# A SIMPLE DERIVATION OF THE INTEGRALS OF PRODUCTS OF LEGENDRE POLYNOMIALS WITH LOGARITHMIC WEIGHT 

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#### Abstract

We explore integrals of products of Legendre polynomials with a logarithmic weight function. More precisely, for Legendre polynomials $P_{m}$ and $P_{n}$ of orders $m$


 and $n$, respectively, we provide simple derivations of the integrals$$
\int_{0}^{1} P_{n}(2 x-1) P_{m}(2 x-1) \log (x) d x
$$

## 1. Introduction

In this short note we, compute the integrals of products of Legendre polynomials with logarithmic weight. Specifically, we are interested in

$$
N_{n, m}:=\int_{0}^{1} P_{n}(2 x-1) P_{m}(2 x-1) \log (x) d x
$$

for $m, n \in \mathbb{N}$. The Legendre polynomials $P_{0}, P_{1}, P_{2} \ldots$, are defined recursively (1], 22.7.10). The initial polynomials are given by

$$
\begin{aligned}
& P_{0}(x)=1, \\
& P_{1}(x)=x .
\end{aligned}
$$

For $n \geqslant 1$ the polynomials $P_{n}$ are defined by the three-term-recurrence

$$
\begin{equation*}
(n+1) P_{n+1}(x)=(2 n+1) x P_{n}(x)-n P_{n-1}(x) \tag{1}
\end{equation*}
$$

In this work, we derive for $n \neq m$,

$$
\begin{equation*}
N_{n, m}=\frac{(-1)^{n+m+1}}{|n-m|(n+m+1)} \tag{2}
\end{equation*}
$$

and for $m=n$

$$
\begin{align*}
(2 n+1) N_{n, n} & =(2 n-1) N_{n-1, n-1}-\frac{2}{(2 n-1) 2 n(2 n+1)} \\
& =-1-2 \sum_{j=1}^{n} \frac{1}{(2 j-1) 2 j(2 j+1)} \tag{3}
\end{align*}
$$

[^0]The solution for the special case $m=0$ and $n>0$ is obtained in [2], and given by

$$
\int_{0}^{1} P_{n}(2 x-1) \log (x) d x=\frac{(-1)^{n+1}}{n(n+1)}
$$

This has been generalized to integrals of single Jacobi polynomials with various weight functions, see for example [3, 4, 5, 7, 8]. In [6] [(4.6)], integrals of products of Legendre polynomials with logarithmic weights are represented by finite sums involving the Gamma function and the Digamma function. The result (22) can be found in [1] $[8.14 .8,8.14 .10]$ as integrals of products of Legendre functions of the first and the second kind. On the other hand $N_{n, n}$ is given in [1] [8.14.9] via the derivative of the Digamma function. Albeit being a special case and easily derivable thereof, the simple representation of (3) seems to be new. Moreover, the elementary derivations of (2) and (3) seem, to our surprise and the best of our knowledge, to be missing in the literature.

## 2. Main Results

We give rigorous derivations of the values of the integrals

$$
N_{n, m}:=\int_{0}^{1} P_{n}(2 x-1) P_{m}(2 x-1) \log (x) d x
$$

for arbitrary $n, m \in \mathbb{N}$.
Theorem 1. For $n, m \in \mathbb{N}, n>m$,

$$
\begin{equation*}
N_{n, m}=\frac{(-1)^{n+m+1}}{(n-m)(n+m+1)} \tag{4}
\end{equation*}
$$

Proof of Theorem 1. We prove the result by induction.
Induction base case
The base case for $m=0$ and $n>m$ is, as discussed above, proven in [2]:

$$
\begin{equation*}
N_{n, 0}=\int_{0}^{1} P_{n}(2 x-1) \log (x) d x=\frac{(-1)^{n+1}}{n(n+1)} \tag{5}
\end{equation*}
$$

For $m=1$ and $n>m$, we observe that $P_{1}(2 x-1)=2 x-1$. It follows from Equation (11) that for $n \geqslant 1$

$$
\begin{equation*}
P_{n}(2 x-1)(2 x-1)=\frac{n+1}{2 n+1} P_{n+1}(2 x-1)+\frac{n}{2 n+1} P_{n-1}(2 x-1) \tag{6}
\end{equation*}
$$

Substituting (6) into the definition of $N_{n, 1}$ gives

$$
\begin{aligned}
N_{n, 1} & :=\int_{0}^{1} P_{n}(2 x-1)(2 x-1) \log (x) d x \\
& =\int_{0}^{1}\left[\frac{n+1}{2 n+1} P_{n+1}(2 x-1)+\frac{n}{2 n+1} P_{n-1}(2 x-1)\right] \log (x) d x \\
& =\frac{n+1}{2 n+1} \int_{0}^{1} P_{n+1}(2 x-1) \log (x) d x+\frac{n}{2 n+1} \int_{0}^{1} P_{n-1}(2 x-1) \log (x) d x \\
& =\frac{n+1}{2 n+1} N_{n+1,0}+\frac{n}{2 n+1} N_{n-1,0}
\end{aligned}
$$

Substituting the results for $N_{n+1,0}$ and $N_{n-1,0}$, see Equation (15), leads to

$$
\begin{aligned}
N_{n, 1} & =\frac{n+1}{2 n+1} \frac{(-1)^{n+2}}{(n+1)(n+2)}+\frac{n}{2 n+1} \frac{(-1)^{n}}{(n-1) n} \\
& =\frac{(-1)^{n+2}}{2 n+1}\left[\frac{1}{n+2}+\frac{1}{n-1}\right]=\frac{(-1)^{n+2}}{2 n+1} \frac{2 n+1}{(n+2)(n-1)} \\
& =\frac{(-1)^{n+2}}{(n+2)(n-1)}
\end{aligned}
$$

which is the base case for $m=1$.
Induction hypothesis


$$
\begin{align*}
& N_{n, m-1}=\frac{(-1)^{n+m}}{(n-m+1)(n+m)} \text { for } n>m-1 \text { and } \\
& N_{n, m-2}=\frac{(-1)^{n+m-1}}{(n-m+2)(n+m-1)} \text { for } n>m-2 \tag{7}
\end{align*}
$$

Induction step
It follows from Equation (11) that for $m>1$

$$
\begin{equation*}
m P_{m}(2 x-1)=(2 m-1)(2 x-1) P_{m-1}(2 x-1)-(m-1) P_{m-2}(2 x-1) \tag{8}
\end{equation*}
$$

4
Multiplying $N_{n, m}$ by $m$ and substituting (8) yields

$$
\begin{aligned}
m N_{n, m} & =\int_{0}^{1} P_{n}(2 x-1) m P_{m}(2 x-1) \log (x) d x \\
& =\int_{0}^{1} P_{n}(2 x-1)(2 m-1)(2 x-1) P_{m-1}(2 x-1) \log (x) d x \\
& -\int_{0}^{1} P_{n}(2 x-1)(m-1) P_{m-2}(2 x-1) \log (x) d x
\end{aligned}
$$

which results in
(9)

$$
\begin{aligned}
& m N_{n, m}= \\
& (2 m-1) \int_{0}^{1}(2 x-1) P_{n}(2 x-1) P_{m-1}(2 x-1) \log (x) d x-(m-1) N_{n, m-2}
\end{aligned}
$$

where we made use of the definition of $N_{n, m-2}$. It follows from Equation (1) that for $n \geqslant 1$

$$
\begin{equation*}
(2 n+1)(2 x-1) P_{n}(2 x-1)=(n+1) P_{n+1}(2 x-1)+n P_{n-1}(2 x-1) \tag{10}
\end{equation*}
$$

Multiplying Equation (19) by $(2 n+1)$ and subsituting (10) leads to

$$
\begin{aligned}
& (2 n+1) m N_{n, m}= \\
& (2 m-1) \int_{0}^{1}\left[(n+1) P_{n+1}(2 x-1)+n P_{n-1}(2 x-1)\right] P_{m-1}(2 x-1) \log (x) d x \\
& -(2 n+1)(m-1) N_{n, m-2}
\end{aligned}
$$

Making use of the definition of $N_{n+1, m-1}$ and $N_{n-1, m-1}$ gives us

$$
\begin{align*}
& (2 n+1) m N_{n, m}= \\
& (2 m-1)(n+1) N_{n+1, m-1}+(2 m-1) n N_{n-1, m-1}-(2 n+1)(m+1) N_{n, m-2} \tag{11}
\end{align*}
$$

The assumption that $n>m$ implies that

$$
\begin{aligned}
n+1 & >m-1 \\
n-1 & >m-1 \text { and } \\
n & >m-2
\end{aligned}
$$

hence the induction hypotheses (7) are valid for $N_{n+1, m-1}, N_{n-1, m-1}$ and $N_{n, m-2}$. Substituting the respective results into Equation (11), we obtain

$$
\begin{align*}
(2 n+1) m N_{n, m}= & (2 m-1)(n+1) \frac{(-1)^{n+m+1}}{(n-m+2)(n+m+1)} \\
& +(2 m-1) n \frac{(-1)^{n+m-1}}{(n-m)(n+m-1)}  \tag{12}\\
& -(2 n+1)(m-1) \frac{(-1)^{n+m-1}}{(n-m+2)(n+m-1)} .
\end{align*}
$$

Bringing the right hand side of (12) on a common denominator shows that

$$
(2 n+1) m N_{n, m}=(-1)^{n+m+1} \frac{(2 n+1) m(n-m+2)(n+m-1)}{(n-m+2)(n+m+1)(n-m)(n+m-1)}
$$

Dividing by $(2 n+1) m$ and simplifying the fraction proves the result (4).
Corollary 1. For $n, m \in \mathbb{N}, n \neq m$,

$$
N_{n, m}=\frac{(-1)^{n+m+1}}{|n-m|(n+m+1)}
$$

Proof. The case of $n>m$ is covered by Theorem In the case of $m>n$, we obtain the result of the Corollary by observing that $N_{n, m}=N_{m, n}$.

## Theorem 2.

$$
\begin{align*}
N_{0,0} & =-1  \tag{13}\\
3 N_{1,1} & =-\frac{4}{3} \tag{14}
\end{align*}
$$

and for $n>1$,

$$
\begin{aligned}
(2 n+1) N_{n, n} & =(2 n-1) N_{n-1, n-1}-\frac{2}{(2 n-1) 2 n(2 n+1)} \\
& =-1-2 \sum_{j=1}^{n} \frac{1}{(2 j-1) 2 j(2 j+1)} .
\end{aligned}
$$

Proof of Theorem 圆. Equations (13) and (14) can be verified by using $P_{0}(2 x-1)=1$ and $P_{1}(2 x-1)=2 x-1$ and by explicitly computing the integrals. For arbitrary $n>1$, it follows from Equation (1) that for $n>1$

$$
\begin{equation*}
P_{n}(2 x-1)=\frac{2 n-1}{n}(2 x-1) P_{n-1}(2 x-1)-\frac{n-1}{n} P_{n-2}(2 x-1) . \tag{15}
\end{equation*}
$$

Replacing one of the factors $P_{n}(2 x-1)$ in the defintion of $N_{n, n}$ by (15) and using the definition of $N_{n, n-2}$ yields

$$
\begin{align*}
N_{n, n} & =\int_{0}^{1} P_{n}(2 x-1) P_{n}(2 x-1) \log (x) d x \\
& =\int_{0}^{1} P_{n}(2 x-1) \frac{2 n-1}{n}(2 x-1) P_{n-1}(2 x-1) \log (x) d x-\frac{n-1}{n} N_{n, n-2} \tag{16}
\end{align*}
$$

It follows from Equation (1) that for $n \geqslant 1$

$$
\begin{equation*}
(2 n+1)(2 x-1) P_{n}(2 x-1)=(n+1) P_{n+1}(2 x-1)+n P_{n-1}(2 x-1) . \tag{17}
\end{equation*}
$$

By multiplication of Equation (16) with $(2 n+1)$ and substituting (17) we obtain

$$
\begin{aligned}
& (2 n+1) N_{n, n}= \\
& (2 n-1) N_{n-1, n-1}+\frac{2 n-1}{n}(n+1) N_{n+1, n-1}-(2 n+1) \frac{n-1}{n} N_{n, n-2}
\end{aligned}
$$

where we made use of the definition of $N_{n+1, n-1}$ and $N_{n, n-2}$. Using Theorem 1 results in

$$
\begin{gathered}
(2 n+1) N_{n, n}=(2 n-1) N_{n-1, n-1}-\frac{(2 n-1)(n+1)}{2 n(2 n+1)}+\frac{(2 n+1)(n-1)}{2 n(2 n-1)}= \\
(2 n-1) N_{n-1, n-1}+\frac{-(2 n-1)^{2}(n+1)+(2 n+1)^{2}(n-1)}{(2 n-1) 2 n(2 n+1)}
\end{gathered}
$$

which can be easily simplified to

$$
(2 n+1) N_{n, n}=(2 n-1) N_{n-1, n-1}-\frac{2}{(2 n-1) 2 n(2 n+1)}
$$

Finally, recursively substituting the last equation yields, together with the result for $N_{0,0}$, that

$$
(2 n+1) N_{n, n}=-1-2 \sum_{j=1}^{n} \frac{1}{(2 j-1) 2 j(2 j+1)},
$$

which concludes the proof.

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