}(\hat{\theta})\right)=\operatorname{tr}\left(z_{k}(\hat{\theta}) z_{k}(\hat{\theta})^{\mathrm{T}}\right)$. It then follows from (57) of Lemma A. 2 that, for all $k \geq 0$, (43) can be written as

$$
J_{k}(\hat{\theta})=\operatorname{tr}\left[\sum_{i=0}^{k} z_{i}(\hat{\theta}) z_{i}(\hat{\theta})^{\mathrm{T}}+\left(\hat{\theta}-\theta_{0}\right) P_{0}^{-1}\left(\hat{\theta}-\theta_{0}\right)^{\mathrm{T}}\right] .
$$

Proposition 5 then implies that, for all $k \geq 0$, (53) and $\theta_{k+1}^{\mathrm{T}}=\theta_{k}^{\mathrm{T}}+P_{k+1} \phi_{k}\left(y_{k}^{\mathrm{T}}-\phi_{k}^{\mathrm{T}} \theta_{k}^{\mathrm{T}}\right)$ hold. Taking the transpose then yields (54).

Example 1. This example is from [10] and uses PCAC for the control of a flexible structure under harmonic and
broadband disturbances. Consider the 4-bay truss show in Figure 2 made of flexible truss elements with unknown mass and stiffness. Two actuators are placed at nodes 3 and 4 with control authority in the $x$-direction and $x$-direction displacement sensors are placed at nodes $5,6,7$, and 8 . The objective is to use PCAC to suppress the effects of harmonic and broadband disturbances without prior knowledge of the truss dynamics. See [10] for further details.

This example has inputs $u_{k} \in \mathbb{R}^{2}$ and outputs $y_{k} \in \mathbb{R}^{4}$. Online identification was done in [10] using vec-permutation, given by (51) and (52), with identity regularization. We replicated the results of [10] using matrix RLS, given by (53) and (54). Table III shows that using matrix RLS resulted in a $97.6 \%$ decrease in computation time needed for system identification per step. Moreover, since system identification is a significant part of PCAC, Table III also shows a $21.4 \%$ decrease in total computation time per step.


Fig. 2: Flexible truss structure from [10] with nodes labeled.
TABLE III: Truss ex. computation time per step: mean $\pm$ std. deviation

|  | Vec-Permutation | Matrix RLS | Change |
| ---: | :--- | :--- | :--- |
| ID Time per Step | $(5.9 \pm 0.5) \mathrm{ms}$ | $(0.14 \pm 0.03) \mathrm{ms}$ | $-97.6 \%$ |
| Total Time per Step | $(28 \pm 6) \mathrm{ms}$ | $(22 \pm 6) \mathrm{ms}$ | $-21.4 \%$ |

## VI. Conclusions

This work derives batch and recursive least squares algorithms for the identification of matrix parameters. Under the assumption of independent, identical column weighting, these methods minimize the same cost function as the vecpermutation approach. It is also shown how, under persistent excitation, convergence guarantees can be extended from the vector case to the matrix case. This approach can be used fast online identification of MIMO systems which is critical in indirect adaptive model predictive control.

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## APPENDIX

Lemma A.1. Let $A \in \mathbb{R}^{k \times l}$ and $B \in \mathbb{R}^{l \times m}$. Then,

$$
\begin{equation*}
\operatorname{vec}(A B)=\left(I_{m} \otimes A\right) \operatorname{vec}(B)=\left(B^{\mathrm{T}} \otimes I_{k}\right) \operatorname{vec}(A) \tag{55}
\end{equation*}
$$

Lemma A.2. Let $x \in \mathbb{R}^{n \times m}$, let $A \in \mathbb{R}^{n \times n}$, and let $B \in$ $\mathbb{R}^{m \times m}$. Then,

$$
\begin{align*}
& \operatorname{vec}(x)^{\mathrm{T}}\left(I_{m} \otimes A\right) \operatorname{vec}(x)=\operatorname{tr}\left(x^{\mathrm{T}} A x\right)  \tag{56}\\
& \operatorname{vec}(x)^{\mathrm{T}}\left(B \otimes I_{n}\right) \operatorname{vec}(x)=\operatorname{tr}\left(x B x^{\mathrm{T}}\right) \tag{57}
\end{align*}
$$

Proof. Note that by Lemma A.1, $\operatorname{vec}(x)^{\mathrm{T}}\left(I_{m} \otimes A\right) \operatorname{vec}(x)=$ $\operatorname{vec}(x)^{\mathrm{T}} \operatorname{vec}(A x)=\operatorname{tr}\left(x^{\mathrm{T}} A x\right)$, proving (56). Next, by Lemma A.1, $\operatorname{vec}(x)^{\mathrm{T}}\left(B \otimes I_{n}\right) \operatorname{vec}(x)=$ $\operatorname{vec}(x)^{\mathrm{T}} \operatorname{vec}\left(x B^{\mathrm{T}}\right)=\operatorname{tr}\left(x^{\mathrm{T}} x B^{\mathrm{T}}\right)=\operatorname{tr}\left(x B^{\mathrm{T}} x^{\mathrm{T}}\right)=$ $\operatorname{tr}\left(x B x^{\mathrm{T}}\right)$.

Lemma A.3. For all $k \geq 0$, let $\phi_{k} \in \mathbb{R}^{p \times n}$, let $y_{k} \in \mathbb{R}^{p}$, and let $\Gamma_{i} \in \mathbb{R}^{p \times p}$ be positive definite. Furthermore, let $\theta_{0} \in \mathbb{R}^{n}$ and let $P_{0} \in \mathbb{R}^{n \times n}$ be positive definite. For all $k \geq 0$, define function $J_{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ as $J_{k}(\hat{\theta})=\sum_{i=0}^{k}\left(y_{i}-\phi_{k} \hat{\theta}\right)^{\mathrm{T}} \Gamma_{i}\left(y_{i}-\right.$ $\left.\phi_{i} \hat{\theta}\right)+\left(\hat{\theta}-\theta_{0}\right)^{\mathrm{T}} P_{0}^{-1}\left(\hat{\theta}-\theta_{0}\right)$. Then, for all $k \geq 0, J_{k}$ has
a unique minimizer, denoted as $\theta_{k+1} \triangleq \arg \min _{\hat{\theta} \in \mathbb{R}^{n}} J_{k}(\hat{\theta})$. Furthermore, for all $k \geq 0$,

$$
\begin{equation*}
\theta_{k+1}=A_{k}^{-1} b_{k} \tag{58}
\end{equation*}
$$

where $A_{k} \triangleq P_{0}^{-1}+\sum_{i=0}^{k} \phi_{i}^{\mathrm{T}} \Gamma_{i} \phi_{i}$ and $b_{k} \triangleq P_{0}^{-1} \theta_{0}+$ $\sum_{i=0}^{k} \phi_{i}^{\mathrm{T}} \Gamma_{i} y_{i}$. Moreover, for all $k \geq 0, \theta_{k+1}$ can be expressed recursively as

$$
\begin{align*}
P_{k+1}^{-1} & =P_{k}^{-1}+\phi_{k}^{\mathrm{T}} \Gamma_{k} \phi_{k}  \tag{59}\\
\theta_{k+1} & =\theta_{k}+P_{k+1} \phi_{k}^{\mathrm{T}} \Gamma_{k}\left(y_{k}-\phi_{k} \theta_{k}\right) \tag{60}
\end{align*}
$$

where, for all $k \geq 0, P_{k} \in \mathbb{R}^{n \times n}$ is positive definite, hence nonsingular. Finally, for all $k \geq 0, P_{k+1}$ can be expressed recursively as

$$
\begin{equation*}
P_{k+1}=P_{k}-P_{k} \phi_{k}^{\mathrm{T}}\left(\Gamma_{k}^{-1}+\phi_{k} P_{k} \phi_{k}^{\mathrm{T}}\right)^{-1} \phi_{k} P_{k} \tag{61}
\end{equation*}
$$

Proof. See [3].


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[^1]:    ${ }^{1}$ Note that since the measurement, regressor, and parameters are all matrices, the results of this work can be easily extended to measurements processes of the form $y_{k}=\theta \phi_{k}$ by rewriting as $y_{k}^{\mathrm{T}}=\phi_{k}^{\mathrm{T}} \theta^{\mathrm{T}}$ and identifying parameters $\theta^{\mathrm{T}}$. For brevity, we leave the details to the reader.

