# Itō and Itō-Wentzell chain rule for flows of conditional laws of continuous semimartingales: an easy approach 

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#### Abstract

We provide a general Itō-Wentzell formula for a random field of maps on the Wasserstein space of probability measures, defined by continuous semimartingales, and evaluated along the flow of conditional distributions of another continuous semimartingale. Our method follows standard arguments of Itō calculus, and thus bypasses the approximation by empirical measures commonly used in the existing literature. As an application, we derive the dynamic programming equation for a mean field stochastic control problem with common noise.


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## 1 Introduction

Let $X=\left(X_{t}\right)_{t \geq 0}$ be a square integrable continuous semimartingale on a filtered probability space $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \geq 0}, \mathbb{P}\right)$. For simplicity, we consider the scalar case as the multidimensional extension does not raise any special difficulties. Denote by $m_{t}:=\mathbb{P} \circ X_{t}^{-1}$ the marginal law of $X_{t}$, which lies in the set $\mathcal{P}_{2}(\mathbb{R})$ of all probability measures with finite second moment. For a function $u: \mathcal{P}_{2}(\mathbb{R}) \longrightarrow \mathbb{R}$, with appropriate regularity, an Itō's chain rule for the map $t \longmapsto u\left(m_{t}\right)$ was established by various methods in the literature after the Lectures of P.L. Lions at the Collège de France, see Buckdahn, Li, Peng \& Rainer [BLPR17] and Chassagneux, Crisan \& Delarue [CCD15] for continuous diffusions, Cavallazzi [Cav22] for a Krylov-type extension of the Itō formula to maps in appropriate Sobolev spaces, Li [Li12] and Burzoni, Ignazio, Reppen \& Soner [BIRS20] for special classes of jump-diffusions with continuous marginals. The case of general càdlàg semimartingales was solved simultaneously by Guo, Pham \& Wei [GPW22] and Talbi, Touzi \& Zhang [TTZ23].

[^0]The Itō chain rule states that, for a map $u$ with appropriate smoothness on the Wasserstein space of probability measures, we have

$$
\begin{equation*}
u\left(m_{t}\right)=u\left(m_{0}\right)+\mathbb{E}\left[\int_{0}^{t} \partial_{\mu} u\left(m_{s}, X_{s}\right) \mathrm{d} X_{s}+\frac{1}{2} \int_{0}^{t} \partial_{x} \partial_{\mu} u\left(m_{s}, X_{s}\right) \mathrm{d}\langle X\rangle_{s}\right], \text { for all } t \geq 0, \tag{1.1}
\end{equation*}
$$

where $\partial_{\mu} u: \mathcal{P}_{2}(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ denotes the so-called Lions derivative, and $\partial_{x}$ is the partial gradient operator with respect to the $x$-variable. See e.g. Carmona \& Delarue [CD18a].

Our objective in this paper is to revisit extensions of the last Itō's rule in two directions: - the measure variable is random and defined as the conditional law $\mu_{t}:=\mathbb{P} \circ\left(X_{t} \mid \mathcal{F}^{0}\right)^{-1}$ of $X_{t}$ given some sub-sigma algebra $\mathcal{F}^{0}$ of $\mathcal{F}$,

- the function $u$ is extended to the context of a dynamic stochastic flow of continuous semimartingales $\left\{u_{t}(x), t \geq 0\right\}$ for all fixed $x \in \mathbb{R}$.

The first extension is motivated by the vibrant research activity on mean field stochastic control with common noise, and the Master equation in the context of mean field games with common noise. The second extension is also motivated by similar stochastic control problems under partial information. The huge interest of the community in this area is enhanced by the wide applications in various questions pertaining to multiple agents decision problems.

Our main emphasis is on the simplicity of our derivations which follow standard arguments in Itō calculus, and which allows to obtain new extensions which were not considered in the existing literature. In order to better explain our approach, let us show how (1.1) can be obtained by means of the following early graduate class level arguments (where the two first steps are simple reminders):

- We first recall from Cardaliaguet, Delarue, Lasry \& Lions [CDLL15] that the Lions derivative $\partial_{\mu}$ is related to the functional linear derivative $\delta_{m}$ by $\partial_{\mu}=\partial_{x} \delta_{m}$, where $\delta_{m} u: \mathcal{P}_{2}(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined for all $m, m^{\prime} \in \mathcal{P}_{2}(\mathbb{R})$ by the following limit, if exists:

$$
\lim _{\varepsilon \searrow 0} \frac{1}{\varepsilon}\left[u\left(m+\varepsilon\left(m^{\prime}-m\right)\right)-u(m)\right]=:\left\langle\delta_{m} u(m), m^{\prime}-m\right\rangle=\int \delta_{m} u(m, x)\left(m^{\prime}-m\right)(\mathrm{d} x)
$$

Notice that this definition is a mix of directional and Gâteaux derivative, and that the map $\delta_{m} u(m):=\delta_{m} u(m,):. \mathbb{R}^{d} \longrightarrow \mathbb{R}$ needs to have quadratic growth, at most, in order for the last integral to be well-defined.

- By standard calculus, and under slight regularity, this definition is equivalent to the requirement of existence of such a function $\delta_{m} u$ satisfying the requirement

$$
u(m)-u\left(m^{\prime}\right)=\int_{0}^{1}\left\langle\delta_{m} u\left((1-\lambda) m+\lambda m^{\prime}\right), m^{\prime}-m\right\rangle \mathrm{d} \lambda, \text { for all } m, m^{\prime} \in \mathcal{P}_{2}(\mathbb{R}),
$$

which is all we need for our subsequent derivation of Ito's formula.

- Given a dense partition $\left(t_{i}^{n}\right)_{i \geq 0}$ of the $[0, \infty)$, denote $s_{i}^{n}:=s \wedge t_{i}^{n}$, for all $s \geq 0$, and use the telescopic decomposition together with the last definition to see that:

$$
\begin{equation*}
u\left(m_{s}\right)-u\left(m_{0}\right)=\sum_{i \geq 1} u\left(m_{s_{i}^{n}}\right)-u\left(m_{s_{i-1}^{n}}\right)=\sum_{i \geq 1} \int_{0}^{1}\left\langle U_{n, i}^{\lambda}, m_{s_{i}^{n}}-m_{s_{i-1}^{n}}\right\rangle \mathrm{d} \lambda, \tag{1.2}
\end{equation*}
$$

with $U_{n, i}^{\lambda}:=\delta_{m} u\left((1-\lambda) m_{s_{i-1}^{n}}+\lambda m_{s_{i}^{n}}\right)$ a scalar map on $\mathbb{R}$. We next observe that

$$
\begin{aligned}
\left\langle U_{n, i}^{\lambda}, m_{s_{i}^{n}}-m_{s_{i-1}^{n}}\right\rangle & =\int U_{n, i}^{\lambda} \mathrm{d}\left(m_{s_{i}^{n}}-m_{s_{i-1}^{n}}\right) \\
& =\mathbb{E}\left[U_{n, i}^{\lambda}\left(X_{s_{i}^{n}}\right)-U_{n, i}^{\lambda}\left(X_{s_{i-1}^{n}}\right)\right] \\
& =\mathbb{E}\left[\int_{s_{i-1}^{n}}^{s_{i}^{n}}\left(U_{n, i}^{\lambda}\right)^{\prime}\left(X_{r}\right) \mathrm{d} X_{r}+\frac{1}{2}\left(U_{n, i}^{\lambda}\right)^{\prime \prime}\left(X_{r}\right) \mathrm{d}\langle X\rangle_{r}\right],
\end{aligned}
$$

where the last equality follows from the standard Ito's formula, under the appropriate regularity assumptions on the map $U_{n, i}^{\lambda}$. Plugging this expression in (1.2), we obtain the required formula (1.1) by standard limiting argument using the dominated convergence theorem.

The last argument is most appealing as it uses the standard intuitive notion of functional linear derivative $\delta_{m}$. Moreover, it completely bypasses the crucial step of projection on empirical measures used in most of the previous literature following the Lectures of P.L. Lions at the Collège de France, see Chassagneux, Crisan \& Delarue [CCD15], Buckdahn, Li, Peng \& Rainer [BLPR17], Carmona \& Delarue [CD18a]. Here, the idea is to approximate the marginal law $m_{t}$ by the corresponding empirical measure $\bar{m}_{t}^{N}:=\frac{1}{N} \sum_{i \leq N} \delta_{X_{t}^{i}}$ of a finite sample of $N$ independent copies $\left(X^{1}, \ldots, X^{N}\right)$, apply the standard Itō's formula to the finite dimensional map $\bar{u}^{N}\left(X_{t}^{1}, \ldots, X_{t}^{N}\right):=u\left(\bar{m}_{t}^{N}\right)$, and finally take limits by using fine results on the convergence of empirical measures.

The simple method outlined above is applied in Talbi, Touzi \& Zhang [TTZ23] in the context of càd-làg semimartingales, see also the parallel paper by Guo, Pham \& Wei [GPW22] which uses a functional analytic extension of an appropriate class of cylindrical maps in order to account for the jumps of the semimartingale.

The main contribution of this paper is to show that the previous simple method also applies to derive an Itō-Wentzell chain rule for conditional laws. This answers in particular a question raised in dos Reis \& Platonov [dRP22], who derive the Itō-Wentzell formula by adapting the technique of projection on empirical measures used by Carmona \& Delarue [CD18b] to derive the Itō formula for conditional laws, see also Cardaliaguet, Delarue, Lasry \& Lions [CDLL15] in the context of the Master equation. We notice that, while the state process in [dRP22] is defined by SDEs driven by Brownian motions and is conditioned by a Brownian motion, we consider in this paper general continuous semimartingales with general conditioning. Moreover, our Itō-Wentzell formula is derived for a random flow continuous semimartingale, extending the case of deterministic function of a process and a conditional law of another process of [dRP22].

The paper is organized as follows. Section 3 provides an Itō's formula for conditional marginal laws of continuous semimartingales. Although this result is a particular case of the subsequent one, we believe that it deserves to be isolated for the sake of clarity. Section 4 contains our general Itō-Wentzell formula in the context where the random field of maps is also defined by continuous semimartingales. Finally, Section 5 provides an application in mean field stochastic control with common noise.

## 2 Notations

We denote $x \cdot y:=\sum_{i} x_{i} y_{i}$ the Euclidean scalar product of two vectors in any finite dimensional space, $A: B:=\operatorname{Tr}[A B]$ and $A^{\otimes 2}=A A^{\top}$ for all matrices of appropriate size.

Throughout this paper, we fix a constant maturity $T>0$, and a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F}=\left\{\mathcal{F}_{t}, 0 \leq t \leq T\right\}$.

Let $\pi: 0=t_{0}<\ldots<t_{n_{\pi}}=T$ be a subdivision of $[0, T]$ with mesh size $|\pi|:=$ $\max _{i \leq n_{\pi}}\left(t_{i}-t_{i-1}\right)$. A sequence of subdivisions $\left(\pi^{n}\right)_{n \geq 0}$ is dense if the sequence of meshes $\left|\pi^{n}\right|$ converges to 0 as $n \rightarrow \infty$.

A stochastic process is said to be piecewise constant along the subdivision $\pi$ if it is constant on each interval $\left(t_{i-1}, t_{i}\right]$. For a process $Y$ valued in $\mathbb{R}^{d}$, we denote the increment to the subdivision by:

$$
\Delta^{\pi} Y_{s}:=\sum_{i=1}^{n_{\pi}}\left(Y_{s \wedge t_{i}}-Y_{s \wedge t_{i-1}}\right), \text { for all } s \geq 0
$$

In other words, for $s \in\left(t_{i_{s}-1}, t_{i_{s}}\right]$, we have $\Delta^{\pi} Y_{s}=Y_{s}-Y_{t_{i_{s}-1}}$, and if the process $Y$ is in addition piecewise constant along $\pi$, we have $\Delta^{\pi} Y_{s}=Y_{t_{i_{s}}}-Y_{t_{i_{s}-1}}$.

The total variation of $Y$ is denoted by

$$
|Y|_{\mathrm{TV}}=\sup _{\pi} \sum_{i=1}^{n_{\pi}-1}\left|Y_{t_{i+1}}-Y_{t_{i}}\right|=\sup _{\pi} \sum_{i=1}^{n_{\pi}-1}\left|\Delta^{\pi} Y_{t_{i+1}}\right|,
$$

where $|$.$| is the Euclidean norm in \mathbb{R}^{d}$.
The quadratic variation of $Y$ is defined as

$$
\langle Y\rangle_{s}=\lim _{|\pi| \rightarrow 0} \sum_{i=1}^{n_{\pi}-1}\left(Y_{t_{i+1}}-Y_{t_{i}}\right)\left(Y_{t_{i+1}}-Y_{t_{i}}\right)^{\top}, \quad \text { for all } \quad s \geq 0,
$$

where the limit is in probability and does not depend on the choice of the subdivisions sequence.
$X$ is said to be a (continuous) semimartingale if it can be written as $X_{s}=X_{0}+A_{s}+$ $M_{s}, s \in[0, T]$ where $A$ is a (continuous) finite-variation process and $M$ is a (continuous) martingale.
$\mathbb{H}^{2}(Y)$ is the collection of all progressively measurable processes $H$, with same dimension as $Y$, such that $\mathbb{E}\left[\int_{0}^{T} H_{s} H_{s}^{\top}: d\langle Y\rangle_{s}\right]<\infty$.

A sequence $\left(H^{n}\right)_{n \geq 0}$ of predictable bounded processes is called a simple approximation of a process $H \in \mathbb{H}^{2}(Y)$ if there exists a dense sequence of subdivisions $\left(\pi^{n}\right)_{n>0}$ such that $H^{n}$ is piecewise constant along $\pi^{n}$, for all $n \geq 0$, and $H^{n} \longrightarrow H$ in $\mathbb{H}^{2}(Y)$, as $n \rightarrow \infty$.

The following (probably well-known) result will be used frequently. As we failed to find a reference for it, we report its proof as a complement in the Appendix section 6.

Lemma 1. Let $X$ be a semimartingale with decomposition $X=A+M$ into a finite variation process $A$ and a martingale $M$ satisfying $\mathbb{E}\left[|A|_{\mathrm{TV}}^{2}+\langle M\rangle_{T}^{2}\right]<\infty$. Let $\left(H^{n}\right)_{n \geq 0}$ be a simple approximation of a matrix-valued bounded progressively measurable $H$ with rows in $\mathbb{H}^{2}(X)$, along some sequence of subdivisions $\left(\pi^{n}: 0=t_{0}^{n}<\ldots<t_{p_{n}}^{n}=T\right)$. Then:

$$
\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}:\left(\Delta^{\pi^{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi^{n}} X_{t_{i}^{n}}\right)^{\top} \longrightarrow \int_{0}^{T} H_{s}: \mathrm{d}\langle X\rangle_{s}, \quad \text { in } \mathbb{L}^{1} \quad \text { as } n \rightarrow \infty
$$

We notice that, if $H$ is continuous, then the last convergence result holds true with $H_{t_{i}^{n}}^{n}=H_{t_{i}^{n}}$, for $i=0, \ldots, n_{p_{n}}$.

The marginal laws considered in this paper lie in the set $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ of all probability measures on $\mathbb{R}^{d}$ with finite second moment. Similarly, our conditional marginal laws are random maps taking values in $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. This set is naturally equipped with the Wasserstein distance

$$
\mathrm{d}\left(m, m^{\prime}\right):=\inf _{\pi \in \Pi\left(m, m^{\prime}\right)} \int|x-x|^{2} \pi\left(\mathrm{~d} x, \mathrm{~d} x^{\prime}\right), \quad \text { for all } \quad m, m^{\prime} \in \mathcal{P}\left(\mathbb{R}^{d}\right) \text {, }
$$

where $\Pi\left(m, m^{\prime}\right)$ is the set of all couplings of ( $m, m^{\prime}$ ), i.e. probability measures on $\mathbb{R}^{d} \times \mathbb{R}^{d}$ with marginals $m$ and $m^{\prime}$.

We say that a function $u: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}$ admits a (first order) functional linear derivative if there exists a map $\delta_{m} u: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that for all $m, m^{\prime} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$,

$$
u\left(m^{\prime}\right)-u(m)=\int_{0}^{1} \int_{\mathbb{R}^{d}} \delta_{m} u\left(\bar{m}^{\lambda}, x\right)\left(m^{\prime}-m\right)(\mathrm{d} x) \mathrm{d} \lambda \text { with } \bar{m}^{\lambda}:=\lambda m^{\prime}+(1-\lambda) m
$$

and $\delta_{m} u$ has quadratic growth in $x$, locally uniformly in $m$, so that the last integral is welldefined. Similarly, the second order functional linear derivative $\delta_{m}^{2}: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \times \mathbb{R}^{d} \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ is such that for all $m, m^{\prime} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, and $x \in \mathbb{R}^{d}$ :

$$
\delta_{m} u\left(m^{\prime}, x\right)-\delta_{m} u(m, x)=\int_{0}^{1} \int_{\mathbb{R}^{d}} \delta_{m}^{2} u\left(\bar{m}^{\lambda}, x, \hat{x}\right)\left(m^{\prime}-m\right)(\mathrm{d} \hat{x}) \mathrm{d} \lambda,
$$

and $\delta_{m}^{2} u$ has quadratic growth in $\hat{x}$, locally uniformly in $m$, for all fixed $x \in \mathbb{R}^{d}$. Notice that under these conditions, $\delta_{m} \partial_{x} \delta_{m} u=\partial_{x} \delta_{m}^{2} u$.

## 3 Itō's formula

Throughout this paper, we consider an $\mathbb{R}^{d}$-valued continuous semimartingale with canonical decomposition

$$
X=X_{0}+M+A,
$$

where $M$ is a martingale and $A$ is a finite-variation process, both started from 0 , and $X_{0} \in \mathbb{L}^{2}\left(\mathcal{F}_{0}\right)$.

For an arbitrary filtration $\mathbb{F}^{0}=\left(\mathcal{F}_{t}^{0}\right)_{t \geq 0}$, we denote by $\mu_{t}=\mathcal{L}\left(X_{t} \mid \mathcal{F}_{T}^{0}\right)$, the law of $X_{t}$ conditional on $\mathcal{F}_{T}^{0}$, for $t \in[0, T]$.

Our first result is the following Itō's formula for flows of conditional law of the continuous semimartingale $X$.
Assumption 3.1. The map $u: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \longrightarrow \mathbb{R}$, and the continuous semimartingale $X$ satisfy:
(I1) $\delta_{m} u, \partial_{x}^{2} \delta_{m} u, \delta_{m}^{2} u, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u$ exist and are continuous in each variable;
(I2) $\partial_{x}^{2} \delta_{m} u, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u$ are bounded;
(I3) $X_{0},|A|_{\text {TV }}$ and $\langle M\rangle_{T}$ are square integrable.
Theorem 1. Let Assumption 3.1 hold true, and let $\hat{X}$ be a copy of $X$ on a copy probability space, with $\mathcal{L}\left(\hat{X} \mid \mathcal{F}_{T}^{0}\right)=\mathcal{L}\left(X \mid \mathcal{F}_{T}^{0}\right)$. Then:

$$
\begin{aligned}
u\left(\mu_{T}\right)-u\left(\mu_{0}\right)= & \mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} u\left(\mu_{s}, X_{s}\right) \cdot \mathrm{d} X_{s}+\frac{1}{2} \partial_{x}^{2} \delta_{m} u\left(\mu_{s}, X_{s}\right): \mathrm{d}\langle X\rangle_{s}\right] \\
& +\mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{T} \frac{1}{2} \partial_{x \hat{x}}^{2} \delta_{m}^{2} u\left(\mu_{s}, X_{s}, \hat{X}_{s}\right): \mathrm{d}\langle X, \hat{X}\rangle_{s}\right], \quad \text { a.s. }
\end{aligned}
$$

where $\mathbb{E}^{0}:=\mathbb{E}\left[\cdot \mid \mathcal{F}_{T}^{0}\right]$ and $\hat{\mathbb{E}}^{0}:=\mathbb{E}\left[\cdot \mid X, \mathcal{F}_{T}^{0}\right]$ denote the conditional expectations in the enlarged space.

Proof. Let us first prove the result when $\partial_{x}^{2} \delta_{m} u$ and $\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u$ are bounded. We organize our arguments in three steps.
Step 1. Let $\pi^{n}: 0=t_{0}^{n}<t_{1}^{n}<\ldots<t_{p_{n}}^{n}=T$ be a dense sequence of partitions of $[0, T]$, and denote $\mu_{t_{i}^{n}}^{\lambda}=\lambda \mu_{t_{i}^{n}}+(1-\lambda) \mu_{t_{i-1}^{n}}^{n}$. By the definition of the linear functional derivative, we have:

$$
\begin{align*}
\delta_{i}^{n} u:=u\left(\mu_{t_{i}^{n}}\right)-u\left(\mu_{t_{i-1}^{n}}\right) & =\int_{0}^{1} \int_{\mathbb{R}^{d}} \delta_{m} u\left(\mu_{t_{i}^{n}}^{\lambda}, x\right)\left(\mu_{t_{i}^{n}}-\mu_{t_{i-1}^{n}}\right)(\mathrm{d} x) \mathrm{d} \lambda \\
& =\int_{0}^{1} \mathbb{E}^{0}\left[\delta_{m} u\left(\mu_{t_{i}^{n}}^{\lambda}, X_{t_{i}^{n}}\right)-\delta_{m} u\left(\mu_{t_{i}^{n}}^{\lambda}, X_{t_{i-1}^{n}}\right)\right] \mathrm{d} \lambda . \tag{3.1}
\end{align*}
$$

By the second order Taylor theorem, we may rewrite this as
$\delta_{i}^{n} u=\int_{0}^{1} \mathbb{E}^{0}\left[\partial_{x} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i-1}^{n}}\right) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}}+\frac{1}{2} \partial_{x}^{2} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}^{n}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda$,
for some r.v. $\xi_{t_{i-1}^{n}}$ lying between $X_{t_{i-1}^{n}}$ and $X_{t_{i}^{n}}$. Let us introduce an independent copy $\hat{X}$ of $X$ conditionally to $\mathbb{F}^{0}$. Using the notation $\{F(\theta, \cdot)\}_{\theta_{1}^{\prime}}^{\theta_{2}^{\prime}}:=F\left(\theta, \theta_{2}^{\prime}\right)-F\left(\theta, \theta_{1}^{\prime}\right)$, for all map $F\left(\theta, \theta^{\prime}\right)$, and denoting $\gamma:=\partial_{x} \delta_{m} u$, we compute that

$$
\begin{aligned}
\left\{\gamma\left(., X_{t_{i-1}^{n}}^{n}\right)\right\}_{\mu_{t_{i-1}^{n}}^{\lambda}}^{\mu_{t_{i-1}^{n}}^{n}} & =\int_{0}^{1} \int \delta_{m} \gamma\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{x}\right)\left(\mu_{t_{i-1}^{n}}^{\lambda}-\mu_{t_{i-1}^{n}}^{n}\right)(\mathrm{d} \hat{x}) \mathrm{d} \lambda^{\prime} \\
& =\lambda \int_{0}^{1} \hat{\mathbb{E}}^{0}\left[\left\{\delta_{m} \gamma\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \cdot\right)\right\}_{\hat{X}_{t_{i-1}^{n}}^{n}}\right] \mathrm{d} \lambda^{\prime} \\
& =\lambda \int_{0}^{1} \hat{\mathbb{E}}^{0}\left[\partial_{\hat{x}} \delta_{m} \gamma\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right) \Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right] \mathrm{d} \lambda^{\prime},
\end{aligned}
$$

for some $\hat{\xi}_{t_{i-1}^{n}}$ between $\hat{X}_{t_{i-1}^{n}}$ and $\hat{X}_{t_{i}^{n}}$. This provides:

$$
\begin{aligned}
\delta_{i}^{n} u= & \mathbb{E}^{0}\left[\partial_{x} \delta_{m} u\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}}\right] \\
& +\mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top} \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda\right] \\
& +\int_{0}^{1} \mathbb{E}^{0}\left[\frac{1}{2} \partial_{x}^{2} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda .
\end{aligned}
$$

Summing (3.1), and denoting by $t^{n}(s)$ the closest subdivision point strictly to the left of $s$, this provides:

$$
\begin{aligned}
U_{T}\left(\mu_{T}\right)- & U_{0}\left(\mu_{0}\right)=\int_{0}^{T} \mathbb{E}^{0}\left[\partial_{x} \delta_{m} u\left(\mu_{t^{n}(s)}, X_{t^{n}(s)}\right) \cdot \mathrm{d} X_{s}\right] \\
& +\sum_{i=1}^{p_{n}} \int_{0}^{1} \mathbb{E}^{0}\left[\frac{1}{2} \partial_{x}^{2} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}^{n}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}{ }^{\top}\right] \mathrm{d} \lambda\right. \\
& +\sum_{i=1}^{p_{n}} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top} \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda\right]
\end{aligned}
$$

Our goal in the subsequent steps is to analyse the convergence of each term in the last decomposition, along a suitable sequence of subdivisions, towards the formula announced in Theorem 1. Proving that one such subsequence exists is enough.
Step 2. In this step, we start with the first term that we denote $U_{1}^{n}$. Namely,
$U_{1}^{n}:=\sum_{i=1}^{p_{n}} \mathbb{E}^{0}\left[\int_{t_{i-1}^{n}}^{t_{i}^{n}} \partial_{x} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{n}, X_{s}\right) \cdot \mathrm{d} X_{s}\right]=\mathbb{E}^{0}\left[\int_{0}^{T} f_{n}(s) \cdot \mathrm{d} X_{s}\right]$ with $f_{n}(s):=\partial_{x} \delta_{m} u\left(\mu_{t^{n}(s)}, X_{s}\right)$,
where $t^{n}(s)$ is the last point of the subdivision $\pi^{n}$ which is strictly to the left of $s$. Since $\mu_{s}$ is continuous in $s, f_{n}(s) \longrightarrow f(s)=\partial_{x} \delta_{m} u\left(\mu_{s}, X_{s}\right)$ almost surely when $n \rightarrow+\infty$. Moreover,
$\left|\partial_{x} \delta_{m} u\left(\mu_{s}, X_{s}\right)\right| \leq C\left(1+\left|X_{s}\right|\right)$, by our assumption on $\partial_{x}^{2} \delta_{m} u$, and as $\mathbb{E}\left[\langle M\rangle_{T}+|A|_{\mathrm{TV}}^{2}\right]<\infty$, we also have $\mathbb{E}\left[\sup _{0 \leq t \leq T}\left|X_{t}\right|^{2}\right]<\infty$. Then, it follows from the BDG inequality that

$$
\mathbb{E}\left[\left|\int_{0}^{T}\left(f_{n}(s)-f(s)\right) \cdot \mathrm{d} X_{s}\right|\right] \leq C_{\mathrm{BDG}} \mathbb{E}\left[\left|\int_{0}^{T}\left(f_{n}(s)-f(s)\right)^{\otimes 2}: \mathrm{d}\langle X\rangle_{s}\right|^{\frac{1}{2}}\right] \longrightarrow 0 \text { in } \mathbb{L}^{2}
$$

by dominated convergence, as

$$
\left|\int_{0}^{T}\left(f_{n}(s)-f(s)\right)^{\otimes 2}: \mathrm{d}\langle X\rangle_{s}\right|^{\frac{1}{2}} \leq C\left(\sup _{s}\left(1+\left|X_{s}\right|\right)^{2}\left|\langle X\rangle_{T}\right|\right)^{\frac{1}{2}} \leq \frac{C}{2}\left(\sup _{s}\left(1+\left|X_{s}\right|\right)^{2}+\left|\langle X\rangle_{T}\right|\right) \in \mathbb{L}^{1} .
$$

This shows that $\int_{0}^{T} \partial_{x} \delta_{m} u\left(\mu_{t^{n}(s)}, X_{s}\right) \cdot \mathrm{d} X_{s} \longrightarrow \int_{0}^{T} \partial_{x} \delta_{m} u\left(M_{s}, X_{s}\right) \cdot \mathrm{d} X_{s}$ in $\mathbb{L}^{1}$, thus implying by the Jensen inequality that

$$
U_{1}^{n} \longrightarrow \mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} u\left(\mu_{s}, X_{s}\right) \cdot \mathrm{d} X_{s}\right] \text { in } \mathbb{L}^{1}, \quad \text { as } n \rightarrow \infty
$$

and therefore almost surely along some subsequence.
Step 3. We next analyse the convergence of the third term

$$
U_{3}^{n}:=\sum_{i=1}^{p_{n}} \int_{0}^{1} \lambda \int_{0}^{1} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\mu_{t_{i}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}^{n}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda,
$$

by arguing in two steps:

- First, substituting $\mu_{t_{i-1}^{n}}$ to $\mu_{t_{i}^{n}}^{\lambda \lambda^{\prime}}$, and $\hat{X}_{t_{i-1}^{n}}$ to $\hat{\xi}_{t_{i-1}^{n}}$, we compute that

$$
\begin{aligned}
\bar{U}_{3}^{n} & :=\sum_{i=1}^{p_{n}} \int_{0}^{1} \lambda \int_{0}^{1} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\partial_{x} \partial_{\hat{x}} \partial_{m}^{2} u\left(\mu_{t_{i}^{n}}, X_{t_{i-1}^{n}}, \hat{X}_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda \\
& =\frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}:\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top}\right], \text { with } H_{s}:=\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\mu_{s}, X_{s}, \hat{X}_{s}\right),
\end{aligned}
$$

which leads by the polarized version of Lemma 1 to:

$$
\bar{U}_{3}^{n} \longrightarrow \frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{T} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\mu_{s}, X_{s}, \hat{X}_{s}\right): \mathrm{d}\langle X, \hat{X}\rangle_{s}\right], \quad \text { in } \quad \mathbb{L}^{1}, \text { as } n \rightarrow \infty
$$

- we next control the error $U_{3}^{n}-\bar{U}_{3}^{n}=\mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\sum_{i=1}^{p_{n}} \varepsilon_{t_{i-1}^{n}}:\left(\Delta^{\pi^{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi^{n}} \hat{X}_{t_{i}^{n}}\right)^{\top}\right]$, with

$$
\varepsilon_{t_{i-1}^{n}}:=\int_{0}^{1} \int_{0}^{1} \lambda\left(\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\mu_{t_{i}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}^{n}, \hat{X}_{t_{i-1}^{n}}^{n}\right)-\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right)\right) \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda .
$$

We write the proof for $d=1$, as the $d$-dimensional case does not raise any difficulty.

$$
\begin{align*}
\left|\sum_{i=1}^{p_{n}} \varepsilon_{t_{i-1}^{n}}\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)\right| & \leq \sum_{i=1}^{p_{n}}\left|\Delta^{\pi_{n}} X_{t_{i}^{n}} \|\left|\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right|\right| \varepsilon_{t_{i-1}^{n}} \mid \\
& \leq \frac{1}{2} \sum_{i=1}^{p_{n}}\left|\varepsilon_{t_{i-1}^{n}}\right|\left|\Delta^{\pi_{n}} X_{t_{i}^{n}}\right|^{2}+\frac{1}{2} \sum_{i=1}^{p_{n}}\left|\varepsilon_{t_{i-1}^{n}}\right|\left|\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right|^{2} \\
& \leq \varepsilon_{\pi^{n}} Q^{n}, \text { with } Q^{n}:=\sum_{i=1}^{p_{n}}\left|\Delta^{\pi_{n}} X_{t_{i}^{n}}\right|^{2}+\sum_{i=1}^{p_{n}}\left|\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right|^{2} \tag{3.2}
\end{align*}
$$

and for $i=1, \ldots, n$,

$$
\left|\varepsilon_{t_{i-1}^{n}}\right| \leq \varepsilon_{\pi^{n}}:=\sup _{0 \leq j \leq n-1} \sup _{t_{j}^{n} \leq s_{1}, s_{2} \leq t_{j+1}^{n}}\left|\partial_{x} \partial_{\hat{x}} \partial_{m}^{2} u\left(\mu_{s_{1}}, X_{s_{2}}, \hat{X}_{t_{j}^{n}}\right)-\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\mu_{t_{j}^{n}}, X_{t_{j}^{n}}, \hat{X}_{t_{j}^{n}}\right)\right|
$$

Notice that the map $g:\left(s_{1}, s_{2}, r\right) \longmapsto \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(M_{s_{1}}, X_{s_{2}}, \hat{X}_{r}\right)$ is a.s. continuous on the compact $[0, T]^{3}$, therefore it is uniformly continuous, and thus

$$
\varepsilon_{\pi^{n}}=\sup _{0 \leq i \leq n-1 t_{i-1}^{n} \leq s_{1}, s_{2} \leq t_{i}^{n}} \sup \left|g\left(s_{1}, s_{2}, t_{i-1}^{n}\right)-g\left(t_{i-1}^{n}, t_{i-1}^{n}, t_{i-1}^{n}\right)\right| \longrightarrow 0, \text { a.s. }
$$

Since $Q^{n} \longrightarrow\langle X\rangle_{T}+\langle\hat{X}\rangle_{T}<+\infty$ in $\mathbb{L}^{1}$, we deduce from (3.2) together with the dominated convergence theorem (using the fact that $\varepsilon_{\pi}$ is uniformly bounded, because $\partial_{x} \partial_{\hat{x}} \partial_{m}^{2} u$ is bounded) that the error term converges towards 0 in $\mathbb{L}^{1}$, and therefore that it is still the case after taking conditional expectations. This last convergence in $\mathbb{L}^{1}$ yields the a.s. convergence along a subsequence.

Step 4. By following the same line of argument as in Step 3, we also obtain the following convergence for the second term

$$
U_{2}^{n}:=\frac{1}{2} \sum_{i=1}^{p_{n}} \int_{0}^{1} \mathbb{E}^{0}\left[\partial_{x}^{2} \delta_{m} u\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda
$$

Indeed, we find that

$$
U_{2}^{n} \longrightarrow \frac{1}{2} \mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x}^{2} \delta_{m}\left(\mu_{s}, X_{s}\right): \mathrm{d}\langle X\rangle_{s}\right] \text { in } \mathbb{L}^{1}
$$

## 4 Itō-Wentzell's formula

In addition to the continuous semimartingale with canonical decomposition

$$
X=X_{0}+M+A,
$$

we now consider the extension from a deterministic function $u$ to a process $U:[0, T] \times$ $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \times \Omega \longrightarrow \mathbb{R}$, with the following dynamics for the random field $U_{t}(m)$ :

$$
\begin{equation*}
U_{t}(m)=U_{0}(m)+\int_{0}^{t} \phi_{r}(m) \cdot \mathrm{d} B_{r}+\psi_{r}(m) \cdot \mathrm{d} N_{r} \tag{4.1}
\end{equation*}
$$

where $B$ is a finite-variation process and $N$ is a martingale. Our main result is the following Itō-Wentzell formula which will be established under the following conditions.

Assumption 4.1. The maps $f \in\left\{U_{0}, \phi_{t}, \psi_{t}, t \in[0, T]\right\}$ defined on $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ satisfy
(IW1) $\delta_{m} f, \partial_{x}^{2} \delta_{m} f, \delta_{m}^{2} f, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} f$ exist and are continuous;
(IW2) $f, \partial_{x}^{2} \delta_{m} f, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} f$ are bounded;
and the processes $X$ together with the driving processes $B, N$ of the random field $U$ satisfy (IW3) $X_{0},|A|_{\mathrm{TV}}$ and $\langle M\rangle_{T}$ are square integrable, and both $|B|_{\mathrm{TV}},\langle N\rangle_{T}$ are bounded.

We observe that the boundedness condition on $|B|_{T V}$ and $\langle N\rangle_{T}$ can be weakened at the price of stronger boundedness conditions on $\partial_{x} \delta_{m} \phi$ and $\partial_{x} \delta_{m} \psi$. We deliberately choose this setup in order to compare to the conditions of dos Reis \& Platonov [dRP22].

Theorem 2. Let Assumption 4.1 hold.
(i) All derivatives $\delta_{m} U, \partial_{x}^{2} \delta_{m} U, \delta_{m}^{2} U, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U$ exist, are continuous a.s., and are semimartingales defined by the decomposition for $i, j=1, \ldots, d$ :

$$
\begin{aligned}
\partial_{x_{i}} \delta_{m} U_{t}(m, x) & =\partial_{x_{i}} \delta_{m} U_{0}(m, x)+\int_{0}^{t} \partial_{x_{i}} \delta_{m} \phi_{s}(m, x) \cdot \mathrm{d} B_{s}+\partial_{x_{i}} \delta_{m} \psi_{s}(m, x) \cdot \mathrm{d} N_{s}, \\
\partial_{x_{i}, x_{j}}^{2} \delta_{m} U_{t}(m, x) & =\partial_{x_{i}, x_{j}}^{2} \delta_{m} U_{0}(m, x)+\int_{0}^{t} \partial_{x_{i}, x_{j}}^{2} \delta_{m} \phi_{s}(m, x) \cdot \mathrm{d} B_{s}+\partial_{x_{i}, x_{j}}^{2} \delta_{m} \psi_{s}(m, x) \cdot \mathrm{d} N_{s} \\
\partial_{x_{i}, \hat{x}_{j}}^{2} \delta_{m}^{2} U_{t}(m, x, \hat{x}) & =\partial_{x}^{i} \partial_{\hat{x}}^{j} \delta_{m}^{2} U_{0}(m, x, \hat{x})+\int_{0}^{t} \partial_{x}^{i} \partial_{\hat{x}}^{i} \delta_{m}^{2} \phi_{s}(m, x, \hat{x}) \cdot \mathrm{d} B_{s}+\partial_{x}^{i} \partial_{\hat{x}}^{i} \delta_{m}^{2} \psi_{s}(m, x, \hat{x}) \cdot \mathrm{d} N_{s} .
\end{aligned}
$$

(ii) Moreover, we have

$$
\begin{aligned}
U_{T}\left(\mu_{T}\right)-U_{0}\left(\mu_{0}\right)= & \mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} U_{s}\left(\mu_{s}, X_{s}\right) \cdot \mathrm{d} X_{s}+\frac{1}{2} \partial_{x}^{2} \delta_{m} U_{s}\left(\mu_{s}, X_{s}\right): \mathrm{d}\langle X\rangle_{s}\right] \\
& +\mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{T} \frac{1}{2} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U_{s}\left(\mu_{s}, X_{s}, \hat{X}_{s}\right): \mathrm{d}\langle X, \hat{X}\rangle_{s}\right] \\
& +\int_{0}^{T} \phi_{s}\left(\mu_{s}\right) \cdot \mathrm{d} B_{s}+\psi_{s}\left(\mu_{s}\right) \cdot \mathrm{d} N_{s}+\mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} \psi_{s}\left(\mu_{s}, X_{s}\right): \mathrm{d}\langle N, M\rangle_{s}\right], \text { a.s. }
\end{aligned}
$$

Proof. We organize the proof in several steps.

1. We start by the existence and continuity of $\delta_{m} U, \partial_{x}^{2} \delta_{m} U, \delta_{m}^{2} U, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U$. We first show that the functional linear derivative $\delta_{m} U_{t}$ exists for all $t \in[0, T]$, and is given by the first
expression in (i), i.e.

$$
\begin{equation*}
\delta_{m} U_{t}(m, x)=\delta_{m} U_{0}(m, x)+\int_{0}^{t} \delta_{m} \phi_{s}(m, x) \cdot \mathrm{d} B_{s}+\delta_{m} \psi_{s}(m, x) \cdot \mathrm{d} N_{s} \tag{4.2}
\end{equation*}
$$

For arbitrary $m, m^{\prime} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, it follows from the decomposition of $U$ and the definition of the linear functional derivative for the maps $U_{0}, \phi_{s}, \psi_{s}$, as guaranteed by Assumption (IW1), that:

$$
\begin{aligned}
& U_{t}\left(m^{\prime}\right)-U_{t}(m)=U_{0}\left(m^{\prime}\right)-U_{0}(m)+\int_{0}^{t}\left(\phi_{s}\left(m^{\prime}\right)-\phi_{s}(m)\right) \cdot \mathrm{d} B_{s}+\int_{0}^{t}\left(\psi_{s}\left(m^{\prime}\right)-\psi_{s}(m)\right) \cdot \mathrm{d} N_{s} \\
&=\int_{0}^{1} \int \delta_{m} U_{0}\left(m^{\lambda}, x\right)\left(m^{\prime}-m\right)(\mathrm{d} x) \mathrm{d} \lambda+\int_{0}^{t}\left(\int_{0}^{1} \int \delta_{m} \phi_{s}\left(m^{\lambda}, x\right)\left(m^{\prime}-m\right)(\mathrm{d} x) \mathrm{d} \lambda\right) \cdot \mathrm{d} B_{s} \\
&+\int_{0}^{t}\left(\int_{0}^{1} \int \delta_{m} \phi_{s}\left(m^{\lambda}, x\right)\left(m^{\prime}-m\right)(\mathrm{d} x) \mathrm{d} \lambda\right) \cdot \mathrm{d} N_{s}
\end{aligned}
$$

By the Fubini theorem, this provides $U_{t}\left(m^{\prime}\right)-U_{t}(m)=\int_{0}^{1} \int F_{s}\left(m^{\lambda}, x\right)\left(m^{\prime}-m\right)(\mathrm{d} x) \mathrm{d} \lambda$, where $F_{t}(m, x)$ is given by the right hand side of (4.2). Moreover, it follows from (IW2) that the maps $\delta_{m} U_{0}, \delta_{m} \phi_{s}, \delta_{m} \psi_{s}$ have linear growth in $x$, uniformly in $m$. As $\mathbb{E}\left[|B|_{\mathrm{TV}}^{2}+\right.$ $\left.\langle N\rangle_{T}\right]<\infty$ by (IW3), this implies that the map $F_{t}(m, x)$ also has linear growth in $x$, uniformly in $m$. Notice also that

- $F_{t}$ inherits the continuity of $\delta_{m} U_{0}, \delta_{m} \phi_{t}$, and $\delta_{m} \psi_{t}$, by the dominated convergence theorem due to their boundedness, uniformly in $t, m$, assumed in (IW2). We may then conclude that $\delta_{m} U_{t}=F_{t}$ by the definition of the linear functional derivative.
- $\partial_{x}^{2} \delta_{m} U$ exists and inherits the continuity of $\partial_{x}^{2} \delta_{m} U_{0}, \partial_{x}^{2} \delta_{m} \phi$, and $\partial_{x}^{2} \delta_{m} \psi$, by the dominated convergence theorem due to their boundedness, uniformly in $t, m$, assumed in (IW2)

Finally, observe that the coefficients of the SDEs driving $U$ and $\delta_{m} U$ (with fixed $x$ ) satisfy the same conditions in (IW1) and (IW2). Applying the previous argument to the process $\delta_{m} U_{t}(m, x)$, for fixed $x$, it follows that $\delta_{m}^{2} U_{t}$ and $\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U_{t}(m, x, \hat{x})$ also exist and are continuous, with decomposition given by the third expression in (i).
2. Let $\pi^{n}: 0=t_{0}^{n}<t_{1}^{n}<\ldots<t_{p_{n}}^{n}=T$ be a dense sequence of partitions of $[0, T]$. As in the proof of Itō's formula, we start from the telescopic decomposition:

$$
\begin{align*}
U_{t_{i}^{n}}\left(\mu_{t_{i}^{n}}\right)-U_{t_{i-1}^{n}}\left(\mu_{t_{i-1}^{n}}\right)=R_{1}+R_{2} \text { where } R_{1} & :=U_{t_{i}^{n}}\left(\mu_{t_{i}^{n}}\right)-U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}\right),  \tag{4.3}\\
R_{2} & :=U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}\right)-U_{t_{i-1}^{n}}\left(\mu_{t_{i-1}^{n}}^{n}\right)=\int_{t_{i-1}^{n}}^{t_{i}^{n}} \mathrm{~d} U_{s}\left(\mu_{t_{i-1}^{n}}\right),
\end{align*}
$$

with dynamics of $\left\{U_{s}(m), s \geq 0\right\}$ given by (4.1). We next further compute $R_{1}$ by using the definition of the functional linear derivative:

$$
R_{1}=\int_{0}^{1}\left(\delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, .\right) \mathrm{d}\left(\mu_{t_{i}^{n}}-\mu_{t_{i-1}^{n}}\right) \mathrm{d} \lambda=\int_{0}^{1} \mathbb{E}^{0}\left[\delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i}^{n}}\right)-\delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i-1}^{n}}^{n}\right)\right] \mathrm{d} \lambda .\right.
$$

By the second order Taylor theorem, we may rewrite this as
$R_{1}=\int_{0}^{1} \mathbb{E}^{0}\left[\partial_{x} \delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i-1}^{n}}^{n}\right) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}}+\frac{1}{2} \partial_{x}^{2} \delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda$, for some r.v. $\xi_{t_{i-1}^{n}}$ lying between $X_{t_{i-1}^{n}}$ and $X_{t_{i}^{n}}$. Denoting $\gamma_{s}:=\partial_{x} \delta_{m} U_{s}$, we compute that

$$
\begin{aligned}
\left\{\gamma_{t_{i}^{n}}\left(., X_{t_{i-1}^{n}}\right)\right\}_{\mu_{t_{i-1}^{\prime}}^{\lambda}}^{\mu_{t-1}^{n}} & =\int_{0}^{1} \delta_{m} \gamma_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{x}\right)\left(\mu_{t_{i-1}^{n}}^{\lambda}-\mu_{t_{i-1}^{n}}\right)(\mathrm{d} \hat{x}) \mathrm{d} \lambda^{\prime} \\
& =\lambda \int_{0}^{1} \hat{\mathbb{E}}^{0}\left[\left\{\delta_{m} \gamma_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \cdot\right)\right\}_{\hat{X}_{t_{i-1}^{n}}^{n}}^{\hat{X}_{t^{n}}^{n}}\right] \mathrm{d} \lambda^{\prime} \\
& =\lambda \int_{0}^{1} \hat{\mathbb{E}}^{0}\left[\partial_{\hat{x}} \delta_{m} \gamma_{t_{i}^{n}}\left(\mu_{t_{i-1}^{\prime}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right) \Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right] \mathrm{d} \lambda^{\prime}
\end{aligned}
$$

for some $\hat{\xi}_{t_{i-1}^{n}}$ between $\hat{X}_{t_{i-1}^{n}}$ and $\hat{X}_{t_{i}^{n}}$. By the regularity results obtained in Step 1 of the present proof, we may also write $\gamma_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{n}, X_{t_{i-1}^{n}}\right)=\gamma_{t_{i-1}^{n}}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right)+\int_{t_{i-1}^{n}}^{t_{n}^{n}} \mathrm{~d} \gamma_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right)$. Substituting back the expression of the map $\gamma$, this provides:

$$
\begin{aligned}
R_{1}= & \mathbb{E}^{0}\left[\partial_{x} \delta_{m} U_{t}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}}\right]+\mathbb{E}^{0}\left[\left(\int_{t_{i-1}^{n}}^{t_{i}^{n}} \mathrm{~d} \gamma_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}^{n}\right)\right) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}}\right] \\
& +\mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top} \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda\right] \\
& +\int_{0}^{1} \mathbb{E}^{0}\left[\frac{1}{2} \partial_{x}^{2} \delta_{m} U_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)^{\top}\right] \mathrm{d} \lambda,
\end{aligned}
$$

where $\mathrm{d} \gamma_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right)=f_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right) \mathrm{d} B_{s}+g_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right) \mathrm{d} N_{s}, s \in\left[t_{i-1}^{n}, t_{i}^{n}\right)$, with

$$
\left(f_{s}, g_{s}\right):=\partial_{x} \delta_{m}\left(\phi_{s}, \psi_{s}\right)
$$

Summing the decomposition (4.3), and denoting by $t^{n}(s)$ the closest subdivision point strictly to the left of $s$, this provides:

$$
\left.\begin{array}{rl}
U_{T}\left(\mu_{T}\right)- & U_{0}\left(\mu_{0}\right)
\end{array}\right)=\int_{0}^{T} \mathrm{~d} U_{s}\left(\mu_{t^{n}(s)}\right)+\int_{0}^{T} \mathbb{E}^{0}\left[\partial_{x} \delta_{m} U_{t^{n}(s)}\left(\mu_{t^{n}(s)}, X_{t^{n}(s)}\right) \cdot \mathrm{d} X_{s}\right] .
$$

3. We now show that the different terms of the last decomposition converge towards the formula announced in the theorem, for a certain dense sequence of subdivisions.
3.1 We first prove that, after possibly passing to a subsequence,

$$
\begin{equation*}
\int_{0}^{T} \mathrm{~d} U_{s}\left(\mu_{t^{n}(s)}\right)=\int_{0}^{T} \phi_{s}\left(\mu_{t^{n}(s)}\right) \cdot \mathrm{d} B_{s}+\psi_{s}\left(\mu_{t^{n}(s)}\right) \cdot \mathrm{d} N_{s} \longrightarrow \int_{0}^{T} \mathrm{~d} U_{s}\left(\mu_{s}\right) \text { a.s. } \tag{4.4}
\end{equation*}
$$

By the dominated convergence theorem for the Stieltjes integral $\int \cdot \mathrm{d} B_{s}$, and the boundedness of $\phi$, we obtain the convergence of the finite variation part

$$
\int_{0}^{T} \phi_{s}\left(\mu_{t^{n}(s)}\right) \cdot \mathrm{d} B_{s} \longrightarrow \int_{0}^{T} \phi_{s}\left(\mu_{s}\right) \cdot \mathrm{d} B_{s}, \text { a.s. }
$$

As for the stochastic integral component, we estimate by the Itō isometry that

$$
\begin{equation*}
\mathbb{E}\left[\left(\int_{0}^{T}\left(\psi_{s}\left(\mu_{t^{n}(s)}\right)-\psi_{s}\left(\mu_{s}\right)\right) \cdot \mathrm{d} N_{s}\right)^{2}\right] \leq \mathbb{E}\left[\int_{0}^{T}\left(\psi_{s}\left(\mu_{t^{n}(s)}\right)-\psi_{s}\left(\mu_{s}\right)\right)^{\otimes 2}: \mathrm{d}\langle N\rangle_{s}\right],( \tag{4.5}
\end{equation*}
$$

Since $\psi$ and $\mu$ are a.s. continuous and $\psi$ is bounded, it follows from dominated convergence for the Stieltjes stochastic integral $\int \cdot \mathrm{d}\langle N\rangle_{s}$ that $\int_{0}^{T}\left(\psi_{s}\left(\mu_{t^{n}(s)}\right)-\psi_{s}\left(\mu_{s}\right)\right)^{\otimes 2}: \mathrm{d}\langle N\rangle_{s} \longrightarrow 0$, a.s. Furthermore, $\left|\int_{0}^{T}\left(\psi_{s}\left(\mu_{t^{n}(s)}\right)-\psi_{s}\left(\mu_{s}\right)\right)^{\otimes 2}: \mathrm{d}\langle N\rangle_{s}\right| \leq 4\|\psi\|_{\infty}^{2} \operatorname{Tr}\left[\langle N\rangle_{T}\right]$, which is in $\mathbb{L}^{1}$. We then deduce from (4.5) and the dominated convergence theorem that

$$
\int_{0}^{T} \psi_{s}\left(\mu_{t^{n}(s)}\right) \cdot \mathrm{d} N_{s} \longrightarrow \int_{0}^{T} \psi_{s}\left(\mu_{s}\right) \cdot \mathrm{d} N_{s} \text { in } \mathbb{L}^{2}, \text { and a.s. along some subsequence, }
$$

thus completing the proof of (4.4).
For the remaining terms, we use the same method as in the proof of Theorem 1 , by arguing that the sequence inside the conditional expectations converges in $\mathbb{L}^{1}$ towards the desired results.
3.2. Denote $H_{s}:=\partial_{x} \delta_{m} U_{s}\left(\mu_{s}, X_{s}\right)$. The convergence of the second term is implied by the following two convergence results:

$$
\begin{equation*}
\int_{0}^{T} H_{t^{n}(s)} \cdot \mathrm{d} A_{s} \longrightarrow \int_{0}^{T} H_{s} \cdot \mathrm{~d} A_{s}, \quad \text { and } \quad \int_{0}^{T} H_{t^{n}(s)} \cdot \mathrm{d} M_{s} \longrightarrow \int_{0}^{T} H_{s} \cdot \mathrm{~d} M_{s}, \quad \text { in } \mathbb{L}^{1} \cdot(4 \tag{4.6}
\end{equation*}
$$

The first convergence of the finite variation part follows from the a.s. pathwise continuity of the process $H$, together with the dominated convergence theorem for the Stieltjes integral $\int \cdot \mathrm{d} A_{s}$ together with the linear growth of $\partial_{x} \delta_{m} U_{s}$ in the $x$-variable, as implied by (IW2). Similarly, it follows from the BDG inequality and the dominated convergence theorem for the Stieltjes integral $\int \cdot \mathrm{d}\langle M\rangle_{s}$ that $\int_{0}^{T} H_{t^{n}(s)} \cdot \mathrm{d} M_{s} \longrightarrow \int_{0}^{T} H_{s} \cdot \mathrm{~d} M_{s}$ in $\mathbb{L}^{1}$.
3.3. In this step, we justify the $\mathbb{L}^{1}$ convergence of the third term. Denoting $F_{t}:=$ $\int_{0}^{t} f_{s}\left(\mu_{s}, X_{s}\right) \mathrm{d} B_{s}$, and $G_{t}:=\int_{0}^{t} g_{s}\left(\mu_{s}, X_{s}\right) \mathrm{d} N_{s}$, we shall now show that

$$
\begin{aligned}
& \Phi^{n}:=\sum_{i=1}^{p_{n}} \int_{t_{i-1}^{n}}^{t_{i}^{n}} f_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right) \mathrm{d} B_{s} \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} \longrightarrow \operatorname{Tr}\left[\langle X, F\rangle_{T}\right]=0, \text { in } \mathbb{L}^{1}, \text { as } n \rightarrow \infty \\
& \Psi^{n}:=\sum_{i=1}^{p_{n}} \int_{t_{i-1}^{n}}^{t_{i}^{n}} g_{s}\left(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}\right) \mathrm{d} N_{s} \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} \longrightarrow \operatorname{Tr}\left[\langle X, G\rangle_{T}\right], \text { in } \mathbb{L}^{1}, \text { as } n \rightarrow \infty
\end{aligned}
$$

where we have $\langle F, X\rangle_{T}=0$, a.s. due to the fact that the process $F$ has finite variation. In order to prove the first convergence, we use Lemma 1 to conclude that, along some subsequence,

$$
\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right) \cdot\left(\Delta^{\pi_{n}} F_{t_{i}^{n}}\right) \longrightarrow \operatorname{Tr}\left[\langle X, F\rangle_{T}\right], \text { in } \mathbb{L}^{1}
$$

As $\partial_{x} f$ and $\partial_{\hat{x}} \delta_{m} f$ are bounded by Condition (IW2), it follows that the process $\delta_{s}^{n}:=$ $f_{s}\left(\mu_{t^{n}(s)}, X_{t^{n}(s)}\right)-f_{s}\left(\mu_{s}, X_{s}\right)$ satisfies

$$
\begin{equation*}
\left|\delta_{s}^{n}\right| \leq C \varpi_{X}^{\pi^{n}}, \quad \text { for some constant } \quad C>0, \text { with } \varpi_{X}^{\pi^{n}}:=\sup _{\left|t-t^{\prime}\right| \leq\left|\pi^{n}\right|}\left|X_{t}-X_{t^{\prime}}\right| \tag{4.7}
\end{equation*}
$$

We then estimate the error term by

$$
\begin{aligned}
\mathbb{E}\left|\Phi^{n}-\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right) \cdot\left(\Delta^{\pi_{n}} F_{t_{i}^{n}}\right)\right| & =\mathbb{E}\left|\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right) \cdot \int_{t_{i-1}^{n}}^{t_{i}^{n}} \delta_{s}^{n} \mathrm{~d} B_{s}\right| \\
& \leq \mathbb{E}\left[\sup _{1 \leq j \leq p_{n}}\left|\Delta^{\pi_{n}} X_{t_{j}^{n}}\right| \int_{0}^{T}\left|\delta_{s}^{n}\right| \mathrm{d}\left|B_{s}\right| \mathrm{TV}\right] \\
& \leq C \mathbb{E}\left[|B|_{\mathrm{TV}}\left(\varpi_{X}^{\pi^{n}}\right)^{2}\right] \\
& \leq C\left\||B|_{\mathrm{TV}}\right\|_{\infty} \mathbb{E}\left[\left(\varpi_{X}^{\pi^{n}}\right)^{2}\right] \longrightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$, by Conditions (IW3).
A similar argument allows to justify the $\mathbb{L}^{1}$ convergence of $\Psi^{n}$ towards $\operatorname{Tr}\left[\langle G, X\rangle_{T}\right]$ in $\mathbb{L}^{1}$. Indeed, using again Lemma 1, we are reduced to the following estimate involving the process $\eta_{s}^{n}:=g_{s}\left(\mu_{t^{n}(s)}, X_{t^{n}(s)}\right)-g_{s}\left(\mu_{s}, X_{s}\right)$ :

$$
\begin{aligned}
\mathbb{E}\left|\sum_{i=1}^{p_{n}} \Delta X_{t_{i}^{n}} \cdot \int_{t_{i-1}^{n}}^{t_{i}^{n}} \eta_{s}^{n} \mathrm{~d} N_{s}\right| & \leq \mathbb{E}\left[\varpi_{X}^{\pi^{n}}\left|\sum_{i=1}^{p_{n}} \int_{t_{i-1}^{n}}^{t_{i}^{n}} \eta_{s}^{n} \mathrm{~d} N_{s}\right|\right] \\
& =\mathbb{E}\left[\varpi_{X}^{\pi^{n}}\left|\int_{0}^{T} \eta_{s}^{n} \mathrm{~d} N_{s}\right|\right] \\
& \leq \mathbb{E}\left[\left(\varpi_{X}^{\pi^{n}}\right)^{2}\right]^{\frac{1}{2}} \mathbb{E}\left[\int_{0}^{T}\left(\eta_{s}^{n}\right)\left(\eta_{s}^{n}\right)^{\top}: \mathrm{d}\langle N\rangle_{s}\right]^{\frac{1}{2}}
\end{aligned}
$$

by the Cauchy-Schwartz inequality and the Itō isometry. Notice that Conditions (IW2) induces the same estimates for the process $\eta^{n}$ as those for $\delta^{n}$ in (4.7). Then, the required convergence result follows from Conditions (IW3).
3.4. We finally analyse the last two terms by applying the same calculations as in the proof of the standard Itō formula, using Lemma 1, and we obtain the convergence as $n \rightarrow \infty$ :

$$
\sum_{i=1}^{p_{n}} \int_{0}^{1} \frac{1}{2} \partial_{x}^{2} \delta_{m} u_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)^{\top} \mathrm{d} \lambda \longrightarrow \frac{1}{2} \int_{0}^{T} \partial_{x}^{2} \delta_{m} u_{s}\left(\mu_{s}, X_{s}\right): \mathrm{d}\langle X\rangle_{s}
$$

and

$$
\begin{gathered}
\sum_{i=1}^{p_{n}} \int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} u_{t_{i}^{n}}\left(\mu_{t_{i-1}^{n}}^{\lambda \lambda^{\prime}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}\right):\left(\Delta^{\pi_{n}} X_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}}\right)^{\top} \mathrm{d} \lambda^{\prime} \mathrm{d} \lambda \\
\quad \longrightarrow \frac{1}{2} \int_{0}^{T} \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} u\left(\mu_{s}, X_{s}, \hat{X}_{s}\right): \mathrm{d}\langle X, \hat{X}\rangle_{s}
\end{gathered}
$$

in $\mathbb{L}^{1}$. This completes the proof.

### 4.1 Examples

### 4.1.1 Brownian Case

Let us consider the special case where the process $X$ and the random field $U$ are Itō processes defined by

$$
\begin{aligned}
\mathrm{d} U_{t}(m) & =\phi_{t}(m) \mathrm{d} t+\psi_{t}(m) \cdot \mathrm{d} W_{t}+\psi_{t}^{0}(m) \cdot \mathrm{d} W_{t}^{0}, \\
\mathrm{~d} X_{t} & =b_{t} \mathrm{~d} t+\sigma_{t} \mathrm{~d} W_{t}+\sigma_{t}^{0} \mathrm{~d} W_{t}^{0}
\end{aligned}
$$

where $W, W^{0}$ are Brownian motions. We choose here $\mathbb{F}^{0}=\left(\mathcal{F}_{t}^{0}\right)_{0 \leq t \leq T}$ to be the filtration generated by $W^{0}$. This setting reduces to that of dos Reis \& Platonov [dRP22]. The conditionally independent copy $\hat{X}$ is defined by

$$
\mathrm{d} \hat{X}_{t}=\hat{b}_{t} \mathrm{~d} t+\hat{\sigma}_{t} \mathrm{~d} \hat{W}_{t}+\hat{\sigma}_{t}^{0} \mathrm{~d} W_{t}^{0}
$$

where $\hat{b}, \hat{\sigma}, \hat{\sigma}^{0}, \hat{W}$ are conditionally independent copies of $b, \sigma, \sigma^{0}, W$, respectively. We rephrase our Theorem 2 in the present setting in order to compare it with the corresponding statement in [dRP22].

Corollary 1. For $f \in\left\{U_{0}, \phi_{t}, \psi_{t}, \psi_{t}^{0}, t \in[0, T]\right\}$, assume:

- $f, \delta_{m} f, \partial_{x}^{2} \delta_{m} f, \delta_{m}^{2} f, \partial_{\hat{x}}^{2} \delta_{m}^{2} f, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} f$ exist and are continuous;
- $\partial_{x}^{2} \delta_{m} f, \partial_{\hat{x}}^{2} \delta_{m}^{2} f, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} f$ are bounded;
- $\mathbb{E}\left[\int_{0}^{T}\left(\left|X_{0}\right|^{2}+\left|b_{s}\right|^{2}+\left|\sigma_{s} \sigma_{s}^{\top}\right|^{2}+\left|\sigma_{s}^{0}\left(\sigma_{s}^{0}\right)^{\top}\right|^{2}\right) \mathrm{d} s\right]<\infty$.

Then $\delta_{m} U, \partial_{x}^{2} \delta_{m} U, \delta_{m}^{2} U, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U$ exist, are continuous a.s., and are Itō processes driven by the Brownian motions $W$ and $W^{0}$, with coefficients defined by the corresponding derivatives
of the coefficients of $U$. Moreover, we have:

$$
\begin{aligned}
U_{T}\left(\mu_{T}\right)-U_{0}\left(\mu_{0}\right)= & \int_{0}^{T} \phi_{s}\left(\mu_{s}\right) \mathrm{d} s+\int_{0}^{T} \psi_{s}\left(\mu_{s}\right) \cdot \mathrm{d} W_{s}+\int_{0}^{T} \psi_{s}^{0}\left(\mu_{s}\right) \cdot \mathrm{d} W_{s}^{0} \\
& +\mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} U_{s}\left(\mu_{s}, X_{s}\right) \cdot b_{s} \mathrm{~d} s+\int_{0}^{T}\left(\sigma_{s}^{0}\right)^{\top} \partial_{x} \delta_{m} U_{s}\left(\mu_{s}, X_{s}\right) \cdot \mathrm{d} W_{s}^{0}\right] \\
& +\frac{1}{2} \mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x}^{2} \delta_{m} U_{s}\left(\mu_{s}, X_{s}\right):\left(\sigma_{s} \sigma_{s}^{\top}+\sigma_{s}^{0}\left(\sigma_{s}^{0}\right)^{\top}\right) \mathrm{d} s\right] \\
& +\mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} \psi_{s}^{0}\left(\mu_{s}, X_{s}\right):\left(\sigma_{s}^{0}\right)^{\top} \mathrm{d} s\right] \\
& +\frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{T} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U_{s}\left(\mu_{s}, X_{s}, \hat{X}_{s}\right): \sigma_{s}^{0}\left(\hat{\sigma}_{s}^{0}\right)^{\top} \mathrm{d} s\right], \text { a.s. }
\end{aligned}
$$

We thus find the same formula as in [dRP22], with the only additional hypothesis that the highest-order derivatives are bounded instead of square-integrable.

### 4.1.2 Semimartingale factor random field model

Suppose that $X$ is a continuous semimartingale:

$$
X_{t}=X_{0}+A_{t}+M_{t} \text { for all } t \in[0, T],
$$

where $\left(A_{t}\right)_{t}$ is a finite-variation process and $\left(M_{t}\right)_{t}$ is a martingale. In this section, we consider the case where the random field is defined by $U_{t}(m):=u\left(t, m, Y_{t}\right)$ for some factor process $Y=\left(Y_{t}\right)_{0 \leq t \leq T}$ which is another continuous semimartingale :

$$
Y_{t}=Y_{0}+V_{t}+S_{t},
$$

with finite-variation process $V$, and a martingale $S$. Here, the deterministic function $u:(t, m, y) \in[0, T] \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \times \mathbb{R} \mapsto u(t, m, y) \in \mathbb{R}$ will be assumed to be sufficiently smooth. For this, we extend naturally the definition of the functional linear derivative by reducing to the standard definition once the variables $(t, y)$ are frozen. The second order functional linear derivative $\delta_{m}^{2}$ is also defined similarly. The following result is a direct restatement of Theorem 2 in the present context.

Corollary 2. Let us suppose that

- $\partial_{t} u, \partial_{y}^{2} u, \delta_{m} u, \partial_{x}^{2} \delta_{m} u, \delta_{m}^{2} u, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u, \delta_{m} \partial_{y} u, \partial_{x} \delta_{m} \partial_{y} u$ exist and continuous;
- $\partial_{y}^{2} u, \partial_{x}^{2} \delta_{m} u, \partial_{x} \partial_{x} \delta_{m}^{2} u, \partial_{x} \delta_{m} \partial_{y} u$ are bounded ;
- $X_{0},|A|_{\mathrm{TV}},|V|_{\mathrm{TV}},\langle M\rangle_{T}$ and $S$ are square integrable.

Then, denoting $\Theta_{t}=\left(t, \mu_{t}, Y_{t}\right)$ for all $t \in[0, T]$, we have:

$$
\begin{aligned}
u\left(\Theta_{T}\right) & -u\left(\Theta_{0}\right)=\int_{0}^{T} \partial_{t} u\left(\Theta_{s}\right) \mathrm{d} s+\partial_{y} u\left(\Theta_{s}\right) \cdot \mathrm{d} Y_{s}+\frac{1}{2} \partial_{y}^{2} u\left(\Theta_{s}\right): \mathrm{d}\langle Y\rangle_{s} \\
& +\mathbb{E}^{0}\left[\int_{0}^{T} \partial_{x} \delta_{m} u\left(\Theta_{s}, X_{s}\right) \cdot \mathrm{d} X_{s}+\frac{1}{2} \partial_{x}^{2} \delta_{m} u\left(\Theta_{s}, X_{s}\right): \mathrm{d}\langle X\rangle_{s}+\partial_{x} \delta_{m} \partial_{y} u\left(\Theta_{s}, X_{s}\right): \mathrm{d}\langle X, Y\rangle_{s}\right] \\
& +\frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0}\left[\int_{0}^{T} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u\left(\Theta_{s}, X_{s}, \hat{X}_{s}\right): \mathrm{d}\langle X, \hat{X}\rangle_{s}\right] \text { a.s. }
\end{aligned}
$$

Proof. By the standard Itō formula for finite-dimension Itō processes, the random field $\left\{U_{t}(m)=u_{t}\left(t, m, Y_{t}\right),(t, m) \in[0, T] \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)\right\}$ has the following semimartingale decomposition:

$$
\mathrm{d} U_{t}(m)=\phi_{t}(m) \cdot \mathrm{d} B_{t}+\psi_{t}(m) \cdot \mathrm{d} N_{t}
$$

with $B_{t}=\left(t, V_{t},\langle S\rangle_{t}\right), N_{t}=S_{t}, \phi_{t}(m)=\left(\partial_{t} u_{t}\left(m, Y_{t}\right), \partial_{y} u_{t}\left(m, Y_{t}\right), \frac{1}{2} \partial_{y}^{2} u_{t}\left(m, Y_{t}\right)\right)$ and $\psi_{t}(m)=\partial_{y} u_{t}\left(m, Y_{t}\right)$. The result is now a direct application of Theorem 2.

## 5 Application to Mean-Field Control: HJB Equation

Let $\Omega=\mathcal{C}^{0}\left(\mathbb{R}_{+}, \mathbb{R}^{d}\right) \times \mathcal{C}^{0}\left(\mathbb{R}_{+}, \mathbb{R}^{d_{0}}\right)$ with canonical process $\left(X_{t}, W_{t}^{0}\right):\left(\omega, \omega^{0}\right) \in \Omega \mapsto$ $\left(\omega, \omega^{0}\right)(t) \in \mathbb{R}^{d} \times \mathbb{R}^{d_{0}}$. The corresponding canonical filtration is denoted by $\mathbb{F}=\left\{\mathcal{F}_{t}, t \geq\right.$ $0\}$. We also introduce the control space $\mathcal{A}$ consisting of all $\mathbb{F}$-progressively measurable processes $\alpha$ with values in a compact subset $A$ of a finite dimensional space.

Let $b, \sigma$, and $\sigma^{0}$ be given bounded maps

$$
\left(b, \sigma, \sigma^{0}\right): \mathbb{R}_{+} \times \mathbb{R}^{d} \times \mathbb{R}^{d} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \times A \longrightarrow \mathbb{R}^{d} \times \mathcal{M}_{d, d}(\mathbb{R}) \times \mathcal{M}_{d, d_{0}}(\mathbb{R}),
$$

and

$$
\left(k, \gamma, \gamma^{0}\right): \mathbb{R}_{+} \times \mathbb{R}^{d} \longrightarrow \mathbb{R}^{d} \times \mathcal{M}_{d, d}(\mathbb{R}) \times \mathcal{M}_{d, d_{0}}(\mathbb{R}) .
$$

For $t \geq 0$ and $m \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, we denote by $\mathcal{P}(t, y, m)$ the collection of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ satisfying:
(i) $W^{0}$ is a $\mathbb{P}$-Brownian Motion;
(ii) The process $Y$ is defined by

$$
Y_{t}=y, \quad \text { and } \quad \mathrm{d} Y_{s}=k\left(s, Y_{s}\right) \mathrm{d} s+\gamma\left(s, Y_{s}\right) \mathrm{d} B_{s}^{\mathbb{P}}+\gamma^{0}\left(s, Y_{s}\right) \mathrm{d} W_{s}^{0}, s \geq t, \mathbb{P}-\mathrm{a} . \mathrm{s}
$$

for some $\mathbb{P}$-Brownian motion $B^{\mathbb{P}}$;
(iii) the conditional marginal law of $X_{t}$ given $W^{0}$ is $\mathbb{P}_{X_{t}}^{W^{0}}:=\mathbb{P} \circ\left(X_{t} \mid W^{0}\right)^{-1}=m$, and there exists a control process $\alpha \in \mathcal{A}$ such that for $s \geq t$ :

$$
\mathrm{d} X_{s}=b\left(s, X_{s}, Y_{s}, \mathbb{P}_{X_{s}}^{W^{0}}, \alpha_{s}\right) \mathrm{d} s+\sigma\left(s, X_{s}, Y_{s}, \mathbb{P}_{X_{s}}^{W^{0}}, \alpha_{s}\right) \mathrm{d} W_{s}^{\mathbb{P}}+\sigma^{0}\left(s, X_{s}, Y_{s}, \mathbb{P}_{X_{s}}^{W^{0}}, \alpha_{s}\right) \mathrm{d} W_{s}^{0}
$$

$\mathbb{P}$-a.s. for some $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$. Here, $\mathbb{P}_{X_{s}}^{W^{0}}$ is the conditional law of $X_{s}$ under $\mathbb{P}$ given $\left\{W_{r}^{0}, r \geq 0\right\}$.

Let us suppose that $\mathcal{P}(t, y, m)$ is compact for every $(t, y, m) \in[0, T] \times \mathbb{R}^{d} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$.
We define the objective function as

$$
J(t, y, m, \mathbb{P}):=\mathbb{E}_{t}\left[\int_{t}^{T} f^{\alpha_{s}}\left(Y_{s}, \mathbb{P}_{X_{s}}^{W^{0}}\right) \mathrm{d} s+g\left(Y_{T}, \mathbb{P}_{X_{T}}^{W^{0}}\right)\right], \text { for all } \mathbb{P} \in \mathcal{P}(t, y, m)
$$

with running reward map $f: A \times \mathbb{R}^{d} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}$, and final reward $g: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}^{d} \times \mathbb{R}$. The dynamic version of the control problem is defined by:

$$
V(t, y, m):=\sup _{\mathbb{P} \in \mathcal{P}(t, y, m)} J(t, y, m, \mathbb{P}) .
$$

We start from the Dynamic Programming Principle (DPP) which holds under fairly general assumptions, see e.g. Djete, Possamaï \& Tan [DPT20]:

$$
V(t, y, m)=\sup _{\mathbb{P} \in \mathcal{P}(t, m)} \mathbb{E}_{t}\left[\int_{t}^{\theta^{\mathbb{P}}} f^{\alpha_{s}}\left(X_{s}, Y_{s}, \mathbb{P}_{X_{s}}^{W^{0}}\right) \mathrm{d} s+V\left(\theta^{\mathbb{P}}, Y_{\theta^{\mathbb{P}}}, \mathbb{P}_{X_{\theta} \mathbb{P}^{W^{0}}}^{W^{0}}\right],\right.
$$

for any family $\left\{\theta^{\mathbb{P}}\right\}_{\mathbb{P} \in \mathcal{P}(t, m)}$ of $[t, T]$-valued stopping times.
Proposition 1. Suppose that

- $\partial_{t} V, \partial_{y} V, \partial_{y}^{2} V, \delta_{m} V, \partial_{x} \delta_{m} V, \partial_{x}^{2} \delta_{m} V, \delta_{m}^{2} V, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} V, \partial_{x} \delta_{m} \partial_{y} V$ exist;
- $\sigma^{0 \top} \partial_{x} \delta_{m} V, \gamma^{\top} \partial_{y} V$ and $\gamma^{0 \top} \partial_{y} V$ are uniformly bounded;
- $\partial_{t} V, \partial_{y} V, \partial_{x} \delta_{m} V, \partial_{x}^{2} \delta_{m} V, \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} V, \partial_{x} \delta_{m} \partial_{y} V$ are Lipschitz in $m$ uniformly in all other arguments.

Then $V$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$
\begin{aligned}
\begin{aligned}
0 & - \\
& -\partial_{t} V-k \cdot \partial_{y} V-\frac{1}{2}\left(\gamma \gamma^{\top}+\gamma^{0} \gamma^{0 \top}\right): \partial_{y y} V \\
& -\sup _{a \in \mathbb{L}^{0}(A)}\left\{\int\left(f^{a}+b^{a} \cdot \partial_{x} \delta_{m} V+\frac{1}{2}\left(\sigma^{a} \sigma^{a^{\top}}+\sigma^{0^{a}} \sigma^{0^{a \top}}\right): \partial_{x x} \delta_{m} V+\sigma^{0^{a}} \gamma^{0^{\top}}: \partial_{x} \delta_{m} \partial_{y} V\right)(., x) m(\mathrm{~d} x)\right. \\
& \left.\quad+\frac{1}{2} \iint \sigma^{0^{a}}(., x) \sigma^{0^{a}}(., \hat{x}): \partial_{x \hat{x}}^{2} \delta_{m}^{2} V(., x, \hat{x}) m(\mathrm{~d} x) m(\mathrm{~d} \hat{x})\right\},
\end{aligned} \\
\left.V\right|_{t=T}=g .
\end{aligned}
$$

where we denoted $\varphi^{a}(., x):=\varphi(t, y, m, x, a(x))$ for all function $\varphi$.

Proof. We write the proof for $d=d_{0}=1$.

- For $t \in[0, T]$, let $\alpha$ be a control process satisfying the definition of $\mathcal{P}(t, y, m)$. By the Dynamic Programming Principle, for any stopping time $\theta$,

$$
V(t, y, m) \geq \mathbb{E}_{t}\left[\int_{t}^{\theta} f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right) \mathrm{d} s+V\left(\theta, Y_{\theta}, \mu_{\theta}\right)\right]
$$

that is,

$$
\begin{equation*}
\mathbb{E}_{t}\left[\int_{t}^{\theta} f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right) \mathrm{d} s+V\left(\theta, Y_{\theta}, \mu_{\theta}\right)-V(t, y, m)\right] \leq 0 . \tag{5.1}
\end{equation*}
$$

By Itō-Wentzell's formula,

$$
\begin{aligned}
V\left(\theta, Y_{\theta}, \mu_{\theta}\right)-V(t, y, m)= & \int_{t}^{\theta} \mathcal{L}^{\alpha} V\left(s, Y_{s}, \mu_{s}\right) \mathrm{d} s+\int_{t}^{\theta} \partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s} \mathrm{~d} B_{s} \\
& +\int_{t}^{\theta}\left(\mathbb{E}^{0}\left[\partial_{x} \delta_{m} V\left(s, Y_{s}, \mu_{s}, X_{s}\right) \sigma_{s}^{0}\right]+\partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s}^{0}\right) \mathrm{d} W_{s}^{0},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}^{\alpha} V(s, y, m) & :=\partial_{t} V(s, y, m)+\partial_{y} V(s, y, m) k_{s}+\frac{1}{2} \partial_{y}^{2}\left(\gamma_{s}^{2}+\left(\gamma_{s}^{0}\right)^{2}\right) V(s, y, m) \\
& +\int \partial_{x} \delta_{m} V(s, y, m, x) b_{s} m(\mathrm{~d} x)+\frac{1}{2} \int \partial_{x}^{2} \delta_{m} V(s, y, m, x)\left(\sigma_{s}^{2}+\left(\sigma_{s}^{0}\right)^{2}\right) m(\mathrm{~d} x) \\
& +\int \partial_{x} \delta_{m} \partial_{y} V(s, y, m, x) \sigma_{s}^{0} \gamma_{s}^{0} m(\mathrm{~d} x)+\frac{1}{2} \iint \partial_{x \hat{x}}^{2} \delta_{m}^{2} V(s, y, m, x, \hat{x}) \sigma_{s}^{0} \hat{\sigma}_{s}^{0} m(\mathrm{~d} x) m(\mathrm{~d} \hat{x}) .
\end{aligned}
$$

Let us now choose $\theta=\theta_{h}:=\inf \left\{s>t,\left|X_{s}-X_{t}\right| \geq 1\right.$ or $\left.\left|Y_{s}-Y_{t}\right| \geq 1\right\} \wedge(t+h)$, for $h>$
0 . Since $X, Y, \sigma^{0}, \gamma, \gamma^{0}, \mu, \partial_{y} V$ and $\partial_{x} \delta_{m} V$ are continuous, $\mathbb{E}^{0}\left[\partial_{x} \delta_{m} V\left(s, Y_{s}, \mu_{s}, X_{s}\right) \sigma_{s}^{0}\right]$, $\partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s}^{0}$ and $\partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s}$ are bounded over $\left[t, \theta_{h}\right]$ and therefore
$\mathbb{E}_{t}\left[\int_{t}^{\theta_{h}}\left(\mathbb{E}^{0}\left[\partial_{x} \delta_{m} V\left(s, Y_{s}, \mu_{s}, X_{s}\right) \sigma_{s}^{0}\right]+\partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s}^{0}\right) \mathrm{d} W_{s}^{0}+\int_{t}^{\theta_{h}} \partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s} \mathrm{~d} B_{s}\right]=0$.
Then, dividing (5.1) by $h>0$ :

$$
\mathbb{E}_{t}\left[\frac{1}{h} \int_{t}^{\theta_{h}}\left(f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{\alpha} V\left(s, Y_{s}, \mu_{s}\right)\right) \mathrm{d} s\right] \leq 0 .
$$

But a.s., for $h$ small enough $\theta_{h}=t+h$ and

$$
\frac{1}{\theta_{h}-t} \int_{t}^{\theta_{h}}\left(f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{\alpha} V\left(s, Y_{s}, \mu_{s}\right)\right) \mathrm{d} s \longrightarrow f^{\alpha_{t}}(y, m)+\mathcal{L}^{\alpha} V(t, y, m) \text { a.s. }
$$

By dominated convergence for the expectation, we then have that

$$
\mathbb{E}_{t}\left[f^{\alpha_{t}}(y, m)+\mathcal{L}^{\alpha} V(t, y, m)\right]=f^{\alpha_{t}}(y, m)+\mathcal{L}^{\alpha} V(t, y, m) \leq 0 .
$$

Since $\alpha$ was taken arbitrarily, we can conclude that

$$
\sup _{a \in \mathbb{L}^{0}(A)} \mathcal{L}^{a} V(t, y, m)+f^{a}(y, m) \leq 0 .
$$

- Now, let $\varepsilon>0$. There are $\mathbb{P}^{\varepsilon} \in \mathcal{P}(t, y, m)$ and an associated control process $\alpha=\alpha^{\varepsilon}$ such that for every deterministic stopping time $\theta>t$,

$$
-\varepsilon+V(t, y, m) \leq \mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta} f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right) \mathrm{d} s+V\left(\theta, y, \mu_{\theta}\right)\right]
$$

Then, by Itō-Wentzell's formula,

$$
\begin{aligned}
\varepsilon & \geq-\mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta} f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right) \mathrm{d} s+\left(V\left(\theta, Y_{\theta}, \mu_{\theta}\right)-V(t, y, m)\right)\right] \\
& =-\mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta} f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right) \mathrm{d} s+\int_{t}^{\theta} \mathcal{L}^{\alpha_{s}} V\left(s, Y_{s}, \mu_{s}\right) \mathrm{d} s\right] \\
& -\mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left(\mathbb{E}^{0}\left[\partial_{x} \delta_{m} V\left(s, Y_{s}, \mu_{s}, X_{s}\right) \sigma_{s}^{0}\right]+\partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s}^{0}\right) \mathrm{d} W_{s}^{0}+\int_{t}^{\theta_{h}} \partial_{y} V\left(s, Y_{s}, \mu_{s}\right) \gamma_{s} \mathrm{~d} B_{s}\right] \\
& =-\mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left(f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{\alpha_{s}} V\left(s, Y_{s}, \mu_{s}\right)\right) \mathrm{d} s\right] .
\end{aligned}
$$

We used the fact that $\left(\partial_{x} \delta_{m} V\right) \sigma^{0}, \partial_{y} V \gamma$ and $\partial_{y} V \gamma^{0}$ are taken to be uniformly bounded.

With

$$
F(s, y, m):=\sup _{a} f^{a}(y, m)+\mathcal{L}^{a} V(s, y, m)
$$

we have that

$$
\begin{aligned}
& \mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left(f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{\alpha_{s}} V\left(s, Y_{s}, \mu_{s}\right)\right) \mathrm{d} s\right] \\
\leq & \sup _{a} \mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left(f^{a}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{a} V\left(s, Y_{s}, \mu_{s}\right) \mathrm{d} s\right)\right] \\
\leq & (\theta-t) F(t, y, m)+\sup _{a} \mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left|f^{a}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{a} V\left(s, Y_{s}, \mu_{s}\right)-\left(f^{a}(y, m)+\mathcal{L}^{a} V(t, y, m)\right)\right| \mathrm{d} s\right] .
\end{aligned}
$$

Let us show that $m \mapsto f^{a}(y, m)+\mathcal{L}^{a} V(s, y, m)$ is Lipschitzian in $m$ uniformly on $(a, y, s) \in \mathcal{A} \times \mathbb{R}^{d} \times[0, T]$.

- By assumption, $f, \partial_{t} V$ and $\partial_{y} V$ are Lipschitzian in $m$ uniformly on $(s, y)$ and $k$ is uniformly bounded;
- for $\left(s, s^{\prime}\right) \in[0, T]^{2},\left(y, y^{\prime}\right) \in\left(\mathbb{R}^{d}\right)^{2},\left(m, m^{\prime}\right) \in\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)\right)^{2}$ and $\left(a, a^{\prime}\right) \in \mathcal{A}^{2}$,

$$
\begin{aligned}
& \left|\int \partial_{x} \delta_{m} V\left(s^{\prime}, y^{\prime}, m^{\prime}, x\right) b\left(s^{\prime}, x, y^{\prime}, m^{\prime}, a^{\prime}\right) \mathrm{d} m^{\prime}(x)-\int \partial_{x} \delta_{m} V(s, y, m, x) b(s, x, y, m, a) \mathrm{d} m(x)\right| \\
= & \left|\mathbb{E}\left[\partial_{x} \delta_{m} V\left(s^{\prime}, y^{\prime}, m^{\prime}, X^{\prime}\right) b\left(s^{\prime}, X^{\prime}, y^{\prime}, m^{\prime}, a^{\prime}\right)-\partial_{x} \delta_{m} V(s, y, m, X) b(s, X, y, m, a)\right]\right|
\end{aligned}
$$

for $X, X^{\prime}$ random variables such that $X \sim m$ and $X^{\prime} \sim m^{\prime}$. Since, by assumption, $\partial_{x} \delta_{m} V$ is Lipschitzian in $m$ uniformly on ( $s, y, x$ ), we have, with $K$ the Lipschitz constant,

$$
\begin{aligned}
& \left|\partial_{x} \delta_{m} V\left(s^{\prime}, y^{\prime}, m^{\prime}, X^{\prime}\right) b\left(s^{\prime}, X^{\prime}, y^{\prime}, m^{\prime}, a^{\prime}\right)-\partial_{x} \delta_{m} V(s, y, m, X) b(s, X, y, m, a)\right| \\
\leq & \|b\|_{\infty}\left|\partial_{x} \delta_{m} V\left(s^{\prime}, y^{\prime}, m^{\prime}, X^{\prime}\right)-\partial_{x} \delta_{m} V(s, y, m, X)\right| \\
\leq & \|b\|_{\infty} K \mathrm{~d}\left(m^{\prime}, m\right),
\end{aligned}
$$

hence the result;

- with similar calculations, $(s, y, m, a) \mapsto \frac{1}{2} \int \partial_{x}^{2} \delta_{m} V(s, y, m, x)\left(\sigma_{s}^{2}+\left(\sigma_{s}^{0}\right)^{2}\right) m(\mathrm{~d} x)$,

$$
\begin{aligned}
& (s, y, m, a) \mapsto \frac{1}{2} \iint \partial_{x \hat{x}}^{2} \delta_{m}^{2} V(s, y, m, x, \hat{x}) \sigma_{s}^{0} \hat{\sigma}_{s}^{0} m(\mathrm{~d} x) m(\mathrm{~d} \hat{x}) \text { and } \\
& (s, y, m, a) \mapsto \int \partial_{x} \delta_{m} \partial_{y} V(s, y, m, x) \sigma_{s}^{0} \gamma_{s}^{0} m(\mathrm{~d} x) \text { are Lipschitzian in } m \text { uniformly } \\
& \text { on }(s, y, a) \text {. }
\end{aligned}
$$

Therefore, there exists $C>0$ such that, for every $(a, s, y, m),\left(a^{\prime}, s^{\prime}, y^{\prime} m^{\prime}\right) \in(\mathcal{A} \times$ $\left.[0, T] \times\left(\mathbb{R}^{d}\right)^{2} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)\right)^{2}$,

$$
\left|f^{a^{\prime}}\left(y^{\prime}, m^{\prime}\right)+\mathcal{L}^{a^{\prime}} V\left(s^{\prime}, y^{\prime}, m^{\prime}\right)-\left(f^{a}(y, m)+\mathcal{L}^{a} V(s, y, m)\right)\right| \leq C \mathrm{~d}\left(m^{\prime}, m\right)
$$

Then

$$
\begin{aligned}
& \mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left(f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{\alpha_{s}} V\left(s, Y_{s}, \mu_{s}\right)\right) \mathrm{d} s\right] \\
\leq & (\theta-t) F(t, y, m)+\sup _{a} \mathbb{E}_{t}^{\varepsilon}\left[C \int_{t}^{\theta}\left|\mathrm{d}\left(\mu_{s}, m\right)\right| \mathrm{d} s\right] .
\end{aligned}
$$

However,

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{d}\left(\mu_{s}, m\right)\right] & =\mathbb{E}\left[\inf \left\{\mathbb{E}^{0}\left[\left|Z_{\mu}-Z_{m}\right|^{2}\right], \mathbb{P}_{Z_{\mu} \mid W^{0}}=\mu_{s}, \mathbb{P}_{Z_{m} \mid W^{0}}=m\right\}^{\frac{1}{2}}\right] \\
& \leq \mathbb{E}\left[\mathbb{E}^{0}\left[\left|X_{s}-X_{t}\right|^{2}\right]^{\frac{1}{2}}\right] \\
& \leq \mathbb{E}\left[\mathbb{E}^{0}\left[\left|X_{s}-X_{t}\right|^{2}\right]\right]^{\frac{1}{2}} \\
& \leq\left(\|b\|_{\infty}^{2}(s-t)^{2}+\left(\|\sigma\|_{\infty}^{2}+\left\|\sigma^{0}\right\|_{\infty}^{2}\right)(s-t)\right)^{\frac{1}{2}} \\
& \leq\|b\|_{\infty}(s-t)+\left(\|\sigma\|_{\infty}^{2}+\left\|\sigma^{0}\right\|_{\infty}^{2}\right)^{\frac{1}{2}} \sqrt{s-t}
\end{aligned}
$$

where we used Cauchy-Schwarz's inequality.
Therefore,

$$
\begin{aligned}
& \mathbb{E}_{t}^{\varepsilon}\left[\int_{t}^{\theta}\left(f^{\alpha_{s}}\left(Y_{s}, \mu_{s}\right)+\mathcal{L}^{\alpha_{s}} V\left(s, Y_{s}, \mu_{s}\right)\right) \mathrm{d} s\right] \\
\leq & (\theta-t) F(t, y, m)+C\left(\|b\|_{\infty} \frac{(\theta-t)^{2}}{2}+\frac{2}{3}\left(\|\sigma\|_{\infty}^{2}+\left\|\sigma^{0}\right\|_{\infty}^{2}\right)^{\frac{1}{2}}(\theta-t)^{3}\right) .
\end{aligned}
$$

With $\theta=t+\sqrt{\varepsilon}, C_{1}=\frac{1}{2} C\|b\|_{\infty}$ and $C_{2}=\frac{2}{3} C\left(\|\sigma\|_{\infty}^{2}+\left\|\sigma^{0}\right\|_{\infty}^{2}\right)^{\frac{1}{2}}$, we obtain that

$$
\varepsilon \geq-\sqrt{\varepsilon} F(t, y, m)-C_{1} \varepsilon-C_{2} \varepsilon^{\frac{3}{2}}
$$

Dividing by $\sqrt{\varepsilon}$ and taking the limit $\varepsilon \longrightarrow 0$ finally yields that $F(t, y, m) \geq 0$.

## 6 Appendix: Proof of Lemma 1

For simplicity, we only report the proof for $d=1$, as the extension to arbitrary dimension does not raise any difficulty. Let us first notice that

$$
\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}\left(\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right) \longrightarrow \int_{0}^{T} H_{s} \mathrm{~d}\langle M\rangle_{s} \text { in } \mathbb{L}^{1}
$$

Now, with transparent notations, it is obvious that

$$
\begin{aligned}
\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}\left(\left(\Delta^{\pi} X_{t_{i}^{n}}\right)^{2}-\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)\right\|_{1} & \leq\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}\left(\Delta^{\pi} A_{t_{i}^{n}}\right)^{2}\right\|_{1} \\
& +\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}\left(\left(\Delta^{\pi} M_{t_{i}^{n}}\right)^{2}-\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)\right\|_{1} \\
& +2\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}} \Delta^{\pi} A_{t_{i}^{n}} \Delta^{\pi} M_{t_{i}^{n}}\right\|_{1} .
\end{aligned}
$$

For the first term on the right-hand side: writing $\|H\|_{\infty}$ for a uniform, deterministic bound on $\left|H^{n}\right|, n \geq 1$,

$$
\left|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)^{2}\right| \leq\|H\|_{\infty} \sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)^{2}=\|H\|_{\infty} \mathrm{QV}_{\pi^{n}}(A)
$$

where $\mathrm{QV}_{\pi^{n}}(A)$ is the quadratic variation of the finite-variation process $A$ along the partition $\pi^{n}$. As $\mathrm{QV}_{\pi^{n}}(A) \longrightarrow 0$ a.s. and $\mathrm{QV}_{\pi^{n}}(A) \leq|A|_{\mathrm{TV}}^{2}$ which is in $\mathbb{L}^{1}$, it follows from the dominated convergence theorem that

$$
\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)^{2}\right\|_{1} \longrightarrow 0
$$

For the middle term, we introduce the martingale defined by $R_{t}^{t_{i-1}^{n}}:=\left(\Delta^{\pi_{n}} M_{t}\right)^{2}-\Delta^{\pi_{n}}\langle M\rangle_{t}$ for $t \geq t_{i-1}^{n}$ and we now show that

$$
\left\|\sum_{i=1}^{p_{n}-1} H_{t_{i-1}^{n}}^{n} R_{t_{i}^{n}}^{t_{i-1}^{n}}\right\|_{2}^{2}=\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}\left(\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2}-\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)\right\|_{2}^{2} \longrightarrow 0
$$

To see this, we directly compute that

$$
\begin{aligned}
\left\|\sum_{i=1}^{p_{n}-1} H_{t_{i-1}^{n}}^{n} R_{t_{i}^{n}}^{t_{i-1}^{n}}\right\|_{2}^{2} & =\mathbb{E}\left[\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}{ }^{2}\left(R_{t_{i}^{n-1}}^{t_{i-1}^{n}}\right)^{2}\right]+2 \mathbb{E}\left[\sum_{0 \leq i<j \leq n-1} H_{t_{i-1}^{n}}^{n} H_{t_{j}^{n}}^{n} R_{t_{i}^{n-1}}^{t_{i-1}^{n}} R_{t_{j+1}^{n}}^{t_{j}^{n}}\right] \\
& =\mathbb{E}\left[\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}{ }^{2}\left(R_{t_{i}^{n-1}}^{t_{i-1}^{n}}\right)^{2}\right]+2 \sum_{0 \leq i<j \leq n-1} \mathbb{E}\left[H_{t_{i-1}^{n}}^{n} H_{t_{j}^{n n}}^{n} R_{t_{i}^{n-1}}^{t_{i-1}^{n}} \mathbb{E}\left[R_{t_{j+1}^{n}}^{t_{j}^{n}} \mid \mathcal{F}_{t_{j}^{n}}\right]\right]
\end{aligned}
$$

As $\mathbb{E}\left[R_{t_{j+1}^{n}}^{t_{j}^{n}} \mid \mathcal{F}_{t_{j}^{n}}\right]=0$, this implies that

$$
\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}\left(\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2}-\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)\right\|_{2}^{2} \leq\|H\|_{\infty}^{2} \sum_{i=1}^{p_{n}} \mathbb{E}\left[\left(R_{t_{i}^{n-1}}^{t_{i n}^{n}}\right)^{2}\right]
$$

We next estimate that

$$
\begin{aligned}
\mathbb{E}\left[\left(R_{t_{i}^{t_{i n}^{n}}}^{t_{1}}\right)^{2}\right] & =\mathbb{E}\left[\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{4}-2\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2} \Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}+\left(\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)^{2}\right] \\
& \leq \mathbb{E}\left[\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{4}\right]+\mathbb{E}\left[\left(\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)^{2}\right] \\
& \leq\left(1+C_{4}\right) \mathbb{E}\left[\left(\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)^{2}\right]
\end{aligned}
$$

for some constant $C_{4}$ induced by the BDG inequality for the order $p=4$. Therefore,

$$
\left\|\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}}^{n}\left(\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2}-\Delta^{\pi_{n}}\langle M\rangle_{t_{i}^{n}}\right)\right\|_{2}^{2} \leq\left(1+C_{4}\right)\|H\|_{\infty}^{2} \mathbb{E}\left[\mathrm{QV}_{\pi}^{n}(\langle M\rangle)\right]
$$

where $\mathrm{QV}_{\pi}^{n}(\langle M\rangle)=\sum_{i=1}^{p_{n}}\left(\langle M\rangle_{t_{i}^{n}}-\langle M\rangle_{t_{i-1}^{n}}\right)^{2}$. Since $\langle M\rangle$ is a finite-variation process, $\mathrm{QV}_{\pi}^{n}(\langle M\rangle) \longrightarrow 0$ almost surely as $n \rightarrow \infty$, and since $\mathrm{QV}_{\pi}^{n}(\langle M\rangle) \leq\langle M\rangle_{T}^{2} \in \mathbb{L}^{1}$ by Condition (IW3), we conclude by dominated convergence.

Finally, for the last term, note that the previous calculations, for $H=1$, show that $\sum_{i}\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2} \longrightarrow\langle M\rangle_{T}$ in $\mathbb{L}^{2}$. Then, by applying the Cauchy-Schwarz inequality twice:

$$
\begin{aligned}
\mathbb{E}\left[\left|\sum_{i=1}^{p_{n}} H_{t_{i-1}}^{n}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)\right|\right] & \leq\|H\|_{\infty} \mathbb{E}\left[\sqrt{\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)^{2}} \sqrt{\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2}}\right] \\
& \leq\|H\|_{\infty}\left\|\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)^{2}\right\|\left\|_{2}\right\| \sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} M_{t_{i}^{n}}\right)^{2} \|_{2} \longrightarrow 0,
\end{aligned}
$$

since $\mathbb{E}\left[\sum_{i=1}^{p_{n}}\left(\Delta^{\pi_{n}} A_{t_{i}^{n}}\right)^{2}\right] \rightarrow 0$ by dominated convergence (see first term calculations).

## References

[BIRS20] Matteo Burzoni, Vincenzo Ignazio, A. Max Reppen, and H. M. Soner. Viscosity solutions for controlled mckean-vlasov jump-diffusions. SIAM Journal on Control and Optimization, 58(3):1676-1699, 2020.
[BLPR17] Rainer Buckdahn, Juan Li, Shige Peng, and Catherine Rainer. Mean-field stochastic differential equations and associated pdes. The Annals of Probability, 45(2):824-878, 2017.
[Cav22] Thomas Cavallazzi. Itô-krylov's formula for a flow of measures, 2022.
[CCD15] Jean-François Chassagneux, Dan Crisan, and François Delarue. A probabilistic approach to classical solutions of the master equation for large population equilibria, 2015.
[CD18a] René Carmona and François Delarue. Probabilistic Theory of Mean Field Games with Applications I Mean Field FBSDEs, Control, and Games, pages 1-695. Probability Theory and Stochastic Modelling. Springer Nature, United States, 2018.
[CD18b] René Carmona and François Delarue. Probabilistic Theory of Mean Field Games with Applications II Mean Field Games with Common Noise and Master Equations, pages 1-679. Probability Theory and Stochastic Modelling. Springer Nature, United States, 2018.
[CDLL15] Pierre Cardaliaguet, François Delarue, Jean-Michel Lasry, and Pierre-Louis Lions. The master equation and the convergence problem in mean field games, 2015.
[DPT20] Mao Fabrice Djete, Dylan Possamaï, and Xiaolu Tan. Mckean-vlasov optimal control: the dynamic programming principle, 2020.
[dRP22] Goncalo dos Reis and Vadim Platonov. Ito-wentzell-lions formula for measure dependent random fields under full and conditional measure flows, 2022.
[GPW22] Xin Guo, Huyên Pham, and Xiaoli Wei. Itô's formula for flows of measures on semimartingales, 2022.
[Li12] Juan Li. Reflected mean-field backward stochastic differential equations. approximation and associated nonlinear pdes, 2012.
[TTZ23] Mehdi Talbi, Nizar Touzi, and Jianfeng Zhang. Dynamic programming equation for the mean field optimal stopping problem, 2023.


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