On the Li–Zheng theorem

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Abstract. By the well-known I. Kotlarski lemma, if ξ_1, ξ_2, and ξ_3 are independent real-valued random variables with nonvanishing characteristic functions, $L_1 = \xi_1 - \xi_3$ and $L_2 = \xi_2 - \xi_3$, then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j up to shift. Siran Li and Xunjie Zheng generalized this result for the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$ assuming that all ξ_j have first and second moments, ξ_2 and ξ_3 are identically distributed, and a_j, b_j satisfy some conditions. In the article, we give a simpler proof of this theorem. In doing so, we also prove that the condition of existence of moments can be omitted. Moreover, we prove an analogue of the Li–Zheng theorem for independent random variables with values in the field of p-adic numbers, in the field of integers modulo p, where $p \neq 2$, and in the discrete field of rational numbers.

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1. Introduction

In the article [5] dedicated to the characterization of the gamma and the Gaussian distribution I. Kotlarski proved the following lemma: Let ξ_1, ξ_2, and ξ_3 be independent real-valued random variables with nonvanishing characteristic functions, and let $L_1 = \xi_1 - \xi_3$ and $L_2 = \xi_2 - \xi_3$. Then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j up to shift. The characterization theorems proved in [5] remained known only to mathematicians who deal with characterization problems of mathematical statistics. At the same time, a number of studies that are far from characterization problems are based on Kotlarski's lemma. See, e.g., [6], where numerous articles with references to Kotlarski's lemma including in economics literature, are mentioned. We especially pay attention to the important article by C.R. Rao [7]. In particular, he considered n independent real-valued random variables ξ_j with nonvanishing characteristic functions, the linear forms $L_1 = a_1\xi_1 + \cdots + a_n\xi_n$, $L_2 = b_1\xi_1 + \cdots + b_n\xi_n$ and proved that under some natural conditions on the coefficients a_j, b_j the characteristic function of the random vector (L_1, L_2) determines the characteristic functions of the random variables ξ_j up to factors of the form $\exp\{P_j(y)\}$, where $P_j(y)$ is a polynomial of degree at most $n-2$. Kotlarski's lemma follows from this Rao theorem. Note also that some generalizations of Rao's theorem were studied in [1], see also [2, §15], for locally compact Abelian groups. The proof of Rao's theorem is based on the following statement on solutions of a functional equation.

Lemma 1.1. *Let a_j, b_j, $j = 1, 2, \dots, n$, be nonzero real numbers such that $a_ib_j \neq a_jb_i$ for all $i \neq j$. Consider the equation*

$$\sum_{j=1}^{n} \psi_j(a_ju + b_jv) = A(u) + B(v), \quad u, v \in \mathbb{R},$$

where $\psi_j(y)$, $A(y)$, and $B(y)$ are continuous complex-valued functions on $\mathbb{R}$. Then $\psi_j(y)$ are polynomial on $\mathbb{R}$ of degree at most n.

This lemma is well known. In fact, the lemma was first used, although it was not explicitly formulated, in the works by Skitovich and Darmois, where the Gaussian distribution on the real line is characterized by the independence of two linear forms of n independent random variables. Two different proofs of the lemma and some generalizations can be found in [4, §1.5].

In article [6], Siran Li and Xunjie Zheng proved the following statement.

Li–Zheng theorem. *Let U and V be random variables with nonvanishing characteristic functions. Assume that*

$$U = X + aZ_1 + bZ_2, \quad V = Y + cZ_1 + dZ_2,$$

where X, Y, Z_1, and Z_2 are independent random variables with well-defined first and second moments, Z_1 and Z_2 are identically distributed, and a, b, c, and d are nonzero real constants which are known. Suppose that $ac \neq -bd$ and $(a, c) \neq (-b, -d)$. Then, the joint distribution of (U, V) uniquely determines the distributions of X, Y, Z_1, and Z_2 up to a change of location.

Our article consists of three parts. In the first part, we give a simpler proof of the Li–Zheng theorem and show that it essentially follows from Lemma 1.1. In doing so, we also prove that the condition of existence of moments of the random variables X, Y, Z_1, and Z_2 can be omitted. In the second part, we prove an analogue of the Li–Zheng theorem for independent random variables with values in the field of p-adic numbers. In the third part, we prove an analogue of the Li–Zheng theorem for independent random variables with values in the field of integers modulo p, where $p \neq 2$, and in the discrete field of rational numbers. For the proof of the corresponding theorems we solve some functional equations on the character group of the additive group of the field.

2. Real-valued random variables

Let us formulate the Li–Zheng theorem in more familiar for us notation. In doing so, we omit the condition of existence of moments of independent random variables.

Theorem 2.1. *Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent real-valued random variables with nonvanishing characteristic functions. Let a_j, b_j, $j = 2, 3$, be nonzero real numbers such that $a_2b_2 \neq -a_3b_3$ and $(a_2, b_2) \neq (-a_3, -b_3)$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. If the random variables ξ_2 and ξ_3 are identically distributed, then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift.*

Proof. 1. Let η_1, η_2, η_3, and η_4 be independent real-valued random variables with nonvanishing characteristic functions. Assume that η_2 and η_3 are identically distributed. Denote by μ_j and ν_j the distributions of the random variables ξ_j and η_j and by $\hat{\mu}_j(y)$ and $\hat{\nu}_j(y)$ their characteristic functions. Put $M_1 = \eta_1 + a_2\eta_2 + a_3\eta_3$ and $M_2 = b_2\eta_2 + b_3\eta_3 + \eta_4$. Suppose that the distributions of the random vectors (L_1, L_2) and (M_1, M_2) coincide. Taking into account that the random variables ξ_j are independent and $\hat{\mu}_j(y) = \mathbf{E}\left[e^{i\xi_j y}\right]$, the characteristic function of the random vector (L_1, L_2) can be represented in the form

$$\begin{aligned}\mathbf{E}\left[e^{i(L_1u+L_2v)}\right] &= \mathbf{E}\left[e^{i((\xi_1+a_2\xi_2+a_3\xi_3)u+(b_2\xi_2+b_3\xi_3+\xi_4)v)}\right]\\ &= \mathbf{E}\left[e^{i\xi_1u}e^{i\xi_2(a_2u+b_2v)}e^{i\xi_3(a_3u+b_3v)}e^{i\xi_4v}\right] = \mathbf{E}\left[e^{i\xi_1u}\right]\mathbf{E}\left[e^{i\xi_2(a_2u+b_2v)}\right]\mathbf{E}\left[e^{i\xi_3(a_3u+b_3v)}\right]\\ &\quad\times \mathbf{E}\left[e^{i\xi_4v}\right] = \hat{\mu}_1(u)\hat{\mu}_2(a_2u+b_2v)\hat{\mu}_3(a_3u+b_3v)\hat{\mu}_4(v), \quad u, v \in \mathbb{R}. \quad (1)\end{aligned}$$

Analogically, the characteristic function of the random vector (M_1, M_2) is of the form

$$\mathbf{E}\left[e^{i(M_1u+M_2v)}\right] = \hat{\nu}_1(u)\hat{\nu}_2(a_2u+b_2v)\hat{\nu}_3(a_3u+b_3v)\hat{\nu}_4(v), \quad u, v \in \mathbb{R}. \tag{2}$$

It follows from (1) and (2) that the random vectors (L_1, L_2) and (M_1, M_2) have the same characteristic functions and hence they are identically distributed if and only if the characteristic functions $\hat{\mu}_j(y)$ and $\hat{\nu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(u)\hat{\mu}_2(a_2u+b_2v)\hat{\mu}_3(a_3u+b_3v)\hat{\mu}_4(v) = \hat{\nu}_1(u)\hat{\nu}_2(a_2u+b_2v)\hat{\nu}_3(a_3u+b_3v)\hat{\nu}_4(v), \quad u, v \in \mathbb{R}. \tag{3}$$

Set

$$f_j(y) = \hat{\nu}_j(y)/\hat{\mu}_j(y), \quad \psi_j(y) = \ln f_j(y), \quad j = 1, 2, 3, 4. \tag{4}$$

Since the characteristic functions $\hat{\mu}_j(y)$ and $\hat{\nu}_j(y)$ do not vanish, (3) is equivalent to the fact that the functions $f_j(y)$ satisfy the equation

$$f_1(u)f_2(a_2u+b_2v)f_3(a_3u+b_3v)f_4(v) = 1, \quad u, v \in \mathbb{R}. \tag{5}$$

Note that we obtained equation (5) without assuming that the random variables ξ_2 and ξ_3 and also η_2 and η_3 are identically distributed.

It follows from (5) that the functions $\psi_j(y)$ satisfy the equation

$$\psi_1(u) + \psi_2(a_2u+b_2v) + \psi_3(a_3u+b_3v) + \psi_4(v) = 0, \quad u, v \in \mathbb{R}. \tag{6}$$

Rewrite this equation in the form

$$\psi_2(a_2u+b_2v) + \psi_3(a_3u+b_3v) = A(u) + B(v), \quad u, v \in \mathbb{R}. \tag{7}$$

2. First assume that $a_2b_3 \neq a_3b_2$. Then by Lemma 1.1, the functions $\psi_2(y)$ and $\psi_3(y)$ are polynomial of degree at most 2. Taking into account that $\psi_2(0) = \psi_3(0) = 0$, we have

$$\psi_j(y) = \sigma_j y^2 + \beta_j y, \quad y \in \mathbb{R}, \ j = 2, 3, \tag{8}$$

where σ_j, β_j are complex numbers. Substituting (8) into (6) and setting first $v = 0$ and then $u = 0$ in the obtained equation, we infer

$$\psi_j(y) = \sigma_j y^2 + \beta_j y, \quad y \in \mathbb{R}, \ j = 1, 4, \tag{9}$$

where σ_j, β_j are complex numbers. Substitute (8) and (9) into (6). We get from the received equation

$$\sigma_1u^2 + \sigma_2(a_2u+b_2v)^2 + \sigma_3(a_3u+b_3v)^2 + \sigma_4v^2 = 0, \quad u, v \in \mathbb{R}. \tag{10}$$

It follows from (10) that

$$\sigma_2a_2b_2 + \sigma_3a_3b_3 = 0. \tag{11}$$

Since ξ_2 and ξ_3 are identically distributed and η_2 and η_3 are also identically distributed, we have $\hat{\mu}_2(y) = \hat{\mu}_3(y)$ and $\hat{\nu}_2(y) = \hat{\nu}_3(y)$, $y \in \mathbb{R}$. Hence $\psi_2(y) = \psi_3(y)$, $y \in \mathbb{R}$. This implies in particular that $\sigma_2 = \sigma_3$. Inasmuch as $a_2b_2 + a_3b_3 \neq 0$ by the conditions of the theorem, we get from (11) that $\sigma_2 = \sigma_3 = 0$. Then it follows from (10) that $\sigma_1 = \sigma_4 = 0$. Thus

$$\psi_j(y) = \beta_j y, \quad y \in \mathbb{R}, \ j = 1, 2, 3, 4. \tag{12}$$

In view of (4), $\psi_j(-y) = \overline{\psi_j(y)}$ for all $y \in \mathbb{R}$ and (12) implies that $\beta_j = i\alpha_j$, where α_j are real numbers. Hence $f_j(y) = e^{i\alpha_j y}$. Thus we proved that

$$\hat{\nu}_j(y) = \hat{\mu}_j(y)e^{i\alpha_j y}, \quad y \in \mathbb{R}.$$

It follows from this that

$$\nu_j = \mu_j * E_{\alpha_j}, \quad j = 1, 2, 3, 4,$$

where E_{α_j} is the degenerate distribution concentrated at the point α_j. So, if $a_2b_3 \neq a_3b_2$, the theorem is proved.

3. Assume now that $a_2b_3 = a_3b_2$. In this case, we can not apply Lemma 1.1 for solving equation (7), but equation (7) can be easily solved directly.

Since ξ_2 and ξ_3 are identically distributed and η_2 and η_3 are also identically distributed, we have $\psi_2(y) = \psi_3(y)$, $y \in \mathbb{R}$. Put

$$\psi(y) = \psi_2(y) = \psi_3(y). \tag{13}$$

In view of $a_2b_3 = a_3b_2$, set $c = b_2/a_2 = b_3/a_3$. Inasmuch as $a_2u + b_2v = a_2(u + cv)$ and $a_3u + b_3v = a_3(u + cv)$, it is easy to see that equation (7) can be rewritten in the form

$$\psi(a_2(u + cv)) + \psi(a_3(u + cv)) = \psi(a_2u) + \psi(a_3u) + \psi(a_2cv) + \psi(a_3cv), \quad u, v \in \mathbb{R}. \tag{14}$$

Let $a_2 = a_3$. Then (14) implies that $\psi(y)$ is a homogeneous linear function. If we consider this fact, it follows from (6) and (13) that $\psi_1(y)$ and $\psi_4(y)$ are also homogeneous linear functions, i.e., (12) is fulfilled. As noted in the proof of the final part of item 2, the statement of the theorem follows from this.

Let $a_2 \neq a_3$. Put

$$\varphi(y) = \psi(a_2y) + \psi(a_3y).$$

It follows from (14) that the function $\varphi(y)$ satisfies the equation

$$\varphi(u + v) = \varphi(u) + \varphi(v), \quad u, v \in \mathbb{R}.$$

Hence there is a complex number a such that $\varphi(y) = ay$ for all $y \in \mathbb{R}$. Consider the function

$$\gamma(y) = \psi(y) - by, \quad y \in \mathbb{R}, \tag{15}$$

where $b = \frac{a}{a_2+a_3}$. Then we have

$$\gamma(a_2y) + \gamma(a_3y) = \psi(a_2y) - ba_2y + \psi(a_3y) - ba_3y = \varphi(y) - ay = 0.$$

This implies that

$$\gamma(y) = -\gamma(ky), \quad y \in \mathbb{R}, \tag{16}$$

where $k = \frac{a_2}{a_3}$. Since $a_2b_3 = a_3b_2$ and $(a_2, b_2) \neq (-a_3, -b_3)$, we have $k \neq -1$. By the condition, $k \neq 1$. For this reason $|k| \neq 1$. Suppose for definiteness that $|k| < 1$. We get from (16)

$$\gamma(y) = (-1)^n\gamma(k^ny), \quad y \in \mathbb{R}, \ n = 1, 2, \dots \tag{17}$$

Obviously, $k^ny \to 0$ as $n \to \infty$ for all $y \in \mathbb{R}$. Taking into account that $\gamma(y)$ is a continuous function and $\gamma(0) = 0$, it follows from (17) that $\gamma(y) = 0$ for all $y \in \mathbb{R}$. Hence (15) implies that $\psi(y)$ is a homogeneous linear function. Then it follows from (6) that $\psi_1(y)$ and $\psi_4(y)$ are also homogeneous linear functions, i.e., (12) is fulfilled. As noted above, the statement of the theorem follows from this. The theorem is completely proved. □

Let a_j, b_j, $j = 2, 3$, be nonzero real numbers. It is obvious that if $a_2b_3 \neq a_3b_2$, then $(a_2, b_2) \neq (-a_3, -b_3)$. Assume that $a_2b_3 = a_3b_2$. This implies that $a_2b_2 \neq -a_3b_3$, and the condition $(a_2, b_2) \neq (-a_3, -b_3)$ is equivalent to the condition that either $|a_2| \neq |a_3|$ or $a_2 = a_3$. Taking this into account, Theorem 2.1 can be reformulated as follows (compare below with Theorems 3.2, 4.1, and 4.3).

Theorem 2.2. *Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in $\mathbb{R}$ with nonvanishing characteristic functions. Suppose that the random variables ξ_2 and ξ_3 are identically distributed. Let a_j, b_j, $j = 2, 3$, be nonzero real numbers. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. Assume that one of the following conditions holds:*

(I) *$a_2b_3 \neq a_3b_2$ and $a_2b_2 \neq -a_3b_3$;*

(II) *$a_2b_3 = a_3b_2$ and $|a_2| \neq |a_3|$;*

(III) *$a_2b_3 = a_3b_2$ and $a_2 = a_3$.*

Then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift.

Remark 2.3. Let us assume that in Theorem 2.2 the coefficients a_j and b_j satisfy the condition

(IV) $a_2b_3 = a_3b_2$ and $a_2 = -a_3$.

We will verify that in this case the distribution of the random vector (L_1, L_2) uniquely determines the distributions of the random variables ξ_1 and ξ_4, i.e., $\nu_1 = \mu_1$ and $\nu_4 = \mu_4$, but need not necessarily determines the distribution of the random variables ξ_2 and ξ_3 up to shift. Taking into account that $|a_2| = |a_3|$ if and only if either $a_2 = a_3$ or $a_2 = -a_3$, it means that we can not strengthen Theorem 2.2 and replace conditions (II) and (III) in Theorem 2.2 by the condition $a_2b_3 = a_3b_2$.

Since the random variables ξ_2 and ξ_3 are identically distributed and η_2 and η_3 are also identically distributed, put $f(y) = f_2(y) = f_3(y)$. In view of (IV), equation (5) takes the form

$$f_1(u)|f(a_2(u + cv))|^2 f_4(v) = 1, \quad u, v \in \mathbb{R}. \tag{18}$$

Set $l(y) = |f(a_2y)|^2$. It follows from (18) that the function $l(y)$ satisfies the equation

$$l(u + v) = l(u)l(v), \quad u, v \in \mathbb{R}.$$

Hence $l(y) = e^{\kappa y}$, where $\kappa \in \mathbb{R}$. Inasmuch as $l(-y) = l(y)$, we have $\kappa = 0$, i.e., $l(y) = 1$ for all $y \in \mathbb{R}$. Taking this into account, we get from equation (18) that $f_1(y) = f_4(y) = 1$ for all $y \in \mathbb{R}$. Hence $\hat{\nu}_1(y) = \hat{\mu}_1(y)$ and $\hat{\nu}_4(y) = \hat{\mu}_4(y)$ for all $y \in \mathbb{R}$. This implies that $\nu_1 = \mu_1$ and $\nu_4 = \mu_4$.

Consider the distributions μ and ν with the characteristic functions

$$\hat{\mu}(y) = \exp\{(e^{iy} - 1)\}, \quad \hat{\nu}(y) = \exp\{(e^{-iy} - 1)\}, \quad y \in \mathbb{R}.$$

Then we have

$$|\hat{\mu}(y)| = |\hat{\nu}(y)| = \exp\{\cos y - 1\}, \quad y \in \mathbb{R}.$$

This implies that

$$|f(y)| = 1, \quad y \in \mathbb{R}. \tag{19}$$

It is obvious that ν is not a shift of μ. Moreover, It is easy to see that there is no a distribution λ such that either $\nu = \mu * \lambda$ or $\mu = \nu * \lambda$.

Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables such that ξ_2 and ξ_3 are identically distributed. Assume that the random variable ξ_j has the distribution μ_j, $j = 1, 2, 3, 4$, where μ_1 and μ_4 are arbitrary distributions with nonvanishing characteristic functions and $\mu_2 = \mu_3 = \mu$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 - a_2\xi_3$ and $L_2 = b_2\xi_2 - b_2\xi_3 + \xi_4$. Let η_1, η_2, η_3, and η_4 be independent random variables such that η_2 and η_3 are identically distributed. Suppose that η_j has the distribution ν_j, $j = 1, 2, 3, 4$, where $\nu_1 = \mu_1$, $\nu_2 = \nu_3 = \nu$, and $\nu_4 = \mu_4$. Put $M_1 = \eta_1 + a_2\eta_2 - a_2\eta_3$ and $M_2 = b_2\eta_2 - b_2\eta_3 + \eta_4$.

Taking into account that $f_1(y) = f_4(y) = 1$ for all $y \in \mathbb{R}$ and (19), we see that the functions $f_1(y)$, $f(y)$, and $f_4(y)$ satisfy equation (18). Hence the random vectors (L_1, L_2) and (M_1, M_2) are identically distributed, while ν is not a shift of μ.

3. Random variables with values in the field of p-adic numbers

Let X be a locally compact Abelian group, Y be its character group. Denote by (x, y) the value of a character $y \in Y$ at an element $x \in X$. Let μ be a distribution on X. Denote by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x), \quad y \in Y, \tag{20}$$

the characteristic function of the distribution μ.

Let $f(y)$ be a function on Y and let h be an element of Y. Denote by Δ_h the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y), \quad y \in Y.$$

A function $f(y)$ on Y is called a polynomial if

$$\Delta_h^{n+1} f(y) = 0$$

for some n and for all $y, h \in Y$.

We need the following well-known statement (for the proof see, e.g., [2, Proposition 1.30]).

Lemma 3.1. *Let Y be a locally compact Abelian group such that all its elements are compact. Then any continuous polynomial on Y is a constant.*

Consider the field of p-adic numbers $\mathbb{Q}_p$. When we say the group $\mathbb{Q}_p$, we mean the additive group of the field $\mathbb{Q}_p$. The group $\mathbb{Q}_p$ is a locally compact Abelian group. Its character group is topologically isomorphic to $\mathbb{Q}_p$ ([3, (25.1)]). Multiplication by a nonzero element of $\mathbb{Q}_p$ is a topological automorphism of the group $\mathbb{Q}_p$. Note that $(ax, y) = (x, ay)$ for all $a, x, y \in \mathbb{Q}_p$. If μ is a distribution on $\mathbb{Q}_p$, the characteristic function $\hat{\mu}(y)$ is defined by formula (20), where $X = Y = \mathbb{Q}_p$. The group $\mathbb{Q}_p$ is totally disconnected and consists of compact elements. Denote by $|\cdot|_p$ the norm in the field $\mathbb{Q}_p$.

In this section, we prove the following analogue of the Li–Zheng theorem for the field $\mathbb{Q}_p$.

Theorem 3.2. *Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in the field $\mathbb{Q}_p$ with nonvanishing characteristic functions. Let a_j, b_j, $j = 2, 3$, be nonzero elements of $\mathbb{Q}_p$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. Assume that one of the following conditions holds:*

(I) *$a_2b_3 \neq a_3b_2$;*

(II) *$a_2b_3 = a_3b_2$, $|a_2|_p \neq |a_3|_p$, and the random variables ξ_2 and ξ_3 are identically distributed.*

Then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift.

Proof. Consider independent random variables η_1, η_2, η_3, and η_4 with values in the field $\mathbb{Q}_p$ with nonvanishing characteristic functions. Put $M_1 = \eta_1 + a_2\eta_2 + a_3\eta_3$ and $M_2 = b_2\eta_2 + b_3\eta_3 + \eta_4$. Suppose that the distributions of the random vectors (L_1, L_2) and (M_1, M_2) coincide. Note that if ξ is a random variable with values in $\mathbb{Q}_p$ and distribution μ, then $\hat{\mu}(y) = \mathbf{E}[(\xi, y)]$. Taking this into account and the fact that $(ax, y) = (x, ay)$ for all $a, x, y \in \mathbb{Q}_p$, we can argue as in item 1 of the proof of Theorem 2.1. Keeping the same notation, we arrive at the equation

$$f_1(u)f_2(a_2u + b_2v)f_3(a_3u + b_3v)f_4(v) = 1, \quad u, v \in \mathbb{Q}_p. \tag{21}$$

1. Assume that condition (I) holds. Since we do not suppose that ξ_2 and ξ_3 are identically distributed and η_2 and η_3 are identically distributed, we can assume, without loss of generality, that $L_1 = \xi_1 + \xi_2 + \xi_3$ and $M_1 = \eta_1 + \eta_2 + \eta_3$. Then equation (21) takes the form

$$f_1(u)f_2(u + b_2v)f_3(u + b_3v)f_4(v) = 1, \quad u, v \in \mathbb{Q}_p, \tag{22}$$

and the condition $a_2b_3 \neq a_3b_2$ is transform to the condition $b_2 \neq b_3$. The group $\mathbb{Q}_p$ is totally disconnected. For this reason, as opposed to the case of the real line, we can not take the logarithm of both sides of equation (22) and pass to the corresponding additive equation.

To solve equation (22) we use a slightly different approach and split the solution of equation (22) into two parts. First we prove that $|f_j(y)| = 1$ for all $y \in \mathbb{Q}_p$, $j = 2, 3$. Then we prove that the functions $f_j(y)$, $j = 1, 2, 3, 4$, are characters of the $\mathbb{Q}_p$. It is obvious that the statement of the theorem follows from this.

Put

$$\theta_j(y) = \ln |f_j(y)|, \quad j = 1, 2, 3, 4.$$

It follows from (22) that the functions $\theta_j(y)$ satisfy the equation

$$\theta_1(u) + \theta_2(u + b_2v) + \theta_3(u + b_3v) + \theta_4(v) = 0, \quad u, v \in \mathbb{Q}_p,$$

which can be written in the form

$$\theta_2(u + b_2v) + \theta_3(u + b_3v) = C(u) + D(v), \quad u, v \in \mathbb{Q}_p. \tag{23}$$

For solving equation (23) we use the finite difference method. The reasoning is standard and the same as in the case of the real line. We present it here for completeness.

Let g be an arbitrary element of the group $\mathbb{Q}_p$. Substitute $u - b_3g$ for u and $v + g$ for v in equation (23). Subtracting (23) from the resulting equation we get

$$\Delta_{(b_2-b_3)g}\theta_2(u + b_2v) = \Delta_{-b_3g}C(u) + \Delta_g D(v), \quad u, v \in \mathbb{Q}_p. \tag{24}$$

Let h be an arbitrary element of the group $\mathbb{Q}_p$. Substitute $u + h$ for u in equation (24). Subtracting (24) from the resulting equation we obtain

$$\Delta_h\Delta_{(b_2-b_3)g}\theta_2(u + b_2v) = \Delta_h\Delta_{-b_3g}C(u), \quad u, v \in \mathbb{Q}_p. \tag{25}$$

Let k be an arbitrary element of the group $\mathbb{Q}_p$. Substitute $v + k$ for v in equation (25). Subtracting (25) from the resulting equation we get

$$\Delta_{b_2k}\Delta_h\Delta_{(b_2-b_3)g}\theta_2(u + b_2v) = 0, \quad u, v \in \mathbb{Q}_p. \tag{26}$$

Substituting $v = 0$ in equation (26), we obtain

$$\Delta_{b_2k}\Delta_h\Delta_{(b_2-b_3)g}\theta_2(u) = 0, \quad u \in \mathbb{Q}_p.$$

Since $b_2 - b_3 \neq 0$ and g, h and k are arbitrary elements of the group $\mathbb{Q}_p$, we conclude that the function $\theta_2(y)$ satisfies the equation

$$\Delta_h^3 \theta_2(y) = 0, \quad y, h \in \mathbb{Q}_p,$$

i.e., is a polynomial on $\mathbb{Q}_p$. Since the group $\mathbb{Q}_p$ consists of compact elements and the polynomial $\theta_2(y)$ is continuous, by Lemma 3.1, $\theta_2(y)$ is a constant. In view of $\theta_2(0) = 0$, we have $\theta_2(y) = 0$ for all $y \in \mathbb{Q}_p$. For the function $\theta_3(y)$ we argue similarly excluding first the function $\theta_2(y)$ from equation (23). Thus we proved that $\theta_2(y) = \theta_3(y) = 0$ and hence

$$|f_2(y)| = |f_3(y)| = 1, \quad y \in \mathbb{Q}_p.$$

Let us prove that the functions $f_j(y)$, $j = 1, 2, 3, 4$, are characters of the group $\mathbb{Q}_p$. Rewrite equation (22) in the form

$$f_2(u + b_2 v) f_3(u + b_3 v) = S(u)T(v), \quad u, v \in \mathbb{Q}_p. \tag{27}$$

To solve equation (27), we apply the method that was used to prove Theorem 3.1 in [1], see also [2, Theorem 15.8].

Let g be an arbitrary element of the group $\mathbb{Q}_p$. Substitute $u - b_3 g$ for u and $v + g$ for v in equation (27). Dividing the resulting equation by equation (27), we get

$$\frac{f_2(u + b_2 v - b_3 g + b_2 g)}{f_2(u + b_2 v)} = \frac{S(u - b_3 g)T(v + g)}{S(u)T(v)}, \quad u, v \in \mathbb{Q}_p. \tag{28}$$

Let h be an arbitrary element of the group $\mathbb{Q}_p$. Substitute $u + h$ for u in equation (28). Dividing the resulting equation by equation (28), we receive

$$\frac{f_2(u + b_2 v - b_3 g + b_2 g + h) f_2(u + b_2 v)}{f_2(u + b_2 v + h) f_2(u + b_2 v - b_3 g + b_2 g)} = \frac{S(u - b_3 g + h) S(u)}{S(u + h) S(u - b_3 g)}, \quad u, v \in \mathbb{Q}_p. \tag{29}$$

Let k be an arbitrary element of the group $\mathbb{Q}_p$. Substitute $v + k$ for v in equation (29). Dividing the resulting equation by equation (29), we obtain

$$\frac{f_2(u + b_2 v - b_3 g + b_2 g + h + b_2 k) f_2(u + b_2 v + b_2 k)}{f_2(u + b_2 v + h + b_2 k) f_2(u + b_2 v - b_3 g + b_2 g + b_2 k)} \times \frac{f_2(u + b_2 v + h) f_2(u + b_2 v - b_3 g + b_2 g)}{f_2(u + b_2 v - b_3 g + b_2 g + h) f_2(u + b_2 v)} = 1, \quad u, v \in \mathbb{Q}_p. \tag{30}$$

Substitute in (30) $u = v = 0$ and $g = -\frac{b_2 k}{b_2 - b_3}$. Then we get

$$\frac{f_2^2(h) f_2(b_2 k) f_2(-b_2 k)}{f_2(h + b_2 k) f_2(h - b_2 k)} = 1, \quad h, k \in \mathbb{Q}_p. \tag{31}$$

Taking into account that

$$f_2(-y) = \overline{f_2(y)}, \quad |f_2(y)| = 1, \quad y \in \mathbb{Q}_p, \tag{32}$$

we have $f_2(b_2 k) f_2(-b_2 k) = 1$ for all $k \in \mathbb{Q}_p$. Then it follows from (31) that the function $f_2(y)$ satisfies the equation

$$f_2^2(u) = f_2(u + v) f_2(u - v), \quad u, v \in \mathbb{Q}_p. \tag{33}$$

In view of (32), we get from (33) that

$$f_2^2(u + v) = f_2^2(u) f_2^2(v), \quad u, v \in \mathbb{Q}_p, \tag{34}$$

i.e., the function $f_2^2(y)$ is a character of the group $\mathbb{Q}_p$. Substituting $u = v = y$ in (33), we obtain

$$f_2^2(y) = f_2(2y), \quad y \in \mathbb{Q}_p. \tag{35}$$

Taking into account that the mapping $y \to 2y$ is a topological automorphism of the group $\mathbb{Q}_p$, it follows from (34) and (35) that the function $f_2(y)$ satisfies the equation

$$f_2(u+v) = f_2(u)f_2(v), \quad u, v \in \mathbb{Q}_p,$$

i.e., is a character of the group $\mathbb{Q}_p$. For the function $f_3(y)$ we argue similarly. By the Pontryagin duality theorem, there are elements $x_2, x_3 \in \mathbb{Q}_p$ such that

$$f_j(y) = (x_j, y), \quad y \in \mathbb{Q}_p, \ j = 2, 3. \tag{36}$$

Substituting (36) into (22) and putting first $v = 0$ and then $u = 0$ in the obtained equation, we get that the functions $f_1(y)$ and $f_4(y)$ are also characters of the group $\mathbb{Q}_p$.

Thus, we have proved that the characteristic functions $\hat{\mu}_j(y)$, $j = 1, 2, 3, 4$, are determined up to multiplication by a character. Hence the distributions μ_j, $j = 1, 2, 3, 4$, are determined up to shift. The theorem is proved if condition (I) is satisfied.

2. Assume that condition (II) holds. Put

$$c = b_2/a_2 = b_3/a_3, \quad g(y) = f_2(a_2 y) f_3(a_3 y), \quad \tau(y) = \ln|g(y)|. \tag{37}$$

2a. We prove in this item that the functions $f_1(y)$, $f_4(y)$, and $g(y)$ are characters of the group $\mathbb{Q}_p$. In doing so, we do not assume that $|a_2|_p \neq |a_3|_p$ and the random variables ξ_2 and ξ_3 are identically distributed.

Since $a_2 u + b_2 v = a_2(u + cv)$ and $a_3 u + b_3 v = a_3(u + cv)$, rewrite equation (21) in the form

$$f_1(u) g(u + cv) f_4(v) = 1, \quad u, v \in \mathbb{Q}_p. \tag{38}$$

Substituting first $v = 0$ and then $u = 0$ in (38), we get that the function $g(y)$ satisfies the equation

$$g(u+v) = g(u)g(v), \quad u, v \in \mathbb{Q}_p. \tag{39}$$

It follows from (39) that the function $\tau(y)$ satisfies the equation

$$\tau(u+v) = \tau(u) + \tau(v), \quad u, v \in \mathbb{Q}_p.$$

Hence $\tau(y)$ is a continuous polynomial. Since the group $\mathbb{Q}_p$ consists of compact elements, by Lemma 3.1, $\tau(y)$ is a constant. In view of $\tau(0) = 0$, we have $\tau(y) = 0$ and hence $|g(y)| = 1$ for all $y \in \mathbb{Q}_p$. Taking into account (39), this means that the function $g(y)$ is a character of the group $\mathbb{Q}_p$. By the Pontryagin duality theorem, there is an element $a \in \mathbb{Q}_p$ such that

$$g(y) = (a, y), \quad y \in \mathbb{Q}_p. \tag{40}$$

Hence

$$g(u + cv) = (a, u + cv), \quad u, v \in \mathbb{Q}_p. \tag{41}$$

Substituting (41) into (38), we find from the resulting equation that the functions $f_1(y)$ and $f_4(y)$ are also characters of the group $\mathbb{Q}_p$.

Note that in the proof we did not use the fact that $|a_2|_p \neq |a_3|_p$ and the random variables ξ_2 and ξ_3 are identically distributed.

2b. Suppose that the random variables η_2 and η_3 are identically distributed. Since the random variables ξ_2 and ξ_3 are also identically distributed, we have $f_2(y) = f_3(y)$. Set

$$f(y) = f_2(y) = f_3(y).$$

Then

$$g(y) = f(a_2 y) f(a_3 y). \tag{42}$$

We will prove now that the function $f(y)$ is a character of the group $\mathbb{Q}_p$. Set

$$\theta(y) = \ln |f(y)|.$$

We have

$$\tau(y) = \ln |g(y)| = \ln |f(a_2 y) f(a_3 y)| = \theta(a_2 y) + \theta(a_3 y), \quad y \in \mathbb{Q}_p.$$

Since $\tau(y) = 0$ for all $y \in \mathbb{Q}_p$, it follows from this that

$$\theta(a_2 y) + \theta(a_3 y) = 0, \quad y \in \mathbb{Q}_p. \tag{43}$$

Inasmuch as $|a_2|_p \neq |a_3|_p$, assume for definiteness that $|a_2|_p < |a_3|_p$. Put $k = \frac{a_2}{a_3}$. Then $|k|_p < 1$. We obtain from (43) that $\theta(y) = -\theta(ky)$, and hence

$$\theta(y) = (-1)^n \theta(k^n y), \quad y \in \mathbb{Q}_p, \ n = 1, 2, \ldots. \tag{44}$$

It is obvious that $|k^n y|_p \to 0$ as $n \to \infty$ for all $y \in \mathbb{Q}_p$. Since $\theta(y)$ is a continuous function and $\theta(0) = 0$, we obtain from (44) that $\theta(y) = 0$ and hence $|f(y)| = 1$ for all $y \in \mathbb{Q}_p$.

Since $|a_2|_p \neq |a_3|_p$, we have $a_2 + a_3 \neq 0$. Put

$$b = \frac{a}{a_2 + a_3}, \quad h(y) = f(y)(-b, y). \tag{45}$$

It follows from (40), (42), and (45) that

$$\begin{aligned} h(a_2 y) h(a_3 y) &= f(a_2 y)(-b, a_2 y) f(a_3 y)(-b, a_3 y) \\ &= g(y)(-b(a_2 + a_3), y) = g(y)(-a, y) = 1, \quad y \in \mathbb{Q}_p. \end{aligned}$$

Hence $h(y) = h^{-1}(ky)$. This implies that

$$h(y) = h^{(-1)^n}(k^n y), \quad y \in \mathbb{Q}_p, \ n = 1, 2, \ldots. \tag{46}$$

We have $|k^n y|_p \to 0$ as $n \to \infty$ for all $y \in \mathbb{Q}_p$. Inasmuch as $h(y)$ is a continuous function and $h(0) = 1$, it follows from (46) that $h(y) = 1$ for all $y \in \mathbb{Q}_p$, and (45) implies that

$$f(y) = (b, y), \quad y \in \mathbb{Q}_p. \tag{47}$$

Taking into account (47) and the fact that $f_1(y)$ and $f_4(y)$ are also characters of the group $\mathbb{Q}_p$, we see that the characteristic functions $\hat{\mu}_j(y)$ are determined up to multiplication by a character. Hence the distributions μ_j are determined up to shift. The theorem is proved if condition (II) is satisfied, and hence is completely proved. $\square$

Corollary 3.3. *Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in the field $\mathbb{Q}_p$ with nonvanishing characteristic functions. Let a_j, b_j, $j = 2, 3$, be nonzero elements of $\mathbb{Q}_p$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. Then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_1 and ξ_4 up to shift.*

Proof. By Theorem 3.2, in the case, when $a_2b_3 \neq a_3b_2$, the statement of the corollary is true. If $a_2b_3 = a_3b_2$, as has been proved in item 2a of the proof of Theorem 3.2 the functions $f_1(y)$ and $f_4(y)$ are characters of the group $\mathbb{Q}_p$. The statement of the corollary follows from this. $\square$

We can not omit the condition $|a_2|_p \neq |a_3|_p$ in condition (II) of Theorem 3.2. Namely, the following statement is true.

Proposition 3.4. *Let a_j, b_j, $j = 2, 3$, be nonzero elements of the field $\mathbb{Q}_p$ such that $a_2b_3 = a_3b_2$. Assume that $|a_2|_p = |a_3|_p$. Then there are independent random variables ξ_1, ξ_2, ξ_3, and ξ_4 with values in $\mathbb{Q}_p$ with nonvanishing characteristic functions such that the following is true:*

(I) *the random variables ξ_2 and ξ_3 are identically distributed;*

(II) *the distribution of the random vector (L_1, L_2), where $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$, need not necessarily determine the distribution of the random variables ξ_2 and ξ_3 up to shift.*

Proof. Let G be a second countable compact Abelian group, H be its character group. Then H is a countable discrete Abelian group. Let $\alpha(h)$ be a real-valued nonvanishing function on the group H satisfying the conditions:

(i) $\alpha(0) = 1$;

(ii) $\alpha(-h) = \alpha(h)$ for all $h \in H$;

(iii) $\sum\limits_{h \in H} |\alpha(h)| < 2$;

Consider on the group G the function

$$\rho(g) = \sum_{h \in H} \alpha(h)\overline{(g, h)}, \quad g \in G.$$

It follows from (i)–(iii) that $\rho(g)$ is the nonnegative density with respect to the Haar distribution on G of a distribution μ on the group G with the characteristic function $\hat{\mu}(h) = \alpha(h)$.

Put

$$\beta(h) = \begin{cases} 1 & \text{if} \quad h = 0, \\ -\alpha(h) & \text{if} \quad h \neq 0. \end{cases}$$

Then $\beta(h)$ is a real-valued nonvanishing function on the group H. Obviously, the function $\beta(h)$ also satisfies conditions (i)–(iii). Hence there is a distribution ν on the group G with the characteristic function $\hat{\nu}(h) = \beta(h)$. It is easy to see that if G is not isomorphic to the additive group of the integers modulo 2, then ν is not a shift of μ.

Let $G = \mathbb{Z}_p$ be the ring of p-adic integers. Then $\mathbb{Z}_p$ is a compact subgroup of the group $\mathbb{Q}_p$. Let μ and ν be the distributions on $\mathbb{Z}_p$ constructed above. Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in $\mathbb{Z}_p$ such that ξ_2 and ξ_3 are identically distributed. Assume that the random variable ξ_j has the distribution μ_j, $j = 1, 2, 3, 4$, where μ_1 and μ_4 are arbitrary distributions on $\mathbb{Z}_p$ with nonvanishing characteristic functions and $\mu_2 = \mu_3 = \mu$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. Let η_1, η_2, η_3, and η_4 be independent random variables with values in $\mathbb{Z}_p$ such that η_2 and η_3 are identically distributed. Suppose that η_j has the distribution ν_j, $j = 1, 2, 3, 4$, where $\nu_1 = \mu_1$, $\nu_2 = \nu_3 = \nu$, and $\nu_4 = \mu_4$. Put $M_1 = \eta_1 + a_2\eta_2 + a_2\eta_3$ and $M_2 = b_2\eta_2 + b_2\eta_3 + \eta_4$.

Consider ξ_j and η_j as independent random variables with values in the field $\mathbb{Q}_p$ and verify that the characteristic functions of the random vectors (L_1, L_2) and (M_1, M_2) coincide. This is equivalent to the fact that the functions $f_j(y) = \hat{\nu}_j(y)/\hat{\mu}_j(y)$ satisfy equation (21), where $f_2(y) = f_3(y) = f(y)$. Taking into account that $f_1(y) = f_4(y) = 1$ for all $y \in \mathbb{Q}_p$, equation (21) takes the form

$$f(a_2(u + cv))f(a_3(u + cv)) = 1, \quad u, v \in \mathbb{Q}_p,$$

where $c = b_2/a_2 = b_3/a_3$, or

$$f(a_2y)f(a_3y) = 1, \quad y \in \mathbb{Q}_p. \tag{48}$$

We consider the distributions μ and ν as distributions on $\mathbb{Q}_p$. It is easy to see that then the function $f(y)$ is of the form

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in p\mathbb{Z}_p, \\ -1 & \text{if} \quad y \notin p\mathbb{Z}_p. \end{cases} \tag{49}$$

It follows from (49) and the condition $|a_2|_p = |a_3|_p$ that the function $f(y)$ satisfies equation (48). Thus, the characteristic functions of the random vectors (L_1, L_2) and (M_1, M_2) coincide. Hence the random vectors (L_1, L_2) and (M_1, M_2) are identically distributed, while ν is not a shift of μ.

We note that if $|a_2|_p \neq |a_3|_p$, then the function $f(y)$ does not satisfy equation (48), i.e., the random vectors (L_1, L_2) and (M_1, M_2) are not identically distributed. □

The condition $|a_2|_p = |a_3|_p$ is an analogue for the field $\mathbb{Q}_p$ of the condition $|a_2| = |a_3|$ for the field $\mathbb{R}$. Comparing the statement of Theorem 2.2 in the case when condition (III) holds and Proposition 3.4, we see that in the field $\mathbb{Q}_p$ do not exist nonzero elements a_j, b_j, $j = 2, 3$, such that $a_2b_3 = a_3b_2$ and $|a_2|_p = |a_3|_p$ and the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_2 and ξ_3 up to shift.

4. Random variables with values in the field of integers modulo p, where $p \neq 2$, and in the discrete field of rational numbers

Let p be a prime number and $\mathbb{Z}(p)$ be the field of integers modulo p. When we say the group $\mathbb{Z}(p)$, we mean the additive group of the field $\mathbb{Z}(p)$, i.e., the additive group of the integers modulo p. The character group of the group $\mathbb{Z}(p)$ is isomorphic to $\mathbb{Z}(p)$. Multiplication by a nonzero element of $\mathbb{Z}(p)$ is an automorphism of the group $\mathbb{Z}(p)$. Note that $(ax, y) = (x, ay)$ for all $a, x, y \in \mathbb{Z}(p)$. If μ is a distribution on $\mathbb{Z}(p)$, the characteristic function $\hat{\mu}(y)$ is defined by formula (20), where $X = Y = \mathbb{Z}(p)$. The group $\mathbb{Z}(p)$ is finite and hence compact.

In this section, we prove analogues of the Li–Zheng theorem for the field of integers modulo p, where $p \neq 2$, and for the discrete field of rational numbers $\mathbb{Q}$.

Theorem 4.1. *Consider the field $\mathbb{Z}(p)$, where $p \neq 2$. Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in $\mathbb{Z}(p)$ with nonvanishing characteristic functions. Let a_j, b_j, $j = 2, 3$, be nonzero elements of $\mathbb{Z}(p)$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. Then the following statements hold.*

1. *If $a_2b_3 \neq a_3b_2$, then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift.*

2. *If $a_2b_3 = a_3b_2$, then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_1 and ξ_4 up to shift.*

Proof. 1. Assume that $a_2b_3 \neq a_3b_2$. The proof of Theorem 3.2 in the case, when condition (I) holds, is based only on the following properties of the group $\mathbb{Q}_p$: the group $\mathbb{Q}_p$ consists of compact elements and the mapping $y \to 2y$ is a topological automorphism of the group $\mathbb{Q}_p$. Both of these properties are also valid for the group $\mathbb{Z}(p)$, where $p \neq 2$. For this reason the proof remains unchanged if, instead of the field $\mathbb{Q}_p$, we consider the filed $\mathbb{Z}(p)$. Thus, statement 1 is valid.

2. Assume that $a_2b_3 = a_3b_2$. The reasoning carried out in item 2a of the proof of Theorem 3.2 is based only on the fact that the group $\mathbb{Q}_p$ consists of compact elements. For this reason the proof remains unchanged if, instead of the field $\mathbb{Q}_p$, we consider the filed $\mathbb{Z}(p)$. Thus, statement 2 is valid. ☐

Theorem 4.1 can not be strengthened. Namely, the following statement is true.

Proposition 4.2. *Let a_j, b_j, $j = 2, 3$, be nonzero elements of the field $\mathbb{Z}(p)$, where $p \neq 2$, such that $a_2b_3 = a_3b_2$. Then there are independent random variables ξ_1, ξ_2, ξ_3, and ξ_4 with values in $\mathbb{Z}(p)$ with nonvanishing characteristic functions such that the following is true:*

(I) *the random variables ξ_2 and ξ_3 are identically distributed;*

(II) *the distribution of the random vector (L_1, L_2), where $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$, need not necessarily determine the distribution of the random variables ξ_2 and ξ_3 up to shift.*

Proof. To prove the proposition, we argue for the field $\mathbb{Z}(p)$ in the same way as in Proposition 3.4 we argued for the field $\mathbb{Q}_p$ and keep the same notation. The only difference is that instead of (49) the function $f(y)$ is of the form

$$f(y) = \begin{cases} 1 & \text{if} \quad y = 0, \\ -1 & \text{if} \quad y \neq 0, \end{cases} \tag{50}$$

because we construct the distributions μ and ν at once on $\mathbb{Z}(p)$. The proposition will be proved if we verify that that the function $f(y)$ satisfies the equation

$$f(a_2y)f(a_3y) = 1, \quad y \in \mathbb{Z}(p).$$

In view of (50), it is obvious. ☐

Proposition 4.2 shows that, unlike Theorems 2.2 and 3.2, in the field $\mathbb{Z}(p)$ do not exist nonzero elements a_j, b_j, $j = 2, 3$, such that $a_2b_3 = a_3b_2$ and the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_2 and up ξ_3 to shift.

Let $\mathbb{Q}$ be the field of rational numbers considering in the discrete topology. When we say the group $\mathbb{Q}$, we mean the additive group of the field $\mathbb{Q}$. The character group of the group $\mathbb{Q}$ is topologically isomorphic to the $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$, where $\boldsymbol{a} = (2, 3, 4, \dots)$ ([3, (25.4)]). The group $\Sigma_{\boldsymbol{a}}$ is compact. Since the multiplication by any nonzero integer is an automorphism of the group $\mathbb{Q}$, the multiplication by any nonzero integer is a topological automorphism of the group $\Sigma_{\boldsymbol{a}}$. Therefore, the multiplication by any nonzero rational number is well-defined and is also a topological automorphism in the group $\Sigma_{\boldsymbol{a}}$. Note that $(ax, y) = (x, ay)$ for all $a, x \in \mathbb{Q}$, $y \in \Sigma_{\boldsymbol{a}}$. If μ is a distribution on the group $\mathbb{Q}$, the characteristic function $\hat{\mu}(y)$ is defined by formula (20), where $X = \mathbb{Q}$, $Y = \Sigma_{\boldsymbol{a}}$. The set $\mathbb{Q}$ is a countable subset of $\mathbb{R}$. For this reason, if a random variable ξ take values in $\mathbb{Q}$ we can consider ξ as a random variable with values in $\mathbb{R}$. This implies, in particular, that Theorem 2.2 is valid for the field $\mathbb{Q}$, i.e., when ξ_j take values in $\mathbb{Q}$ and $a_j, b_j \in \mathbb{Q}$. However, the fact that random variables ξ_j can be considered as random variables taking values in the discrete field $\mathbb{Q}$ allows us to strengthen Theorem 2.2 for the field $\mathbb{Q}$.

Theorem 4.3. *Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in the field $\mathbb{Q}$ with nonvanishing characteristic functions. Let a_j, b_j, $j = 2, 3$, be nonzero elements of $\mathbb{Q}$. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. Assume that one of the following conditions holds:*

(I) $a_2b_3 \neq a_3b_2$;

(II) $a_2b_3 = a_3b_2$, $|a_2| \neq |a_3|$, *and the random variables ξ_2 and ξ_3 are identically distributed;*

(III) $a_2b_3 = a_3b_2$, $a_2 = a_3$, *and the random variables ξ_2 and ξ_3 are identically distributed.*

Then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift.

Proof. Assume that condition (I) holds. The proof of Theorem 3.2 in the case, when condition (I) holds, is based only on the following properties of the group $\mathbb{Q}_p$: the group $\mathbb{Q}_p$ consists of compact elements and the mapping $y \to 2y$ is a topological automorphism of the group $\mathbb{Q}_p$. Both of these properties are also valid for the group $\Sigma_{\boldsymbol{a}}$, where $\boldsymbol{a} = (2, 3, 4, \dots)$. For this reason the proof remains unchanged if, instead of the field $\mathbb{Q}_p$, we consider the filed $\mathbb{Q}$.

Note that if $a_2b_3 \neq a_3b_2$, then it follows from Theorem 2.2 that the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift, only if additional conditions are satisfied, namely $a_2b_2 \neq -a_3b_3$ and the random variables ξ_2 and ξ_3 are identically distributed.

In the case, when condition (II) or (III) is satisfied, the corresponding statements follows from the corresponding statements of Theorem 2.2 for the field $\mathbb{R}$. $\square$

Remark 4.4. The distributions μ and ν constructed in Remark 2.3 in fact are the distributions on the group of integers. We can consider μ and ν as as distributions on $\mathbb{Q}$. Hence Remark 2.3 is valid for the random variables with values in the field $\mathbb{Q}$ and it shows that we can not strengthen Theorem 4.3 and replace conditions (II) and (III) in Theorem 4.3 by the condition $a_2b_3 = a_3b_2$.

In Theorems 2.2, 3.2, 4.1, and 4.3 independent random variables take values in a locally compact field, namely in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Z}(p)$, and $\mathbb{Q}$. In doing so, coefficients of the linear forms are elements of the field. We can also study a more general problem, when independent random variables take values in a locally compact Abelian group X, and coefficients of the forms are continuous endomorphisms of X. Taking this into account, we formulate the following problem.

Let X be a second countable locally compact Abelian group. Let ξ_1, ξ_2, ξ_3, and ξ_4 be independent random variables with values in X with nonvanishing characteristic functions. Let a_j, b_j, $j = 2, 3$, be continuous endomorphisms of the group X. Consider the linear forms $L_1 = \xi_1 + a_2\xi_2 + a_3\xi_3$ and $L_2 = b_2\xi_2 + b_3\xi_3 + \xi_4$. What are the conditions on a_j, b_j, and ξ_j to guarantee that the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to shift?

It follows from the results of the article that these conditions depend on the group X.

A more general problem can also be formulated.

Let us assume that we know the distribution of the random vector (L_1, L_2). How uniquely does this distribution determine the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$?

In connection with this problem, we note that in the case when X is an arbitrary $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$, it follows from Theorem 3.1 in [1], see also [2, Theorem 15.8], that if $L_1 = \xi_1 + \xi_2 + \xi_3$ and b_2, b_3, and $b_2 - b_3$ are topological automorphisms of the group $\Sigma_{\boldsymbol{a}}$, then the distribution of the random vector (L_1, L_2) determines the distributions of the random variables ξ_j, $j = 1, 2, 3, 4$, up to convolution with a Gaussian distribution on $\Sigma_{\boldsymbol{a}}$.

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