THE LEIBNIZ PROP IS A CROSSED PRESIMPLICIAL ALGEBRA

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ABSTRACT. We prove that the Leibniz PROP is isomorphic (as k-linear categories) to the symmetric crossed presimplicial algebra $k[(\Delta^+)^{op}\mathbb{S}]$ where Δ^+ is the skeletal category of finite well-ordered sets with surjections, but the distributive law between $(\Delta^+)^{op}$ and the symmetric groups $\mathbb{S} = \bigsqcup_{n \ge 1} S_n$ is not the standard one.

INTRODUCTION

Leibniz algebras are non-commutative analogues of Lie algebras. Like any other algebraic structure, there is an operad, or better yet a PRO or a PROP, that parametrize all algebraic operations on the finite tensor powers of a Leibniz algebra. The PRO(P)s for algebras of different types (magmatic, associative, commutative, Lie, Leibniz, Poisson, Jacobian etc.) come in different flavors (plain set, linear, piecewise linear, topological, homotopical, simplicial, differential graded etc.), but are always considered as symmetric or braided monoidal categories. In this paper, we show that when we drop the monoidality assumption, the parametrizing category Leib for Leibniz algebras is isomorphic to a twisted product of the opposite skeletal category of finite well-ordered sets with surjections $(\Delta^+)^{op}$ (which parametrizes operations for not-necessarily unital associative algebras) and the group ring $\Bbbk[\mathbb{S}]$ of the collection of symmetric groups $\mathbb{S} = \bigsqcup_{n \ge 1} S_n$ considered as a category. The twisted product is determined by a distributive law between $(\Delta^+)^{op}$ and $\Bbbk[\mathbb{S}]$, but a non-standard one.

The parametrizing categories we are going to consider in this paper are as follows:

- (a) Mag for magmatic algebras, i.e. algebras with a binary operation with no other condition.
- (b) Simp for (not necessarily unital) associative algebras.
- (c) Sym for graded vector spaces with a compatible actions of symmetric groups $\mathbb{S} = \bigsqcup_{n \ge 1} S_n$.
- (d) Braid for graded vector spaces with a compatible actions of Artin braid groups $\mathbb{B} = \bigsqcup_{n \ge 1} B_n$.
- (e) Leib and Leib^{op} for left and right Leibniz algebras.

We define each of these categories with explicit generators and relations, and describe the distributive laws between relevant categories on the set of generators explicitly. The proofs we give in this paper are highly algebro-combinatorial in nature, and in theory, should yield themselves for machine verification as in [6, 5], but we meticulously check them by hand aided by string diagrams. Our main result (Theorem 3.9) is that there is a distributive law of the form ω : Simp \otimes Sym \rightarrow Sym \otimes Simp and Leib is isomorphic to the twisted product Sym \otimes_{ω} Simp.

Plan of the article. In Section 1, we introduce the non-monoidal PRO(P)s that we call *combinatorial operations categories*, and then we define all of the combinatorial operations categories we use by explicit generators and relations. Section 2 deals with the distributive laws between these operations categories in all different variations that we are going to need. In Section 3, we define the combinatorial operations category for Leibniz algebras, and finally give our main result Theorem 3.9. Appendix contains some basic facts about operads and PRO(P)s that we need to put our combinatorial operations categories into their proper context. See Remark A.3.

Notation and conventions. We use \cup for ordinary union of sets while \sqcup will denote the disjoint union. We use a base field k with no assumption on the characteristic. All unadorned tensor products are over k. Throughout the article, C will be a k-linear strict symmetric monoidal category with a monoidal product \odot , and a unit object I. We will also assume that C is fibered over the category of k-vector spaces, i.e. there is a faithful k-linear functor $C \rightarrow Mod$ -k which is not necessarily monoidal. For each natural number $n \ge 1$, we use S_n to denote the symmetric group on n-letters, B_n to denote the Artin braid group on n-strands. We use Δ for the skeletal category of finite well-ordered sets with set maps that preserve order. The subcategories of injective and surjective maps are respectively denoted by Δ^- and Δ^+ . For a positive integer n, we will use [n] for the (well-ordered) set $\{1, \ldots, n\}$. So, in particular, $[0] = \emptyset$. For two non-zero positive integers n and m, an m-composition of n is a sequence of positive integers (n_1, \ldots, n_m) such that each $n_i > 0$ and $n = n_1 + \cdots + n_m$.

1. COMBINATORIAL OPERATIONS CATEGORIES

1.1. K-bimodules and K-algebras. We denote the smallest k-linear PRO by K whose non-zero morphisms consist of the constant multiples of identities on each object. With this definition one can see that every k-linear PRO contains K as a subcategory. There exists inclusion functor $1_{\mathcal{C}}$: $\mathbb{K} \to \mathcal{C}$ for every k-linear PRO \mathcal{C} .

One can also describe \mathbb{K} as the locally unital algebra $\mathbb{K} = \mathbb{k}^{\oplus \mathbb{N}}$ spanned algebraically by countably many vectors 1_n for every $n \in \mathbb{N}$ subject to the condition that $1_n \cdot 1_m = \delta_{nm}$ for every $n, m \in \mathbb{N}$. With this definition at hand, one can now define a \mathbb{K} -module as a countable collection $(V_n)_{n \in \mathbb{N}}$ of vector spaces indexed by \mathbb{N} . Then a \mathbb{K} -bimodule $(V_{n,m})_{n,m \in \mathbb{N}}$ is a collection of vector spaces doubly indexed by \mathbb{N} . A \mathbb{K} -(bi)module V is called *locally finite* if for every $x \in V$ there are only finitely many $n, m \in \mathbb{N}$ such that $1_n \cdot x$ and $x \cdot 1_m$ are non-zero. A locally finite $x \in V$ is called *faithful* if $\sum_{n \in \mathbb{N}} 1_n \cdot x$ is well-defined and is equal to x. A similar condition holds for bimodules. This condition is equivalent to the fact that $V = \bigoplus_{n \in \mathbb{N}} 1_n \cdot V$ for a module and $V = \bigoplus_{n,m \in \mathbb{N}} 1_n \cdot V \cdot 1_m$ for a bimodule. Note that a \mathbb{K} -bimodule $V = (V_{n,m})_{n,m \in \mathbb{N}}$ is faithful is equivalent to the fact that

$$\mathbb{K}\otimes_{\mathbb{K}} V \cong V \cong V \otimes_{\mathbb{K}} \mathbb{K}$$

Note that technically we should call \mathbb{K} -(bi)modules as \mathbb{K} -algebras since \mathbb{K} is a PRO. However, later on we are going to need to work with monoid objects in the category of \mathbb{K} -bimodules, and we would have had to refer to what we call \mathbb{K} -algebras as "algebras in the category of \mathbb{K} -algebras" which is needlessly more confusing.

Proposition 1.1. The category of faithful \mathbb{K} -bimodules is strictly monoidal with \mathbb{K} being the unit object and a product defined on the objects as

$$V\otimes_{\mathbb{K}}W=\left(igoplus_{\mathfrak{m}\in\mathbb{N}}V_{\mathfrak{n},\mathfrak{m}}\otimes W_{\mathfrak{m},\ell}
ight)_{\mathfrak{n},\ell\in\mathbb{N}}$$

Proof. Assume the faithful \mathbb{K} -bimodules V and W have bases $B_{n,m}$ and $C_{n,m}$, respectively. Then the bigraded vector space $(V \otimes_{\mathbb{K}} W)_{n,\ell}$ has the basis $\bigsqcup_{m \in \mathbb{N}} B_{n,m} \times C_{m,\ell}$, and thus, it is a faithful \mathbb{K} -bimodule.

A faithful K-bimodule together with a unital associative operation $\mu: V \otimes_{\mathbb{K}} V \to V$ making the following diagrams commutative is called a K-algebra.



If V is a K-algebra such that each $V_{n,m}$ is finite dimensional, we will call V as a locally finite dimensional K-algebra. We will use $Alg(\mathbb{K})$ and $alg(\mathbb{K})$ to denote the category of K-algebras and locally finite dimensional K-algebras, respectively.

1.2. Combinatorial operations categories. A small k-linear category C is called *a combinatorial operations category* if the object set is \mathbb{N} . A combinatorial operations category C is called *finite* if C(n, m) is finite dimensional for every $n, m \in \mathbb{N}$. A combinatorial operations category C is called *symmetric* (resp. braided) if each C([n], [m]) is a S_m - S_n (resp. B_m - B_n) bimodule. We denote the category of (resp. symmetric or braided) combinatorial operations categories, and their functors as CO_k (resp. as sym CO_k , or $brCO_k$).

Proposition 1.2. *The category of combinatorial operations categories and the category of* \mathbb{K} *-algebras are equivalent.*

Proof. Given a $\mathcal{C} \in \mathbf{CO}_{\Bbbk}$, we can define the \mathbb{K} -bimodule $\tilde{\mathcal{C}} = \bigoplus_{i,j \ge 0} C_1^{ij}$. The composition of the category induces the product $\mu_{\tilde{\mathcal{C}}} \colon \tilde{\mathcal{C}} \otimes_{\mathbb{K}} \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ and thus we get a \mathbb{K} -algebra. Functors between combinatorial operations categories naturally induce \mathbb{K} -algebra morphisms.

For the other way, given a \mathbb{K} -algebra $\tilde{\mathcal{C}}$ which is also a bigraded \mathbb{k} -vector space $\bigoplus_{i,j\geq 0} C_{i,j}$, we can define $\mathcal{C} \in \mathbf{CO}_{\mathbb{k}}$ where we define categorical composition of two morphisms by the opposite product of the \mathbb{K} -algebra $\tilde{\mathcal{C}}$. Namely for morphisms $c: [i] \to [j]$ and $c': [j] \to [k]$ we define $c' \circ c$ to be $\mu_{\tilde{\mathcal{C}}}^{op}(c',c) = \mu_{\tilde{\mathcal{C}}}(c,c')$. Similarly as above \mathbb{K} -algebra morphisms induce functors between the combinatorial operations categories. It is trivial to check that compositions of these functors yield identity on both object sets.

Based on Proposition 1.2, we are going to use the terms *combinatorial operations category* and \mathbb{K} -algebra interchangeably.

1.3. Free K-algebras. The free associative K-algebra T(V) from a faithful K-bimodule V is defined as

$$T(V) := \mathbb{K} \oplus \bigoplus_{n \geqslant 1} \overbrace{V \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V}^{n\text{-times}}$$

where the multiplication on T(V) is given by concatenation. Note that any K-algebra can be written as a quotient of a free associative K-algebra of a faithful K-bimodule.

1.4. The canonical free algebras. Consider the faithful K-bimodule $\partial = \bigoplus_{n \ge 0} {}^{n+1} \partial^n$ where ${}^{n+1} \partial^n = \operatorname{Span}_{\mathbb{k}}(\partial_j^n \mid 0 \le j \le n)$ and rest of the bigraded parts are assumed to be zero. This definition implies that $\mathbf{1}_{n+1} \cdot \partial_j^n = \partial_j^n \cdot \mathbf{1}_n = \partial_j^n$ and the rest of the left or right actions of K yield zero.

Next, consider the faithful K-bimodule $\chi = \bigoplus_{n \ge 1} {}^n \chi^n$ where ${}^n \chi^n = \text{Span}_{\Bbbk}(\chi_i^n | 0 \le i \le n-1)$ and rest of the bigraded parts are assumed to be zero. This definition implies that $1_n \cdot \chi_j^n = \chi_j^n \cdot 1_n = \chi_j^n$ and the rest of the left or right actions of K yield zero.

In the Sections below, we are going to consider the free algebras $T(\partial)$ and $T(\chi)$, and their various quotients.

1.5. Mag. Because any K-algebra \mathcal{A} is defined from a K-bimodule we can define ideals generated by its elements. We will use $\langle a_i | i \in I \rangle$ to denote a bilateral ideal generated by an indexed family of elements a_i for $i \in I$.

Now, after [14, Section 1.2.5] and [11, 8], we define *the magmatic* \mathbb{K} -algebra Mag as the quotient of T(∂) by the ideal I_{Mag} where

(1.1)
$$I_{Mag} = \langle \partial_i^{n+1} \partial_j^n - \partial_{j+1}^{n+1} \partial_i^n \mid 0 \le i < j \le n \text{ and } n \ge 0 \rangle$$

The same structure is called as *pseudo-simplicial structure* in [11], and as *almost-simplicial structure* in [8].

The generator ∂_i^n can be depicted as follows:



The following string diagram represents the element with the smallest indices $\partial_0^2 \partial_1^1 - \partial_2^2 \partial_0^1$ in the ideal I_{Mag} .



We read the diagram above from bottom to top and we read the generators from right to left, as we read compositions of morphisms in a category.

Proposition 1.3. The magmatic \mathbb{K} -algebra Mag has a k-basis of trivial monomials 1_n , and nontrivial monomials of the form $\partial_{i_m}^m \cdots \partial_{i_n}^n$ with $m \ge n$ and $i_m \ge \cdots \ge i_n$ with $0 \le i_j \le j$ for all $j = n, \ldots, m$.

Proof. It is clear that Mag has a basis consisting of monomials in ∂_i^{ℓ} 's with $0 \leq i \leq \ell$. Since the defining identities in Mag are difference of two monomials of the same length, we can write a preferred basis by replacing certain submonomials (of length 2) with other submonomials (of length 2). We will refer the operation as *to straighten* from now on. Based on the relations in I_{Mag} , if $i_{j+1} < i_j$ occurs in a monomial $\partial_{i_{n+\ell}}^{n+\ell} \cdots \partial_{i_n}^n$ we can swap these indices using the identity $\partial_{i_{j+1}}^{j+1} \partial_{i_j}^j = \partial_{i_{j+1}}^{j+1} \partial_{i_{j+1}}^j$ in Mag. In other words, we can write a basis of monomials in which subscripts are non-increasing $i_{n+\ell} \geq \cdots \geq i_n$.

Remark 1.4. Consider the normalized basis of Mag in terms of the monomials of the form

(1.3)
$$\partial_{j_m}^m \cdots \partial_{j_n}^n$$
 with $j_m \ge \cdots \ge j_n$ and $0 \le j_u \le u$ for all $u = n, \dots, m$

If we let n = 0 then the indexing sequences of integers we used above are called *parking functions* [4, Chapter 13]. For notational convenience, we write $m \ge n - 1$. The case m = n - 1 corresponds to the case we have the trivial monomial 1_n from Mag, and the case m = n corresponds to the case we have only one term ∂_i^n . In other words m - n + 1 is the length of words in ∂_i^n 's.

1.6. Simp. We define the *presimplicial* K-algebra Simp as the quotient $T(\partial)/I_{Simp}$ where

(1.4)
$$I_{\text{Simp}} = \langle \partial_i^{n+1} \partial_j^n - \partial_{j+1}^{n+1} \partial_i^n \mid 0 \leq i \leq j \leq n, \text{ and } n \geq 0 \rangle$$

Note that Simp can also be defined as a quotient of the magmatic K-algebra Mag by the bilateral ideal $\langle \partial_i^{n+1} \partial_i^n - \partial_{i+1}^{n+1} \partial_i^n | 0 \leq i \leq n \rangle.$

Proposition 1.5. *The presimplicial* K-algebra Simp has a k-basis of consisting of monomials of the form

(1.5)
$$\partial_{i_m}^m \cdots \partial_{i_n}^n \text{ with } i_m > \cdots > i_n \text{ and } 0 \leq i_j \leq j$$

for every $j = n, \ldots, m$.

Proof. Recall from Remark 1.4 that we have a preferred basis for Mag of the form (1.3). Now, in Simp we replace monomials of the form $\partial_{\ell}^{j+1}\partial_{\ell}^{j}$ with $\partial_{\ell+1}^{j+1}\partial_{\ell}^{j}$. Then the fact that in the basis monomials we must have $i_m > \cdots > i_n$ easily follows.

Proposition 1.6. Simp is isomorphic to the categorical algebra of $(\Delta^+)^{\text{op}}$.

Proof. The maps $\sigma_j^n:\{0,\ldots,n+1\}\to\{0,\ldots,n\}$ are defined as the order preserving surjections that sends both j and j + 1 to j for $0 \le j \le n$ and $n \ge 0$. These surjections are subject to relations $\sigma_{j}^{n} \circ \sigma_{i}^{n+1} = \sigma_{i}^{n} \circ \sigma_{j+1}^{n+1} \text{ for } 0 \leqslant i \leqslant j \leqslant n \text{ and for all } n \geqslant 0. \text{ So any morphism } \varphi \colon \{0, \ldots, m\} \to 0$ $\{0, \ldots, n\}$ in Δ^+ can be uniquely decomposed as

$$\phi = \sigma_{i_n}^n \circ \cdots \circ \sigma_{i_{m-1}}^{m-1}$$

where $i_n < \cdots < i_{m-1}$. Notice that the monomials of the form (1.5) are in bijection with these unique compositions in the reverse order. This finishes the proof.

1.7. Braid. We define the *braid* \mathbb{K} -algebra Braid as the quotient of $T(\chi)$ by the ideal I_{Braid} where

$$I_{Braid} = \langle \chi_i^n \chi_j^n - \chi_j^n \chi_i^n, \ \chi_i^n \chi_{i+1}^n \chi_i^n - \chi_{i+1}^n \chi_i^n \chi_{i+1}^n \mid 2 \leq |i-j|, \ 2 \leq n, \ 0 \leq i \leq n-2 \rangle$$

The relations above describe the braid groups B_{n+1} on n+1 strands for $n \ge 1$. The generator χ_j^n can be depicted as follows:

0 1

(1.6)
$$\begin{array}{c|c} 0 & 1 & j & j+1 & n-1 & n \\ & & & \ddots & \searrow & \ddots & & \\ 0 & 1 & & j & j+1 & n-1 & n \end{array}$$

Then two defining relations for i < j in Braid can be depicted as follows:



1.8. Sym. We define the symmetric $\mathbb K\text{-algebra Sym}$ as the quotient $\mathsf{T}(\chi)/I_{Sym}$ where

$$I_{Sym} = I_{Braid} + \langle \chi_i^n \chi_i^n - 1_n \mid 0 \leqslant i \leqslant n - 1, \text{ and } 1 \leqslant n \rangle$$

One can equivalently define Sym as the quotient Braid/ $\langle \chi_i^n \chi_i^n - 1_n | 0 \le i \le n - 1$, and $1 \le n \rangle$. The string diagram for the extra relation defining Sym is depicted as follows:

2. DISTRIBUTIVE LAWS BETWEEN COMBINATORIAL OPERATIONS CATEGORIES

In this Section, we are going to define a distributive law on a free combinatorial operations category, and then extend it to the other cases we are interested in. The distributive laws we are going to define can be considered as distributive laws of operads or PROPs. There are examples of such distributive laws in the literature [12, 6, 5], but our cases are different and we build them from ground up using explicit generators and relations.

2.1. **Transpositions.** For a \mathbb{K} -algebra \mathcal{B} and \mathbb{K} -bimodule \mathcal{A} , a morphism of \mathbb{K} -bimodules $\omega : \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ is called a right transposition for \mathcal{B} if the following diagram commutes:

A right transposition is called *unital* if it satisfies the unitality condition:

One can also define (unital) left transpositions similarly.

2.2. **Distributive laws.** Let \mathcal{B} and \mathcal{A} be two \mathbb{K} -algebras. A morphism of \mathbb{K} -bimodules $\omega : \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ is called a distributive law if it is a unital left transposition for \mathcal{A} and it is a unital right transposition for \mathcal{B} . A distributive law ω is called *balanced* if ω is invertible and ω^{-1} is also a distributive law.

Proposition 2.1. A distributive law for two \mathbb{K} -algebras $\omega \colon \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ induces a unital associative algebra structure on the \mathbb{K} -bimodule $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ with the multiplication $(\mu_{\mathcal{A}} \otimes_{\mathbb{K}} \mu_{\mathcal{B}})(\mathcal{A} \otimes_{\mathbb{K}} \omega \otimes_{\mathbb{K}} \mathcal{B})$. We denote the resulting \mathbb{K} -algebra by $\mathcal{A} \otimes_{\omega} \mathcal{B}$.

Proof. The functors $(\cdot) \otimes_{\mathbb{K}} \mathcal{A}$ and $(\cdot) \otimes_{\mathbb{K}} \mathcal{B}$ are monads on the category of \mathbb{K} -modules. Then $(\cdot) \otimes_{\mathbb{K}} \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ is the composite endofunctor which is a monad by [3, Sect.1.].

The string diagram of this twisted product algebra structure is as follows:



Proposition 2.2. Let C be a \mathbb{K} -algebra and A, and \mathcal{B} be two \mathbb{K} -subalgebras. If C is isomorphic to $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ as a \mathbb{k} -vector space then there is a unique distributive law $\omega \colon \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ which makes $\mathcal{A} \otimes_{\omega} \mathcal{B}$ isomorphic to C as a \mathbb{K} -algebra.

Proof. The distributive law $\omega: \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ comes from the multiplication in \mathcal{C} and the fact that \mathcal{C} is isomorphic to $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ as vector spaces. The fact that ω is a unital distributive law (commutativity of the Diagram (2.2)) comes from the fact that $1_{\mathcal{A}} = 1_{\mathcal{B}}$, and the fact that ω is a right transposition (i.e. that the Diagram (2.1) commutes) can be described as $(bb')\alpha = b(b'\alpha)$, while the fact that ω is a left transposition (the dual diagram of (2.1)) can be described as $b(\alpha\alpha') = (b\alpha)\alpha'$ for every $b, b' \in \mathcal{B}$ and $\alpha, \alpha' \in \mathcal{A}$.

2.3. Distributive laws on free algebras.

Proposition 2.3. Any morphism of faithful \mathbb{K} -bimodules $\omega \colon W \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} W$ can be extended into a morphism of the type

$$\omega_{\mathfrak{n}}^{\mathfrak{m}} \colon W^{\otimes_{\mathbb{K}} \mathfrak{m}} \otimes_{\mathbb{K}} V^{\otimes_{\mathbb{K}} \mathfrak{n}} \to V^{\otimes_{\mathbb{K}} \mathfrak{n}} \otimes_{\mathbb{K}} W^{\otimes_{\mathbb{K}} \mathfrak{m}}$$

for any $n, m \ge 1$ by applying ω successively while keeping other components the same.

Proof. Extension exists and the difference in the order of application can be boiled down to the commutativity of the following diagram for the case n, $m \ge 2$ which is obvious.

We can take the direct sum of such extensions and write the ultimate version ω_*^* : $T(W) \otimes_{\mathbb{K}} T(V) \to T(V) \otimes_{\mathbb{K}} T(W)$.

Proposition 2.4. *Extension of any morphism of faithful* \mathbb{K} *-bimodules* $\omega \colon W \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} W$ *to their free* \mathbb{K} *-algebras* $\omega_*^* \colon T(W) \otimes_{\mathbb{K}} T(V) \to T(V) \otimes_{\mathbb{K}} T(W)$ *is a unital distributive law.*

2.4. The canonical distributive law.

Lemma 2.5. The \mathbb{K} -bimodule morphism ζ : $T(\vartheta) \otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}} T(\vartheta)$ defined on the generators of $T(\vartheta) \otimes_{\mathbb{K}} T(\chi)$ as

(2.3)
$$\zeta\left(\partial_{i}^{n}\otimes\chi_{j}^{n}\right) = \begin{cases} \chi_{j+1}^{n+1}\otimes\partial_{i}^{n} & \text{if } i < j\\ \chi_{i+1}^{n+1}\chi_{i}^{n+1}\otimes\partial_{i+1}^{n} & \text{if } i = j\\ \chi_{i-1}^{n+1}\chi_{i}^{n+1}\otimes\partial_{i-1}^{n} & \text{if } i = j+1\\ \chi_{j}^{n+1}\otimes\partial_{i}^{n} & \text{if } i > j+1 \end{cases}$$

defines a distributive law of free \mathbb{K} -algebras of the form ζ : $T(\mathfrak{d}) \otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}} T(\mathfrak{d})$.

The representation of this distributive law in terms of string diagrams for the second and third cases are shown below:



Remark 2.6. In the remaining sections below, we are going to show that the distributive law ζ : $T(\partial) \otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}} T(\partial)$ induces four different distributive laws of \mathbb{K} -algebras of the form

 $(2.4) Mag \otimes_{\mathbb{K}} Braid \longrightarrow Braid \otimes_{\mathbb{K}} Mag$

 $(2.5) \qquad \qquad \operatorname{Simp} \otimes_{\mathbb{K}} \operatorname{Braid} \longrightarrow \operatorname{Braid} \otimes_{\mathbb{K}} \operatorname{Simp}$

$$(2.7) \qquad \qquad \operatorname{Simp} \otimes_{\mathbb{K}} \operatorname{Sym} \longrightarrow \operatorname{Sym} \otimes_{\mathbb{K}} \operatorname{Simp}$$

Below, we will also display the string diagrams of *some* of the identities. The choice of including the string diagram just indicates that we display the diagram as a guide for the reader if the identity requires lengthy algebraic manipulations.

But before we proceed, we are going to need the following Lemma:

Lemma 2.7. Let $\mathcal{A} = T(V)/I_{\mathcal{A}}$ and $\mathcal{B} = T(W)/I_{\mathcal{B}}$ be two \mathbb{K} -algebras. Let $\omega : W \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} W$ be a morphism of faithful \mathbb{K} -bimodules that extends to a left transposition for \mathcal{A} of the form $\omega : T(W) \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} T(W)$ and to a right transposition for \mathcal{B} as $\omega : \mathcal{B} \otimes_{\mathbb{K}} T(V) \to T(V) \otimes_{\mathbb{K}} \mathcal{B}$. Then ω extends to a unital distributive law of the form $\omega : \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$.

Proof. The only thing we need to prove is that we have a well-defined morphism of \mathbb{K} -bimodules of the form $\omega \colon \mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ The assumption that ω extends to a right transposition for \mathcal{B} and a left transposition for \mathcal{A} implies that

$$\omega(I_{\mathcal{B}} \otimes_{\mathbb{K}} \mathsf{T}(\mathsf{V})) \subset \mathsf{T}(\mathsf{V}) \otimes_{\mathbb{K}} I_{\mathcal{B}} \quad \text{and} \quad \omega(\mathsf{T}(W) \otimes_{\mathbb{K}} I_{\mathcal{A}}) \subset I_{\mathcal{A}} \otimes_{\mathbb{K}} \mathsf{T}(W)$$

Since we have

$$\mathcal{B} \otimes_{\mathbb{K}} \mathcal{A} := \frac{\mathsf{T}(\mathsf{W}) \otimes_{\mathbb{K}} \mathsf{T}(\mathsf{V})}{\mathsf{I}_{\mathcal{B}} \otimes_{\mathbb{K}} \mathsf{T}(\mathsf{V}) + \mathsf{T}(\mathsf{W}) \otimes_{\mathbb{K}} \mathsf{I}_{\mathcal{A}}} \quad \text{and} \quad \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B} := \frac{\mathsf{T}(\mathsf{V}) \otimes_{\mathbb{K}} \mathsf{T}(\mathsf{W})}{\mathsf{I}_{\mathcal{A}} \otimes_{\mathbb{K}} \mathsf{T}(\mathsf{W}) + \mathsf{T}(\mathsf{V}) \otimes_{\mathbb{K}} \mathsf{I}_{\mathcal{B}}}$$

we see that we have a well-defined extension.

2.5. Mag $\otimes_{\mathbb{K}}$ Braid \to Braid $\otimes_{\mathbb{K}}$ Mag. We need to show that the conditions stated in Lemma 2.7 are satisfied for ζ . Firstly, we are going to show that $\zeta : T(\partial) \otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}} T(\partial)$ extends to a left transposition of the form $\zeta : T(\partial) \otimes_{\mathbb{K}} Braid \to Braid \otimes_{\mathbb{K}} T(\partial)$. We need to show that the following diagram commutes:

$$\begin{array}{c} \Gamma(\partial) \otimes_{\mathbb{K}} \operatorname{Braid} \otimes_{\mathbb{K}} \operatorname{Braid} \xrightarrow{\zeta \otimes \operatorname{Braid}} \operatorname{Braid} \otimes_{\mathbb{K}} T(\partial) \otimes_{\mathbb{K}} \operatorname{Braid} \xrightarrow{\operatorname{Braid} \otimes_{\zeta}} \operatorname{Braid} \otimes_{\mathbb{K}} \operatorname{Braid} \otimes_{\mathbb{K}} T(\partial) \\ & \downarrow^{\mu_{\operatorname{Braid}} \otimes T(\partial)} \\ T(\partial) \otimes_{\mathbb{K}} \operatorname{Braid} \xrightarrow{\zeta} & \operatorname{Braid} \otimes_{\mathbb{K}} T(\partial) \end{array}$$

We check if this is well-defined, i.e. if $\zeta(T(\partial) \otimes_{\mathbb{K}} \mathcal{I}_{Braid}) \subset \mathcal{I}_{Braid} \otimes_{\mathbb{K}} T(\partial)$. For this, we have to check that the defining relations of \mathcal{I}_{Braid} are preserved.

We check the first relation in \mathcal{I}_{Braid} , namely,

$$\zeta(\mathfrak{d}^{\mathfrak{n}}_{\mathfrak{i}}\otimes\chi^{\mathfrak{n}}_{\mathfrak{j}}\chi^{\mathfrak{n}}_{k})=\zeta(\mathfrak{d}^{\mathfrak{n}}_{\mathfrak{i}}\otimes\chi^{\mathfrak{n}}_{k}\chi^{\mathfrak{n}}_{\mathfrak{j}})$$

in Braid $\otimes_{\mathbb{K}} T(\partial)$. We need to verify this equality for all $0 \leq i \leq n$ and $0 \leq j, k \leq n - 1$ with $|j - k| \geq 2$. We examine this in 7 cases as follows:

Case (1): i < j and i < k:

$$\begin{split} \zeta(\partial_{i}^{n}\otimes\chi_{j}^{n}\chi_{k}^{n}) =& \chi_{j+1}^{n+1}\chi_{k+1}^{n+1}\otimes\partial_{i}^{n} \\ =& \chi_{k+1}^{n+1}\chi_{j+1}^{n+1}\otimes\partial_{i}^{n} = \zeta(\partial_{i}^{n}\otimes\chi_{k}^{n}\chi_{j}^{n}) \end{split}$$

since $|(j + 1) - (k + 1)| = |j - k| \ge 2$.

Case (2): i < j and i = k. Since $|(j + 1) - (i + 1)| = |(j + 1) - (k + 1)| = |j - k| = |j - i| \ge 2$ and clearly $|(j + 1) - i| \ge |j - i|$, we get

$$\begin{split} \zeta(\partial_i^n \otimes \chi_j^n \chi_i^n) = & \chi_{j+1}^{n+1} \chi_{i+1}^{n+1} \chi_i^{n+1} \otimes \partial_{i+1}^n \\ = & \chi_{i+1}^{n+1} \chi_i^{n+1} \chi_{j+1}^{n+1} \otimes \partial_{i+1}^n = \zeta(\partial_i^n \otimes \chi_i^n \chi_j^n) \end{split}$$

We can depict the equality above with the string diagrams below.



Since all derivations are reversible, this case is equivalent to the case where i < k and i = j. Case (3): i < j and i = k + 1 (or k = i - 1.) Conditions imply that k < i < j, therefore $(j + 1) - i = (j + 1) - (k + 1) = |j - k| \ge 2$ and clearly $(j + 1) - (i - 1) = ((j + 1) - i) + 1 \ge 3$. So, we have

$$\begin{split} \zeta(\boldsymbol{\vartheta}_{i}^{n}\otimes\boldsymbol{\chi}_{j}^{n}\boldsymbol{\chi}_{i-1}^{n}) = & \chi_{j+1}^{n+1}\boldsymbol{\chi}_{i-1}^{n+1}\boldsymbol{\chi}_{i}^{n+1}\otimes\boldsymbol{\vartheta}_{i-1}^{n} \\ = & \chi_{i-1}^{n+1}\boldsymbol{\chi}_{i}^{n+1}\boldsymbol{\chi}_{j+1}^{n+1}\otimes\boldsymbol{\vartheta}_{i-1}^{n} = \zeta(\boldsymbol{\vartheta}_{i}^{n}\otimes\boldsymbol{\chi}_{i-1}^{n}\boldsymbol{\chi}_{j}^{n}) \end{split}$$

One can depict the equality above by the following string diagrams:



As before, since all derivations are reversible, this case is equivalent to the case where i < k and i = j + 1.

Case (4): i < j and i > k + 1. Since k < i < j we have $j - k \ge 2$, and therefore, $(j + 1) - k \ge 2$. Then $\chi_{j+1}^{n+1}\chi_k^{n+1} = \chi_k^{n+1}\chi_{j+1}^{n+1}$ which implies

$$\begin{split} \zeta(\partial_{i}^{n}\otimes\chi_{j}^{n}\chi_{k}^{n}) =& \chi_{j+1}^{n+1}\chi_{k}^{n+1}\otimes\partial_{i}^{n} \\ =& \chi_{k}^{n+1}\chi_{j+1}^{n+1}\otimes\partial_{i}^{n} = \zeta(\partial_{i}^{n}\otimes\chi_{k}^{n}\chi_{j}^{n}) \end{split}$$

Case (5): i = j and i > k + 1. Since k < k + 1 < i = j we have $|(i + 1) - k| = |(j + 1) - k| = j - k + 1 \ge 2$ and clearly $|i - k| = |j - k| \ge 2$. Then

$$\begin{split} \zeta(\partial_{i}^{n}\otimes\chi_{i}^{n}\chi_{k}^{n}) =& \chi_{i+1}^{n+1}\chi_{i}^{n+1}\chi_{k}^{n+1}\otimes\partial_{i+1}^{n} \\ =& \chi_{k}^{n+1}\chi_{i+1}^{n+1}\chi_{i}^{n+1}\otimes\partial_{i+1}^{n} = \zeta(\partial_{i}^{n}\otimes\chi_{k}^{n}\chi_{i}^{n}) \end{split}$$

We can represent the equality above with the string diagram below:



These derivations are reversible as before. Thus this case is equivalent to the case where i < k and i > j + 1.

Case (6): i = j + 1 (or j = i - 1) and i > k + 1. Since i > j > k with $j - k \ge 2$ in this case, we have $i - k > (i - 1) - k = j - k \ge 2$. Then

$$\begin{split} \zeta(\partial_{i}^{n}\otimes\chi_{i-1}^{n}\chi_{k}^{n}) =& \chi_{i-1}^{n+1}\chi_{i}^{n+1}\chi_{k}^{n+1}\otimes\partial_{i-1}^{n} \\ =& \chi_{k}^{n+1}\chi_{i-1}^{n+1}\chi_{i}^{n+1}\otimes\partial_{i-1}^{n} = \zeta(\partial_{i}^{n}\otimes\chi_{k}^{n}\chi_{i-1}^{n}) \end{split}$$

The string diagram depicting the equality above is as follows:



This case is equivalent to the case where i > j + 1 and i = k + 1 since the derivations are reversible.

Case (7): i > j + 1 and i > k + 1

$$\begin{split} \zeta(\partial_i^n \otimes \chi_j^n \chi_k^n) = & \chi_j^{n+1} \chi_k^{n+1} \otimes \partial_i^n \\ = & \chi_k^{n+1} \chi_j^{n+1} \otimes \partial_i^n = \zeta(\partial_i^n \otimes \chi_k^n \chi_j^n) \end{split}$$

We have to check the remaining relation in \mathcal{I}_{Braid} , namely,

$$\zeta(\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{n}}\otimes\chi_{\mathfrak{j}}^{\mathfrak{n}}\chi_{\mathfrak{j}+1}^{\mathfrak{n}}\chi_{\mathfrak{j}}^{\mathfrak{n}})=\zeta(\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{n}}\otimes\chi_{\mathfrak{j}+1}^{\mathfrak{n}}\chi_{\mathfrak{j}}^{\mathfrak{n}}\chi_{\mathfrak{j}+1}^{\mathfrak{n}})$$

in Braid $\otimes_{\mathbb{K}} T(\partial)$. We need to verify this equality for all $0 \le i \le n$ and $0 \le j \le n-2$ where $n \ge 2$. We examine this in 5 cases as follows: Case (1): i < j

$$\begin{split} \zeta(\partial_i^n \otimes \chi_j^n \chi_{j+1}^n \chi_j^n) =& \chi_{j+1}^{n+1} \chi_{j+2}^{n+1} \chi_{j+1}^{n+1} \otimes \partial_i^n \\ =& \chi_{j+2}^{n+1} \chi_{j+1}^{n+1} \chi_{j+2}^{n+1} \otimes \partial_i^n = \zeta(\partial_i^n \otimes \chi_{j+1}^n \chi_j^n \chi_{j+1}^n) \end{split}$$

Case (2): i > j + 2

$$\begin{aligned} \zeta(\partial_i^n \otimes \chi_j^n \chi_{j+1}^n \chi_j^n) = & \chi_j^{n+1} \chi_{j+1}^{n+1} \chi_j^{n+1} \otimes \partial_i^n \\ = & \chi_{j+1}^{n+1} \chi_j^{n+1} \chi_{j+1}^{n+1} \otimes \partial_i^n = \zeta(\partial_i^n \otimes \chi_{j+1}^n \chi_j^n \chi_{j+1}^n) \end{aligned}$$

Case (3): i = j + 2. We have

$$\zeta(\partial_{j+2}^{n} \otimes \chi_{j+1}^{n} \chi_{j}^{n} \chi_{j+1}^{n}) = \chi_{j+1}^{n+1} \chi_{j+2}^{n+1} \chi_{j+1}^{n+1} \chi_{j+1}^{n+1} \chi_{j+2}^{n+1} \otimes \partial_{j}^{n}$$

On the other hand, we have

$$\begin{split} \chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1} = & \chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\\ = & \chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1} = \chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1} \end{split}$$

Thus

$$\begin{split} \chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\otimes \partial_{j}^{n} = & \chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\otimes \partial_{j}^{n} \\ = & \zeta(\partial_{j+2}^{n}\otimes\chi_{j}^{n}\chi_{j+1}^{n}\chi_{j}^{n}) \end{split}$$

These identities can be seen more easily from the following diagrams:



Case (4): i = j + 1. We have

$$\zeta(\mathfrak{d}_{j+1}^{n} \otimes \chi_{j+1}^{n} \chi_{j}^{n} \chi_{j+1}^{n}) = \chi_{j+2}^{n+1} \chi_{j+1}^{n+1} \chi_{j}^{n+1} \chi_{j+1}^{n+1} \chi_{j+2}^{n+1} \otimes \mathfrak{d}_{j+1}^{n}$$

On the other hand we have

$$\begin{split} \chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1} = & \chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+2}^{n+1}\\ = & \chi_{j}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1} = \chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1} \end{split}$$

Thus

$$\begin{split} \chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\otimes \mathfrak{d}_{j+1}^{n} = &\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\otimes \mathfrak{d}_{j+1}^{n} \\ = &\zeta(\mathfrak{d}_{j+1}^{n}\otimes\chi_{j}^{n}\chi_{j+1}^{n}\chi_{j}^{n}) \end{split}$$

Diagrammatic interpretation is the following for this case:



Case (5): i = j

$$\zeta(\partial_{j}^{n} \otimes \chi_{j+1}^{n} \chi_{j}^{n} \chi_{j+1}^{n}) = \chi_{j+2}^{n+1} \chi_{j+1}^{n+1} \chi_{j}^{n+1} \chi_{j+2}^{n+1} \chi_{j+1}^{n+1} \otimes \partial_{j+2}^{n}$$

On the other hand we have

$$\begin{split} \chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1} = & \chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1} = \chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j+1}^{n+1} \\ = & \chi_{j+1}^{n+1}\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1} = \chi_{j+1}^{n+1}\chi_{j}^{n}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n} \end{split}$$

Thus

$$\begin{split} \chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\otimes \mathfrak{d}_{j+2}^{n} = &\chi_{j+1}^{n+1}\chi_{j}^{n}\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n}\otimes \mathfrak{d}_{j+2}^{n} \\ = &\zeta(\mathfrak{d}_{j}^{n}\otimes\chi_{j}^{n}\chi_{j+1}^{n}\chi_{j}^{n}) \end{split}$$



We also need to check if $\zeta : T(\vartheta) \otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}} T(\vartheta)$ extends to a right transposition for Mag:

$$\begin{array}{cccc} Mag \otimes_{\mathbb{K}} Mag \otimes_{\mathbb{K}} \mathsf{T}(\chi) & \xrightarrow{(Mag \otimes \zeta)} & Mag \otimes_{\mathbb{K}} \mathsf{T}(\chi) \otimes_{\mathbb{K}} Mag & \xrightarrow{\zeta \otimes Mag} & \mathsf{T}(\chi) \otimes_{\mathbb{K}} Mag \otimes_{\mathbb{K}} \mathsf{Mag} \\ & & \downarrow^{\mu_{Mag} \otimes \mathsf{T}(\chi)} & & \mathsf{T}(\chi) \otimes_{\mathbb{K}} \mathsf{Mag} \downarrow \\ & & \mathsf{Mag} \otimes_{\mathbb{K}} \mathsf{T}(\chi) & \xrightarrow{\zeta} & \mathsf{T}(\chi) \otimes_{\mathbb{K}} \mathsf{Mag} \end{array}$$

For this, in $T(\chi) \otimes_{\mathbb{K}} Mag$ we need to verify

$$\zeta(\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{n}+1}\mathfrak{d}_{\mathfrak{j}}^{\mathfrak{n}}\otimes\chi_{k}^{\mathfrak{n}})=\zeta(\mathfrak{d}_{\mathfrak{j}+1}^{\mathfrak{n}+1}\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{n}}\otimes\chi_{k}^{\mathfrak{n}})$$

for all $0 \le i < j \le n$ and $0 \le k \le n - 1$ where $n \ge 1$. We examine this in 7 cases as follows: Case (1): k < i - 1 < i < j. Since j > k + 1, we also have j + 1 > k + 1 which implies

$$\begin{split} \zeta(\partial_{i}^{n+1}\partial_{j}^{n}\otimes\chi_{k}^{n})=&\chi_{k}^{n+2}\otimes\partial_{i}^{n+1}\partial_{j}^{n}\\ =&\chi_{k}^{n+2}\otimes\partial_{j+1}^{n+1}\partial_{i}^{n}=\zeta(\partial_{j+1}^{n+1}\partial_{i}^{n}\otimes\chi_{k}^{n}) \end{split}$$

Case (2): k = i - 1 < i < j $\zeta \left(\partial_i^{n+1} \partial_j^n \otimes \chi_{i-1}^n \right) = \chi_{i-1}^{n+2} \chi_i^{n+2} \otimes \partial_{i-1}^{n+1} \partial_j^n$ $= \chi_{i-1}^{n+2} \chi_i^{n+2} \otimes \partial_{j+1}^{n+1} \partial_{i-1}^n = \zeta \left(\partial_{j+1}^{n+1} \partial_i^n \otimes \chi_{i-1}^n \right)$

Case (3): k = i < i + 1 < j

$$\begin{split} \zeta \left(\partial_i^{n+1} \partial_j^n \otimes \chi_i^n \right) = & \chi_{i+1}^{n+2} \chi_i^{n+2} \otimes \partial_{i+1}^{n+1} \partial_j^n \\ = & \chi_{i+1}^{n+2} \chi_i^{n+2} \otimes \partial_{j+1}^{n+1} \partial_{i+1}^n = \zeta \left(\partial_{j+1}^{n+1} \partial_i^n \otimes \chi_i^n \right) \end{split}$$

Case (4): k = i < i + 1 = j

$$\begin{split} \zeta \left(\partial_i^{n+1} \partial_{i+1}^n \otimes \chi_i^n \right) = & \chi_{i+1}^{n+2} \chi_i^{n+2} \chi_{i+2}^{n+2} \chi_{i+1}^{n+2} \otimes \partial_{i+2}^{n+1} \partial_i^n \\ = & \chi_{i+1}^{n+2} \chi_{i+2}^{n+2} \chi_i^{n+2} \chi_{i+1}^{n+2} \otimes \partial_i^{n+1} \partial_{i+1}^n = \zeta \left(\partial_{i+2}^{n+1} \partial_i^n \otimes \chi_i^n \right) \end{split}$$

The string diagrams for the equality above are as follows:



Case (5): i < k < j - 1 < j

$$\begin{split} \zeta \left(\partial_i^{n+1} \partial_j^n \otimes \chi_k^n \right) = & \chi_{k+1}^{n+2} \otimes \partial_i^{n+1} \partial_j^n \\ = & \chi_{k+1}^{n+2} \otimes \partial_{j+1}^{n+1} \partial_i^n = \zeta \left(\partial_{j+1}^{n+1} \partial_i^n \otimes \chi_k^n \right) \end{split}$$

Case (6): i < k = j - 1 < j

$$\begin{split} \zeta \left(\partial_i^{n+1} \partial_j^n \otimes \chi_{j-1}^n \right) = & \chi_j^{n+2} \chi_{j+1}^{n+2} \otimes \partial_i^{n+1} \partial_{j-1}^n \\ = & \chi_j^{n+2} \chi_{j+1}^{n+2} \otimes \partial_j^{n+1} \partial_i^n = \zeta \left(\partial_{j+1}^{n+1} \partial_i^n \otimes \chi_{j-1}^n \right) \end{split}$$

Case (7): i < k = j

$$\begin{split} \zeta \left(\partial_i^{n+1} \partial_j^n \otimes \chi_j^n \right) = & \chi_{j+2}^{n+2} \chi_{j+1}^{n+2} \otimes \partial_i^{n+1} \partial_{j+1}^n \\ = & \chi_{j+2}^{n+2} \chi_{j+1}^{n+2} \otimes \partial_{j+2}^{n+1} \partial_i^n = \zeta \left(\partial_{j+1}^{n+1} \partial_i^n \otimes \chi_j^n \right) \end{split}$$

Case (8): i < j < k.

$$\begin{split} \zeta(\vartheta_i^{n+1}\vartheta_j^n\otimes\chi_k^n) =& \chi_{k+2}^{n+2}\otimes\vartheta_i^{n+1}\vartheta_j^n \\ =& \chi_{k+2}^{n+2}\otimes\vartheta_{j+1}^{n+1}\vartheta_i^n = \zeta(\vartheta_{j+1}^{n+1}\vartheta_i^n\otimes\chi_k^n) \end{split}$$

2.6. Simp $\otimes_{\mathbb{K}}$ Braid \rightarrow Braid $\otimes_{\mathbb{K}}$ Simp. We have already established that $\zeta : T(\partial) \otimes_{\mathbb{K}} T(\chi) \rightarrow T(\chi) \otimes_{\mathbb{K}} T(\partial)$ extends to a left transposition of the form $\zeta : T(\partial) \otimes_{\mathbb{K}} Braid \rightarrow Braid \otimes_{\mathbb{K}} T(\partial)$. We only need to show that $\zeta : T(\partial) \otimes_{\mathbb{K}} T(\chi) \rightarrow T(\chi) \otimes_{\mathbb{K}} T(\partial)$ also extends to a right transposition for Simp, ζ : Simp $\otimes_{\mathbb{K}} T(\chi) \rightarrow T(\chi) \otimes_{\mathbb{K}} Simp$. Observe that most of the work is also done here because we showed that the distributive law ζ extends from Mag to Simp. Also, note that an equivalence of

elements in Mag implies their equivalence in Simp. The only remaining relation we check in \mathcal{I}_{Simp} is this:

$$\zeta(\partial_{i}^{n+1}\partial_{i}^{n}\otimes\chi_{k}^{n})=\zeta(\partial_{i+1}^{n+1}\partial_{i}^{n}\otimes\chi_{k}^{n})$$

for all $0 \le i \le n$ and $0 \le k \le n-1$ where $n \ge 1$. We examine this in 4 cases as follows:

Case(1): i < k. Since i < k, we also have i + 1 < k + 1 which implies

$$\begin{split} \zeta \left(\partial_{i}^{n+1} \partial_{i}^{n} \otimes \chi_{k}^{n} \right) = & \chi_{k+2}^{n+2} \otimes \partial_{i}^{n+1} \partial_{i}^{n} \\ = & \chi_{k+2}^{n+2} \otimes \partial_{i+1}^{n+1} \partial_{i}^{n} = \zeta \left(\partial_{i+1}^{n+1} \partial_{i}^{n} \otimes \chi_{k}^{n} \right) \end{split}$$

Case(2): i > k + 1

$$\begin{split} \zeta \left(\partial_i^{n+1} \partial_i^n \otimes \chi_k^n \right) = & \chi_k^{n+2} \otimes \partial_i^{n+1} \partial_i^n \\ = & \chi_k^{n+2} \otimes \partial_{i+1}^{n+1} \partial_i^n = \zeta \left(\partial_{i+1}^{n+1} \partial_i^n \otimes \chi_k^n \right) \end{split}$$

Case(3): i = k

$$\begin{split} \zeta \left(\partial_{i}^{n+1} \partial_{i}^{n} \otimes \chi_{i}^{n} \right) = & \chi_{i+2}^{n+2} \chi_{i+1}^{n+2} \chi_{i}^{n+2} \otimes \partial_{i+1}^{n+1} \partial_{i+1}^{n} \\ = & \chi_{i+2}^{n+2} \chi_{i+1}^{n+2} \chi_{i}^{n+2} \otimes \partial_{i+2}^{n+1} \partial_{i+1}^{n} = \zeta \left(\partial_{i+1}^{n+1} \partial_{i}^{n} \otimes \chi_{i}^{n} \right) \end{split}$$



Case(4): i = k + 1

$$\begin{split} \zeta \left(\partial_{i}^{n+1} \partial_{i}^{n} \otimes \chi_{i-1}^{n} \right) = & \chi_{i-1}^{n+2} \chi_{i}^{n+2} \chi_{i+1}^{n+2} \otimes \partial_{i-1}^{n+1} \partial_{i-1}^{n} \\ = & \chi_{i-1}^{n+2} \chi_{i}^{n+2} \chi_{i+1}^{n+2} \otimes \partial_{i}^{n+1} \partial_{i-1}^{n} = \zeta \left(\partial_{i+1}^{n+1} \partial_{i}^{n} \otimes \chi_{i-1}^{n} \right) \end{split}$$



2.7. Mag $\otimes_{\mathbb{K}}$ Sym \to Sym $\otimes_{\mathbb{K}}$ Mag. Similar to the previous case, most of the things we need to show are done in Section 2.5. The only remaining relation we check in \mathcal{I}_{Sym} is

$$\zeta(\partial_{i}^{n}\otimes\chi_{i}^{n}\chi_{i}^{n})=\zeta(\partial_{i}^{n}\otimes\mathbf{1}_{n})$$

for all $0 \le i \le n$ and $0 \le j \le n - 1$ where $n \ge 1$. We examine this in 4 cases as follows:

Case(1): i < j

$$\begin{split} \zeta \left(\partial_{i}^{n} \otimes \chi_{j}^{n} \chi_{j}^{n} \right) = & \chi_{j+1}^{n+1} \chi_{j+1}^{n+1} \otimes \partial_{i}^{n} \\ = & \mathbf{1}_{n+1} \otimes \partial_{i}^{n} = \zeta \left(\partial_{i}^{n} \otimes \mathbf{1}_{n} \right) \end{split}$$

Case(2): i > j + 1

$$\begin{split} \zeta \left(\partial_{i}^{n} \otimes \chi_{j}^{n} \chi_{j}^{n} \right) = & \chi_{j}^{n+1} \chi_{j}^{n+1} \otimes \partial_{i}^{n} \\ = & \mathbf{1}_{n+1} \otimes \partial_{i}^{n} = \zeta \left(\partial_{i}^{n} \otimes \mathbf{1}_{n} \right) \end{split}$$

Case(3): i = j

$$\begin{aligned} \zeta \left(\partial_{i}^{n} \otimes \chi_{i}^{n} \chi_{i}^{n} \right) = & \chi_{i+1}^{n+1} \chi_{i}^{n+1} \chi_{i}^{n+1} \chi_{i+1}^{n+1} \otimes \partial_{i}^{n} = \chi_{i+1}^{n+1} \chi_{i+1}^{n+1} \otimes \partial_{i}^{n} \\ = & \mathbf{1}_{n+1} \otimes \partial_{i}^{n} = \zeta \left(\partial_{i}^{n} \otimes \mathbf{1}_{n} \right) \end{aligned}$$

Case(4): i = j + 1

$$\begin{aligned} \zeta \left(\partial_i^n \otimes \chi_{i-1}^n \chi_{i-1}^n \right) = & \chi_{i-1}^{n+1} \chi_i^{n+1} \chi_i^{n+1} \chi_{i-1}^{n+1} \otimes \partial_i^n = \chi_{i-1}^{n+1} \chi_{i-1}^{n+1} \otimes \partial_i^n \\ = & \mathbf{1}_{n+1} \otimes \partial_i^n = \zeta \left(\partial_i^n \otimes \mathbf{1}_n \right) \end{aligned}$$

2.8. Simp $\otimes_{\mathbb{K}}$ Sym \to Sym $\otimes_{\mathbb{K}}$ Simp. We have already established that $\zeta : T(\partial) \otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}} T(\partial)$ extends to both left transposition of the form $\zeta : T(\partial) \otimes_{\mathbb{K}}$ Sym \to Sym $\otimes_{\mathbb{K}} T(\partial)$ and right transposition $\zeta :$ Simp $\otimes_{\mathbb{K}} T(\chi) \to T(\chi) \otimes_{\mathbb{K}}$ Simp. By Lemma 2.7, ζ induces a distributive law Simp $\otimes_{\mathbb{K}}$ Sym \to Sym $\otimes_{\mathbb{K}}$ Simp as well.

3. THE LEIBNIZ COMBINATORIAL OPERATIONS CATEGORY

From here on, by abuse of notation, we will write ∂_j^n for $1_{n+1} \otimes \partial_j^n$ and similarly χ_j^n for $\chi_j^n \otimes 1_n$ in Braid \otimes_{ζ} Mag, or in any of its quotients.

3.1. Leib **and** Leib^{op}. Let \mathcal{G} be Braid or Sym. Recall that $\mathcal{G} \otimes_{\zeta}$ Mag is a \mathbb{K} -algebra where the multiplication is determined by the distributive law ζ as described in Proposition 5.16. We define two ideals of $\mathcal{G} \otimes_{\zeta}$ Mag as follows:

(3.1)
$$I_{\text{Leib}} := \left\langle \partial_{j+1}^{n+1} \partial_{j}^{n} - \left(\mathbf{1}_{n+2} - \chi_{j+1}^{n+2} \right) \partial_{j}^{n+1} \partial_{j}^{n} \mid 0 \leq j \leq n, \ n \geq 0 \right\rangle$$

and

$$(3.2) I_{\text{Leib}^{\text{op}}} := \left\langle \partial_j^{n+1} \partial_j^n - \left(\mathbf{1}_{n+2} - \chi_{j+1}^{n+2} \right) \partial_{j+1}^{n+1} \partial_j^n \mid 0 \leqslant j \leqslant n, \ n \geqslant 0 \right\rangle$$

Then we define \mathbb{K} -algebras Leib and Leib^{op} as the quotients $(\mathcal{G} \otimes_{\zeta} Mag)/I_{Leib}$ and $(\mathcal{G} \otimes_{\zeta} Mag)/I_{Leib^{op}}$, respectively. In the case $\mathcal{G} =$ Sym these \mathbb{K} -algebras are called *left Leibniz* and *right Leibniz* \mathbb{K} -algebras, respectively. In the case $\mathcal{G} =$ Braid, we call them *braided left Leibniz* and *braided right Leibniz* \mathbb{K} -algebras. Unless otherwise stated we will use the unbraided versions of the Leibniz \mathbb{K} -algebras below.

The element of the I_{Leib} with the smallest superscript n = 0 is represented by the following string diagram:



Proposition 3.1. Assume $\mathcal{G} = \text{Braid}$ or $\mathcal{G} = \text{Sym.}$ Consider the automorphism $\alpha : \mathcal{G} \otimes_{\zeta} \text{Mag} \rightarrow \mathcal{G} \otimes_{\zeta} \text{Mag}$ of \mathbb{K} -bimodules defined on the generators

$$\alpha(1_n) = 1_n, \quad \alpha(\chi_j^n) = \chi_j^n \quad and \quad \alpha(\partial_j^n) = \chi_j^{n+1}\partial_j^n$$

for any $n \ge 0$ and $0 \le j \le n$. Then α extends to an automorphism of \mathbb{K} -algebras.

Proof. If you would like to follow the equations below by drawing corresponding string diagrams for α , the nontrivial parts depicted after relevant calculations. The diagram for nonidentity part of α on the generators can be seen as follows:



We must prove that α preserves the relations in Braid \otimes_{ζ} Mag. We start with the relations in Mag: for i < j we obtain

$$\begin{aligned} \alpha(\partial_{i}^{n+1}\partial_{j}^{n}) =& \chi_{i}^{n+2}\partial_{i}^{n+1}\chi_{j}^{n+1}\partial_{j}^{n} = \chi_{i}^{n+2}\chi_{j+1}^{n+2}\partial_{i}^{n+1}\partial_{j}^{n} \\ =& \chi_{j+1}^{n+2}\chi_{i}^{n+2}\partial_{j+1}^{n+1}\partial_{i}^{n} = \chi_{j+1}^{n+2}\partial_{j+1}^{n+1}\chi_{i}^{n+1}\partial_{i}^{n} \\ =& \alpha(\partial_{i+1}^{n+1}\partial_{i}^{n}) \end{aligned}$$

by using the distributive law ζ . As for the interaction between χ_i^n and ∂_j^n , we consider 4 different cases:

Case (1) i < j.

$$\begin{aligned} \alpha(\partial_i^n \chi_j^n) = & \chi_i^{n+1} \partial_i^n \chi_j^n = \chi_i^{n+1} \chi_{j+1}^n \partial_i^n \\ = & \chi_{j+1}^{n+1} \chi_i^n \partial_i^n = \alpha(\chi_{j+1}^{n+1} \partial_i^n) \end{aligned}$$

Case (2) i = j.

$$\begin{aligned} \alpha(\partial_{i}^{n}\chi_{i}^{n}) = & \chi_{i}^{n+1}\partial_{i}^{n}\chi_{i}^{n} = \chi_{i}^{n+1}\chi_{i+1}^{n+1}\chi_{i}^{n+1}\partial_{i+1}^{n} \\ = & \chi_{i+1}^{n+1}\chi_{i}^{n+1}\chi_{i+1}^{n+1}\partial_{i+1}^{n} = \alpha(\chi_{i+1}^{n+1}\chi_{i}^{n+1}\partial_{i+1}^{n}) \end{aligned}$$



Case (3) i = j + 1.

$$\begin{aligned} \alpha(\partial_{j+1}^{n}\chi_{j}^{n}) = & \chi_{j+1}^{n+1}\partial_{j+1}^{n}\chi_{j}^{n} = \chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\partial_{j}^{n} \\ = & \chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j}^{n+1}\partial_{j}^{n} = \alpha(\chi_{j}^{n+1}\chi_{j+1}^{n+1}\partial_{j}^{n}) \end{aligned}$$



Case (4) i > j + 1.

$$\alpha(\partial_i^n \chi_j^n) = \chi_i^{n+1} \partial_i^n \chi_j^n = \chi_i^{n+1} \chi_j^{n+1} \partial_i^n$$
$$= \chi_j^{n+1} \chi_i^{n+1} \partial_i^n = \alpha(\chi_j^{n+1} \partial_i^n)$$

The result follows.

Proposition 3.2. The left and right Leibniz \mathbb{K} -algebras Leib and Leib^{op} are isomorphic via α we defined in Proposition 3.1.

Proof. We know that α is an automorphism of the K-algebra Sym \otimes_{ζ} Mag. We will show that α maps the ideals I_{Leib} and $I_{\text{Leib}^{\text{op}}}$ to each other thus prove our statement. So we apply α to each generator of I_{Leib}

$$\begin{aligned} \alpha(\partial_{j+1}^{n+1}\partial_{j}^{n} - (\mathbf{1}_{n+2} - \chi_{j+1}^{n+2})\partial_{j}^{n+1}\partial_{j}^{n}) \\ = & \chi_{j+1}^{n+2}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\partial_{j}^{n+1}\partial_{j}^{n} - (\mathbf{1}_{n+2} - \chi_{j+1}^{n+2})\chi_{j}^{n+2}\chi_{j+1}^{n+2}\chi_{j}^{n+2}\partial_{j+1}^{n+1}\partial_{j}^{n} \\ = & \chi_{j}^{n+2}\chi_{j+1}^{n+1}\chi_{j}^{n+1} \left(\partial_{j}^{n+1}\partial_{j}^{n} - (\mathbf{1}_{n+2} - \chi_{j}^{n+2})\partial_{j+1}^{n+1}\partial_{j}^{n}\right) \\ \leqslant n. \end{aligned}$$

for any $0 \leq j \leq n$.

3.2. A different presentation for Leib.

Proposition 3.3. Assume $\mathcal{G} = \text{Braid}$ or $\mathcal{G} = \text{Sym}$. The product $\mathcal{G} \otimes_{\zeta} \text{Mag}$ has a basis that consists of monomials of the form

(3.3)
$$\tau_{m+1}\partial_{j_m}^m\cdots\partial_{j_r}^n$$

where $m \ge n - 1$, $\tau_{m+1} \in B_{m+2}$ (resp. $\tau_{m+1} \in S_{m+2}$) and $j_m \ge \cdots \ge j_n$. Based on this fact, $\mathcal{G} \otimes_{\zeta} \text{Simp has a basis that consists of monomials of given in Equation 3.3 but this time for <math>j_m > \cdots > j_n$.

Proof. We will give the proof for Mag for the case $\mathcal{G} = Braid$. The other cases are similar. Note that the relations of the distributive law given in Equation (2.3) indicate that we can straighten the arbitrary mixed monomials of χ_i^n and ∂_j^n where χ_i^n 's move to the left and ∂_j^n 's move to the right. Once this is done, we can straighten ∂_j^n 's using Propositions 1.3 and 1.5. We regroup the monomials of χ_i^n on the left and call it τ_{m+1} . Hence we can always obtain monomials of the form $\tau_{m+1}\partial_{j_m}^m \cdots \partial_{j_n}^n$ with particular subindices described above.

Proposition 3.4. Leib^{op} and Sym \otimes_{ζ} Simp have the same \Bbbk -basis.

Proof. By applying the new relations we get from $I_{Leib^{op}}$, we need to straighten the basis monomials with monomials conforming to the condition stated above. Notice that we do not need to straighten the Braid group part, but the Mag part. We will write the proof by induction on the length of the

monomials coming from Mag. For m = n (the trivial monomial 1_n) and m = n + 1 the statement is trivial. So, the base case is when m = n + 2. If we have a monomial of the form $\partial_{j_{n+1}}^{n+1} \partial_{j_n}^n$ with $j_{n+1} \ge j_n$. The only case where this monomial has to be replaced is when $j_{n+1} = j_n$. In that case we replace $\partial_{j_n}^{n+1} \partial_{j_n}^n$ with $(1 - \chi_{j_n+1}^{n+2})\partial_{j_n+1}^{n+1}\partial_{j_n}^n$ and since $j_n + 1 > j_n$, the new monomial conforms to the statement. Assume any monomial of length ℓ can be straighten to conform to the statement. Assume $\partial_{j_m}^m \cdots \partial_{j_n}^n$ with $m - n = \ell$ is a monomial in Leib of length $\ell + 1$. Notice that the relations in Leib^{op} indicate that if a length 2 part $\partial_{j_{u+1}}^{u+1} \partial_{j_u}^u$ of a monomial $\partial_{j_m}^m \cdots \partial_{j_n}^n$ is replaced

$$\underbrace{\partial_{j_m}^m \cdots \partial_{j_{u+1}}^{u+1}}_{\text{affected region}} \underbrace{\partial_{j_u}^u \cdots \partial_{j_n}^n}_{\text{unaffected region}}$$

the part of the monomial to the right of $\partial_{j_u}^u$ stays unaffected. Thus if the monomial already satisfies $j_{n+1} > j_n$, we can straighten $\partial_{j_m}^m \cdots \partial_{j_{n+1}}^{n+1}$ and then attach $\partial_{j_n}^n$ after the fact. If, on the other hand, $j_{n+1} = j_n$ then $\partial_{j_m}^m \cdots \partial_{j_n}^n$ is replaced with

$$\partial_{j_m}^{\mathfrak{m}}\cdots\partial_{j_{n+2}}^{\mathfrak{n}+2}(1-\chi_{j_n+1}^{\mathfrak{n}+2})\partial_{j_n+1}^{\mathfrak{n}+1}\partial_{j_n}^{\mathfrak{n}}$$

and we can move the braid group elements all the way to the left, straighten the part to the left of $\partial_{j_n}^n$ in Mag and then apply induction hypothesis. The result follows.

Definition 3.5. Let $\rho_{-1}^n := 0$ in Leib and then recursively define

$$\rho_{j+1}^{n} := \vartheta_{j+1}^{n} + \chi_{j+1}^{n+1} \rho_{j}^{n}$$

for any $0 \leq j \leq n-1$. One can also define ρ_i^n non-recursively as

$$\rho_{j}^{n} = \vartheta_{j}^{n} + \sum_{\alpha=1}^{j} \chi_{j}^{n+1} \cdots \chi_{\alpha}^{n+1} \vartheta_{\alpha-1}^{n}$$

for any $0 \leq j \leq n$.

Lemma 3.6. We have the following relationship between χ_j^n and ρ_i^n in Leib:

$$(3.4) \qquad \rho_{i}^{n}\chi_{j}^{n} = \begin{cases} \chi_{j+1}^{n+1}\rho_{i}^{n} & \text{if } j > i \\ \chi_{i+1}^{n+1}\chi_{i}^{n+1}\rho_{i+1}^{n} - \chi_{i+1}^{n+1}\chi_{i}^{n+1}\chi_{i+1}^{n+1}\rho_{i}^{n} + \chi_{i}^{n+1}\chi_{i+1}^{n+1}\rho_{i-1}^{n} & \text{if } j = i \\ \chi_{j}^{n+1}\rho_{i}^{n} & \text{if } j < i \end{cases}$$

for any $n \ge 1$ and for any $0 \le i \le n$ and $0 \le j \le n - 1$.

Proof. To prove this lemma, we will use a \mathbb{K} -algebra endomorphism $(\cdot)[+1]$: Leib \rightarrow Leib and its variants. This endomorphism shifts up the right and left degrees on generators by 1:

(3.5)
$$1_{n}[+1] = 1_{n+1}$$
 $\chi_{i}^{n}[+1] = \chi_{i}^{n+1}$ $\partial_{j}^{n}[+1] = \partial_{j}^{n+1}$

for $0 \le i \le n - 1$ and $0 \le j \le n$. Observe that this endomorphism can be generalized to greater shifts in degree. In particular, for all $0 \le j \le n$ we have $\rho_j^n = \rho_j^j [n - j]$. Note also that, this is a K-algebra morphism because multiplication structure is determined solely by the lower indices.

When we visualize every elementary tensor in terms of their diagrammatic representation, we see that the endomorphism just adds strands to the right of the diagram that does not interfere with the multiplication. On the opposite side, we can remove idle strands in the diagram when we multiply elements and then add them later. So it is a naturally arising endomorphism.

We prove the lemma by using case-by-case analysis:

Case (1) j > i. Under this assumption, we can reduce the case of calculation of $\rho_i^n \chi_j^n$ to $\rho_i^{j+1} \chi_j^{j+1}$ because we can adjust the upper indices as follows:

$$\rho_i^n \chi_j^n = \left(\rho_i^{j+1} \chi_j^{j+1}\right) [n-j-1]$$

So, it is enough to consider products of the form $\rho_i^n\chi_{n-1}^n$ for $0\leqslant i< n.$

$$\rho_{i}^{n}\chi_{n-1}^{n} = \partial_{i}^{n}\chi_{n-1}^{n} + \sum_{a=1}^{i}\chi_{i}^{n+1}\cdots\chi_{a}^{n+1}\partial_{a-1}^{n}\chi_{n-1}^{n}$$
$$= \chi_{n}^{n+1}\partial_{i}^{n} + \sum_{a=1}^{i}\chi_{i}^{n+1}\cdots\chi_{a}^{n+1}\chi_{n}^{n+1}\partial_{a-1}^{n}$$
$$= \chi_{n}^{n+1}\rho_{i}^{n}$$

Having this equality, we can go back and complete the remaining cases as follows:

$$\rho_{i}^{n}\chi_{j}^{n} = (\rho_{i}^{j+1}\chi_{j}^{j+1})[n-j-1] = (\chi_{j+1}^{j+2}\rho_{i}^{j+1})[n-j-1] = \chi_{j+1}^{n+1}\rho_{i}^{n}$$

Case (2) i = j. As it is done above, we can reduce the case to $\rho_{n-1}^n \chi_{n-1}^n$.

$$\rho_{n-1}^{n}\chi_{n-1}^{n} = \vartheta_{n-1}^{n}\chi_{n-1}^{n} + \chi_{n-1}^{n+1}\vartheta_{n-2}^{n}\chi_{n-1}^{n} + \sum_{a=1}^{n-2}\chi_{n-1}^{n+1}\chi_{n-2}^{n+1}\cdots\chi_{a}^{n+1}\vartheta_{a-1}^{n}\chi_{n-1}^{n}$$
$$= \chi_{n}^{n+1}\chi_{n-1}^{n+1}\vartheta_{n}^{n} + \chi_{n-1}^{n+1}\chi_{n}^{n+1}\vartheta_{n-2}^{n} + \sum_{a=1}^{n-2}\chi_{n-1}^{n+1}\chi_{n-2}^{n+1}\chi_{a}^{n+1}\chi_{n}^{n+1}\vartheta_{a-1}^{n}$$
$$= \chi_{n}^{n+1}\chi_{n-1}^{n+1}\rho_{n}^{n} - \chi_{n}^{n+1}\chi_{n-1}^{n+1}\chi_{n}^{n+1}\rho_{n-1}^{n} + \chi_{n-1}^{n+1}\chi_{n}^{n+1}\rho_{n-2}^{n}$$

Case (3) i = j + 1. We can reduce this case to $\rho_n^n \chi_{n-1}^n$.

$$\begin{split} \rho_{n}^{n}\chi_{n-1}^{n} &= \partial_{n}^{n}\chi_{n-1}^{n} + \chi_{n}^{n+1}\partial_{n-1}^{n}\chi_{n-1}^{n} + \sum_{a=1}^{n-1}\chi_{n}^{n+1}\chi_{n-1}^{n+1}\chi_{n-2}^{n+1}\cdots\chi_{a}^{n+1}\partial_{a-1}^{n}\chi_{n-1}^{n} \\ &= \chi_{n-1}^{n+1}\chi_{n}^{n+1}\partial_{n-1}^{n} + \chi_{n}^{n+1}\chi_{n}^{n+1}\chi_{n-1}^{n+1}\partial_{n}^{n} + \sum_{a=1}^{n-1}\chi_{n}^{n+1}\chi_{n-1}^{n+1}\chi_{n-2}^{n+1}\cdots\chi_{a}^{n+1}\chi_{n}^{n+1}\partial_{a-1}^{n} \\ &= \chi_{n-1}^{n+1}\chi_{n}^{n+1}\partial_{n-1}^{n} + \chi_{n-1}^{n+1}\partial_{n}^{n} + \sum_{a=1}^{n-1}\chi_{n-1}^{n+1}\chi_{n-1}^{n+1}\chi_{n-2}^{n+1}\cdots\chi_{a}^{n+1}\partial_{a-1}^{n} \\ &= \chi_{n-1}^{n+1}\chi_{n}^{n+1}\partial_{n-1}^{n} + \chi_{n-1}^{n+1}\partial_{n}^{n} + \sum_{a=1}^{n-1}\chi_{n-1}^{n+1}\chi_{n-1}^{n+1}\chi_{n-2}^{n+1}\cdots\chi_{a}^{n+1}\partial_{a-1}^{n} \\ &= \chi_{n-1}^{n+1}\rho_{n}^{n} \end{split}$$

$$\begin{split} \rho_{n}^{n}\chi_{j}^{n} =& \chi_{j}^{n+1}\partial_{n}^{n} + \sum_{a=1}^{j-1}\chi_{n}^{n+1}\cdots\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\chi_{j-1}^{n+1}\cdots\chi_{a}^{n+1}\partial_{a-1}^{n} \\ &+\chi_{n}^{n+1}\cdots\chi_{j+1}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\partial_{j-1}^{n} + \chi_{n}^{n+1}\cdots\chi_{j+2}^{n+1}\chi_{j+1}^{n+1}\chi_{j+1}^{n+1}\partial_{j+1}^{n} \\ &+\chi_{n}^{n+1}\cdots\chi_{j+2}^{n+1}\chi_{j}^{n+1}\chi_{j+1}^{n+1}\partial_{j}^{n} + \sum_{a=j+3}^{n}\chi_{n}^{n+1}\cdots\chi_{a}^{n+1}\chi_{j}^{n+1}\partial_{a-1}^{n} \\ &=\chi_{j}^{n+1}\left(\partial_{n}^{n} + \sum_{a=1}^{n}\chi_{n}^{n+1}\cdots\chi_{a}^{n+1}\partial_{a-1}^{n}\right) \\ &=\chi_{j}^{n+1}\rho_{n}^{n} \end{split}$$

Hence we are done.

Lemma 3.7. We have the following relations satisfied in Leib:

$$\rho_i^{n+1}\rho_j^n = \rho_{j+1}^{n+1}\rho_i^n$$

for any $n \ge 0$ *and* $0 \le i \le j \le n$.

Proof. Since we have

(3.6)
$$\rho_i^{n+1}\rho_j^n = (\rho_i^{j+1}\rho_j^j)[n-j] = \rho_i^{j+1}[n-j] \cdot \rho_j^j[n-j]$$

the statement reduces to proving $\rho_i^{n+1}\rho_n^n = \rho_{n+1}^{n+1}\rho_i^n$ for every $i \le n$. We will use induction on n to prove the statement. For the base case n = 0 we have

$$\rho_1^1 \rho_0^0 - \rho_0^1 \rho_0^0 = \vartheta_1^1 \vartheta_0^0 + \chi_1^2 \vartheta_0^1 \vartheta_0^0 - \vartheta_0^1 \vartheta_0^0 = \vartheta_1^1 \vartheta_0^0 - (1_2 - \chi_1^2) \vartheta_0^1 \vartheta_0^0$$

Right hand side of the equation lies in I_{Leib} , meaning that $\rho_0^1 \rho_0^0 = \rho_1^1 \rho_0^0$ in Leib. Now, assume as our induction hypothesis that we have $\rho_i^n \rho_{n-1}^{n-1} = \rho_n^n \rho_i^{n-1}$ hold for any $0 \le i \le n-1$. We need to show $\rho_i^{n+1} \rho_n^n = \rho_{n+1}^{n+1} \rho_i^n$ holds for any $0 \le i \le n$. We divide this in 3 cases.

Case (1) $0 \leq i < n - 1$

$$\begin{split} \rho_{i}^{n+1}\rho_{n}^{n} &= \rho_{i}^{n+1}\partial_{n}^{n} + \rho_{i}^{n+1}\chi_{n}^{n+1}\left(\rho_{n-1}^{n-1}\right)[+1] \\ &= \partial_{n+1}^{n+1}\rho_{i}^{n} + \chi_{n+1}^{n+2}\left(\rho_{n}^{n}\rho_{i}^{n-1}\right)[+1] \\ &= \partial_{n+1}^{n+1}\rho_{i}^{n} + \chi_{n+1}^{n+2}\rho_{n}^{n+1}\rho_{i}^{n} \\ &= \rho_{n+1}^{n+1}\rho_{i}^{n} \end{split}$$

where we use two main identities $\rho_i^{n+1}\chi_n^{n+1} = \chi_{n+1}^{n+2}\rho_i^{n+1}$ from Lemma 3.6 and $\rho_i^n\rho_{n-1}^{n-1} = \rho_n^n\rho_i^{n-1}$ from the induction hypothesis. We also use $\rho_i^{n+1}\partial_n^n = \partial_{n+1}^{n+1}\rho_i^n$ which is straightforward to show, using the non-recursive definition of ρ_i^{n+1} and relations in Mag.

Case (2) i = n - 1

$$\begin{split} \rho_{n-1}^{n+1}\rho_n^n &= \rho_{n-1}^{n+1}\partial_n^n + \rho_{n-1}^{n+1}\chi_n^{n+1}\rho_{n-1}^n \\ &= \partial_{n+1}^{n+1}\rho_{n-1}^n + \chi_{n+1}^{n+1}\rho_{n-1}^{n+1}\rho_{n-1}^n \\ &= \partial_{n+1}^{n+1}\rho_{n-1}^n + \chi_{n+1}^{n+1}\rho_n^{n+1}\rho_{n-1}^n \\ &= \rho_{n+1}^{n+1}\rho_{n-1}^n \end{split}$$

We use similar identities as in the previous case, namely $\rho_{n-1}^{n+1}\partial_n^n = \partial_{n+1}^{n+1}\rho_{n-1}^n$ and $\rho_{n-1}^{n+1}\chi_n^{n+1} = \chi_{n+1}^{n+1}\rho_n^{n+1}$.

Case (3) i = n

First we note that $\partial_n^{n+1}\partial_n^n = \partial_{n+1}^{n+1}\partial_n^n + \chi_{n+1}^{n+2}\partial_n^{n+1}\partial_n^n$ in Leib by the relations in the ideal I_{Leib} . Moreover, we use the identities $\partial_n^{n+1}\chi_n^{n+1} = \chi_{n+1}^{n+2}\chi_n^{n+2}\partial_{n+1}^{n+1}$ and $\partial_{n+1}^{n+1}\rho_{n-1}^n = \rho_{n-1}^{n+1}\partial_n^n$ and $\chi_n^{n+2}\rho_{n+1}^{n+1} = \rho_{n+1}^{n+1}\chi_n^{n+1}$. So we have:

$$\begin{split} \rho_{n}^{n+1}\rho_{n}^{n} &= \vartheta_{n}^{n+1}\vartheta_{n}^{n} + \vartheta_{n}^{n+1}\chi_{n}^{n+1}\rho_{n-1}^{n} + \chi_{n}^{n+2}\rho_{n-1}^{n+1}\rho_{n}^{n} \\ &= \vartheta_{n+1}^{n+1}\vartheta_{n}^{n} + \chi_{n+1}^{n+2}\vartheta_{n}^{n+1}\vartheta_{n}^{n} + \chi_{n+1}^{n+2}\chi_{n}^{n+2}\vartheta_{n+1}^{n+1}\rho_{n-1}^{n} + \chi_{n}^{n+2}\rho_{n+1}^{n+1}\rho_{n-1}^{n} \\ &= \vartheta_{n+1}^{n+1}\vartheta_{n}^{n} + \chi_{n+1}^{n+2}\vartheta_{n}^{n+1}\vartheta_{n}^{n} + \chi_{n+1}^{n+2}\chi_{n}^{n+2}\rho_{n-1}^{n+1}\vartheta_{n}^{n} + \chi_{n}^{n+2}\rho_{n+1}^{n+1}\rho_{n-1}^{n} \\ &= \rho_{n+1}^{n+1}\vartheta_{n}^{n} + \chi_{n}^{n+2}\rho_{n+1}^{n+1}\rho_{n-1}^{n} \\ &= \rho_{n+1}^{n+1}\vartheta_{n}^{n} + \rho_{n+1}^{n+1}\chi_{n}^{n+1}\rho_{n-1}^{n} \\ &= \rho_{n+1}^{n+1}\vartheta_{n}^{n} + \rho_{n+1}^{n+1}\chi_{n}^{n+1}\rho_{n-1}^{n} \\ &= \rho_{n+1}^{n+1}\partial_{n}^{n} + \rho_{n+1}^{n+1}\chi_{n}^{n+1}\rho_{n-1}^{n} \end{split}$$

Hence we are done.

Remark 3.8. In Lemma 3.7 we showed that the the elements ρ_i^n 's satisfy the same relations satisfied by ∂_i^n 's in Simp. So, if we depict ρ_i^n 's the same way we depict ∂_i^n 's in our string diagrams we get the following string diagram for the second case of the relation in Lemma 3.6:



3.3. Leib is a crossed presimplicial algebra.

Theorem 3.9. Equation (3.4) describes a distributive law of the form

 $\omega\colon Simp\otimes_{\mathbb{K}}Sym\to Sym\otimes_{\mathbb{K}}Simp$

Consequently, Leib is isomorphic to Sym \otimes_{ω} Simp as \mathbb{K} -algebras.

Proof. Lemma 3.6 together with Lemma 3.7 ensures that any element of Leib can be written (not necessarily uniquely) as a k-linear combination of monomials of the following form:

$$\tau_{m+1}\rho_{j_m}^m\cdots\rho_{j_n}^n$$

where $\tau_{m+1} \in S_{m+2}$ and $j_m > \cdots > j_n$. We also know that Leib is isomorphic to Leib^{op}, which in turn is isomorphic to Sym $\otimes_{\mathbb{K}}$ Simp as k-vector spaces. Therefore the monomials above must form a k-vector space basis for Leib. Note that this bijection of the bases works in a bigraded level, so these bijections are in fact between bigraded finite sets. Now, by Proposition 2.2 we have a unique distributive law ω : Simp $\otimes_{\mathbb{K}}$ Sym \rightarrow Sym $\otimes_{\mathbb{K}}$ Simp such that Leib is isomorphic to Sym \otimes_{ω} Simp as a \mathbb{K} -algebra. The inherent algebra structure in Leib forces ω to be the one explicitly described distributive law in Equation (3.4).

APPENDIX A. OPERADS AND PRO(P)S

In order to put our definitions of combinatorial operation categories and \mathbb{K} -algebras into their proper context, we need to recall some basic facts about operads, PROs, and PROPs. See Remark A.3 at the end of this section. Our main references are [19, 13, 18, 10, 15].

A.1. Nonsymmetric operads. [15, Section 5.9.3]

A nonsymmetric operad \mathcal{P} in \mathcal{C} consists of collection of objects \mathcal{P}_n in \mathcal{C} for $n \ge 0$, a unit morphism $\iota : \mathbb{I} \to \mathcal{P}_1$, and a collection of composition morphisms

$$\gamma(\mathfrak{m};\mathfrak{n}_{1},\ldots,\mathfrak{n}_{\mathfrak{m}})\colon\mathcal{P}_{\mathfrak{m}}\odot\mathcal{P}_{\mathfrak{n}_{1}}\odot\cdots\odot\mathcal{P}_{\mathfrak{n}_{\mathfrak{m}}}\longrightarrow\mathcal{P}_{\mathfrak{n}}$$

for every $n \ge 1$ and m-composition $n_1 + \cdots + n_m = n$ of n. These morphisms, together with the unit morphism $\iota : \mathbb{I} \to \mathcal{P}_1$ must satisfy unitality



and associativity axioms.

γ

where $1 \leq k \leq m$, $\ell = \ell_1 + \dots + \ell_m$, $\ell_k = \ell_{k,1} + \dots + \ell_{k,n_k}$, and $n = n_1 + \dots + n_m$. Note that $\ell = \sum_{i=1}^k \sum_{j=1}^{n_i} \ell_{i,j}$ is an n-composition of ℓ because $n = n_1 + \dots + n_m$.

As for the morphisms, we will say $f: \mathcal{P} \to \mathcal{Q}$ is a morphism of operads if f is a collection of morphisms $f_n: \mathcal{P}_n \to \mathcal{Q}_n$ in \mathcal{C} such that

$$\begin{array}{c|c} \mathcal{P}_{\mathfrak{m}} \odot \mathcal{P}_{\mathfrak{n}_{1}} \odot \cdots \odot \mathcal{P}_{\mathfrak{n}_{\mathfrak{m}}} & \xrightarrow{\gamma(\mathfrak{m};\mathfrak{n}_{1},\ldots,\mathfrak{n}_{\mathfrak{m}})} & \mathcal{P}_{\mathfrak{n}} \\ f_{\mathfrak{m}} \odot f_{\mathfrak{n}_{1}} \odot \cdots \odot f_{\mathfrak{n}_{\mathfrak{m}}} & & \downarrow \\ \mathcal{Q}_{\mathfrak{m}} \odot \mathcal{Q}_{\mathfrak{n}_{1}} \odot \cdots \odot \mathcal{Q}_{\mathfrak{n}_{\mathfrak{m}}} & \xrightarrow{\gamma(\mathfrak{m};\mathfrak{n}_{1},\ldots,\mathfrak{n}_{\mathfrak{m}})} & \mathcal{Q}_{\mathfrak{n}} \end{array}$$

commutes for every n and for every m-composition $n_1 + \dots + n_m = n$.

We use OP(C) to denote the category of operads in C.

A.2. Symmetric groups as an operad. Consider the collection of symmetric groups S_n for $n \ge 1$ which we denote by S. We define an operad using S in the category of finite sets as follows: let $n_1 + \cdots + n_m = n$ be a m-composition of n. Define the composition maps

$$\gamma(\mathfrak{m};\mathfrak{n}_1,\ldots,\mathfrak{n}_\mathfrak{m})\colon S_\mathfrak{m}\times S_{\mathfrak{n}_1}\times\cdots\times S_{\mathfrak{n}_\mathfrak{m}}\to S_\mathfrak{n}$$

by

$$\gamma(\mathfrak{m};\mathfrak{n}_1,\ldots,\mathfrak{n}_{\mathfrak{m}})(\mu,\sigma_1,\ldots,\sigma_{\mathfrak{m}})=\sigma_{\mu^{-1}(1)}\oplus\cdots\oplus\sigma_{\mu^{-1}(\mathfrak{m})}$$

for every $\mu \in S_m$ and $\sigma_i \in S_{n_i}$ for i = 1, ..., m where the sum $\sigma \oplus \sigma'$ of two permutation $\sigma \in S_a$ and $\sigma' \in S_b$ is defined as

(A.1)
$$(\sigma \oplus \sigma')(\ell) = \begin{cases} \sigma(\ell) & \text{if } \ell \leqslant a \\ \sigma'(\ell - n) & \text{if } a < \ell \leqslant a + b \end{cases}$$

for every $a, b \in \mathbb{N}$ and $1 \leq \ell \leq a + b$. See [9, Example 1.5].

A.3. Symmetric operads. An operad \mathcal{P} is called *symmetric* if S_n has a right action on \mathcal{P}_n for all $n \in \mathbb{N}$. The action comes with two equivariance conditions [19]. The first equivariance condition is given by the following diagram:



for $\sigma \in S_m$, and for any composition $n_1 + \cdots + n_m = n$. The permutation $\sigma(n_1, \ldots, n_m) \in S_{n_1 + \cdots + n_m}$ permutes the blocks $(1, \ldots, n_1), \ldots, (n_{m-1} + 1, \ldots, n_m)$ as σ permutes $\{1, \ldots, m\}$ with a left action. The second equivariance condition is as follows:



for given $\sigma_i \in S_{n_i}$ for $1 \le i \le m$ and their block sum $\sigma_1 \oplus \cdots \oplus \sigma_m \in S_{n_1 + \cdots + n_m}$. Because of these two equivariance conditions, the γ maps are defined as the sum of the following morphisms

$$\gamma(\mathfrak{m};\mathfrak{n}_{1},\ldots,\mathfrak{n}_{\mathfrak{m}})\colon\mathcal{P}_{\mathfrak{m}}\odot_{S_{\mathfrak{m}}}\mathrm{Ind}_{S_{\mathfrak{n}_{1}}\times\cdots\times S_{\mathfrak{n}_{\mathfrak{m}}}}^{S_{\mathfrak{n}}}(\mathcal{P}_{\mathfrak{n}_{1}}\odot\cdots\odot\mathcal{P}_{\mathfrak{n}_{\mathfrak{m}}})\longrightarrow\mathcal{P}_{\mathfrak{n}}$$

over all compositions of the form $n_1 + \cdots + n_m = n$. Tensor product over S_n comes from the first equivariance condition, while the induced module comes from the second.

A.4. The endomorphism operad. Let A be an object in C. There is a canonical operad associated with A which we denote by $\mathcal{O}(A)$ where

$$\mathcal{O}_{\mathfrak{n}}(\mathsf{A}) = \mathcal{C}(\mathsf{A}^{\odot\mathfrak{n}},\mathsf{A})$$

The operadic composition law is defined as follows: let $n_1 + \cdots + n_m = n$ be an m-composition of n and define

$$\mathcal{O}_{\mathfrak{m}}(\mathsf{A}) \odot \mathcal{O}_{\mathfrak{n}_1}(\mathsf{A}) \odot \cdots \odot \mathcal{O}_{\mathfrak{n}_{\mathfrak{m}}}(\mathsf{A}) \to \mathcal{O}_{\mathfrak{n}}(\mathsf{A})$$

via

$$\mathcal{C}(A^{\odot \mathfrak{m}}, A) \otimes \mathcal{C}(A^{\odot \mathfrak{n}_{1}}, A) \otimes \cdots \otimes \mathcal{C}(A^{\odot \mathfrak{n}_{\mathfrak{m}}}, A)$$

$$\downarrow^{\mathcal{C}(A^{\odot \mathfrak{m}}, A) \otimes \odot}$$

$$\mathcal{C}(A^{\odot \mathfrak{m}}, A) \otimes \mathcal{C}(A^{\odot \mathfrak{n}}, A^{\odot \mathfrak{m}})$$

$$\downarrow^{\circ}$$

$$\mathcal{C}(A^{\odot \mathfrak{n}}, A)$$

The operad $\mathcal{O}(A)$ is called the *endomorphism operad* associated with A. Note that if we assume C is symmetric (resp. braided) monoidal, then the endomorphism operad is naturally symmetric (resp. braided).

A.5. Algebras over operads. Let \mathcal{P} be a operad in \mathcal{C} . We call an object $A \in \mathcal{C}$ as a \mathcal{P} -algebra, if there is a morphism of operads of the form $\lambda_A : \mathcal{P} \to \mathcal{O}(A)$. Given two \mathcal{P} -algebras A and B, a morphism f: $A \to B$ is called a morphism of \mathcal{P} -algebras if the following diagram

(A.2)
$$\begin{array}{c} A^{\odot n} \xrightarrow{\lambda_{A}(\alpha)} \\ A^{\odot n} \downarrow \\ f^{\odot n} \downarrow \\ B^{\odot n} \xrightarrow{\lambda_{F}(\alpha)} \\ B \end{array} \xrightarrow{} B$$

commutes for all $n \ge 1$ and $\alpha \in \mathcal{P}_n$. Algebras over \mathcal{P} together with \mathcal{P} -algebra morphisms makes a category denoted by $Alg_{\mathcal{C}}(\mathcal{P})$.

A.6. **PROs.** Our main references for this Section are [10, Definition 2.2.2] and [16, Chapter 5].

A PRO (*PRoduct Operations*) in C is a strict monoidal category \mathcal{P} enriched in C with the objects [n] for $n \in \mathbb{N}$ where morphisms between objects are objects in C. \mathcal{P} is equipped with a monoidal product \oplus given on the objects by the sum of natural numbers.

PROs can be thought of as categories that model algebraic operations with multiple inputs and multiple outputs, whereas operads model multiple inputs but one output. Thus if \mathcal{P} is a PRO(P) in \mathcal{C} , then $\operatorname{res}\mathcal{P}_n := \mathcal{P}(n, 1)$ is an operad in \mathcal{C} .

A.7. Symmetric groups as a PRO. Collection of symmetric groups $\mathbb{S} = \bigsqcup_{n \ge 1} S_n$ forms a monoidal category as follows: $Ob(\mathbb{S}) = \mathbb{N} \setminus \{0\}$ and

$$\mathbb{S}(n,m) = \begin{cases} S_n & \text{if } n = m \\ \emptyset & \text{otherwise} \end{cases}$$

The monoidal product \oplus on objects is just addition of natural numbers. The monoidal product on permutations is defined in Equation (A.1).

Note that \mathbb{S} is a strict symmetric monoidal category where the switch is defined as

$$\begin{array}{c} n \oplus m \xrightarrow{\sigma \oplus \sigma'} n \oplus m \\ \downarrow_{\tau_{n,m}} & & \downarrow_{\tau_{n,m}} \\ m \oplus n \xrightarrow{\sigma' \oplus \sigma} m \oplus n \end{array}$$

using the permutation $\tau_{n,m}$

$$\tau_{n,m}(\mathfrak{i}) = \begin{cases} \ell+m & \text{ if } \ell \leqslant n \\ \ell-n & \text{ if } n < \ell \leqslant n+m \end{cases}$$

for every $n, m \in \mathbb{N}$ and $1 \leq \ell \leq n + m$.

A.8. The PRO(P) associated to an operad. Our main references for this Section are [20, Section 4.1], [13], [10, Section 2.2.6], and [2, Section 3].

Assume \mathcal{O} is an operad over \mathcal{C} , and let us define a PRO cat \mathcal{O} . Since we are defining a PRO, the objects of the category cat \mathcal{O} are [n] for any $n \in \mathbb{N}$, and the monoidal product on the objects is given by addition of natural numbers. The morphisms in cat \mathcal{O} are given by

(A.3)
$$\operatorname{cat}\mathcal{O}([n],[m]) = \bigoplus_{n_1 + \dots + n_m = n} \mathcal{O}_{n_1} \odot \dots \odot \mathcal{O}_{n_m}$$

Here the sum in Equation (A.3) is taken over all m-compositions of n. Since we implicitly assume $\mathcal{O}_0 = 0$, one can take the sum over all order preserving surjections of the form f: $[n] \to [m]$ where each $n_i = |f^{-1}(i)|$. Thus a (homogeneous) morphism $f \in cat\mathcal{O}([n], [m])$ is of the form (f, b) where f: $[n] \to [m]$ is an order preserving surjection, and $b = (b_1 \odot \cdots \odot b_m) \in \mathcal{O}_{n_1} \odot \cdots \odot \mathcal{O}_{n_m}$ where $n_i = |f^{-1}(i)|$ for $i = 1, \ldots, m$. For two homogeneous morphisms (g, b): $[m] \to [\ell]$ and (f, a): $[n] \to [m]$ in $cat\mathcal{O}$, their composition $g \circ f$: $[n] \to [\ell]$ is defined via

(A.4)
$$(g,b) \circ (f,a) = (g \circ f, \gamma(b_1, \odot_{g(j_1)=1} a_{j_1}) \odot \cdots \odot \gamma(b_\ell, \odot_{g(j_\ell)=\ell} a_{j_\ell}))$$

using the composition in the operad \mathcal{O} . The monoidal product of morphisms is taken in the monoidal category (\mathcal{C}, \odot) since morphisms are defined in this category.

In order to extend the PRO catO to a PROP, we need to add a symmetric group actions. Thus we write

(A.5)
$$\operatorname{cat}\mathcal{O}([n],[m]) = \bigoplus_{n_1 + \dots + n_m = n} (\mathcal{O}_{n_1} \odot \dots \odot \mathcal{O}_{n_m}) \otimes_{S_{n_1} \times \dots \times S_{n_m}} S_n$$

The right action of S_m comes from the symmetric monoidal structure on C permuting the terms $\mathcal{O}_{n_1} \odot \cdots \odot \mathcal{O}_{n_m}$. The left S_n action now comes from the induced action from $S_{n_1} \times \cdots \times S_{n_m}$ to S_n . More extensive account on this symmetric structure is given in [2, Section 3].

A.9. Variations in the literature. The prototypical example of the PROP associated with an operad is Segal's category Γ [22] where Γ appears as the PROP for the commutative operad in the category of based pointed sets with smash product as the monoidal product. Our definition of the PROP associated with an operad O comes from [2, Section 3]. In [20, Section 4.1] May defines the PROP associated with an operad with different Hom objects, and in [21, Section 10.2] it is shown that these Hom objects are isomorphic to the Hom objects

(A.6)
$$\operatorname{cat}\mathcal{O}([\mathfrak{n}],[\mathfrak{m}]) = \bigoplus_{f:[\mathfrak{n}]\to[\mathfrak{m}]} \mathcal{O}_{f^{-1}(1)} \odot \cdots \odot \mathcal{O}_{f^{-1}(\mathfrak{m})}$$

where the index runs through all surjective set functions. Leinster [13, pp.21] uses the same Hom objects, while Adams [1, pp.42] and Markl [17, Example 60] use the definition of the Hom objects given in Equation (A.5).

Note that May's description of the Hom objects is equivalent to writing the sum over compositions as in Equation (A.5). This is because for every surjective function $f: [n] \rightarrow [m]$, there is a unique order preserving map $\tilde{f}: [n] \rightarrow [m]$ and a (not necessarily unique) permutation $\sigma \in S_n$ such that $\tilde{f} = f \circ \sigma$. The permutation determines a unique right coset $(S_{n_1} \times \cdots \times S_{n_m})\sigma$ of the subgroup $S_{n_1} \times \cdots \times S_{n_m}$ of S_n coming from block sum of permutations. Thus May's PROP comes with a canonical symmetric structure which is the same as ours.

For two homogeneous morphisms (g, b): $[m] \rightarrow [\ell]$ and (f, a): $[n] \rightarrow [m]$ in cat \mathcal{O} May defines the composition as

(A.7)
$$(g \circ f, \gamma(b_1; \odot_{g(k)=1}a_k) \cdot \sigma_1, \cdots, \gamma(b_i; \odot_{g(k)=i}a_k) \cdot \sigma_i, \cdots, \gamma(b_m; \odot_{g(k)=m}a_k) \cdot \sigma_m)$$

where $\sigma_i \in S_{(g \circ f)^{-1}(i)}$ are the permutations that makes $(g \circ f)(\sigma_1 \oplus \cdots \oplus \sigma_m)$ order preserving. Again, this is equivalent to our composition law in the symmetric case. Adams [1, pp.42] and Markl [17, Example 60] do not explicitly define the compositions for their PROPs. Leinster [13, pp.22] claims he uses "closely related but slightly different" composition compared to May [20], but without an explicit description.

A.10. The PRO(P) associated to an operad is as good as the operad itself. We will call a PRO \mathcal{P} as *reducible* if $\mathcal{P} = cat(res \mathcal{P})$, i.e.

$$\mathcal{P}([n],[m]) = \bigoplus_{n_1 + \dots + n_m = m} \mathcal{P}([n_1],[1]) \odot \dots \odot \mathcal{P}([n_m],[1])$$

for every n and m.

Proposition A.1 ([10, Section 2.2.6] and [2, Proposition 3.1]). The functors

cat: **OP**(C) **PRO**(C): *res*

are an adjoint pair. Moreover, they induce an equivalence between OP(C) and redPRO(C).

Proof. We must show $PRO(cat\mathcal{O}, \mathcal{P}) \cong OP(\mathcal{O}, res\mathcal{P})$ for every operad \mathcal{O} in \mathcal{C} . We first note that morphisms of PROs are functors which are identity on the set of objects. Moreover, since functors in $PRO(cat\mathcal{O}, \mathcal{P})$ are monoidal and morphisms in $cat\mathcal{O}$ are obtained from \mathcal{O} , any functor of the form F: $cat\mathcal{O} \to \mathcal{P}$ on morphisms is determined by their image $\mathcal{O}_n \to \mathcal{P}([n], [1])$. Thus we have shown that the monoidal category associated with an operad is a (reducible) PRO, and $PRO(cat\mathcal{O}, \mathcal{P}) \cong OP(\mathcal{O}, res\mathcal{P})$. On the opposite side, if we only consider the morphisms from [n] to [1] in a PRO, $res\mathcal{P}_n := \mathcal{P}([n], [1])$ is an operad. When \mathcal{P} is reducible, every morphism $[m] \to [n]$ in \mathcal{P} can be written as a sum of monoidal products of morphisms of the form $[m_1] \to [1], \ldots, [m_n] \to [1]$ where $m = m_1 + \cdots + m_n$. In such cases, the (monoidal) category associated to $res\mathcal{P}$ is \mathcal{P} itself since \mathcal{P} is assumed to be reducible.

Proposition A.2 ([13, Theorem 1.6.1]). Let \mathcal{O} be an operad in \mathcal{C} , and let $\operatorname{Alg}_{\mathcal{C}}(\mathcal{O})$ be the category of \mathcal{O} -algebras in \mathcal{C} . Then there is an equivalence of categories between $\operatorname{Alg}_{\mathcal{C}}(\mathcal{O})$ and the category of monoidal functors $\operatorname{Mon}(\operatorname{cat}\mathcal{O},\mathcal{C})$ from $\operatorname{cat}\mathcal{O}$ to \mathcal{C} . The same equivalence can be written for a symmetric (resp. braided) operads and a symmetric (resp. braided) monoidal functors.

Proof. Let $A \in C$ be a \mathcal{O} -algebra with the structure morphisms $\lambda_A^n \colon \mathcal{O}_n \to \mathcal{C}(A^{\odot n}, A)$. We define (symmetric, resp. braided) monoidal functor $\phi_A \colon cat\mathcal{O} \to \mathcal{C}$ by letting $\phi_A([n]) = A^{\odot n}$ on the set of objects. Note that since \mathcal{C} is a strict (symmetric, resp. braided) monoidal category, we get

 $A^{\odot(n+m)} = A^{\odot n} \odot A^{\odot m}$ and we get a strict (symmetric, resp. braided) monoidal functor. Since we have $cat\mathcal{O}([n], [1]) = \mathcal{O}_n$, a morphism $f: [n] \to [1]$ in $cat\mathcal{O}$ corresponds to an element $f \in \mathcal{O}_n$ and we have $\varphi_A(f) = \lambda_A^n(f): A^{\odot n} \to A$. This extends to all morphisms in $cat\mathcal{O}$ because every morphism $[n] \to [m]$ in $cat\mathcal{O}$ can be written as a sum of finite monoidal products of morphisms of the form $[n_i] \to [1]$ determined by m-compositions $n_1 + \cdots + n_m$ of n.

Now, if $f: A \to B$ is a morphism of \mathcal{O} -algebras, we must show that there is a natural transformation of the form $\varphi_f: \varphi_A \to \varphi_B$ making diagrams of the form



commutative for all $\beta \in cat\mathcal{O}$. However, since the morphisms in $cat\mathcal{O}$ is generated by morphisms of the form $[\ell] \rightarrow [1]$ via the monoidal product, in order to verify the natural transformation conditions, it is enough to consider diagrams of the form (A.2) which all commute since we consider \mathcal{O} -algebras. If we assume \mathcal{O} is symmetric (resp. braided) the category $cat\mathcal{O}$ is symmetric (resp. braided). This finishes the one side of the correspondence.

On the other hand, assume we have a strict (symmetric, resp. braided) monoidal functor of the form $\phi: \operatorname{cat} \mathcal{O} \to \mathcal{C}$. Then we have $\phi([n]) = A^{\odot n}$ where $A = \phi([1])$ for each $n \in \mathbb{N}$. Also note that since ϕ is a strict (symmetric, resp. braided) functor, we have (S_n -equivariant, resp. B_n -equivariant) maps $\phi_{n,1}: \mathcal{O}_n \to \mathcal{C}(A^{\odot n}, A)$ since $\operatorname{cat} \mathcal{O}([n], [1]) := \mathcal{O}_n$. The definition of a monoidal functor, together with the definition of composition of morphisms in $\operatorname{cat} \mathcal{O}$ ensures that the collection $(\phi_{\cdot, [1]})_{n \in \mathbb{N}}$ indeed defines a morphism of operads.

It is easy to see that these two constructions are mutual inverses on the set of objects, and morphisms.

A.11. A non-linear example. Consider the skeletal category of finite sets with all set maps and with the Cartesian product as a strict symmetric monoidal category. The unit object is the set [1]. There is a unique operad Comm in this category which is defined as $\text{Comm}_n = [1]$ for every $n \ge 1$.

Let us first consider the *non-unital* version where $\text{Comm}_0 = \emptyset$. The PROP associated with Comm then is the skeletal category of finite sets with surjections since

$$\bigsqcup_{n_{1}+\dots+n_{m}=n} (\operatorname{Comm}_{n_{1}} \times \dots \times \operatorname{Comm}_{n_{m}}) \times_{S_{n_{1}} \times \dots \times S_{n_{m}}} \times S_{n}$$
$$= \bigsqcup_{n_{1}+\dots+n_{m}=n} \underbrace{([1] \times \dots \times [1])}_{\text{m-times}} \times_{S_{n_{1}} \times \dots \times S_{n_{m}}} \times S_{n}$$
$$= \bigsqcup_{n_{1}+\dots+n_{m}=n} (S_{n_{1}} \times \dots \times S_{n_{m}}) \setminus S_{n}$$

and because surjective set maps of the form $f: [n] \to [m]$ with $n_i = f^{-1}(i)$ are in bijective correspondence with the set of right cosets of $S_{n_1} \times \cdots \times S_{n_m}$ in S_n since both sets have the same size $\frac{n!}{n_1!n_2!\cdots n_m!}$ where $n = n_1 + \cdots + n_m$ for each $n_i > 0$.

On the other hand, if we forgo the symmetric structure, we get the opposite category of the skeletal category of well-ordered finite sets with order preserving surjections since

$$\bigsqcup_{n_1+\dots+n_m=n} \operatorname{Comm}_{n_1} \times \dots \times \operatorname{Comm}_{n_m} = \bigsqcup_{n_1+\dots+n_m=n} [1]$$

and because the set of m-compositions of n are in bijective correspondence with order preserving surjective maps of the form $f: [n] \to [m]$ where the bijection from functions to compositions is given by $f \mapsto (f^{-1}(1), \ldots, f^{-1}(m))$. Note that since we assume each $|f^{-1}(i)| = n_i > 0$, we have surjections.

Now, let us consider the *unital* version where $Comm_0 = [1]$, and where we allow $n_i = 0$ in a composition of an integer in the indices we use in the unions. This means we now consider all maps $f: [n] \rightarrow [m]$ since we now allow $f^{-1}(i)$ to be empty for some $i \in [m]$. Then for the symmetric case we get the skeletal category of finite sets with all maps, and for the non-symmetric case we get the skeletal category of well-ordered finite sets with order preserving maps. The former is an unbased analogue of the opposite of Segal's category Γ while the latter is the simplex category Δ .

In terms of parametrizing categories, the symmetric version of the non-unital Comm parametrizes commutative semi-groups while the non-symmetric version parametrizes all semi-groups. The unital version of the symmetric Comm then parametrizes commutative monoids while non-symmetric unital Comm parametrizes all monoids.

On the other hand, if we use the operad we defined Section A.2 and let $Assoc_n = S_n$ for $n \ge 1$ the symmetric PROP associated with this operad is

$$\bigsqcup_{n_1+\dots+n_m=n} (S_{n_1}\times\dots\times S_{n_m})\times_{S_{n_1}\times\dots\times S_{n_m}} S_m$$

In the unital case we get the crossed simplicial group ΔS [7], while in the non-unital case we get Δ^+S the subcategory of epimorphisms of ΔS . These combinatorial operations categories also model semigroups and monoids, respectively.

As another variation, instead of the symmetric groups, we could have easily used the braid groups $\mathbb{B} = \bigsqcup_{n \ge 1} B_n$, and we would have obtained crossed simplicial groups $\Delta \mathbb{B}$ and $\Delta^+ \mathbb{B}$ for braided monoids and braided semigroups, respectively.

Remark A.3. If C is a PRO, we can forget the monoidal structure and obtain a combinatorial operations category, and thus obtain a forgetful functor of the form $PRO_{\Bbbk} \to CO_{\Bbbk}$. Now, we have the following functors

$$\mathbf{OP}_{\Bbbk} \xrightarrow{\simeq} \operatorname{red}\mathbf{PRO}_{\Bbbk} \to \mathbf{PRO}_{\Bbbk} \to \mathbf{CO}_{\Bbbk} \xrightarrow{\simeq} \mathbf{Alg}(\mathbb{K})$$

There are the corresponding functors for symmetric and braided flavors.

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