Itō and Itō-Wentzell chain rule for flows of conditional laws of continuous semimartingales: an easy approach

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April 19, 2024

Abstract

We provide a general Itō-Wentzell formula for a random field of maps on the Wasserstein space of probability measures, defined by continuous semimartingales, and evaluated along the flow of conditional distributions of another continuous semimartingale. Our method follows standard arguments of Itō calculus, and thus bypasses the approximation by empirical measures commonly used in the existing literature. As an application, we derive the dynamic programming equation for a mean field stochastic control problem with common noise.

MSC2020. 60G40, 49N80, 35Q89, 60H30

Keywords. Itō's formula on the Wasserstein space, and its Itō-Wentzell extension, mean field optimal control.

1 Introduction

Let $X = (X_t)_{t\geq 0}$ be a square integrable continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For simplicity, we consider the scalar case as the multidimensional extension does not raise any special difficulties. Denote by $m_t := \mathbb{P} \circ X_t^{-1}$ the marginal law of X_t , which lies in the set $\mathcal{P}_2(\mathbb{R})$ of all probability measures with finite second moment. For a function $u : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$, with appropriate regularity, an Itō's chain rule for the map $t \longmapsto u(m_t)$ was established by various methods in the literature after the Lectures of P.L. Lions at the Collège de France, see Buckdahn, Li, Peng & Rainer [BLPR17] and Chassagneux, Crisan & Delarue [CCD15] for continuous diffusions, Cavallazzi [Cav22] for a Krylov-type extension of the Itō formula to maps in appropriate Sobolev spaces, Li [Li12] and Burzoni, Ignazio, Reppen & Soner [BIRS20] for special classes of jump-diffusions with continuous marginals. The case of general càdlàg semimartingales was solved simultaneously by Guo, Pham & Wei [GPW22] and Talbi, Touzi & Zhang [TTZ23].

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The Itō chain rule states that, for a map u with appropriate smoothness on the Wasserstein space of probability measures, we have

$$u(m_t) = u(m_0) + \mathbb{E}\Big[\int_0^t \partial_\mu u(m_s, X_s) \, \mathrm{d}X_s + \frac{1}{2} \int_0^t \partial_x \partial_\mu u(m_s, X_s) \, \mathrm{d}\langle X \rangle_s\Big], \text{ for all } t \ge 0,$$
(1.1)

where $\partial_{\mu} u : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ denotes the so-called Lions derivative, and ∂_x is the partial gradient operator with respect to the *x*-variable. See e.g. Carmona & Delarue [CD18a].

Our objective in this paper is to revisit extensions of the last Itō's rule in two directions: - the measure variable is random and defined as the conditional law $\mu_t := \mathbb{P} \circ (X_t | \mathcal{F}^0)^{-1}$ of X_t given some sub-sigma algebra \mathcal{F}^0 of \mathcal{F} ,

- the function u is extended to the context of a dynamic stochastic flow of continuous semimartingales $\{u_t(x), t \ge 0\}$ for all fixed $x \in \mathbb{R}$.

The first extension is motivated by the vibrant research activity on mean field stochastic control with common noise, and the Master equation in the context of mean field games with common noise. The second extension is also motivated by similar stochastic control problems under partial information. The huge interest of the community in this area is enhanced by the wide applications in various questions pertaining to multiple agents decision problems.

Our main emphasis is on the simplicity of our derivations which follow standard arguments in Itō calculus, and which allows to obtain new extensions which were not considered in the existing literature. In order to better explain our approach, let us show how (1.1) can be obtained by means of the following early graduate class level arguments (where the two first steps are simple reminders):

• We first recall from Cardaliaguet, Delarue, Lasry & Lions [CDLL15] that the Lions derivative ∂_{μ} is related to the functional linear derivative δ_m by $\partial_{\mu} = \partial_x \delta_m$, where $\delta_m u : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined for all $m, m' \in \mathcal{P}_2(\mathbb{R})$ by the following limit, if exists:

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left[u \left(m + \varepsilon (m' - m) \right) - u(m) \right] =: \langle \delta_m u(m), m' - m \rangle = \int \delta_m u(m, x) (m' - m) (\mathrm{d}x).$$

Notice that this definition is a mix of directional and Gâteaux derivative, and that the map $\delta_m u(m) := \delta_m u(m, .) : \mathbb{R}^d \longrightarrow \mathbb{R}$ needs to have quadratic growth, at most, in order for the last integral to be well-defined.

• By standard calculus, and under slight regularity, this definition is equivalent to the requirement of existence of such a function $\delta_m u$ satisfying the requirement

$$u(m) - u(m') = \int_0^1 \left\langle \delta_m u \big((1 - \lambda)m + \lambda m' \big), m' - m \right\rangle d\lambda, \text{ for all } m, m' \in \mathcal{P}_2(\mathbb{R}),$$

which is all we need for our subsequent derivation of Ito's formula.

• Given a dense partition $(t_i^n)_{i\geq 0}$ of the $[0,\infty)$, denote $s_i^n := s \wedge t_i^n$, for all $s \geq 0$, and use the telescopic decomposition together with the last definition to see that:

$$u(m_s) - u(m_0) = \sum_{i \ge 1} u(m_{s_i^n}) - u(m_{s_{i-1}^n}) = \sum_{i \ge 1} \int_0^1 \langle U_{n,i}^\lambda, m_{s_i^n} - m_{s_{i-1}^n} \rangle \,\mathrm{d}\lambda, \quad (1.2)$$

with $U_{n,i}^{\lambda} := \delta_m u ((1-\lambda)m_{s_{i-1}^n} + \lambda m_{s_i^n})$ a scalar map on \mathbb{R} . We next observe that

$$\begin{aligned} \langle U_{n,i}^{\lambda}, m_{s_{i}^{n}} - m_{s_{i-1}^{n}} \rangle &= \int U_{n,i}^{\lambda} d(m_{s_{i}^{n}} - m_{s_{i-1}^{n}}) \\ &= \mathbb{E} \big[U_{n,i}^{\lambda} (X_{s_{i}^{n}}) - U_{n,i}^{\lambda} (X_{s_{i-1}^{n}}) \big] \\ &= \mathbb{E} \Big[\int_{s_{i-1}^{s_{i}^{n}}}^{s_{i}^{n}} (U_{n,i}^{\lambda})'(X_{r}) \, \mathrm{d}X_{r} + \frac{1}{2} (U_{n,i}^{\lambda})''(X_{r}) \, \mathrm{d}\langle X \rangle_{r} \Big], \end{aligned}$$

where the last equality follows from the standard Itō's formula, under the appropriate regularity assumptions on the map $U_{n,i}^{\lambda}$. Plugging this expression in (1.2), we obtain the required formula (1.1) by standard limiting argument using the dominated convergence theorem.

The last argument is most appealing as it uses the standard intuitive notion of functional linear derivative δ_m . Moreover, it completely bypasses the crucial step of projection on empirical measures used in most of the previous literature following the Lectures of P.L. Lions at the Collège de France, see Chassagneux, Crisan & Delarue [CCD15], Buckdahn, Li, Peng & Rainer [BLPR17], Carmona & Delarue [CD18a]. Here, the idea is to approximate the marginal law m_t by the corresponding empirical measure $\overline{m}_t^N := \frac{1}{N} \sum_{i \leq N} \delta_{X_t^i}$ of a finite sample of N independent copies (X^1, \ldots, X^N) , apply the standard Itō's formula to the finite dimensional map $\overline{u}^N(X_t^1, \ldots, X_t^N) := u(\overline{m}_t^N)$, and finally take limits by using fine results on the convergence of empirical measures.

The simple method outlined above is applied in Talbi, Touzi & Zhang [TTZ23] in the context of càd-làg semimartingales, see also the parallel paper by Guo, Pham & Wei [GPW22] which uses a functional analytic extension of an appropriate class of cylindrical maps in order to account for the jumps of the semimartingale.

The main contribution of this paper is to show that the previous simple method also applies to derive an Itō-Wentzell chain rule for conditional laws. This answers in particular a question raised in dos Reis & Platonov [dRP22], who derive the Itō-Wentzell formula by adapting the technique of projection on empirical measures used by Carmona & Delarue [CD18b] to derive the Itō formula for conditional laws, see also Cardaliaguet, Delarue, Lasry & Lions [CDLL15] in the context of the Master equation. We notice that, while the state process in [dRP22] is defined by SDEs driven by Brownian motions and is conditioned by a Brownian motion, we consider in this paper general continuous semimartingales with general conditioning. Moreover, our Itō-Wentzell formula is derived for a random flow continuous semimartingale, extending the case of deterministic function of a process and a conditional law of another process of [dRP22]. The paper is organized as follows. Section 3 provides an Itō's formula for conditional marginal laws of continuous semimartingales. Although this result is a particular case of the subsequent one, we believe that it deserves to be isolated for the sake of clarity. Section 4 contains our general Itō-Wentzell formula in the context where the random field of maps is also defined by continuous semimartingales. Finally, Section 5 provides an application in mean field stochastic control with common noise.

2 Notations

We denote $x \cdot y := \sum_i x_i y_i$ the Euclidean scalar product of two vectors in any finite dimensional space, A:B := Tr[AB] and $A^{\otimes 2} = AA^{\intercal}$ for all matrices of appropriate size.

Throughout this paper, we fix a constant maturity T > 0, and a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$.

Let $\pi : 0 = t_0 < \ldots < t_{n_{\pi}} = T$ be a subdivision of [0,T] with mesh size $|\pi| := \max_{i \leq n_{\pi}} (t_i - t_{i-1})$. A sequence of subdivisions $(\pi^n)_{n \geq 0}$ is dense if the sequence of meshes $|\pi^n|$ converges to 0 as $n \to \infty$.

A stochastic process is said to be piecewise constant along the subdivision π if it is constant on each interval $(t_{i-1}, t_i]$. For a process Y valued in \mathbb{R}^d , we denote the increment to the subdivision by:

$$\Delta^{\pi} Y_s := \sum_{i=1}^{n_{\pi}} (Y_{s \wedge t_i} - Y_{s \wedge t_{i-1}}), \text{ for all } s \ge 0.$$

In other words, for $s \in (t_{i_s-1}, t_{i_s}]$, we have $\Delta^{\pi} Y_s = Y_s - Y_{t_{i_s-1}}$, and if the process Y is in addition piecewise constant along π , we have $\Delta^{\pi} Y_s = Y_{t_{i_s}} - Y_{t_{i_s-1}}$.

The total variation of Y is denoted by

$$|Y|_{\text{TV}} = \sup_{\pi} \sum_{i=1}^{n_{\pi}-1} |Y_{t_{i+1}} - Y_{t_i}| = \sup_{\pi} \sum_{i=1}^{n_{\pi}-1} |\Delta^{\pi} Y_{t_{i+1}}|.$$

where |.| is the Euclidean norm in \mathbb{R}^d .

The quadratic variation of Y is defined as

$$\langle Y \rangle_s = \lim_{|\pi| \to 0} \sum_{i=1}^{n_{\pi}-1} (Y_{t_{i+1}} - Y_{t_i}) (Y_{t_{i+1}} - Y_{t_i})^{\mathsf{T}}, \text{ for all } s \ge 0,$$

where the limit is in probability and does not depend on the choice of the subdivisions sequence.

X is said to be a (continuous) semimartingale if it can be written as $X_s = X_0 + A_s + M_s$, $s \in [0, T]$ where A is a (continuous) finite-variation process and M is a (continuous) martingale.

 $\mathbb{H}^2(Y)$ is the collection of all progressively measurable processes H, with same dimension as Y, such that $\mathbb{E}\left[\int_0^T H_s H_s^{\intercal} : d\langle Y \rangle_s\right] < \infty$.

A sequence $(H^n)_{n\geq 0}$ of predictable bounded processes is called a simple approximation of a process $H \in \mathbb{H}^2(Y)$ if there exists a dense sequence of subdivisions $(\pi^n)_{n\geq 0}$ such that H^n is piecewise constant along π^n , for all $n \geq 0$, and $H^n \longrightarrow H$ in $\mathbb{H}^2(Y)$, as $n \to \infty$.

The following (probably well-known) result will be used frequently. As we failed to find a reference for it, we report its proof as a complement in the Appendix section 6.

Lemma 1. Let X be a semimartingale with decomposition X = A + M into a finite variation process A and a martingale M satisfying $\mathbb{E}\left[|A|_{TV}^2 + \langle M \rangle_T^2\right] < \infty$. Let $(H^n)_{n\geq 0}$ be a simple approximation of a matrix-valued bounded progressively measurable H with rows in $\mathbb{H}^2(X)$, along some sequence of subdivisions $(\pi^n : 0 = t_0^n < \ldots < t_{p_n}^n = T)$. Then:

$$\sum_{i=1}^{p_n} H^n_{t^n_{i-1}} : (\Delta^{\pi^n} X_{t^n_i}) (\Delta^{\pi^n} X_{t^n_i})^{\mathsf{T}} \longrightarrow \int_0^T H_s : \mathrm{d}\langle X \rangle_s, \quad in \quad \mathbb{L}^1 \quad as \quad n \to \infty$$

We notice that, if H is continuous, then the last convergence result holds true with $H_{t_i^n}^n = H_{t_i^n}$, for $i = 0, \ldots, n_{p_n}$.

The marginal laws considered in this paper lie in the set $\mathcal{P}_2(\mathbb{R}^d)$ of all probability measures on \mathbb{R}^d with finite second moment. Similarly, our conditional marginal laws are random maps taking values in $\mathcal{P}_2(\mathbb{R}^d)$. This set is naturally equipped with the Wasserstein distance

$$d(m,m') := \inf_{\pi \in \Pi(m,m')} \int |x-x|^2 \pi(\mathrm{d}x,\mathrm{d}x'), \text{ for all } m,m' \in \mathcal{P}(\mathbb{R}^d),$$

where $\Pi(m, m')$ is the set of all couplings of (m, m'), i.e. probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals m and m'.

We say that a function $u: \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ admits a (first order) functional linear derivative if there exists a map $\delta_m u: \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that for all $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$u(m') - u(m) = \int_0^1 \int_{\mathbb{R}^d} \delta_m u(\bar{m}^\lambda, x)(m' - m)(\mathrm{d}x) \,\mathrm{d}\lambda \text{ with } \bar{m}^\lambda := \lambda m' + (1 - \lambda)m,$$

and $\delta_m u$ has quadratic growth in x, locally uniformly in m, so that the last integral is welldefined. Similarly, the second order functional linear derivative $\delta_m^2 : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that for all $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$:

$$\delta_m u(m', x) - \delta_m u(m, x) = \int_0^1 \int_{\mathbb{R}^d} \delta_m^2 u(\bar{m}^\lambda, x, \hat{x})(m' - m)(\mathrm{d}\hat{x}) \,\mathrm{d}\lambda,$$

and $\delta_m^2 u$ has quadratic growth in \hat{x} , locally uniformly in m, for all fixed $x \in \mathbb{R}^d$. Notice that under these conditions, $\delta_m \partial_x \delta_m u = \partial_x \delta_m^2 u$.

3 Itō's formula

Throughout this paper, we consider an \mathbb{R}^d -valued continuous semimartingale with canonical decomposition

$$X = X_0 + M + A,$$

where M is a martingale and A is a finite-variation process, both started from 0, and $X_0 \in \mathbb{L}^2(\mathcal{F}_0)$.

For an arbitrary filtration $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t \ge 0}$, we denote by $\mu_t = \mathcal{L}(X_t | \mathcal{F}^0_T)$, the law of X_t conditional on \mathcal{F}^0_T , for $t \in [0, T]$.

Our first result is the following Itō's formula for flows of conditional law of the continuous semimartingale X.

Assumption 3.1. The map $u : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$, and the continuous semimartingale X satisfy:

- (I1) $\delta_m u, \partial_x^2 \delta_m u, \delta_m^2 u, \partial_x \partial_x \delta_m^2 u$ exist and are continuous in each variable;
- (I2) $\partial_x^2 \delta_m u, \partial_x \partial_{\hat{x}} \delta_m^2 u$ are bounded;
- (I3) X_0 , $|A|_{\text{TV}}$ and $\langle M \rangle_T$ are square integrable.

Theorem 1. Let Assumption 3.1 hold true, and let \hat{X} be a copy of X on a copy probability space, with $\mathcal{L}(\hat{X}|\mathcal{F}_T^0) = \mathcal{L}(X|\mathcal{F}_T^0)$. Then:

$$u(\mu_T) - u(\mu_0) = \mathbb{E}^0 \left[\int_0^T \partial_x \delta_m u(\mu_s, X_s) \cdot \mathrm{d}X_s + \frac{1}{2} \partial_x^2 \delta_m u(\mu_s, X_s) \cdot \mathrm{d}\langle X \rangle_s \right] + \mathbb{E}^0 \hat{\mathbb{E}}^0 \left[\int_0^T \frac{1}{2} \partial_{x\hat{x}}^2 \delta_m^2 u(\mu_s, X_s, \hat{X}_s) \cdot \mathrm{d}\langle X, \hat{X} \rangle_s \right], \quad a.s.$$

where $\mathbb{E}^0 := \mathbb{E}[\cdot|\mathcal{F}_T^0]$ and $\hat{\mathbb{E}}^0 := \mathbb{E}[\cdot|X, \mathcal{F}_T^0]$ denote the conditional expectations in the enlarged space.

Proof. Let us first prove the result when $\partial_x^2 \delta_m u$ and $\partial_x \partial_x \delta_m^2 u$ are bounded. We organize our arguments in three steps.

Step 1. Let $\pi^n : 0 = t_0^n < t_1^n < \ldots < t_{p_n}^n = T$ be a dense sequence of partitions of [0, T], and denote $\mu_{t_i^n}^{\lambda} = \lambda \mu_{t_i^n} + (1 - \lambda) \mu_{t_{i-1}^n}$. By the definition of the linear functional derivative, we have:

$$\delta_{i}^{n} u := u(\mu_{t_{i}^{n}}) - u(\mu_{t_{i-1}^{n}}) = \int_{0}^{1} \int_{\mathbb{R}^{d}} \delta_{m} u(\mu_{t_{i}^{n}}^{\lambda}, x)(\mu_{t_{i}^{n}} - \mu_{t_{i-1}^{n}})(\mathrm{d}x) \,\mathrm{d}\lambda$$
$$= \int_{0}^{1} \mathbb{E}^{0} \left[\delta_{m} u(\mu_{t_{i}^{n}}^{\lambda}, X_{t_{i}^{n}}) - \delta_{m} u(\mu_{t_{i}^{n}}^{\lambda}, X_{t_{i-1}^{n}}) \right] \mathrm{d}\lambda.$$
(3.1)

By the second order Taylor theorem, we may rewrite this as

$$\delta_{i}^{n} u = \int_{0}^{1} \mathbb{E}^{0} \left[\partial_{x} \delta_{m} u(\mu_{t_{i-1}}^{\lambda}, X_{t_{i-1}}) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} + \frac{1}{2} \partial_{x}^{2} \delta_{m} u(\mu_{t_{i-1}}^{\lambda}, \xi_{t_{i-1}}) \cdot (\Delta^{\pi_{n}} X_{t_{i}^{n}}) (\Delta^{\pi_{n}} X_{t_{i}^{n}})^{\mathsf{T}} \right] \mathrm{d}\lambda,$$

for some r.v. $\xi_{t_{i-1}^n}$ lying between $X_{t_{i-1}^n}$ and $X_{t_i^n}$. Let us introduce an independent copy \hat{X} of X conditionally to \mathbb{F}^0 . Using the notation $\{F(\theta, \cdot)\}_{\theta'_1}^{\theta'_2} := F(\theta, \theta'_2) - F(\theta, \theta'_1)$, for all map $F(\theta, \theta')$, and denoting $\gamma := \partial_x \delta_m u$, we compute that

$$\begin{split} \left\{ \gamma(., X_{t_{i-1}^{n}}) \right\}_{\mu_{t_{i-1}^{n}}}^{\mu_{t_{i-1}^{n}}} &= \int_{0}^{1} \int \delta_{m} \gamma(\mu_{t_{i-1}^{\lambda\lambda'}}^{\lambda\lambda'}, X_{t_{i-1}^{n}}, \hat{x})(\mu_{t_{i-1}^{n}}^{\lambda} - \mu_{t_{i-1}^{n}})(\mathrm{d}\hat{x}) \,\mathrm{d}\lambda' \\ &= \lambda \int_{0}^{1} \hat{\mathbb{E}}^{0} \left[\left\{ \delta_{m} \gamma(\mu_{t_{i-1}^{n}}^{\lambda\lambda'}, X_{t_{i-1}^{n}}, \cdot) \right\}_{\hat{X}_{t_{i-1}^{n}}}^{\hat{X}_{t_{i-1}^{n}}} \right] \mathrm{d}\lambda' \\ &= \lambda \int_{0}^{1} \hat{\mathbb{E}}^{0} \left[\partial_{\hat{x}} \delta_{m} \gamma(\mu_{t_{i-1}^{n}}^{\lambda\lambda'}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}) \Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}} \right] \mathrm{d}\lambda', \end{split}$$

for some $\hat{\xi}_{t_{i-1}^n}$ between $\hat{X}_{t_{i-1}^n}$ and $\hat{X}_{t_i^n}$. This provides:

$$\begin{split} \delta_i^n u &= \mathbb{E}^0 \left[\partial_x \delta_m u(\mu_{t_{i-1}^n}, X_{t_{i-1}^n}) \cdot \Delta^{\pi_n} X_{t_i^n} \right] \\ &+ \mathbb{E}^0 \hat{\mathbb{E}}^0 \left[\int_0^1 \int_0^1 \lambda \partial_x \delta_m \partial_x \delta_m u(\mu_{t_{i-1}^n}^{\lambda \lambda'}, X_{t_{i-1}^n}, \hat{\xi}_{t_{i-1}^n}) : (\Delta^{\pi_n} X_{t_i^n}) (\Delta^{\pi_n} \hat{X}_{t_i^n})^{\mathsf{T}} \, \mathrm{d}\lambda' \, \mathrm{d}\lambda \right] \\ &+ \int_0^1 \mathbb{E}^0 \left[\frac{1}{2} \partial_x^2 \delta_m u(\mu_{t_{i-1}^n}^\lambda, \xi_{t_{i-1}^n}) : (\Delta^{\pi_n} X_{t_i^n}) (\Delta^{\pi_n} X_{t_i^n})^{\mathsf{T}} \right] \mathrm{d}\lambda. \end{split}$$

Summing (3.1), and denoting by $t^n(s)$ the closest subdivision point strictly to the left of s, this provides:

$$\begin{aligned} U_{T}(\mu_{T}) - U_{0}(\mu_{0}) &= \int_{0}^{T} \mathbb{E}^{0} \left[\partial_{x} \delta_{m} u(\mu_{t^{n}(s)}, X_{t^{n}(s)}) \cdot \mathrm{d}X_{s} \right] \\ &+ \sum_{i=1}^{p_{n}} \int_{0}^{1} \mathbb{E}^{0} \left[\frac{1}{2} \partial_{x}^{2} \delta_{m} u(\mu_{t^{n}_{i-1}}^{\lambda}, \xi_{t^{n}_{i-1}}) : (\Delta^{\pi_{n}} X_{t^{n}_{i}}) (\Delta^{\pi_{n}} X_{t^{n}_{i}})^{\mathsf{T}} \right] \mathrm{d}\lambda \\ &+ \sum_{i=1}^{p_{n}} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} u(\mu_{t^{n}_{i-1}}^{\lambda\lambda'}, X_{t^{n}_{i-1}}, \hat{\xi}_{t^{n}_{i-1}}) : (\Delta^{\pi_{n}} X_{t^{n}_{i}}) (\Delta^{\pi_{n}} \hat{X}_{t^{n}_{i}})^{\mathsf{T}} \mathrm{d}\lambda' \mathrm{d}\lambda \right]. \end{aligned}$$

Our goal in the subsequent steps is to analyse the convergence of each term in the last decomposition, along a suitable sequence of subdivisions, towards the formula announced in Theorem 1. Proving that one such subsequence exists is enough.

Step 2. In this step, we start with the first term that we denote U_1^n . Namely,

$$U_1^n := \sum_{i=1}^{p_n} \mathbb{E}^0 \left[\int_{t_{i-1}^n}^{t_i^n} \partial_x \delta_m u(\mu_{t_{i-1}^n}, X_s) \cdot \mathrm{d}X_s \right] = \mathbb{E}^0 \left[\int_0^T f_n(s) \cdot \mathrm{d}X_s \right] \text{ with } f_n(s) := \partial_x \delta_m u(\mu_{t^n(s)}, X_s),$$

where $t^n(s)$ is the last point of the subdivision π^n which is strictly to the left of s. Since μ_s is continuous in $s, f_n(s) \longrightarrow f(s) = \partial_x \delta_m u(\mu_s, X_s)$ almost surely when $n \to +\infty$. Moreover,

 $|\partial_x \delta_m u(\mu_s, X_s)| \leq C(1+|X_s|)$, by our assumption on $\partial_x^2 \delta_m u$, and as $\mathbb{E}[\langle M \rangle_T + |A|_{TV}^2] < \infty$, we also have $\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^2\right] < \infty$. Then, it follows from the BDG inequality that

$$\mathbb{E}\left[\left|\int_{0}^{T} (f_{n}(s) - f(s)) \cdot \mathrm{d}X_{s}\right|\right] \leq C_{\mathrm{BDG}} \mathbb{E}\left[\left|\int_{0}^{T} (f_{n}(s) - f(s))^{\otimes 2} \cdot \mathrm{d}\langle X \rangle_{s}\right|^{\frac{1}{2}}\right] \longrightarrow 0 \text{ in } \mathbb{L}^{2},$$

by dominated convergence, as

$$\left|\int_0^T (f_n(s) - f(s))^{\otimes 2} : \mathrm{d}\langle X \rangle_s \right|^{\frac{1}{2}} \le C \left(\sup_s (1 + |X_s|)^2 |\langle X \rangle_T | \right)^{\frac{1}{2}} \le \frac{C}{2} \left(\sup_s (1 + |X_s|)^2 + |\langle X \rangle_T | \right) \in \mathbb{L}^1.$$

This shows that $\int_0^T \partial_x \delta_m u(\mu_{t^n(s)}, X_s) \cdot \mathrm{d}X_s \longrightarrow \int_0^T \partial_x \delta_m u(M_s, X_s) \cdot \mathrm{d}X_s$ in \mathbb{L}^1 , thus implying by the Jensen inequality that

$$U_1^n \longrightarrow \mathbb{E}^0 \left[\int_0^T \partial_x \delta_m u(\mu_s, X_s) \cdot \mathrm{d}X_s \right] \quad \text{in} \quad \mathbb{L}^1, \quad \text{as} \quad n \to \infty,$$

and therefore almost surely along some subsequence.

Step 3. We next analyse the convergence of the third term

$$U_{3}^{n} := \sum_{i=1}^{p_{n}} \int_{0}^{1} \lambda \int_{0}^{1} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u(\mu_{t_{i}^{n}}^{\lambda\lambda'}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}) : (\Delta^{\pi_{n}} X_{t_{i}^{n}}) (\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}})^{\mathsf{T}} \right] \mathrm{d}\lambda' \mathrm{d}\lambda,$$

by arguing in two steps:

• First, substituting $\mu_{t_{i-1}^n}$ to $\mu_{t_i^n}^{\lambda\lambda'}$, and $\hat{X}_{t_{i-1}^n}$ to $\hat{\xi}_{t_{i-1}^n}$, we compute that

$$\overline{U}_{3}^{n} := \sum_{i=1}^{p_{n}} \int_{0}^{1} \lambda \int_{0}^{1} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u(\mu_{t_{i}^{n}}, X_{t_{i-1}^{n}}, \hat{X}_{t_{i-1}^{n}}) : (\Delta^{\pi_{n}} X_{t_{i}^{n}}) (\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}})^{\mathsf{T}} \right] \mathrm{d}\lambda' \mathrm{d}\lambda \\
= \frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\sum_{i=1}^{p_{n}} H_{t_{i-1}^{n}} : (\Delta^{\pi_{n}} X_{t_{i}^{n}}) (\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}})^{\mathsf{T}} \right], \text{ with } H_{s} := \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u(\mu_{s}, X_{s}, \hat{X}_{s}),$$

which leads by the polarized version of Lemma 1 to:

$$\overline{U}_{3}^{n} \longrightarrow \frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\int_{0}^{T} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u(\mu_{s}, X_{s}, \hat{X}_{s}) : \mathrm{d}\langle X, \hat{X} \rangle_{s} \right], \quad \mathrm{in} \quad \mathbb{L}^{1}, \text{ as } n \to \infty$$

• we next control the error $U_3^n - \overline{U}_3^n = \mathbb{E}^0 \hat{\mathbb{E}}^0 \left[\sum_{i=1}^{p_n} \varepsilon_{t_{i-1}^n} : (\Delta^{\pi^n} X_{t_i^n}) (\Delta^{\pi^n} \hat{X}_{t_i^n})^{\mathsf{T}} \right]$, with

$$\varepsilon_{t_{i-1}^{n}} := \int_{0}^{1} \int_{0}^{1} \lambda \left(\partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u(\mu_{t_{i}^{n}}^{\lambda \lambda'}, X_{t_{i-1}^{n}}, \hat{X}_{t_{i-1}^{n}}) - \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} u(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}) \right) \mathrm{d}\lambda' \, \mathrm{d}\lambda.$$

We write the proof for d = 1, as the *d*-dimensional case does not raise any difficulty.

$$\begin{aligned} \left| \sum_{i=1}^{p_n} \varepsilon_{t_{i-1}^n} (\Delta^{\pi_n} X_{t_i^n}) (\Delta^{\pi_n} \hat{X}_{t_i^n}) \right| &\leq \sum_{i=1}^{p_n} |\Delta^{\pi_n} X_{t_i^n}| |\Delta^{\pi_n} \hat{X}_{t_i^n}| |\varepsilon_{t_{i-1}^n}| \\ &\leq \frac{1}{2} \sum_{i=1}^{p_n} \left| \varepsilon_{t_{i-1}^n} \right| |\Delta^{\pi_n} X_{t_i^n}|^2 + \frac{1}{2} \sum_{i=1}^{p_n} \left| \varepsilon_{t_{i-1}^n} \right| |\Delta^{\pi_n} \hat{X}_{t_i^n}|^2 \\ &\leq \varepsilon_{\pi^n} Q^n, \text{ with } Q^n := \sum_{i=1}^{p_n} |\Delta^{\pi_n} X_{t_i^n}|^2 + \sum_{i=1}^{p_n} |\Delta^{\pi_n} \hat{X}_{t_i^n}|^2, \quad (3.2) \end{aligned}$$

and for $i = 1, \ldots, n$,

$$\left|\varepsilon_{t_{i-1}^n}\right| \leq \varepsilon_{\pi^n} := \sup_{0 \leq j \leq n-1} \sup_{t_j^n \leq s_1, s_2 \leq t_{j+1}^n} \left|\partial_x \partial_{\hat{x}} \delta_m^2 u(\mu_{s_1}, X_{s_2}, \hat{X}_{t_j^n}) - \partial_x \partial_{\hat{x}} \delta_m^2 u(\mu_{t_j^n}, X_{t_j^n}, \hat{X}_{t_j^n})\right|.$$

Notice that the map $g: (s_1, s_2, r) \mapsto \partial_x \partial_x \partial_x^2 \partial_m^2 u(M_{s_1}, X_{s_2}, \hat{X}_r)$ is a.s. continuous on the compact $[0, T]^3$, therefore it is uniformly continuous, and thus

$$\varepsilon_{\pi^n} = \sup_{0 \le i \le n-1} \sup_{t_{i-1}^n \le s_1, s_2 \le t_i^n} |g(s_1, s_2, t_{i-1}^n) - g(t_{i-1}^n, t_{i-1}^n, t_{i-1}^n)| \longrightarrow 0, \text{ a.s.}$$

Since $Q^n \longrightarrow \langle X \rangle_T + \langle \hat{X} \rangle_T < +\infty$ in \mathbb{L}^1 , we deduce from (3.2) together with the dominated convergence theorem (using the fact that ε_{π} is uniformly bounded, because $\partial_x \partial_{\hat{x}} \delta_m^2 u$ is bounded) that the error term converges towards 0 in \mathbb{L}^1 , and therefore that it is still the case after taking conditional expectations. This last convergence in \mathbb{L}^1 yields the a.s. convergence along a subsequence.

Step 4. By following the same line of argument as in Step 3, we also obtain the following convergence for the second term

$$U_2^n := \frac{1}{2} \sum_{i=1}^{p_n} \int_0^1 \mathbb{E}^0 \Big[\partial_x^2 \delta_m u(\mu_{t_{i-1}}^{\lambda}, \xi_{t_{i-1}}) : (\Delta^{\pi_n} X_{t_i}) (\Delta^{\pi_n} X_{t_i})^{\mathsf{T}} \Big] \, \mathrm{d}\lambda.$$

Indeed, we find that

$$U_2^n \longrightarrow \frac{1}{2} \mathbb{E}^0 \left[\int_0^T \partial_x^2 \delta_m(\mu_s, X_s) : \mathrm{d} \langle X \rangle_s \right] \text{ in } \mathbb{L}^1.$$

 \Box

4 Itō-Wentzell's formula

In addition to the continuous semimartingale with canonical decomposition

$$X = X_0 + M + A,$$

we now consider the extension from a deterministic function u to a process $U : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \Omega \longrightarrow \mathbb{R}$, with the following dynamics for the random field $U_t(m)$:

$$U_t(m) = U_0(m) + \int_0^t \phi_r(m) \cdot dB_r + \psi_r(m) \cdot dN_r, \qquad (4.1)$$

where B is a finite-variation process and N is a martingale. Our main result is the following Itō-Wentzell formula which will be established under the following conditions.

Assumption 4.1. The maps $f \in \{U_0, \phi_t, \psi_t, t \in [0, T]\}$ defined on $\mathcal{P}_2(\mathbb{R}^d)$ satisfy

- (IW1) $\delta_m f, \partial_x^2 \delta_m f, \delta_m^2 f, \partial_x \partial_{\hat{x}} \delta_m^2 f$ exist and are continuous;
- (IW2) $f, \partial_x^2 \delta_m f, \partial_x \partial_{\hat{x}} \delta_m^2 f$ are bounded;

and the processes X together with the driving processes B, N of the random field U satisfy

(IW3) $X_0, |A|_{\text{TV}}$ and $\langle M \rangle_T$ are square integrable, and both $|B|_{\text{TV}}, \langle N \rangle_T$ are bounded.

We observe that the boundedness condition on $|B|_{\text{TV}}$ and $\langle N \rangle_T$ can be weakened at the price of stronger boundedness conditions on $\partial_x \delta_m \phi$ and $\partial_x \delta_m \psi$. We deliberately choose this setup in order to compare to the conditions of dos Reis & Platonov [dRP22].

Theorem 2. Let Assumption 4.1 hold.

(i) All derivatives $\delta_m U$, $\partial_x^2 \delta_m U$, $\delta_m^2 U$, $\partial_x \partial_x \delta_m^2 U$ exist, are continuous a.s., and are semimartingales defined by the decomposition for i, j = 1, ..., d:

$$\partial_{x_i} \delta_m U_t(m, x) = \partial_{x_i} \delta_m U_0(m, x) + \int_0^t \partial_{x_i} \delta_m \phi_s(m, x) \cdot \mathrm{d}B_s + \partial_{x_i} \delta_m \psi_s(m, x) \cdot \mathrm{d}N_s,$$

$$\partial_{x_i, x_j}^2 \delta_m U_t(m, x) = \partial_{x_i, x_j}^2 \delta_m U_0(m, x) + \int_0^t \partial_{x_i, x_j}^2 \delta_m \phi_s(m, x) \cdot \mathrm{d}B_s + \partial_{x_i, x_j}^2 \delta_m \psi_s(m, x) \cdot \mathrm{d}N_s$$

$$\partial_{x_i, \hat{x}_j}^2 \delta_m^2 U_t(m, x, \hat{x}) = \partial_x^i \partial_{\hat{x}}^j \delta_m^2 U_0(m, x, \hat{x}) + \int_0^t \partial_x^i \partial_{\hat{x}}^j \delta_m^2 \phi_s(m, x, \hat{x}) \cdot \mathrm{d}B_s + \partial_x^i \partial_{\hat{x}}^j \delta_m^2 \psi_s(m, x, \hat{x}) \cdot \mathrm{d}N_s$$

(ii) Moreover, we have

$$U_{T}(\mu_{T}) - U_{0}(\mu_{0}) = \mathbb{E}^{0} \left[\int_{0}^{T} \partial_{x} \delta_{m} U_{s}(\mu_{s}, X_{s}) \cdot \mathrm{d}X_{s} + \frac{1}{2} \partial_{x}^{2} \delta_{m} U_{s}(\mu_{s}, X_{s}) : \mathrm{d}\langle X \rangle_{s} \right] \\ + \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\int_{0}^{T} \frac{1}{2} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U_{s}(\mu_{s}, X_{s}, \hat{X}_{s}) : \mathrm{d}\langle X, \hat{X} \rangle_{s} \right] \\ + \int_{0}^{T} \phi_{s}(\mu_{s}) \cdot \mathrm{d}B_{s} + \psi_{s}(\mu_{s}) \cdot \mathrm{d}N_{s} + \mathbb{E}^{0} \left[\int_{0}^{T} \partial_{x} \delta_{m} \psi_{s}(\mu_{s}, X_{s}) : \mathrm{d}\langle N, M \rangle_{s} \right], \text{ a.s.}$$

Proof. We organize the proof in several steps.

1. We start by the existence and continuity of $\delta_m U$, $\partial_x^2 \delta_m U$, $\delta_m^2 U$, $\partial_x \partial_x \delta_m^2 U$. We first show that the functional linear derivative $\delta_m U_t$ exists for all $t \in [0, T]$, and is given by the first

expression in (i), i.e.

$$\delta_m U_t(m,x) = \delta_m U_0(m,x) + \int_0^t \delta_m \phi_s(m,x) \cdot \mathrm{d}B_s + \delta_m \psi_s(m,x) \cdot \mathrm{d}N_s.$$
(4.2)

For arbitrary $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$, it follows from the decomposition of U and the definition of the linear functional derivative for the maps U_0, ϕ_s, ψ_s , as guaranteed by Assumption (IW1), that:

$$\begin{aligned} U_{t}(m') - U_{t}(m) &= U_{0}(m') - U_{0}(m) + \int_{0}^{t} (\phi_{s}(m') - \phi_{s}(m)) \cdot \mathrm{d}B_{s} + \int_{0}^{t} (\psi_{s}(m') - \psi_{s}(m)) \cdot \mathrm{d}N_{s} \\ &= \int_{0}^{1} \int \delta_{m} U_{0}(m^{\lambda}, x)(m' - m)(\mathrm{d}x) \, \mathrm{d}\lambda + \int_{0}^{t} \left(\int_{0}^{1} \int \delta_{m} \phi_{s}(m^{\lambda}, x)(m' - m)(\mathrm{d}x) \, \mathrm{d}\lambda \right) \cdot \mathrm{d}B_{s} \\ &+ \int_{0}^{t} \left(\int_{0}^{1} \int \delta_{m} \phi_{s}(m^{\lambda}, x)(m' - m)(\mathrm{d}x) \, \mathrm{d}\lambda \right) \cdot \mathrm{d}N_{s}, \end{aligned}$$

By the Fubini theorem, this provides $U_t(m') - U_t(m) = \int_0^1 \int F_s(m^\lambda, x)(m' - m)(\mathrm{d}x) \mathrm{d}\lambda$, where $F_t(m, x)$ is given by the right hand side of (4.2). Moreover, it follows from (IW2) that the maps $\delta_m U_0, \delta_m \phi_s, \delta_m \psi_s$ have linear growth in x, uniformly in m. As $\mathbb{E}[|B|_{\mathrm{TV}}^2 + \langle N \rangle_T] < \infty$ by (IW3), this implies that the map $F_t(m, x)$ also has linear growth in x, uniformly in m. Notice also that

- F_t inherits the continuity of $\delta_m U_0$, $\delta_m \phi_t$, and $\delta_m \psi_t$, by the dominated convergence theorem due to their boundedness, uniformly in t, m, assumed in (IW2). We may then conclude that $\delta_m U_t = F_t$ by the definition of the linear functional derivative.
- $\partial_x^2 \delta_m U$ exists and inherits the continuity of $\partial_x^2 \delta_m U_0$, $\partial_x^2 \delta_m \phi$, and $\partial_x^2 \delta_m \psi$, by the dominated convergence theorem due to their boundedness, uniformly in t, m, assumed in (IW2)

Finally, observe that the coefficients of the SDEs driving U and $\delta_m U$ (with fixed x) satisfy the same conditions in (IW1) and (IW2). Applying the previous argument to the process $\delta_m U_t(m, x)$, for fixed x, it follows that $\delta_m^2 U_t$ and $\partial_x \partial_x \delta_m^2 U_t(m, x, \hat{x})$ also exist and are continuous, with decomposition given by the third expression in (i).

2. Let $\pi^n : 0 = t_0^n < t_1^n < \ldots < t_{p_n}^n = T$ be a dense sequence of partitions of [0, T]. As in the proof of Itō's formula, we start from the telescopic decomposition:

$$U_{t_{i}^{n}}(\mu_{t_{i}^{n}}) - U_{t_{i-1}^{n}}(\mu_{t_{i-1}^{n}}) = R_{1} + R_{2} \text{ where } R_{1} := U_{t_{i}^{n}}(\mu_{t_{i}^{n}}) - U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}), \tag{4.3}$$
$$R_{2} := U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}) - U_{t_{i-1}^{n}}(\mu_{t_{i-1}^{n}}) = \int_{t_{i-1}^{n}}^{t_{i}^{n}} \mathrm{d}U_{s}(\mu_{t_{i-1}^{n}}),$$

with dynamics of $\{U_s(m), s \ge 0\}$ given by (4.1). We next further compute R_1 by using the definition of the functional linear derivative:

$$R_{1} = \int_{0}^{1} \int \delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda}, .) \mathrm{d}(\mu_{t_{i}^{n}} - \mu_{t_{i-1}^{n}}) \mathrm{d}\lambda = \int_{0}^{1} \mathbb{E}^{0} \left[\delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i}^{n}}) - \delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i-1}^{n}}) \right] \mathrm{d}\lambda.$$

By the second order Taylor theorem, we may rewrite this as

$$R_{1} = \int_{0}^{1} \mathbb{E}^{0} \left[\partial_{x} \delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda}, X_{t_{i-1}^{n}}) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} + \frac{1}{2} \partial_{x}^{2} \delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}) \cdot (\Delta^{\pi_{n}} X_{t_{i}^{n}})^{\mathsf{T}} \right] \mathrm{d}\lambda,$$

for some r.v. $\xi_{t_{i-1}^n}$ lying between $X_{t_{i-1}^n}$ and $X_{t_i^n}$. Denoting $\gamma_s := \partial_x \delta_m U_s$, we compute that

$$\begin{split} \left\{ \gamma_{t_{i}^{n}}(.,X_{t_{i-1}^{n}}) \right\}_{\mu_{t_{i-1}^{n}}}^{\mu_{t_{i-1}^{n}}} &= \int_{0}^{1} \int \delta_{m} \gamma_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda\lambda'},X_{t_{i-1}^{n}},\hat{x})(\mu_{t_{i-1}^{n}}^{\lambda}-\mu_{t_{i-1}^{n}})(\mathrm{d}\hat{x}) \,\mathrm{d}\lambda' \\ &= \lambda \int_{0}^{1} \hat{\mathbb{E}}^{0} \left[\left\{ \delta_{m} \gamma_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda\lambda'},X_{t_{i-1}^{n}},\cdot) \right\}_{\hat{X}_{t_{i-1}^{n}}}^{\hat{X}_{t_{i}^{n}}} \right] \mathrm{d}\lambda' \\ &= \lambda \int_{0}^{1} \hat{\mathbb{E}}^{0} \left[\partial_{\hat{x}} \delta_{m} \gamma_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda\lambda'},X_{t_{i-1}^{n}},\hat{\xi}_{t_{i-1}^{n}}) \Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}} \right] \mathrm{d}\lambda', \end{split}$$

for some $\hat{\xi}_{t_{i-1}^n}$ between $\hat{X}_{t_{i-1}^n}$ and $\hat{X}_{t_i^n}$. By the regularity results obtained in Step 1 of the present proof, we may also write $\gamma_{t_i^n}(\mu_{t_{i-1}^n}, X_{t_{i-1}^n}) = \gamma_{t_{i-1}^n}(\mu_{t_{i-1}^n}, X_{t_{i-1}^n}) + \int_{t_{i-1}^n}^{t_i^n} d\gamma_s(\mu_{t_{i-1}^n}, X_{t_{i-1}^n})$. Substituting back the expression of the map γ , this provides:

$$R_{1} = \mathbb{E}^{0} \left[\partial_{x} \delta_{m} U_{t}(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} \right] + \mathbb{E}^{0} \left[\left(\int_{t_{i-1}^{n}}^{t_{i}^{n}} \mathrm{d}\gamma_{s}(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}) \right) \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} \right] \\ + \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda\lambda'}, X_{t_{i-1}^{n}}, \hat{\xi}_{t_{i-1}^{n}}) : (\Delta^{\pi_{n}} X_{t_{i}^{n}}) (\Delta^{\pi_{n}} \hat{X}_{t_{i}^{n}})^{\mathsf{T}} \mathrm{d}\lambda' \mathrm{d}\lambda \right] \\ + \int_{0}^{1} \mathbb{E}^{0} \left[\frac{1}{2} \partial_{x}^{2} \delta_{m} U_{t_{i}^{n}}(\mu_{t_{i-1}^{n}}^{\lambda}, \xi_{t_{i-1}^{n}}) : (\Delta^{\pi_{n}} X_{t_{i}^{n}}) (\Delta^{\pi_{n}} X_{t_{i}^{n}})^{\mathsf{T}} \right] \mathrm{d}\lambda,$$

where $d\gamma_s(\mu_{t_{i-1}^n}, X_{t_{i-1}^n}) = f_s(\mu_{t_{i-1}^n}, X_{t_{i-1}^n}) dB_s + g_s(\mu_{t_{i-1}^n}, X_{t_{i-1}^n}) dN_s, s \in [t_{i-1}^n, t_i^n)$, with $(f_s, g_s) := \partial_x \delta_m(\phi_s, \psi_s).$

Summing the decomposition (4.3), and denoting by $t^n(s)$ the closest subdivision point strictly to the left of s, this provides:

$$\begin{aligned} U_{T}(\mu_{T}) - U_{0}(\mu_{0}) &= \int_{0}^{T} \mathrm{d}U_{s}(\mu_{t^{n}(s)}) + \int_{0}^{T} \mathbb{E}^{0} \left[\partial_{x} \delta_{m} U_{t^{n}(s)}(\mu_{t^{n}(s)}, X_{t^{n}(s)}) \cdot \mathrm{d}X_{s} \right] \\ &+ \sum_{i=1}^{p_{n}} \mathbb{E}^{0} \left[\left(\int_{t^{n}_{i-1}}^{t^{n}_{i}} f_{s}(\mu_{t^{n}_{i-1}}, X_{t^{n}_{i-1}}) \, \mathrm{d}B_{s} + g_{s}(\mu_{t^{n}_{i-1}}, X_{t^{n}_{i-1}}) \, \mathrm{d}N_{s} \right) \cdot \Delta^{\pi_{n}} X_{t^{n}_{i}} \right] \\ &+ \sum_{i=1}^{p_{n}} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\int_{0}^{1} \int_{0}^{1} \lambda \partial_{\hat{x}} \delta_{m} \partial_{x} \delta_{m} U_{t^{n}_{i}}(\mu^{\lambda\lambda'}_{t^{n}_{i-1}}, X_{t^{n}_{i-1}}, \hat{\xi}_{t^{n}_{i-1}}) : (\Delta^{\pi_{n}} X_{t^{n}_{i}}) (\Delta^{\pi_{n}} \hat{X}_{t^{n}_{i}})^{\mathsf{T}} \, \mathrm{d}\lambda' \, \mathrm{d}\lambda \right] \\ &+ \sum_{i=1}^{p_{n}} \int_{0}^{1} \mathbb{E}^{0} \left[\frac{1}{2} \partial_{x}^{2} \delta_{m} U_{t^{n}_{i}}(\mu^{\lambda}_{t^{n}_{i-1}}, \xi_{t^{n}_{i-1}}) : (\Delta^{\pi_{n}} X_{t^{n}_{i}})^{\mathsf{T}} \, \mathrm{d}\lambda. \end{aligned}$$

3. We now show that the different terms of the last decomposition converge towards the formula announced in the theorem, for a certain dense sequence of subdivisions.

3.1 We first prove that, after possibly passing to a subsequence,

$$\int_0^T \mathrm{d}U_s(\mu_{t^n(s)}) = \int_0^T \phi_s(\mu_{t^n(s)}) \cdot \mathrm{d}B_s + \psi_s(\mu_{t^n(s)}) \cdot \mathrm{d}N_s \longrightarrow \int_0^T \mathrm{d}U_s(\mu_s) \text{ a.s.}$$
(4.4)

By the dominated convergence theorem for the Stieltjes integral $\int \cdot dB_s$, and the boundedness of ϕ , we obtain the convergence of the finite variation part

$$\int_0^T \phi_s(\mu_{t^n(s)}) \cdot \mathrm{d}B_s \quad \longrightarrow \quad \int_0^T \phi_s(\mu_s) \cdot \mathrm{d}B_s, \text{ a.s.}$$

As for the stochastic integral component, we estimate by the $It\bar{o}$ isometry that

$$\mathbb{E}\left[\left(\int_0^T (\psi_s(\mu_{t^n(s)}) - \psi_s(\mu_s)) \cdot \mathrm{d}N_s\right)^2\right] \leq \mathbb{E}\left[\int_0^T (\psi_s(\mu_{t^n(s)}) - \psi_s(\mu_s))^{\otimes 2} \cdot \mathrm{d}\langle N \rangle_s\right], (4.5)$$

Since ψ and μ are a.s. continuous and ψ is bounded, it follows from dominated convergence for the Stieltjes stochastic integral $\int \cdot d\langle N \rangle_s$ that $\int_0^T \left(\psi_s(\mu_{t^n(s)}) - \psi_s(\mu_s) \right)^{\otimes 2} : d\langle N \rangle_s \longrightarrow 0$, a.s. Furthermore, $|\int_0^T \left(\psi_s(\mu_{t^n(s)}) - \psi_s(\mu_s) \right)^{\otimes 2} : d\langle N \rangle_s| \le 4 ||\psi||_{\infty}^2 \operatorname{Tr}[\langle N \rangle_T]$, which is in \mathbb{L}^1 . We then deduce from (4.5) and the dominated convergence theorem that

$$\int_0^T \psi_s(\mu_{t^n(s)}) \cdot \mathrm{d}N_s \longrightarrow \int_0^T \psi_s(\mu_s) \cdot \mathrm{d}N_s \text{ in } \mathbb{L}^2, \text{ and a.s. along some subsequence,}$$

thus completing the proof of (4.4).

For the remaining terms, we use the same method as in the proof of Theorem 1, by arguing that the sequence inside the conditional expectations converges in \mathbb{L}^1 towards the desired results.

3.2. Denote $H_s := \partial_x \delta_m U_s(\mu_s, X_s)$. The convergence of the second term is implied by the following two convergence results:

$$\int_0^T H_{t^n(s)} \cdot \mathrm{d}A_s \longrightarrow \int_0^T H_s \cdot \mathrm{d}A_s, \quad \text{and} \quad \int_0^T H_{t^n(s)} \cdot \mathrm{d}M_s \longrightarrow \int_0^T H_s \cdot \mathrm{d}M_s, \quad \text{in } \mathbb{L}^1(4.6)$$

The first convergence of the finite variation part follows from the a.s. pathwise continuity of the process H, together with the dominated convergence theorem for the Stieltjes integral $\int \cdot dA_s$ together with the linear growth of $\partial_x \delta_m U_s$ in the x-variable, as implied by (IW2). Similarly, it follows from the BDG inequality and the dominated convergence theorem for the Stieltjes integral $\int \cdot d\langle M \rangle_s$ that $\int_0^T H_{t^n(s)} \cdot dM_s \longrightarrow \int_0^T H_s \cdot dM_s$ in \mathbb{L}^1 .

3.3. In this step, we justify the \mathbb{L}^1 convergence of the third term. Denoting $F_t := \int_0^t f_s(\mu_s, X_s) \, \mathrm{d}B_s$, and $G_t := \int_0^t g_s(\mu_s, X_s) \, \mathrm{d}N_s$, we shall now show that

$$\Phi^{n} := \sum_{i=1}^{p_{n}} \int_{t_{i-1}^{n}}^{t_{i}^{n}} f_{s}(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}) \, \mathrm{d}B_{s} \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} \longrightarrow \operatorname{Tr}[\langle X, F \rangle_{T}] = 0, \text{ in } \mathbb{L}^{1}, \text{ as } n \to \infty$$
$$\Psi^{n} := \sum_{i=1}^{p_{n}} \int_{t_{i-1}^{n}}^{t_{i}^{n}} g_{s}(\mu_{t_{i-1}^{n}}, X_{t_{i-1}^{n}}) \, \mathrm{d}N_{s} \cdot \Delta^{\pi_{n}} X_{t_{i}^{n}} \longrightarrow \operatorname{Tr}[\langle X, G \rangle_{T}], \text{ in } \mathbb{L}^{1}, \text{ as } n \to \infty,$$

where we have $\langle F, X \rangle_T = 0$, a.s. due to the fact that the process F has finite variation. In order to prove the first convergence, we use Lemma 1 to conclude that, along some subsequence,

$$\sum_{i=1}^{p_n} (\Delta^{\pi_n} X_{t_i^n}) \cdot (\Delta^{\pi_n} F_{t_i^n}) \longrightarrow \operatorname{Tr}[\langle X, F \rangle_T], \text{ in } \mathbb{L}^1.$$

As $\partial_x f$ and $\partial_x \delta_m f$ are bounded by Condition (IW2), it follows that the process $\delta_s^n := f_s(\mu_{t^n(s)}, X_{t^n(s)}) - f_s(\mu_s, X_s)$ satisfies

$$|\delta_s^n| \leq C \, \varpi_X^{\pi^n}$$
, for some constant $C > 0$, with $\varpi_X^{\pi^n} := \sup_{|t-t'| \leq |\pi^n|} |X_t - X_{t'}|$. (4.7)

We then estimate the error term by

$$\begin{split} \mathbb{E} \left| \Phi^{n} - \sum_{i=1}^{p_{n}} (\Delta^{\pi_{n}} X_{t_{i}^{n}}) \cdot (\Delta^{\pi_{n}} F_{t_{i}^{n}}) \right| &= \mathbb{E} \left| \sum_{i=1}^{p_{n}} (\Delta^{\pi_{n}} X_{t_{i}^{n}}) \cdot \int_{t_{i-1}^{n}}^{t_{i}^{n}} \delta_{s}^{n} \, \mathrm{d}B_{s} \right| \\ &\leq \mathbb{E} \left[\sup_{1 \leq j \leq p_{n}} |\Delta^{\pi_{n}} X_{t_{j}^{n}}| \int_{0}^{T} |\delta_{s}^{n}| \, \mathrm{d}|B_{s}|_{\mathrm{TV}} \right] \\ &\leq C \mathbb{E} \left[|B|_{\mathrm{TV}} (\varpi_{X}^{\pi^{n}})^{2} \right] \\ &\leq C \||B|_{\mathrm{TV}} \|_{\infty} \mathbb{E} \left[(\varpi_{X}^{\pi^{n}})^{2} \right] \longrightarrow 0, \end{split}$$

as $n \to \infty$, by Conditions (IW3).

A similar argument allows to justify the \mathbb{L}^1 convergence of Ψ^n towards $\operatorname{Tr}[\langle G, X \rangle_T]$ in \mathbb{L}^1 . Indeed, using again Lemma 1, we are reduced to the following estimate involving the process $\eta_s^n := g_s(\mu_{t^n(s)}, X_{t^n(s)}) - g_s(\mu_s, X_s)$:

$$\begin{split} \mathbb{E}\Big|\sum_{i=1}^{p_n} \Delta X_{t_i^n} \cdot \int_{t_{i-1}^n}^{t_i^n} \eta_s^n \, \mathrm{d}N_s\Big| &\leq \mathbb{E}\Big[\varpi_X^{\pi^n}\Big|\sum_{i=1}^{p_n} \int_{t_{i-1}^n}^{t_i^n} \eta_s^n \, \mathrm{d}N_s\Big|\Big] \\ &= \mathbb{E}\Big[\varpi_X^{\pi^n}\Big|\int_0^T \eta_s^n \, \mathrm{d}N_s\Big|\Big] \\ &\leq \mathbb{E}\Big[\big(\varpi_X^{\pi^n}\big)^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\int_0^T (\eta_s^n)(\eta_s^n)^\intercal : \mathrm{d}\langle N\rangle_s\Big]^{\frac{1}{2}}, \end{split}$$

by the Cauchy-Schwartz inequality and the Itō isometry. Notice that Conditions (IW2) induces the same estimates for the process η^n as those for δ^n in (4.7). Then, the required convergence result follows from Conditions (IW3).

3.4. We finally analyse the last two terms by applying the same calculations as in the proof of the standard Itō formula, using Lemma 1, and we obtain the convergence as $n \to \infty$:

$$\sum_{i=1}^{p_n} \int_0^1 \frac{1}{2} \partial_x^2 \delta_m u_{t_i^n}(\mu_{t_{i-1}^n}^\lambda, \xi_{t_{i-1}^n}) : (\Delta^{\pi_n} X_{t_i^n})(\Delta^{\pi_n} X_{t_i^n})^{\mathsf{T}} \, \mathrm{d}\lambda \longrightarrow \frac{1}{2} \int_0^T \partial_x^2 \delta_m u_s(\mu_s, X_s) : \mathrm{d}\langle X \rangle_s$$

and

$$\sum_{i=1}^{p_n} \int_0^1 \int_0^1 \lambda \partial_{\hat{x}} \delta_m \partial_x \delta_m u_{t_i^n}(\mu_{t_{i-1}^n}^{\lambda\lambda'}, X_{t_{i-1}^n}, \hat{\xi}_{t_{i-1}^n}) : (\Delta^{\pi_n} X_{t_i^n}) (\Delta^{\pi_n} \hat{X}_{t_i^n})^{\mathsf{T}} \, \mathrm{d}\lambda' \, \mathrm{d}\lambda$$
$$\longrightarrow \frac{1}{2} \int_0^T \partial_{\hat{x}} \delta_m \partial_x \delta_m u(\mu_s, X_s, \hat{X}_s) : \mathrm{d}\langle X, \hat{X} \rangle_s$$

in \mathbb{L}^1 . This completes the proof.

4.1 Examples

4.1.1**Brownian** Case

Let us consider the special case where the process X and the random field U are Ito processes defined by

$$dU_t(m) = \phi_t(m) dt + \psi_t(m) \cdot dW_t + \psi_t^0(m) \cdot dW_t^0,$$

$$dX_t = b_t dt + \sigma_t dW_t + \sigma_t^0 dW_t^0$$

where W, W^0 are Brownian motions. We choose here $\mathbb{F}^0 = (\mathcal{F}^0_t)_{0 \le t \le T}$ to be the filtration generated by W^0 . This setting reduces to that of dos Reis & Platonov [dRP22]. The conditionally independent copy \hat{X} is defined by

$$d\hat{X}_t = \hat{b}_t dt + \hat{\sigma}_t d\hat{W}_t + \hat{\sigma}_t^0 dW_t^0$$

where $\hat{b}, \hat{\sigma}, \hat{\sigma}^0, \hat{W}$ are conditionally independent copies of b, σ, σ^0, W , respectively. We rephrase our Theorem 2 in the present setting in order to compare it with the corresponding statement in [dRP22].

Corollary 1. For $f \in \{U_0, \phi_t, \psi_t, \psi_t^0, t \in [0, T]\}$, assume:

- $f, \delta_m f, \partial_x^2 \delta_m f, \delta_m^2 f, \partial_{\hat{\tau}}^2 \delta_m^2 f, \partial_x \partial_{\hat{x}} \delta_m^2 f$ exist and are continuous;
- $\partial_x^2 \delta_m f, \partial_{\hat{x}}^2 \delta_m^2 f, \partial_x \partial_{\hat{x}} \delta_m^2 f$ are bounded;

• $\mathbb{E}\left[\int_{0}^{T}(|X_{0}|^{2}+|b_{s}|^{2}+|\sigma_{s}\sigma_{s}^{\mathsf{T}}|^{2}+|\sigma_{s}^{0}(\sigma_{s}^{0})^{\mathsf{T}}|^{2})\,\mathrm{d}s\right]<\infty.$ Then $\delta_{m}U, \partial_{x}^{2}\delta_{m}U, \delta_{m}^{2}U, \partial_{x}\partial_{\hat{x}}\delta_{m}^{2}U$ exist, are continuous a.s., and are Itō processes driven by the Brownian motions W and W^0 , with coefficients defined by the corresponding derivatives

of the coefficients of U. Moreover, we have:

$$\begin{aligned} U_{T}(\mu_{T}) - U_{0}(\mu_{0}) &= \int_{0}^{T} \phi_{s}(\mu_{s}) \,\mathrm{d}s + \int_{0}^{T} \psi_{s}(\mu_{s}) \cdot \mathrm{d}W_{s} + \int_{0}^{T} \psi_{s}^{0}(\mu_{s}) \cdot \mathrm{d}W_{s}^{0} \\ &+ \mathbb{E}^{0} \left[\int_{0}^{T} \partial_{x} \delta_{m} U_{s}(\mu_{s}, X_{s}) \cdot b_{s} \,\mathrm{d}s + \int_{0}^{T} (\sigma_{s}^{0})^{\intercal} \partial_{x} \delta_{m} U_{s}(\mu_{s}, X_{s}) \cdot \mathrm{d}W_{s}^{0} \right] \\ &+ \frac{1}{2} \mathbb{E}^{0} \left[\int_{0}^{T} \partial_{x}^{2} \delta_{m} U_{s}(\mu_{s}, X_{s}) : (\sigma_{s} \sigma_{s}^{\intercal} + \sigma_{s}^{0} (\sigma_{s}^{0})^{\intercal}) \,\mathrm{d}s \right] \\ &+ \mathbb{E}^{0} \left[\int_{0}^{T} \partial_{x} \delta_{m} \psi_{s}^{0}(\mu_{s}, X_{s}) : (\sigma_{s}^{0})^{\intercal} \,\mathrm{d}s \right] \\ &+ \frac{1}{2} \mathbb{E}^{0} \hat{\mathbb{E}}^{0} \left[\int_{0}^{T} \partial_{x} \partial_{\hat{x}} \delta_{m}^{2} U_{s}(\mu_{s}, X_{s}, \hat{X}_{s}) : \sigma_{s}^{0} (\hat{\sigma}_{s}^{0})^{\intercal} \,\mathrm{d}s \right], \ a.s. \end{aligned}$$

We thus find the same formula as in [dRP22], with the only additional hypothesis that the highest-order derivatives are bounded instead of square-integrable.

4.1.2 Semimartingale factor random field model

Suppose that X is a continuous semimartingale:

$$X_t = X_0 + A_t + M_t$$
 for all $t \in [0, T]$,

where $(A_t)_t$ is a finite-variation process and $(M_t)_t$ is a martingale. In this section, we consider the case where the random field is defined by $U_t(m) := u(t, m, Y_t)$ for some factor process $Y = (Y_t)_{0 \le t \le T}$ which is another continuous semimartingale :

$$Y_t = Y_0 + V_t + S_t,$$

with finite-variation process V, and a martingale S. Here, the deterministic function $u : (t, m, y) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \mapsto u(t, m, y) \in \mathbb{R}$ will be assumed to be sufficiently smooth. For this, we extend naturally the definition of the functional linear derivative by reducing to the standard definition once the variables (t, y) are frozen. The second order functional linear derivative δ_m^2 is also defined similarly. The following result is a direct restatement of Theorem 2 in the present context.

Corollary 2. Let us suppose that

- $\partial_t u, \partial_y^2 u, \delta_m u, \partial_x^2 \delta_m u, \delta_m^2 u, \partial_x \partial_x \delta_m^2 u, \delta_m \partial_y u, \partial_x \delta_m \partial_y u$ exist and continuous;
- $\partial_y^2 u, \partial_x^2 \delta_m u, \partial_x \partial_x \delta_m^2 u, \partial_x \delta_m \partial_y u$ are bounded ;
- $X_0, |A|_{\text{TV}}, |V|_{\text{TV}}, \langle M \rangle_T$ and S are square integrable.

Then, denoting $\Theta_t = (t, \mu_t, Y_t)$ for all $t \in [0, T]$, we have:

$$\begin{split} u(\Theta_T) - u(\Theta_0) &= \int_0^T \partial_t u(\Theta_s) \, \mathrm{d}s + \partial_y u(\Theta_s) \cdot \mathrm{d}Y_s + \frac{1}{2} \partial_y^2 u(\Theta_s) : \mathrm{d}\langle Y \rangle_s \\ &+ \mathbb{E}^0 \left[\int_0^T \partial_x \delta_m u(\Theta_s, X_s) \cdot \mathrm{d}X_s + \frac{1}{2} \partial_x^2 \delta_m u(\Theta_s, X_s) : \mathrm{d}\langle X \rangle_s + \partial_x \delta_m \partial_y u(\Theta_s, X_s) : \mathrm{d}\langle X, Y \rangle_s \right] \\ &+ \frac{1}{2} \mathbb{E}^0 \hat{\mathbb{E}}^0 \left[\int_0^T \partial_x \partial_x \delta_m^2 u(\Theta_s, X_s, \hat{X}_s) : \mathrm{d}\langle X, \hat{X} \rangle_s \right] \quad a.s. \end{split}$$

Proof. By the standard Itō formula for finite-dimension Itō processes, the random field $\{U_t(m) = u_t(t, m, Y_t), (t, m) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\}$ has the following semimartingale decomposition:

$$dU_t(m) = \phi_t(m) \cdot dB_t + \psi_t(m) \cdot dN_t$$

with $B_t = (t, V_t, \langle S \rangle_t), N_t = S_t, \phi_t(m) = \left(\partial_t u_t(m, Y_t), \partial_y u_t(m, Y_t), \frac{1}{2} \partial_y^2 u_t(m, Y_t)\right)$ and $\psi_t(m) = \partial_y u_t(m, Y_t)$. The result is now a direct application of Theorem 2.

5 Application to Mean-Field Control: HJB Equation

Let $\Omega = \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^{d_0})$ with canonical process $(X_t, W_t^0) : (\omega, \omega^0) \in \Omega \mapsto (\omega, \omega^0)(t) \in \mathbb{R}^d \times \mathbb{R}^{d_0}$. The corresponding canonical filtration is denoted by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$. We also introduce the control space \mathcal{A} consisting of all \mathbb{F} -progressively measurable processes α with values in a compact subset A of a finite dimensional space.

Let b, σ , and σ^0 be given bounded maps

$$(b,\sigma,\sigma^0): \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \longrightarrow \mathbb{R}^d \times \mathcal{M}_{d,d}(\mathbb{R}) \times \mathcal{M}_{d,d_0}(\mathbb{R}),$$

and

$$(k,\gamma,\gamma^0): \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathcal{M}_{d,d}(\mathbb{R}) \times \mathcal{M}_{d,d_0}(\mathbb{R}).$$

For $t \geq 0$ and $m \in \mathcal{P}_2(\mathbb{R}^d)$, we denote by $\mathcal{P}(t, y, m)$ the collection of all probability measures \mathbb{P} on (Ω, \mathcal{F}) satisfying:

- (i) W^0 is a \mathbb{P} -Brownian Motion;
- (ii) The process Y is defined by

$$Y_t = y$$
, and $dY_s = k(s, Y_s) ds + \gamma(s, Y_s) dB_s^{\mathbb{P}} + \gamma^0(s, Y_s) dW_s^0$, $s \ge t$, \mathbb{P} - a.s

for some \mathbb{P} -Brownian motion $B^{\mathbb{P}}$;

(iii) the conditional marginal law of X_t given W^0 is $\mathbb{P}_{X_t}^{W^0} := \mathbb{P} \circ (X_t | W^0)^{-1} = m$, and there exists a control process $\alpha \in \mathcal{A}$ such that for $s \geq t$:

$$dX_s = b(s, X_s, Y_s, \mathbb{P}_{X_s}^{W^0}, \alpha_s) ds + \sigma(s, X_s, Y_s, \mathbb{P}_{X_s}^{W^0}, \alpha_s) dW_s^{\mathbb{P}} + \sigma^0(s, X_s, Y_s, \mathbb{P}_{X_s}^{W^0}, \alpha_s) dW_s^0,$$

 \mathbb{P} -a.s. for some \mathbb{P} -Brownian motion $W^{\mathbb{P}}$. Here, $\mathbb{P}_{X_s}^{W^0}$ is the conditional law of X_s under \mathbb{P} given $\{W_r^0, r \ge 0\}$.

Let us suppose that $\mathcal{P}(t, y, m)$ is compact for every $(t, y, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

We define the objective function as

$$J(t, y, m, \mathbb{P}) := \mathbb{E}_t \left[\int_t^T f^{\alpha_s}(Y_s, \mathbb{P}_{X_s}^{W^0}) \,\mathrm{d}s + g(Y_T, \mathbb{P}_{X_T}^{W^0}) \right], \quad \text{for all} \quad \mathbb{P} \in \mathcal{P}(t, y, m),$$

with running reward map $f: A \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, and final reward $g: \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}$. The dynamic version of the control problem is defined by:

$$V(t, y, m) := \sup_{\mathbb{P} \in \mathcal{P}(t, y, m)} J(t, y, m, \mathbb{P}).$$

We start from the Dynamic Programming Principle (DPP) which holds under fairly general assumptions, see e.g. Djete, Possamaï & Tan [DPT20]:

$$V(t,y,m) = \sup_{\mathbb{P}\in\mathcal{P}(t,m)} \mathbb{E}_t \left[\int_t^{\theta^{\mathbb{P}}} f^{\alpha_s}(X_s,Y_s,\mathbb{P}^{W^0}_{X_s}) \,\mathrm{d}s + V(\theta^{\mathbb{P}},Y_{\theta^{\mathbb{P}}},\mathbb{P}^{W^0}_{X_{\theta^{\mathbb{P}}}}) \right],$$

for any family $\{\theta^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}(t,m)}$ of [t,T]-valued stopping times.

Proposition 1. Suppose that

- $\partial_t V, \partial_y V, \partial_y^2 V, \delta_m V, \partial_x \delta_m V, \partial_x^2 \delta_m V, \delta_m^2 V, \partial_x \partial_x \delta_m^2 V, \partial_x \delta_m \partial_y V$ exist;
- $\sigma^{0^{\intercal}} \partial_x \delta_m V$, $\gamma^{\intercal} \partial_y V$ and $\gamma^{0^{\intercal}} \partial_y V$ are uniformly bounded;
- $\partial_t V, \partial_y V, \partial_x \delta_m V, \partial_x^2 \delta_m V, \partial_x \partial_{\hat{x}} \delta_m^2 V, \partial_x \delta_m \partial_y V$ are Lipschitz in m uniformly in all other arguments.

Then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = -\partial_t V - k \cdot \partial_y V - \frac{1}{2} (\gamma \gamma^{\mathsf{T}} + \gamma^0 \gamma^{0^{\mathsf{T}}}) : \partial_{yy} V - \sup_{a \in \mathbb{L}^0(A)} \left\{ \int \left(f^a + b^a \cdot \partial_x \delta_m V + \frac{1}{2} \left(\sigma^a \sigma^{a^{\mathsf{T}}} + \sigma^{0^a} \sigma^{0^{\mathsf{T}}} \right) : \partial_{xx} \delta_m V + \sigma^{0^a} \gamma^{0^{\mathsf{T}}} : \partial_x \delta_m \partial_y V \right) (., x) m(\mathrm{d}x) + \frac{1}{2} \iint \sigma^{0^a} (., x) \sigma^{0^a} (., \hat{x}) : \partial_{x\hat{x}}^2 \delta_m^2 V (., x, \hat{x}) m(\mathrm{d}x) m(\mathrm{d}\hat{x}) \right\}, V|_{t=T} = g.$$

where we denoted $\varphi^a(.,x) := \varphi(t,y,m,x,a(x))$ for all function φ .

Proof. We write the proof for $d = d_0 = 1$.

• For $t \in [0, T]$, let α be a control process satisfying the definition of $\mathcal{P}(t, y, m)$. By the Dynamic Programming Principle, for any stopping time θ ,

$$V(t, y, m) \ge \mathbb{E}_t \left[\int_t^{\theta} f^{\alpha_s}(Y_s, \mu_s) \, \mathrm{d}s + V(\theta, Y_{\theta}, \mu_{\theta}) \right],$$

that is,

$$\mathbb{E}_t \left[\int_t^{\theta} f^{\alpha_s}(Y_s, \mu_s) \,\mathrm{d}s + V(\theta, Y_\theta, \mu_\theta) - V(t, y, m) \right] \le 0.$$
(5.1)

By Itō-Wentzell's formula,

$$\begin{split} V(\theta, Y_{\theta}, \mu_{\theta}) - V(t, y, m) &= \int_{t}^{\theta} \mathcal{L}^{\alpha} V(s, Y_{s}, \mu_{s}) \, \mathrm{d}s + \int_{t}^{\theta} \partial_{y} V(s, Y_{s}, \mu_{s}) \gamma_{s} \, \mathrm{d}B_{s} \\ &+ \int_{t}^{\theta} \left(\mathbb{E}^{0} \left[\partial_{x} \delta_{m} V(s, Y_{s}, \mu_{s}, X_{s}) \sigma_{s}^{0} \right] + \partial_{y} V(s, Y_{s}, \mu_{s}) \gamma_{s}^{0} \right) \mathrm{d}W_{s}^{0}, \end{split}$$

where

$$\mathcal{L}^{\alpha}V(s,y,m) := \partial_{t}V(s,y,m) + \partial_{y}V(s,y,m)k_{s} + \frac{1}{2}\partial_{y}^{2}(\gamma_{s}^{2} + (\gamma_{s}^{0})^{2})V(s,y,m) \\ + \int \partial_{x}\delta_{m}V(s,y,m,x)b_{s}m(\mathrm{d}x) + \frac{1}{2}\int \partial_{x}^{2}\delta_{m}V(s,y,m,x)(\sigma_{s}^{2} + (\sigma_{s}^{0})^{2})m(\mathrm{d}x) \\ + \int \partial_{x}\delta_{m}\partial_{y}V(s,y,m,x)\sigma_{s}^{0}\gamma_{s}^{0}m(\mathrm{d}x) + \frac{1}{2}\int \int \partial_{x\hat{x}}^{2}\delta_{m}^{2}V(s,y,m,x,\hat{x})\sigma_{s}^{0}\hat{\sigma}_{s}^{0}m(\mathrm{d}x)m(\mathrm{d}\hat{x})$$

Let us now choose $\theta = \theta_h := \inf\{s > t, |X_s - X_t| \ge 1 \text{ or } |Y_s - Y_t| \ge 1\} \land (t+h), \text{ for } h > 0$. Since $X, Y, \sigma^0, \gamma, \gamma^0, \mu, \partial_y V$ and $\partial_x \delta_m V$ are continuous, $\mathbb{E}^0 \left[\partial_x \delta_m V(s, Y_s, \mu_s, X_s) \sigma_s^0\right], \partial_y V(s, Y_s, \mu_s) \gamma_s^0$ and $\partial_y V(s, Y_s, \mu_s) \gamma_s$ are bounded over $[t, \theta_h]$ and therefore

$$\mathbb{E}_t \left[\int_t^{\theta_h} \left(\mathbb{E}^0 \left[\partial_x \delta_m V(s, Y_s, \mu_s, X_s) \sigma_s^0 \right] + \partial_y V(s, Y_s, \mu_s) \gamma_s^0 \right) \mathrm{d}W_s^0 + \int_t^{\theta_h} \partial_y V(s, Y_s, \mu_s) \gamma_s \, \mathrm{d}B_s \right] = 0.$$

Then, dividing (5.1) by h > 0:

$$\mathbb{E}_t \left[\frac{1}{h} \int_t^{\theta_h} (f^{\alpha_s}(Y_s, \mu_s) + \mathcal{L}^{\alpha} V(s, Y_s, \mu_s)) \, \mathrm{d}s \right] \le 0.$$

But a.s., for h small enough $\theta_h = t + h$ and

$$\frac{1}{\theta_h - t} \int_t^{\theta_h} (f^{\alpha_s}(Y_s, \mu_s) + \mathcal{L}^{\alpha} V(s, Y_s, \mu_s)) \,\mathrm{d}s \longrightarrow f^{\alpha_t}(y, m) + \mathcal{L}^{\alpha} V(t, y, m) \text{ a.s.}$$

By dominated convergence for the expectation, we then have that

$$\mathbb{E}_t \left[f^{\alpha_t}(y,m) + \mathcal{L}^{\alpha} V(t,y,m) \right] = f^{\alpha_t}(y,m) + \mathcal{L}^{\alpha} V(t,y,m) \le 0.$$

Since α was taken arbitrarily, we can conclude that

$$\sup_{a \in \mathbb{L}^0(A)} \mathcal{L}^a V(t, y, m) + f^a(y, m) \le 0.$$

• Now, let $\varepsilon > 0$. There are $\mathbb{P}^{\varepsilon} \in \mathcal{P}(t, y, m)$ and an associated control process $\alpha = \alpha^{\varepsilon}$ such that for every deterministic stopping time $\theta > t$,

$$-\varepsilon + V(t, y, m) \le \mathbb{E}_t^{\varepsilon} \left[\int_t^{\theta} f^{\alpha_s}(Y_s, \mu_s) \, \mathrm{d}s + V(\theta, y, \mu_{\theta}) \right].$$

Then, by Itō-Wentzell's formula,

$$\begin{split} \varepsilon &\geq -\mathbb{E}_t^{\varepsilon} \left[\int_t^{\theta} f^{\alpha_s}(Y_s, \mu_s) \,\mathrm{d}s + (V(\theta, Y_{\theta}, \mu_{\theta}) - V(t, y, m)) \right] \\ &= -\mathbb{E}_t^{\varepsilon} \left[\int_t^{\theta} f^{\alpha_s}(Y_s, \mu_s) \,\mathrm{d}s + \int_t^{\theta} \mathcal{L}^{\alpha_s} V(s, Y_s, \mu_s) \,\mathrm{d}s \right] \\ &- \mathbb{E}_t^{\varepsilon} \left[\int_t^{\theta} \left(\mathbb{E}^0 \left[\partial_x \delta_m V(s, Y_s, \mu_s, X_s) \sigma_s^0 \right] + \partial_y V(s, Y_s, \mu_s) \gamma_s^0 \right) \mathrm{d}W_s^0 + \int_t^{\theta_h} \partial_y V(s, Y_s, \mu_s) \gamma_s \,\mathrm{d}B_s \right] \\ &= -\mathbb{E}_t^{\varepsilon} \left[\int_t^{\theta} (f^{\alpha_s}(Y_s, \mu_s) + \mathcal{L}^{\alpha_s} V(s, Y_s, \mu_s)) \,\mathrm{d}s \right]. \end{split}$$

We used the fact that $(\partial_x \delta_m V) \sigma^0, \partial_y V \gamma$ and $\partial_y V \gamma^0$ are taken to be uniformly bounded.

With

$$F(s, y, m) := \sup_{a} f^{a}(y, m) + \mathcal{L}^{a}V(s, y, m),$$

we have that

$$\begin{split} & \mathbb{E}_{t}^{\varepsilon} \left[\int_{t}^{\theta} \left(f^{\alpha_{s}}(Y_{s},\mu_{s}) + \mathcal{L}^{\alpha_{s}}V(s,Y_{s},\mu_{s}) \right) \mathrm{d}s \right] \\ & \leq \sup_{a} \mathbb{E}_{t}^{\varepsilon} \left[\int_{t}^{\theta} \left(f^{a}(Y_{s},\mu_{s}) + \mathcal{L}^{a}V(s,Y_{s},\mu_{s}) \,\mathrm{d}s \right) \right] \\ & \leq (\theta - t)F(t,y,m) + \sup_{a} \mathbb{E}_{t}^{\varepsilon} \left[\int_{t}^{\theta} \left| f^{a}(Y_{s},\mu_{s}) + \mathcal{L}^{a}V(s,Y_{s},\mu_{s}) - \left(f^{a}(y,m) + \mathcal{L}^{a}V(t,y,m) \right) \right| \mathrm{d}s \right] \end{split}$$

Let us show that $m \mapsto f^a(y,m) + \mathcal{L}^a V(s,y,m)$ is Lipschitzian in m uniformly on $(a, y, s) \in \mathcal{A} \times \mathbb{R}^d \times [0, T].$

- By assumption, f, $\partial_t V$ and $\partial_y V$ are Lipschitzian in m uniformly on (s, y) and k is uniformly bounded;
- for $(s,s') \in [0,T]^2$, $(y,y') \in (\mathbb{R}^d)^2$, $(m,m') \in (\mathcal{P}_2(\mathbb{R}^d))^2$ and $(a,a') \in \mathcal{A}^2$,

$$\left| \int \partial_x \delta_m V(s', y', m', x) b(s', x, y', m', a') \, \mathrm{d}m'(x) - \int \partial_x \delta_m V(s, y, m, x) b(s, x, y, m, a) \, \mathrm{d}m(x) \right|$$
$$= \left| \mathbb{E} \left[\partial_x \delta_m V(s', y', m', X') b(s', X', y', m', a') - \partial_x \delta_m V(s, y, m, X) b(s, X, y, m, a) \right] \right|$$

for X, X' random variables such that $X \sim m$ and $X' \sim m'$. Since, by assumption, $\partial_x \delta_m V$ is Lipschitzian in m uniformly on (s, y, x), we have, with K the Lipschitz constant,

$$\begin{aligned} &|\partial_x \delta_m V(s', y', m', X') b(s', X', y', m', a') - \partial_x \delta_m V(s, y, m, X) b(s, X, y, m, a)| \\ &\leq ||b||_{\infty} |\partial_x \delta_m V(s', y', m', X') - \partial_x \delta_m V(s, y, m, X)| \\ &\leq ||b||_{\infty} K d(m', m), \end{aligned}$$

hence the result;

 $\begin{array}{l} - \text{ with similar calculations, } (s,y,m,a) \mapsto \frac{1}{2} \int \partial_x^2 \delta_m V(s,y,m,x) (\sigma_s^2 + (\sigma_s^0)^2) m(\mathrm{d}x), \\ (s,y,m,a) \mapsto \frac{1}{2} \int \int \partial_{x\hat{x}}^2 \delta_m^2 V(s,y,m,x,\hat{x}) \sigma_s^0 \hat{\sigma}_s^0 m(\mathrm{d}x) m(\mathrm{d}\hat{x}) \text{ and} \\ (s,y,m,a) \mapsto \int \partial_x \delta_m \partial_y V(s,y,m,x) \sigma_s^0 \gamma_s^0 m(\mathrm{d}x) \text{ are Lipschitzian in } m \text{ uniformly} \\ \mathrm{on } (s,y,a). \end{array}$

Therefore, there exists C > 0 such that, for every $(a, s, y, m), (a', s', y'm') \in (\mathcal{A} \times [0, T] \times (\mathbb{R}^d)^2 \times \mathcal{P}_2(\mathbb{R}^d))^2$,

$$|f^{a'}(y',m') + \mathcal{L}^{a'}V(s',y',m') - (f^{a}(y,m) + \mathcal{L}^{a}V(s,y,m))| \le Cd(m',m).$$

Then

$$\mathbb{E}_{t}^{\varepsilon} \left[\int_{t}^{\theta} \left(f^{\alpha_{s}}(Y_{s}, \mu_{s}) + \mathcal{L}^{\alpha_{s}}V(s, Y_{s}, \mu_{s}) \right) \mathrm{d}s \right]$$

$$\leq (\theta - t)F(t, y, m) + \sup_{a} \mathbb{E}_{t}^{\varepsilon} \left[C \int_{t}^{\theta} |\mathrm{d}(\mu_{s}, m)| \,\mathrm{d}s \right].$$

However,

$$\mathbb{E} \left[\mathrm{d}(\mu_{s}, m) \right] = \mathbb{E} \left[\inf \left\{ \mathbb{E}^{0} \left[|Z_{\mu} - Z_{m}|^{2} \right], \mathbb{P}_{Z_{\mu}|W^{0}} = \mu_{s}, \mathbb{P}_{Z_{m}|W^{0}} = m \right\}^{\frac{1}{2}} \right] \\ \leq \mathbb{E} \left[\mathbb{E}^{0} \left[|X_{s} - X_{t}|^{2} \right]^{\frac{1}{2}} \right] \\ \leq \mathbb{E} \left[\mathbb{E}^{0} \left[|X_{s} - X_{t}|^{2} \right] \right]^{\frac{1}{2}} \\ \leq \left(||b||_{\infty}^{2} (s - t)^{2} + \left(||\sigma||_{\infty}^{2} + ||\sigma^{0}||_{\infty}^{2} \right) (s - t) \right)^{\frac{1}{2}} \\ \leq ||b||_{\infty} (s - t) + \left(||\sigma||_{\infty}^{2} + ||\sigma^{0}||_{\infty}^{2} \right)^{\frac{1}{2}} \sqrt{s - t},$$

where we used Cauchy-Schwarz's inequality. Therefore,

$$\mathbb{E}_{t}^{\varepsilon} \left[\int_{t}^{\theta} \left(f^{\alpha_{s}}(Y_{s},\mu_{s}) + \mathcal{L}^{\alpha_{s}}V(s,Y_{s},\mu_{s}) \right) \mathrm{d}s \right] \\ \leq (\theta-t)F(t,y,m) + C \left(||b||_{\infty} \frac{(\theta-t)^{2}}{2} + \frac{2}{3}(||\sigma||_{\infty}^{2} + ||\sigma^{0}||_{\infty}^{2})^{\frac{1}{2}}(\theta-t)^{3} \right)$$

With $\theta = t + \sqrt{\varepsilon}$, $C_1 = \frac{1}{2}C||b||_{\infty}$ and $C_2 = \frac{2}{3}C(||\sigma||_{\infty}^2 + ||\sigma^0||_{\infty}^2)^{\frac{1}{2}}$, we obtain that $\varepsilon \ge -\sqrt{\varepsilon}F(t, y, m) - C_1\varepsilon - C_2\varepsilon^{\frac{3}{2}}.$

Dividing by $\sqrt{\varepsilon}$ and taking the limit $\varepsilon \longrightarrow 0$ finally yields that $F(t, y, m) \ge 0$.

6 Appendix: Proof of Lemma 1

For simplicity, we only report the proof for d = 1, as the extension to arbitrary dimension does not raise any difficulty. Let us first notice that

$$\sum_{i=1}^{p_n} H^n_{t^n_{i-1}}(\Delta^{\pi_n} \langle M \rangle_{t^n_i}) \longrightarrow \int_0^T H_s \, \mathrm{d} \langle M \rangle_s \text{ in } \mathbb{L}^1$$

Now, with transparent notations, it is obvious that

$$\begin{aligned} \left\| \left\| \sum_{i=1}^{p_n} H_{t_{i-1}^n} \left((\Delta^{\pi} X_{t_i^n})^2 - \Delta^{\pi_n} \langle M \rangle_{t_i^n} \right) \right\|_1 &\leq \left\| \left\| \sum_{i=1}^{p_n} H_{t_{i-1}^n} (\Delta^{\pi} A_{t_i^n})^2 \right\|_1 \\ &+ \left\| \left\| \sum_{i=1}^{p_n} H_{t_{i-1}^n} \left((\Delta^{\pi} M_{t_i^n})^2 - \Delta^{\pi_n} \langle M \rangle_{t_i^n} \right) \right\|_1 \\ &+ 2 \left\| \left\| \sum_{i=1}^{p_n} H_{t_{i-1}^n} \Delta^{\pi} A_{t_i^n} \Delta^{\pi} M_{t_i^n} \right\|_1. \end{aligned} \right\|$$

For the first term on the right-hand side: writing $||H||_{\infty}$ for a uniform, deterministic bound on $|H^n|, n \ge 1$,

$$\left|\sum_{i=1}^{p_n} H^n_{t^n_{i-1}}(\Delta^{\pi_n} A_{t^n_i})^2\right| \le ||H||_{\infty} \sum_{i=1}^{p_n} (\Delta^{\pi_n} A_{t^n_i})^2 = ||H||_{\infty} QV_{\pi^n}(A)$$

where $QV_{\pi^n}(A)$ is the quadratic variation of the finite-variation process A along the partition π^n . As $QV_{\pi^n}(A) \longrightarrow 0$ a.s. and $QV_{\pi^n}(A) \leq |A|_{TV}^2$ which is in \mathbb{L}^1 , it follows from the dominated convergence theorem that

$$\left\| \sum_{i=1}^{p_n} H^n_{t^n_{i-1}} (\Delta^{\pi_n} A_{t^n_i})^2 \right\|_1 \longrightarrow 0.$$

For the middle term, we introduce the martingale defined by $R_t^{t_{i-1}^n} := (\Delta^{\pi_n} M_t)^2 - \Delta^{\pi_n} \langle M \rangle_t$ for $t \ge t_{i-1}^n$ and we now show that

$$\left\| \sum_{i=1}^{p_n-1} H_{t_{i-1}^n}^n R_{t_i^n}^{t_{i-1}^n} \right\|_2^2 = \left\| \sum_{i=1}^{p_n} H_{t_{i-1}^n}^n \left((\Delta^{\pi_n} M_{t_i^n})^2 - \Delta^{\pi_n} \langle M \rangle_{t_i^n} \right) \right\|_2^2 \longrightarrow 0.$$

To see this, we directly compute that

$$\begin{split} \left| \left| \sum_{i=1}^{p_n-1} H_{t_{i-1}^n}^n R_{t_i^n}^{t_{i-1}^n} \right| \right|_2^2 &= \mathbb{E} \left[\sum_{i=1}^{p_n} H_{t_{i-1}^n}^n (R_{t_i^n}^{t_{i-1}^n})^2 \right] + 2\mathbb{E} \left[\sum_{0 \le i < j \le n-1} H_{t_{i-1}^n}^n H_{t_j^n}^n R_{t_i^n}^{t_{i-1}^n} R_{t_j^n}^{t_j^n} \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{p_n} H_{t_{i-1}^n}^n (R_{t_i^n}^{t_{i-1}^n})^2 \right] + 2 \sum_{0 \le i < j \le n-1} \mathbb{E} \left[H_{t_{i-1}^n}^n H_{t_j^n}^n R_{t_i^n}^{t_{i-1}^n} \mathbb{E} \left[R_{t_{j+1}^n}^n |\mathcal{F}_{t_j^n} \right] \right] . \end{split}$$

As $\mathbb{E}\left[R_{t_{j+1}^n}^{t_j^n}|\mathcal{F}_{t_j^n}\right] = 0$, this implies that

$$\left\| \sum_{i=1}^{p_n} H^n_{t^n_{i-1}} ((\Delta^{\pi_n} M_{t^n_i})^2 - \Delta^{\pi_n} \langle M \rangle_{t^n_i}) \right\|_2^2 \le \|H\|_{\infty}^2 \sum_{i=1}^{p_n} \mathbb{E} \left[(R^{t^n_{i-1}}_{t^n_i})^2 \right].$$

We next estimate that

$$\mathbb{E}\left[(R_{t_i^n}^{t_{i-1}^n})^2 \right] = \mathbb{E}\left[(\Delta^{\pi_n} M_{t_i^n})^4 - 2(\Delta^{\pi_n} M_{t_i^n})^2 \Delta^{\pi_n} \langle M \rangle_{t_i^n} + (\Delta^{\pi_n} \langle M \rangle_{t_i^n})^2 \right]$$

$$\leq \mathbb{E}\left[(\Delta^{\pi_n} M_{t_i^n})^4 \right] + \mathbb{E}\left[(\Delta^{\pi_n} \langle M \rangle_{t_i^n})^2 \right]$$

$$\leq (1 + C_4) \mathbb{E}\left[(\Delta^{\pi_n} \langle M \rangle_{t_i^n})^2 \right]$$

for some constant C_4 induced by the BDG inequality for the order p = 4. Therefore,

$$\left\| \sum_{i=1}^{p_n} H_{t_{i-1}^n}^n \left((\Delta^{\pi_n} M_{t_i^n})^2 - \Delta^{\pi_n} \langle M \rangle_{t_i^n} \right) \right\|_2^2 \le (1 + C_4) ||H||_{\infty}^2 \mathbb{E} \left[\mathrm{QV}_{\pi}^n(\langle M \rangle) \right]$$

where $\operatorname{QV}_{\pi}^{n}(\langle M \rangle) = \sum_{i=1}^{p_{n}} (\langle M \rangle_{t_{i}^{n}} - \langle M \rangle_{t_{i-1}^{n}})^{2}$. Since $\langle M \rangle$ is a finite-variation process, $\operatorname{QV}_{\pi}^{n}(\langle M \rangle) \longrightarrow 0$ almost surely as $n \to \infty$, and since $\operatorname{QV}_{\pi}^{n}(\langle M \rangle) \leq \langle M \rangle_{T}^{2} \in \mathbb{L}^{1}$ by Condition (IW3), we conclude by dominated convergence.

Finally, for the last term, note that the previous calculations, for H = 1, show that $\sum_i (\Delta^{\pi_n} M_{t_i^n})^2 \longrightarrow \langle M \rangle_T$ in \mathbb{L}^2 . Then, by applying the Cauchy-Schwarz inequality twice:

$$\mathbb{E}\left[\left|\sum_{i=1}^{p_n} H_{t_{i-1}^n}^n(\Delta^{\pi_n} A_{t_i^n})(\Delta^{\pi_n} M_{t_i^n})\right|\right] \le ||H||_{\infty} \mathbb{E}\left[\sqrt{\sum_{i=1}^{p_n} (\Delta^{\pi_n} A_{t_i^n})^2} \sqrt{\sum_{i=1}^{p_n} (\Delta^{\pi_n} M_{t_i^n})^2}\right] \le ||H||_{\infty} \left\|\left|\sum_{i=1}^{p_n} (\Delta^{\pi_n} A_{t_i^n})^2\right|\right\|_2 \left\|\sum_{i=1}^{p_n} (\Delta^{\pi_n} M_{t_i^n})^2\right\|_2 \longrightarrow 0,$$

since $\mathbb{E}\left[\sum_{i=1}^{p_n} (\Delta^{\pi_n} A_{t_i^n})^2\right] \to 0$ by dominated convergence (see first term calculations).

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