# REPRESENTATIONS OF $S L_{2}(F)$ 

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#### Abstract

Let $p$ be a prime number, $F$ a non-archimedean local field with residue characteristic $p$, and $R$ an algebraically closed field of characteristic different from $p$. We thoroughly investigate the irreducible smooth $R$-representations of $S L_{2}(F)$. The components of an irreducible smooth $R$-representation $\Pi$ of $G L_{2}(F)$ restricted to $S L_{2}(F)$ form an $L$-packet $L(\Pi)$. We use the classification of such $\Pi$ to determine the cardinality of $L(\Pi)$, which is 1,2 or 4 . When $p=2$ we have to use the Langlands correspondence for $G L_{2}(F)$. When $\ell$ is a prime number distinct from $p$ and $R=\mathbb{Q}_{\ell}^{a c}$, we establish the behaviour of an integral $L$-packet under reduction modulo $\ell$. We prove a Langlands correspondence for $S L_{2}(F)$, and even an enhanced one when the characteristic of $R$ is not 2 . Finally, pursuing a theme of Henniart-Vignéras23, which studied the case of inner forms of $G L_{n}(F)$, we show that near identity an irreducible smooth R-representation of $S L_{2}(F)$ is, up to a finite dimensional representation, isomorphic to a sum of 1,2 or 4 representations in an $L$-packet of size 4 (when $p$ is odd there is only one such $L$-packet).


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## 1. Introduction

1.1. Let $F$ be a locally compact non-archimedean field with residue characteristic $p$ and $R$ an algebraically closed field of characteristic char ${ }_{R}$ different from $p$. We thoroughly investigate the irreducible smooth $R$-representations of $S L_{2}(F)$. Although when $R=\mathbb{C}$ and $p$ odd the first investigations appeared in the 1960's, in work of Gelfand-Graev and Shalika, the study of the modular case (i.e. when $\operatorname{char}_{R}>0$ ) started only recently [ui20, Cui-Lanard-Lu24] when $\operatorname{char}_{F} \neq 2$ and $\operatorname{char}_{R} \neq 2$. Here we give a complete treatment and we make NO assumption on $p, \operatorname{char}_{F}, \operatorname{char}_{R}$, apart from $\operatorname{char}_{R} \neq p$.

As Labesse and Langlands did in the 1970's when $R=\mathbb{C}$ and $\operatorname{char}_{F}=0$, we use the restriction of smooth $R$-representations from $G=G L_{2}(F)$ to $G^{\prime}=S L_{2}(F)$. We prove that an irreducible smooth $R$-representation of $G^{\prime}$ extends to a smooth representation of an open subgroup $H$ of $G$ containing $Z G^{\prime}$ where $Z$ is the centre of $G$, and appears in the restriction to $G^{\prime}$ of a smooth irreducible $R$-representation of $G$, unique up to isomorphims and twists by smooth $R$-characters of $G / G^{\prime}$. When $\operatorname{char}_{F} \neq 2$ we can simply take $H=Z G^{\prime}$, but not when $\operatorname{char}_{F}=2$ because the compact quotient $G / Z G^{\prime}$ is infinite. Those results follow from general facts about $R$-representations, which appear in Section 2. They apply to more general reductive groups over $F$, as we show in Section 3,

In $\S 4.2$, using Whittaker models, we show that the restriction to $G^{\prime}$ of an irreducible smooth $R$-representation $\Pi$ of $G$ has finite length and multiplicity one. Its irreducible components form an $L$-Lacket $L(\Pi)$. An $L$-packet $L(\Pi)$ is called cuspidal when $\Pi$ is cuspidal, supercuspidal when $\Pi$ is supercuspidal, of level 0 if $\Pi$ can be chosen to have level 0 (that is, having non-zero fixed vectors under $1+M_{2}\left(P_{F}\right)$ ), and of positive level otherwise.
Theorem 1.1. The size of an L-packet is 1,2 or 4 .
When $p$ is odd that follows rather easily from $\left|G / Z G^{\prime}\right|=4$, but it is also true when $p=2$, in which case the proof is completed only in Propositions 4.23, 4.29, and uses the Langlands correspondence for $G$, which we recall in $\S 4.4$.
Proposition 1.2. (Corollary 4.30, Proposition 4.23) The L-packets of size 4 are cuspidal and in bijection the biquadratic separable extensions of $F$.

When $p$ is odd there is just one $L$-packet of cardinality 4 but when $\operatorname{char}_{F}=2$, there are infinitely many.
Proposition 1.3. (Proposition 4.8) When $p$ is odd, the supercuspidal L-packets have size 2. When $p=2$, the cuspidal L-packets of level 0 have size 2 .

Proposition 1.4. (Proposition 4.2g) There is a cuspidal non-supercuspidal L-packet if and only if $q+1=0$ in $R$. It is unique of level 0 , and size 4 when $\operatorname{char}_{R}=2$, and size 2 when $\operatorname{char}_{R} \neq 2$.

When $\operatorname{char}_{R} \neq 2$, the non-cuspidal L-packets of size 2 are in bijection with the quadratic separable extensions of $F$. The other non-cuspidal L-packets are singletons.

When $\operatorname{char}_{R}=2$, the non-cuspidal L-packets are singletons.
The bijections in Propositions 1.2 and 1.4 are described in the proofs.

From the Langlands $R$-correspondence for $G L(2, F)$, we get a bijection from the set of $L$ packets to the set of conjugacy classes of Deligne morphisms of $W_{F}$ into $P G L_{2}(R)$, the dual group of $S L_{2}$ over $R$. When $\operatorname{char}_{R} \neq 2$, we even get an enhanced Langlands correspondence, in that we parametrize the elements in an $L$-packet $L(\Pi)$ by the characters of the group $S_{\Pi}$ of connected components of the centralizer $C_{\Pi}$ of the image of the corresponding Deligne morphism in $P G L_{2}(R)$. But when $\operatorname{char}_{R}=2, C_{\Pi}$ is always connected but a supercuspidal $L$-packet is never a singleton.

Theorem 1.5. (Theorem 5.3) Let $\Pi$ be an irreducible smooth $R$-representation of $G L_{2}(F)$. When $\operatorname{char}_{R} \neq 2$, the L-packet $L(\Pi)$ can be parametrized by the $R$-characters of the component group $S_{\Pi}$ of $C_{\Pi}$, and $S_{\Pi}$ is isomorphic to $\{1\}, \mathbb{Z} / 2 \mathbb{Z}$ or $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}$.

When $\operatorname{char}_{R}=2, C_{\Pi}$ is connected for each $\Pi$, but the cardinality of the L-packet $L(\Pi)$ is

1 if $\Pi$ is not cuspidal,
2 if $\Pi$ is supercuspidal,
4 if $\Pi$ is cuspidal not supercuspidal.
We determine explicitely $C_{\Pi}$ for each $\Pi$. When $L(\Pi)$ is not a singleton, we take as a base point the element having a non-zero Whittaker model with respect to a non-trivial smooth $R$-character of $F$, and we describe the parametrization.

When $\operatorname{char}_{R}=2$, Treuman and Venkatesh introduced a "linkage" between irreducible smooth $R$-representations of $G$ and $G^{\prime}$. In $\$ 5.0 .3$ we interpret this notion in terms of dual groups, thus proving their conjectures in a special case.

Let $\ell \neq p$ be a prime number, and $\mathbb{Q}_{\ell}^{a c}$ an algebraic closure of $\mathbb{Q}_{\ell}$ with residue field $\mathbb{F}_{\ell}^{a c}$. Each irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation of $G L_{2}(F)$ lifts to a smooth $\mathbb{Q}_{\ell}^{a c}$-representation. We show that this remains true for $S L_{2}(F)$.
Proposition 1.6. (Corollary 4.25, Proposition 4.31) Each irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\pi$ of $S L_{2}(F)$ is the reduction modulo $\ell$ of an integral irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $S L_{2}(F)$.

An equivalent formulation is that each irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\Pi$ of $G L_{2}(F)$ is the reduction modulo $\ell$ of an integral irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\Pi}$ of $G L_{2}(F)$ such that

$$
|L(\Pi)|=|L(\tilde{\Pi})| .
$$

The reduction modulo $\ell$ of each integral supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation of $G L_{2}(F)$ is irreducible, but this is not true for $S L_{2}(F)$. Each supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $S L_{2}(F)$ is integral and we determine all the cases of reducibility. We choose a supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\Pi}$ of $G L_{2}(F)$ such that $\tilde{\pi} \in L(\tilde{\Pi})$ and denote by $\sigma_{\tilde{\Pi}}$ the irreducible 2dimensional $\mathbb{Q}_{\ell}^{a c}$-representation of $W_{F}$ image of $\tilde{\Pi}$ by the local Langlands correspondence.
Proposition 1.7. (Corollary 4.25) The reduction modulo $\ell$ of $\tilde{\pi}$ has length $\leq 2$. The length is 2 if and only if
$p=2, \sigma_{\tilde{\Pi}}=\operatorname{ind}_{W_{E}}^{W_{F}} \tilde{\xi}, \tilde{\xi}(b) \neq 1, \tilde{\xi}(b)^{\ell^{s}}=1, \ell^{s}$ divides $q+1$, the order of $\left.\left(\tilde{\xi}^{\tau} / \tilde{\xi}\right)\right|_{1+P_{E}}$ is 2 ,
where $b$ is a root of unity of order $q+1$ in a quadratic unramified extension $E / F, \tilde{\xi}$ is a smooth $\mathbb{Q}_{\ell}^{a c}$-character of $E^{*}$ (of $W_{E}$ via class field theory), and $\tau \in \operatorname{Gal}(E / F)$ is not trivial.

Finally we study for $G^{\prime}$ the problem that we treated in Henniart-Vignéras23 for inner forms of $G L_{n}(F)$. For an infinite dimensional irreducible smooth $R$-representation $\pi$ of $G^{\prime}$, we investigate the possible behaviour of the restriction of $\pi$ to sufficiently small open subgroups $K$ of $G^{\prime}$. In fact we show that up to finitely many trivial $R$-characters, $\pi$ is isomorphic near the identity to the sum of 1,2 or 4 elements of an $L$-packet of size 4 .

Theorem 1.8. (Theorem 6.18) Let $\pi$ be an infinite dimensional irreducible smooth $R$ representation of $G^{\prime}$. There are irreducible smooth $R$-representations $\left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$ of $G^{\prime}$ forming an L-packet, and an integer $a_{0}$, such that on a small enough compact open subgroup $K$ of $G^{\prime}$ we have

$$
\pi \simeq a_{0} 1+\sum_{i=1}^{4 / r} \tau_{i}
$$

where $r$ is the size of the $L$-packet containing $\pi$.
For $R=\mathbb{C}$ and $p$ odd, Monica Nevins has similar results which are more precise in that the subgroup $K$ is large. We show that her results carry over to any $R$ (§6.2.8).

As in (loc.cit.) we first deal with the case where $R=\mathbb{C}$, using a germ expansion near the identity à la Harish-Chandra, in terms of nilpotent orbital integrals. However, when $\operatorname{char}_{F}=2$, such an expansion is not available, so we work instead with a complex representation $\pi$ of an open subgroup $H$ of $G$ containing $Z G^{\prime}$. For such a group a germ expansion has been obtained by Lemaire [Lemaire04]. Adapting Moeglin-Waldspurger87] and Varma14] (who assumed $\operatorname{char}_{F}=0$ ) we compute the germ expansion in terms of the dimensions of the different Whittaker models of $\pi$, and express it in terms of $L$-packets of size 4. Theorem 1.8 easily transfers to any $R$ with $\operatorname{char}_{R}=0$, in particular $R=\mathbb{Q}_{\ell}^{a c}$. From our complete classification of irreducible smooth $R$-representations of $G^{\prime}$, and in particular that the $\mathbb{F}_{\ell}^{a c}$-representations of $G^{\prime}$ lift to characteristic 0 when $\ell \neq p$ (Proposition 1.6), we get Theorem 1.8 for $R=\mathbb{F}_{\ell}^{a c}$ and transfer it to any $R$ with $\operatorname{char}_{R}=\ell$.

We think that Theorem 1.8 will extend in the same way to inner forms of $S L_{n}$, using the work of Hiraga-Saito12. We expect that if $\operatorname{char}_{F}=0$ and $R=\mathbb{C}$, a variant of the theorem is true for any connected reductive $F$-group $\underline{H}$, because of the Harish-Chandra germ expansion and of the work of Moeglin-Waldspurger and Varma. But when $\ell \neq p$, it is not known in general if virtual finite length $\mathbb{F}_{\ell}^{a c}$-representations lift to characteristic 0 and it is unlikely that cuspidal irreducible $\mathbb{F}_{\ell}^{a c}$-representations lift. The reason is the the first point has a positive answer when $G$ is a finite group and the answer to the second is negative in general for finite reductive groups. Moreover when $\operatorname{char}_{F}=p$ and $R=\mathbb{C}$, we have to face the problem that a germ expansion in terms of nilpotent orbital integrals might not exist. It is not clear how to define such integrals for bad primes, and sometimes the number of unipotent orbits in $H$ and of nilpotent orbits in Lie $(H)$ are not the same, even over an algebraic closure of $F$. Given our investigation of the case $S L_{2}(F)$, which
uses $L$-indistinguishability, one may wonder about the role of endoscopy and stability in analogous results for a general $H$.

The study of $R$-representations of $G^{\prime}$ has a long history, especially when $R=\mathbb{C}$. Inevitably some of our proofs are adapted from previous papers. However, because we make only the assumption that $\operatorname{char}_{R} \neq p$, we have usually preferred to give complete proofs in that general setting.

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## 2. Generalities

2.1. Let $R$ be a field, $G$ a group, $H$ a subgroup of $G, V$ an $R$-representation of $G$. We denote $\operatorname{char}_{R}$ the characteristic of $R$, and $\left.V\right|_{H}$ the restriction of $V$ to $H$.
2.1.1. When $H$ has finite index in $G$, any irreducible $R$-representation of $H$ is contained in the restriction to $H$ of an irreducible $R$-representation of $G$ Henniart01, Proposition $2.2]$.
2.1.2. If $H$ is normal of finite index in $G$ and $V$ is irreducible, then $\left.V\right|_{H}$ is semisimple of finite length (loc.cit.Proposition 2.1).
2.1.3. If $H$ is normal in $G, V$ is irreducible and $\left.V\right|_{H}$ contains an irreducible subrepresentation, then $\left.V\right|_{H}$ is semisimple and its isotypic components are $G$-conjugate with the same multiplicity.
Proof. Let $W$ be an irreducible subrepresentation of $\left.V\right|_{H}$. Since $H$ is normal in $G$, for $g \in G, H$ acts irreducibly on $g W$ by $(h, g w) \rightarrow h g h^{-1} h w$. The subspace $\sum_{g \in G} g W$ is a nonzero subrepresentation of $V$. Since $V$ is irreducible, it is equal to $V$. Since a representation which is a sum of irreducible subrepresentations is semi-simple BourbakiA8, §4.1 Corollary 1], $\left.V\right|_{H}$ is semisimple.
2.1.4. Assume $H$ normal of finite index in $G$ and let $\pi$ be an irreducible $R$-representation of $H$. We saw that there is an irreducible $R$-representation $\Pi$ of $G$ whose restriction to $H$ (which is semisimple of finite length) contains $\pi$. Clearly if $\chi$ is a $R$-character of $G$ trivial on $H$ then the restriction of $\chi \otimes \Pi$ to $H$ contains $\pi$.
Lemma 2.1. Assume $R$ algebraically closed and $G / H$ abelian. Any irreducible $R$-representation $\Pi^{\prime}$ of $G$ containing $\pi$ is isomorphic to $\Pi \otimes \chi$ for some $R$-character $\chi$ of $G$ trivial on $H$.
Proof. 1 We have $\operatorname{Hom}_{H}\left(\left.\Pi^{\prime}\right|_{H},\left.\Pi\right|_{H}\right) \neq 0$. The right adjoint of the restriction from $G$ to $H$ is the induction $\operatorname{Ind}_{H}^{G}$ from $H$ to $G$, therefore $\Pi^{\prime}$ is isomorphic to an irreducible

[^0]subrepresentation of $\operatorname{Ind}_{H}^{G}\left(\left.\Pi\right|_{H}\right)$. We have $\left.\operatorname{Ind}_{H}^{G}\left(\left.\Pi\right|_{H}\right)\right) \simeq\left(\operatorname{Ind}_{H}^{G} 1\right) \otimes \Pi$ because $G / H$ is finite, and the irreducible subquotients of $\operatorname{Ind}_{H}^{G} 1$ are the characters $\chi$ of $G$ trivial on $H$ because $R$ is algebraically closed. Therefore, there exists $\chi$ such that $\Pi^{\prime} \simeq \Pi \otimes \chi$.
2.2. We suppose that $H$ is a closed subgroup of a locally profinite group $G$ and $V$ is an $R$-representation of $G$.

If the index of $H$ in $G$ is finite, then $H$ is open. Conversely, if $H$ is open cocompact in $G$, then the index of $H$ in $G$ is finite. If $V$ is smooth (i..e. the $G$-stabilizer of any vector is open) then $\left.V\right|_{H}$ is smooth. Conversely, if $H$ is open in $G$ and $\left.V\right|_{H}$ is smooth (resp. admissible i.e. smooth and the dimension of the space $V^{K}$ of $K$-fixed vectors of $V$ is finite, for any open compact subgroup $K \subset H$ ), then $V$ is smooth (resp. admissible).

We suppose also from now on that $H$ is normal in $G$ with a compact quotient $G / H$ and that $V$ is smooth (so $\left.V\right|_{H}$ is smooth).
2.2.1. If $V$ is finitely generated then $\left.V\right|_{H}$ is finitely generated [Henniart01, Lemma 4.1].
2.2.2. If $V$ is irreducible, any irreducible subrepresentation of $\left.V\right|_{H}$ (when there exists one) extends to a (smooth and irreducible) representation of an open subgroup of $G$ of finite index which is admissible if $V$ is (loc.cit.Proposition 4.4).
2.2.3. If $V$ is irreducible and $\left.V\right|_{H}$ contains an irreducible subrepresentation or is noetherian (any subrepresentation is finitely generated), then $\left.V\right|_{H}$ is semisimple of finite length (loc.cit. Théorème 4.2).

We introduce the two properties :
Any finitely generated admissible $R$-representation of $G$ has finite length Any finitely generated smooth $R$-representation of $H$ is noetherian
2.2.4. Let $W$ be an admissible irreducible $R$-representation of $H$.

1) If (2.1) and (2.2) are true, then $W$ is contained in some irreducible admissible $R$ representation of $G$ restricted to $H$ (loc.cit. Corollaire 4.6).
2) If (2.1)) is true, then $W$ is a quotient of some irreducible admissible $R$-representation of $G$ restricted to $H$ (loc.cit. Théorème 4.5).

We give a simple proof of 2) adapted from Tadic92, Proposition 2.2]. The smooth induction $\operatorname{Ind}_{H}^{G} W$ of $W$ to $G$ is admissible as $W$ is and $G / H$ is compact Vignéras96, I.5.6]. A finitely generated subrepresentation of $\operatorname{Ind}_{H}^{G} W$ is admissible, hence of finite length by (2.1). So $\operatorname{Ind}_{H}^{G} W$ contains an irreducible admissible representation $U$. The restriction to $H$ is the left adjoint of the induction $\operatorname{Ind}_{H}^{G}$ hence $W$ is a quotient of $\left.U\right|_{H}$.
2.2.5. We denote by $V \otimes \chi$ the twist of the representation $V$ of $G$ by an $R$-character $\chi$ of $G$, and by
$X_{V}$ the group of $R$-characters $\chi$ of $G$ trivial on $H$ such that $V \otimes \chi \simeq V$.
The characters in $X_{V}$ are smooth by the following lemma.
Lemma 2.2. $V \otimes \chi$ is smooth if and only if $\chi$ is smooth.
Proof. Let $v \in V$ a non-zero element. An open subgroup $K \subset G$ fixing $v$ in $V$, fixes $v$ in $V \otimes \chi$ if and only if $\chi$ is trivial on $K$. The lemma follows because $V$ is smooth.
2.2.6. Assume also that $V$ is irreducible and $\left.V\right|_{H}$ has finite length (semi-simple by (2.2.3) and its isotypic components are $G$-conjugate).

Let $W$ be an irreducible component of $\left.V\right|_{H}, \pi$ its isomorphism class, $G_{\pi}$ the $G$-stabilizer of $\pi$. Let $V_{\pi}$ be the $\pi$-isotypic component of $\left.V\right|_{H}$. The $G$-stabilizer of $V_{\pi}$ is $G_{\pi}$. The $G$-stabilizer of $W$ is open in $G$ (because it contains the $G$-stabilizer of $v \in W$ non 0 and $V$ is smooth) and is contained in $G_{\pi}$. Both have finite index in $G(G / H$ is compact) and

$$
V=\operatorname{Ind}_{G_{\pi}}^{G}\left(V_{\pi}\right)
$$

by Clifford's theory. The representation of $G_{\pi}$ on $V_{\pi}$ is irreducible and the length of $\left.V\right|_{H}$ is

$$
\lg \left(\left.V\right|_{H}\right)=\left[G: G_{\pi}\right] \lg \left(\left.V_{\pi}\right|_{H}\right)
$$

Lemma 2.3. Assume that $G / H$ is abelian. Then:

1) $G_{\pi}$ is normal in $G$ and does not depend on the choice of $\pi$ in $V_{H}$. The smooth $R$-characters of $G$ trivial on $G_{\pi}$ are in $X_{V}$.
2) Assume $R$ algebraically closed.
a) Any irreducible subquotient of the smooth induction $\operatorname{Ind}_{H}^{G} 1$ is a smooth $R$-character $\chi$ of $G$ trivial on $H$.
b) Any irreducible $R$-representation of $G$ containing $\pi$ is a twist $\chi \otimes V$ of $V$ by some smooth $R$-character $\chi$ of $G$ trivial on $H$.
3) When $\left.V\right|_{H}$ has multiplicity 1 , then $W=V_{\pi}$, for a smooth $R$-character $\chi$ of $G$ trivial on $H, \chi \otimes V \simeq V$ if and only if $\chi$ is trivial on $G_{\pi}$, and $G_{\pi}$ is the largest subgroup $I$ of $G$ containing $H$ such that $\lg \left(\left.V\right|_{I}\right)=\lg \left(\left.V\right|_{H}\right)$.
4) When $R$ is algebraically closed and $\left.V\right|_{H}$ has multiplicity 1 , then

$$
\left|X_{V}\right|= \begin{cases}{\left[G: G_{\pi}\right]} & \text { if } \operatorname{char}_{R}=0 \\ {\left[G: G_{\pi, \ell}\right]} & \text { if } \operatorname{char}_{R}=\ell>0\end{cases}
$$

where $G_{\pi, \ell}$ is the smallest subgroup of $G$ containing $G_{\pi}$ such that $\left[G: G_{\pi, \ell}\right]$ is prime to $\ell$.
Proof. 1) The isotypic components of $\left.\Pi\right|_{H}$ are $G$-conjugate, their $G$-stabilizers are $G$ conjugate and contain $H$ hence they are equal because $G / H$ is abelian.

The smooth $R$-characters of $G$ trivial on $G(\pi)$ are in $X_{V}$ because $\chi \otimes V \simeq \operatorname{Ind}_{G_{\pi}}^{G}\left(\left.\chi\right|_{G_{\pi}} \otimes\right.$ $V_{\pi}$ ) for any smooth $R$-character $\chi$ of $G$.
2) a) For any closed subgroup $Q$ of $G$ and a smooth $R$-representation $X$ of $Q$, the representation $\operatorname{Ind}_{Q}^{G} X$ is the space of functions $f: G \rightarrow X$ satisfying $f(q g k)=q f(g)$ for $q \in Q, g \in G, k \in K_{f}$ for some open subgroup $K_{f}$ of $G$, with the action of $G$ by right translation, and $\operatorname{ind}_{Q}^{G} 1$ is the subrepresentation on the subspace of functions of compact support modulo $Q$. When $G / Q$ is compact, $\operatorname{Ind}_{Q}^{G} X=\operatorname{ind}_{Q}^{G} X$.

Let $V \supset U$ be $G$-stable subspaces with $V / U$ irreducible. We can suppose $V$ generated by an element $f$ (indeed $V^{\prime} / U^{\prime} \simeq V / U$ for the $G$-stable space $V^{\prime}$ generated by a $f \in V \backslash U$ and the kernel $U^{\prime}$ of $\left.V^{\prime} \rightarrow V / U\right)$. There is an open subgroup $K$ of $G$ which fixes $f$. We have $U \subset V \subset \operatorname{ind}_{K}^{G} 1$ and one is reduced to the case where $G / H$ is finite.
b) The proof of Lemma 2.1 remains valid with the smooth induction $\operatorname{Ind}_{H}^{G}$ which is the smooth compact induction $\operatorname{ind}_{H}^{G} 1$ because $G / H$ is compact, so that $\operatorname{ind}_{H}^{G}\left(\left.\Pi\right|_{H}\right)=$ $\Pi \otimes \operatorname{ind}_{H}^{G} 1$.
3) Any smooth character $\chi$ of $G$ trivial on $H$ such that $\operatorname{ind}_{G_{\pi}}^{G}\left(V_{\pi}\right) \simeq \operatorname{ind}_{G_{\pi}}^{G}\left(\left.\chi\right|_{G_{\pi}} \otimes V_{\pi}\right)$ is trivial on $G_{\pi}$. Indeed, restricting to $G_{\pi}$ we see that $\left.\chi\right|_{G_{\pi}} \otimes V_{\pi}$ is conjugate to $V_{\pi}$ by some $g \in G$. Restricting to $H$ gives that $\pi \simeq \pi^{g}$ so $g \in G_{\pi}$ hence $\left.\chi\right|_{G_{\pi}} \otimes V_{\pi} \simeq V_{\pi}$. As $\operatorname{Ker}(\chi)$ is open in $G$ and $G / H$ is compact, $J=\operatorname{Ker}(\chi) \cap G_{\pi}$ has finite index in $G_{\pi}$. If $\chi$ is not trivial on $G_{\pi}$ then the action of $J$ on $V_{\pi}$ is reducible. Indeed, $\operatorname{ind}_{J}^{G_{\pi}}(1)$ contains 1 and $\left.\chi\right|_{G_{\pi}}$ as subrepresentations and by Frobenius reciprocity $\operatorname{End}_{J}\left(\left.V_{\pi}\right|_{J}\right)$ is equal to $\operatorname{Hom}_{G_{\pi}}\left(V_{\pi}, \operatorname{ind}_{J}^{G_{\pi}}\left(\left.V_{\pi}\right|_{J}\right)\right)=\operatorname{Hom}_{G_{\pi}}\left(V_{\pi}, V_{\pi} \otimes \operatorname{ind}_{J}^{G_{\pi}}(1)\right)$. Hence $\operatorname{dim}\left(\operatorname{End}_{J_{\pi}}\left(\left.V_{\pi}\right|_{J}\right)\right) \geq 2$ and $\left.V_{\pi}\right|_{J}$ is reducible. But by hypothesis of multiplicity $1,\left.V_{\pi}\right|_{H}$ is irreducible hence $\left.V_{\pi}\right|_{J}$ is reducible as $H \subset J$. So $\chi$ is trivial on $G_{\pi}$.

The group $G_{\pi}$ is a subgroup $I$ of $G$ containing $H$ with $\lg \left(\left.V\right|_{I}\right)=\lg \left(\left.V\right|_{H}\right)$. If $I$ has this property, the restriction to $H$ of any irreducible component on $\left.V\right|_{I}$ is irreducible hence $I$ is contained in $G_{\pi}$.
4) follows from 3).

Remark 2.4. Assume that $\left.V\right|_{H}$ has multiplicity 1. The $G$-stabilizer of any irreducible component of $V$ is $G_{\pi}$. Denote $G_{\pi}=G_{V}$. Let $I$ be a subgroup of $G$ containing $H$. The number of orbits of $I$ in the irreducible components of $\left.V\right|_{G_{V}}$ is $\lg \left(\left.V\right|_{I}\right)$. This number is the same for $I$ and $I G_{V}$ hence $\lg \left(\left.V\right|_{I}\right)=\lg \left(\left.V\right|_{I G_{V}}\right)$. We deduce that $G_{V} \subset I$ if $\left.V\right|_{I}$ is reducible and $|G / I|$ is a prime number.

Let $\theta$ be a smooth $R$-representation of a closed subgroup $U \subset H$. We consider the property:

The functor $\operatorname{Hom}_{U}(-, \theta)$ is exact on smooth $R$-representations of $H$.
Lemma 2.5. If (2.3) is true and $\operatorname{dim}_{\operatorname{Hom}_{U}}(V, \theta)=1$, then $\left.V\right|_{H}$ has multiplicity 1 .
Proof. We denote by $m_{V}(\pi)$ the multiplicity of any irreducible smooth $R$-representation $\pi$ of $H$ in $\left.V\right|_{H}$. By (2.3),

$$
\sum_{\pi} m_{V}(\pi) \operatorname{dim} \operatorname{Hom}_{U}(\pi, \theta)=\operatorname{dim} \operatorname{Hom}_{U}(V, \theta)=1
$$

There is a single $\pi$ with $m_{V}(\pi)=\operatorname{dim} \operatorname{Hom}_{U}(V, \theta)=1$.

## 3. $p$-ADIC REDUCTIVE GROUP

We suppose now that $G$ is a $p$-adic reductive group, that is, the group of rational points $\underline{G}(F)$ of reductive connected $F$-group $\underline{G}$, where $F$ is a local non archimedean field of residual characteristic $p$, of ring of integers $O_{F}$, uniformizer $p_{F}$, maximal ideal $P_{F}$, residue field $k_{F}=O_{F} / P_{F}$ with $q$ elements, and absolute value $|x|_{F}=q^{-v a l(x)},\left|p_{F}\right|_{F}=q^{-1}$ (we do not suppose that the characteristic of $F$ is 0 ).

For an algebraic group $\underline{X}$ over $F$, we denote by the corresponding lightfacee letter $X=\underline{X}(F)$ the group its $F$-points.

Let $R$ be a field of characteristic $\operatorname{char}_{R} \neq p$. Any irreducible smooth $R$-representation of $G$ is admissible Henniart-Vignéras19, and the properties (2.1) and (2.2) hold for $G$. For (2.1) see Vignéras96, II.5.10], Vignéras22, §5], and for (2.2) see [Dat09], Dat-Helm-Kurinczuk-Moss23].

Lemma 3.1. Let $f: \underline{H} \rightarrow \underline{G}$ be an $F$-morphism of reductive connected $F$-groups. Then the subgroup $f(H)$ of $G$ is closed.

Proof. The morphism $f$ induces a constructible action of $H$ on $G$ Bernstein-Zelevinski 77, 6.15 Theorem A], in particular the group $f(H)$, which is the $H$-orbit of the unit of $G$, is locally closed (loc.cit. (6.8) Proposition), $f(H)$ is equal to its closure in $G$ (the closure of $f(H)$ in $G$ is a subgroup containing $f(H)$ as an open hence closed, subgroup). Note that $f(H)$ is open in $G$ when char $_{F}=0$ Platonov-Rapinchuk91, §3.1 Corollary1].
Theorem 3.2. Let $f: \underline{H} \rightarrow \underline{G}$ be an $F$-morphism of reductive connected $F$-groups such that $f(H)$ is a normal subgroup of $G$ of compact quotient $G / f(H)$. Then, the restriction to $f(H)$ of any irreducible admissible $R$-representation of $G$ is semisimple of finite length. Any irreducible admissible $R$-representation of $f(H)$ is contained in some irreducible admissible $R$-representation of $G$ restricted to $f(H)$, and extends to an irreducible admissible representation of some open subgroup of $G$ of finite index.

Proof. $G$ satisfies (2.1) and $f(H)$, satisfies the property (2.2) because $H$ does. Apply the results of $\$ 2.2$.

We now give two examples where we can apply Theorem 3.2,
Proposition 3.3. Let $f: \underline{H} \rightarrow \underline{G}$ be a surjective central $F$-morphism of connected reductive $F$-groups. Then, the subgroup $f(H)$ of $G$ is normal of abelian compact quotient $G / f(H)$.
Proof. There is an $F$-morphism $\kappa: \underline{G} \times \underline{G} \rightarrow \underline{H}$ such that $\kappa(f(x), f(y))=x h x^{-1} y^{-1}$ for all $x, y \in \underline{H}$ Borel-Tits72, 2.2]. So for all $u, v \in G$ we have $u v u^{-1} v^{-1}=f \circ \kappa(u, v) \in f(H)$. The subgroup $f(H)$ of $H$ is closed (Lemma 3.1), normal with abelian quotient $G / f(H)$ (loc.cit. Proposition (2.7)).

The compacity $G / H$ is stated in Silberger79 without proof and in Lemaire19, Proposition A.2.1] with indications for the proof. The idea is to reduce to a connected reductive $F$-anisotropic modulo the center $F$-group.

Let $\underline{S}$ be a maximal $F$-split subtorus of $\underline{G}$, and $\underline{B}$ a parabolic $F$-subgroup of $\underline{G}$ containing $\underline{S}$. The $\underline{G}$-centralizer $\underline{M}$ of $\underline{S}$ is compact modulo its center and is a Levi component of $\underline{B}$.

Let $\underline{U}$ the unipotent radical of $\underline{B}$. By [Borel91, 22.6]. the inverse image $\underline{S}^{\prime}$ of $\underline{S}$ in $\underline{H}$ is a maximal $F$-split torus in $\underline{H}$, and the inverse image $\underline{B}^{\prime}$ of $\underline{B}$ is a parabolic $F$-subgroup of $\underline{H}$ Put $\underline{M}^{\prime}$ for the $\underline{H}$-centralizer of $\underline{S}^{\prime}$ and $\underline{U}^{\prime}$ for the unipotent radical of $\underline{B}^{\prime}$. From loc.cit., $f$ induces a surjective central $F$-morphism $\underline{M}^{\prime} \rightarrow \underline{M}$ and an $F$-isomorphism $\underline{U}^{\prime} \rightarrow \underline{U}$. On the other hand, we have the Iwasawa decomposition $G=K B$ for an open compact subgroup $K$ of $G$. The product map $K \times B \rightarrow G$ gives a surjective map $K \times B / f\left(B^{\prime}\right) \rightarrow G / f(H)$. We have $B / f\left(B^{\prime}\right)=M / f\left(M^{\prime}\right)$, so we just need to prove the compactness of $M / f\left(M^{\prime}\right)$.

Let $X^{*}(\underline{S})$ denote the group of algebraic characters of $\underline{S}$ and $\underline{S}\left(p_{F}\right)=\operatorname{Hom}\left(X^{*}(\underline{S}), p_{F}^{\mathbb{Z}}\right)$. The subgroup $\underline{S}\left(p_{F}\right)$ of $S$ is free abelian of finite rank with a compact quotient $S / \underline{S}\left(p_{F}\right)$. On the other hand, $f$ induces a surjective $F$-morphism $\underline{S}^{\prime} \rightarrow \underline{S}$ sending $\underline{S}^{\prime}\left(p_{F}\right)$ onto a sub-lattice of $\underline{S}\left(p_{F}\right)$. Hence $S / f\left(S^{\prime}\right)$ is finite. So $M / f\left(S^{\prime}\right)$ is compact as $M / S$ is compact, a fortiori $M / f\left(M^{\prime}\right)$ is compact.

Remark 3.4. The condition that $f$ is central in Proposition 3.3 is necessary. Indeed, assume
 Frobenius ${ }^{2}$. The $F$-morphism $f$ is surjective but not central. Let $G=G L_{2}(F), G^{\prime}=$ $S L(2, F), T^{\prime}$ the diagonal torus of $G^{\prime}, U$ the group of unipotent upper triangular matrices in $G^{\prime}$. Then $f(G)=T^{\prime} \varphi\left(G^{\prime}\right)$ is closed but not normal and not cocompact in $G^{\prime}$ (as $\varphi(U)=U \cap T^{\prime} \varphi\left(G^{\prime}\right)$ and $U / \varphi(U)$ homeomorphic to $F / F^{2}$ is not compact).

Corollary 3.5. Assume $R$ algebraically closed. Let $f: \underline{H} \rightarrow \underline{G}$ be a F-morphism of connected reductive $F$-groups which induces a central $F$-isogeny $\underline{H^{\text {der }}} \rightarrow \underline{G}^{\text {der }}$ between the derived groups. Then the conclusions of Theorem 3.2 apply to $f(H)$.

Proof. The $F$-isogeny $\underline{H}^{d e r} \rightarrow \underline{G}^{d e r}$ is surjective with finite kernel contained in the center of $\underline{H}^{d e r}$ Springer98, 12.2.6]. If $\underline{Z}$ is the connected centre of $\underline{G}$, the natural map $\underline{Z} \times \underline{G}^{d e r} \rightarrow \underline{G}$ is surjective [Springer98, 8.1.6 Corollary] Hence the obvious map $\underline{Z} \times \underline{H} \rightarrow \underline{G}$ is surjective and central. Proposition 3.3 applies to $Z f(H)$. But $R$ being algebraically closed, $Z$ acts by a character in any irreducible smooth $R$-representations of $G$, and we get the corollary.

Remark 3.6. There is a more elementary proof that the restriction to $f(H)$ of any irreducible admissible $R$-representation of $G$ is semisimple of finite length in Silberger79.

## 4. Restriction to $S L_{2}(F)$ of Representations of $G L_{2}(F)$

Let $F$ be a local non archimedean field of residue field $k_{F}$ of characteristic $p$ as in $\S 3$, and $R$ an algebraically closed field of characteristic different from $p$.

Let $G=G L_{2}(F)$, and let $B$ (resp. $B^{-}$) denote the subgroup of upper (resp. lower) triangular matrices, $T=$ the subgroup of diagonal matrices, $U$ (resp. $U^{-}$) the subgroup of upper (resp. lower) triangular unipotent matrices, and $Z$ the center of $G$.

Let $G^{\prime}=S L_{2}(F)$. The subgroup $H=Z G^{\prime}$ of $G$ is closed normal of compact abelian quotient $G / Z G^{\prime}$ isomorphic via the determinant to $F^{*} /\left(F^{*}\right)^{2}$, which is a $\mathbb{F}_{2}$-vector space

[^1]of dimension [Neukirch99, Corollary 5.8]
\[

\operatorname{dim}_{\mathbb{F}_{2}} F^{*} /\left(F^{*}\right)^{2}=\left\{$$
\begin{array}{l}
2+e \text { if } \operatorname{char}_{F} \neq 2  \tag{4.1}\\
\infty \text { if } \operatorname{char}_{F} \neq 2
\end{array}
$$ \quad, where 2 O_{F}=P_{F}^{e}\right.
\]

Note that $Z G^{\prime}$ is open in $G$ if and only if $\operatorname{char}_{F} \neq 2$.
For a subset $X \subset G$, put $X^{\prime}=X \cap G^{\prime}$. Write $x=\left(x_{i, j}\right)$ a matrix in $G$ or Lie $G=M_{2}(F)$.
We fix a separable closure $F^{s c}$ of $F$ and will consider only extensions of $F$ contained in $F^{s c}$. We write $W_{F}$ for the Weil group of $F^{s c} / F$ and $\mathrm{Gal}_{F}$ for the Galois group of $F^{s c} / F$. For a field $k$, we denote by $k^{a c}$ an algebraic closure of $k$, and if $k \subset R$ we suppose $k^{a c} \subset R$.

We fix an additive $R$-character $\psi$ of $F$ trivial on $O_{F}$ but not on $P_{F}^{-1}$.
4.1. Whittaker spaces. The smooth $R$-characters of $U$ have the form

$$
\begin{equation*}
\theta_{Y}(u)=\psi \circ \operatorname{tr}(Y(u-1))=\psi\left(Y_{2,1} u_{1,2}\right), \quad u \in U \tag{4.2}
\end{equation*}
$$

for a lower triangular nilpotent matrix $Y$ in $M_{2}(F)$. The case $Y=0$ gives the trivial character of $U$, the cases with $Y \neq 0$ give the non-degenerate characters of $U$.

Notation 4.1. When $Y_{2,1}=1$ we denote $\theta_{Y}=\theta$.
The normalizer of $U$ in $G$ is $T U$. By conjugation, $U$ acts trivially on $U$ and its characters, and a diagonal matrix $t=\operatorname{diag}\left(t_{1}, t_{2}\right)$ acts on $u \in U$ by $\left(t u t^{-1}\right)_{1,2}=\left(t_{1} / t_{2}\right) u_{1,2}$. Also, $t$ acts on a lower triangular nilpotent matrix $Y$ by $\left(t Y t^{-1}\right)_{2,1}=\left(t_{2} / t_{1}\right) Y_{2,1}$. It follows that $T$ acts transitively on the non-degenerate characters of $U$, the quotient $T / Z$ acting simply transitively. By the same formulas, two non-trivial characters $\theta_{Y}$ and $\theta_{Y^{\prime}}$ of $U$ are conjugate in $G^{\prime}$ if and only if they are conjugate by an element of $T^{\prime}$ if and only if $Y_{1,2}$ and $Y_{1,2}^{\prime}$ differ by a square in $F^{*}$.

The $T$-normalizer of $\theta_{Y}$ is equal to $Z$ if $Y \neq 0$ and to $T$ if $Y=0$. The $\theta_{Y}$-coinvariants functor $\tau \mapsto W_{Y}(\tau)$ from the smooth $R$-representations $\tau$ of $U$ to the smooth $R$-representations of the $T$-normalizer of $\theta_{Y}$ is exact. A smooth $R$-representation $\tau$ of $U$ is called degenerate when $W_{Y}(\tau)=0$ for all $Y \neq 0$, and non-degenerate otherwise. A smooth $R$-representation of $G$ or of $G^{\prime}$ is called degenerate (or non-degenerate) if its restriction to $U$ is.

The finite dimensional irreducible smooth $R$-representations of $G$ are of the form $\chi \circ$ det for a smooth $R$-character $\chi$ of $F^{*}$ and are degenerate. If $\Pi$ is an infinite dimensional irreducible smooth $R$-representation of $G$. By the uniqueness of Whittaker models, $\operatorname{dim} W_{Y}(\Pi)=1$ for all $Y \neq 0\left(\right.$ Vignéras96, III.5.10] when $\left.\operatorname{char}_{R}>0\right)$.
4.2. $L$-packets. We will classify the irreducible smooth $R$-representations of $G^{\prime}$ by restricting to $G^{\prime}$ the irreducible smooth $R$-representations $\Pi$ of $G$. The set $L(\Pi)$ of (isomorphism classes of) $\left.\Pi\right|_{G^{\prime}}$ is called an $L$-packet. A parametrization along these lines was obtained when $\operatorname{char}_{F}=0$ and $\operatorname{char}_{R}=\mathbb{C}$ in Labesse-Langlands79. When char $F \neq 2$ and $\operatorname{char}_{R} \neq 2$, this question is studied for supercuspidal representations in the recent work Cui-Lanard-Lu24, §6.2 and §6.3].

Applying Lemma 2.3, Remark [2.4, Lemma 2.5, Theorem 3.2, Corollary 3.5, we have:
Any irreducible smooth $R$-representation of $G^{\prime}$ belongs to a unique $L$-packet.

For two irreducible smooth $R$-representations $\Pi_{1}, \Pi_{2}$ of $G$,

$$
\begin{equation*}
L\left(\Pi_{1}\right)=L\left(\Pi_{2}\right) \Leftrightarrow \Pi_{1}=(\chi \circ \text { det }) \otimes \Pi_{2} \text { for some } R \text {-character } \chi \circ \text { det of } G . \tag{4.4}
\end{equation*}
$$

The trivial character of $G^{\prime}$ is the unique finite dimensional irreducible smooth $R$-representation of $G^{\prime}$, it is degenerate and forms an $L$-packet $L(1)=L(\chi \circ$ det) for any smooth $R$-character $\chi$ of $F^{*}$.

If $\Pi$ is an irreducible smooth $R$-representation of $G$,
(4.5) The restriction of $\Pi$ to $G^{\prime}$ is semi-simple of finite length and multiplicity 1.

The irreducible constituents of $\left.\Pi\right|_{G^{\prime}}$ are $G$-conjugate (even $B$-conjugate as $G=B G^{\prime}$ ), and form an $L$-packet $L(\Pi)$ of cardinality the length of $\left.\Pi\right|_{G^{\prime}}$. The $G$-stabilizer of $\pi \in L(\Pi)$ does not depend on the choice of $\pi$ in $L(\Pi)$ and is denoted $G_{\Pi}$. By $\$ 2.2 .6, G_{\Pi}$ is an open normal subgroup of $G$ containing $G^{\prime} Z$, determined by the subgroup $\operatorname{det}\left(G_{\Pi}\right)$ of $F^{*}$ containing $\left(F^{*}\right)^{2}$. The order of the quotient $G / G_{\Pi} \simeq F^{*} / \operatorname{det}\left(G_{\Pi}\right)$ is a power of 2 . When $\operatorname{char}_{F} \neq 2,\left|G / G_{\Pi}\right|$ divides $\left|F^{*} /\left(F^{*}\right)^{2}\right|=2^{2+e}$ with $e$ defined in (4.1).

$$
\begin{equation*}
G_{\Pi} \text { is the largest subgroup } I \text { of } G \text { containing } H \text { such that } \lg \left(\left.\Pi\right|_{I}\right)=\lg \left(\left.\Pi\right|_{G^{\prime}}\right), \tag{4.6}
\end{equation*}
$$

$$
\begin{equation*}
\Pi=\operatorname{ind}_{G_{\Pi}}^{G} V_{\pi} \text { where } V_{\pi} \text { is the space of } \pi \tag{4.7}
\end{equation*}
$$

The cardinality of $L(\Pi)$ is $\left|G / G_{\Pi}\right|=\left|F^{*} / \operatorname{det}\left(G_{\Pi}\right)\right|$.

$$
\begin{equation*}
|L(\Pi)| \text { is a power of } 2, \text { and }|L(\Pi)| \text { divides } 2^{2+e} \text { when } \operatorname{char}_{F} \neq 2 \tag{4.8}
\end{equation*}
$$

When $p$ is odd since $\left|F^{*} /\left(F^{*}\right)^{2}\right|=4$ we deduce:
Proposition 4.2. When $p$ is odd, the cardinality of an L-packet is 1,2 or 4 .
When $p=2$ we will prove that this remains true using the local Langlands correspondence.

By class field theory, any open subgroup of $F^{*}$ of index 2 is equal to $N_{E / F}\left(E^{*}\right)$ for a unique quadratic separable extension $E / F$ of relative norm $N_{E / F}: E^{*} \rightarrow F^{*}$, and conversely. Any open subgroup of $F^{*}$ of index 4 containing $\left(F^{*}\right)^{2}$ is equal to $N_{K / F}\left(K^{*}\right)$ for a unique biquadratic separable extension $K / F$ of relative norm $N_{K / F}: K^{*} \rightarrow F^{*}$, and conversely.

When $p$ is odd, each quadratic extension of $F$ is separable and tamely ramified, and there is a unique biquadratic separable extension of $F$.

When $p=2$, if $\operatorname{char}_{F}=0$, there are $2^{e+2}$ quadratic separable extensions of $F$ and $2^{e+1}\left(2^{e+2}-1\right)$ biquadratic separable extensions of $F$ (formula (4.1)). If $\operatorname{char}_{F}=2$, there are infinitely many quadratic, resp. biquadratic, separable extensions of $F$.
Definition 4.3. If $\left|F^{*} / \operatorname{det}\left(G_{\Pi}\right)\right|=2$, resp.4, there is a (unique) quadratic, resp.biquadratic, separable extension $E / F$ such that $\operatorname{det}\left(G_{\Pi}\right)=N_{E / F}\left((E)^{*}\right)$, we denote this extension by $E_{\Pi}$.

We denote by
$X_{\Pi}$ the group of characters $\chi \circ \operatorname{det}$ of $G$ such that $\Pi \otimes(\chi \circ \operatorname{det}) \simeq \Pi$.
A character of $X_{\Pi}$ is smooth (Lemma (2.2) of trivial square. So $X_{\Pi}=\{1\}$ if $\operatorname{char}_{R}=2$.

Notation 4.4. When $\operatorname{char}_{R} \neq 2$, the non-trivial smooth $R$-characters of $F^{*}$ of trivial square are the $R$-characters $\eta_{E}$ of $F^{*}$ of kernel $N_{E / F}\left(E^{*}\right)$, for quadratic separable extensions $E / F$. The modulus $q^{ \pm v a l}$ of $F^{*}$ is equal to $\eta_{E}$ if and only if $E / F$ is unramified and $q+1=0$ in $R$.

By Lemma 2.3 and the formula (4.8),
$X_{\Pi}$ is the group of $R$-characters of $G$ trivial on $G_{\Pi}$,
When $\operatorname{char}_{R} \neq 2$, the cardinality of $L(\Pi)$ is $\left|X_{\Pi}\right|$.
It is known that $\left|X_{\Pi}\right|=1,2$ or 4 when
a) $R=\mathbb{C}$ and char ${ }_{F}=0$ [Labesse-Langlands79 [Shelstad79],
b) $\operatorname{char}_{F} \neq 2, \operatorname{char}_{R} \neq 2$ [Cui-Lanard-Lu24, Proposition 6.6].

When $\operatorname{char}_{R} \neq 2$ we will prove that $\left|X_{\Pi}\right|=1,2$ or 4 using the local Langlands correspondence, therefore $\left|L_{\Pi}\right|=1,2$ or 4 when $p=2$.

For a lower triangular matrix $Y \neq 0$, we have

$$
\sum_{\pi \in L(\Pi)} \operatorname{dim}_{R} W_{Y}(\pi)=\operatorname{dim}_{R} W_{Y}(\Pi)
$$

As $\operatorname{dim}_{R} W_{Y}(\pi)=1$, we have $\operatorname{dim}_{R} W_{Y}(\pi)=0$ or 1 , and there is a single $\pi \in L(\Pi)$ with $W_{Y}(\pi) \neq 0$.

Notation 4.5. In an $L$-packet of size $>1$, an upper index + will designate the unique element with a non-zero $\theta$-coinvariants $\pi_{\theta}$ (notation 4.1).
4.3. Representations. We denote by $\operatorname{Gr}_{R}^{\infty}(G)$ the Grothendieck group of finite length smooth $R$-representations of $G$ and by $[\tau]$ the image in $\operatorname{Gr}_{R}^{\infty}(G)$ of a finite length smooth $R$-representation $\tau$ of $G$. Similarly for $G^{\prime}$.

### 4.3.1. Parabolic induction.

The smooth parabolic induction $\operatorname{ind}_{B}^{G}(\sigma)$ of a smooth $R$-representation $(\sigma, V)$ of $T$ is the space of functions $f: G \rightarrow V$ such that $f(t u g k)=\sigma(t) f(g)$ for $t \in T, u \in U, g \in G$ and an open compact subgroup $K_{f} \subset G$, with the action of $G$ by right translation. The functor $\operatorname{ind}_{B}^{G}$ is exact with the $U$-coinvariant functor $(-)_{U}$ as left adjoint, and $(-)_{\bar{U}} \otimes \delta$ as rignt adjoint where $\delta$ is the homomorphism of $T$ :

$$
\delta(\operatorname{diag}(a, d))=q^{-\operatorname{val}(a / d)}: T \rightarrow q^{\mathbb{Z}} \quad\left(a, d \in F^{*}\right)
$$

[Dat-Helm-Kurinczuk-Moss23, Corollary 1.3]. The modulus $\left|\left.\right|_{F}\right.$ of $F^{*}$ is $q^{- \text {val }}$ and the modulus of $B$ is the inflation of $\delta$. We choose a square root $q^{1 / 2}$ of $q$ in $R^{*}$ to define the square root of $\delta$,

$$
\begin{equation*}
\nu(\operatorname{diag}(a, d))=\left(q^{1 / 2}\right)^{-\operatorname{val}(a / d)}: T \rightarrow\left(q^{1 / 2}\right)^{\mathbb{Z}} \quad\left(a, d \in F^{*}\right) \tag{4.13}
\end{equation*}
$$

and the normalized parabolic induction $i_{B}^{G}(\sigma)=\operatorname{ind}_{B}^{G}(\sigma \nu)$. For a smooth $R$-character $\chi \circ$ det of $G$ we have

$$
(\chi \circ \operatorname{det}) \otimes \operatorname{ind}_{B}^{G}(\sigma) \simeq \operatorname{ind}_{B}^{G}(\chi \otimes \sigma), \quad(\chi \circ \operatorname{det}) \otimes i_{B}^{G} \sigma \simeq i_{B}^{G}(\chi \otimes \sigma)
$$

Similarly for $G^{\prime}$, we define the parabolic induction $\operatorname{ind}_{B^{\prime}}^{G^{\prime}}$ from the smooth $R$-representation $\sigma$ of $T^{\prime}$ to those of $G^{\prime}$ and the normalized parabolic induction $i_{B^{\prime}}^{G^{\prime}}$

$$
i_{B^{\prime}}^{G^{\prime}}(\sigma)=\operatorname{ind}_{B^{\prime}}^{G^{\prime}}\left(\nu^{\prime} \sigma\right), \quad \nu^{\prime}\left(\operatorname{diag}\left(a, a^{-1}\right)\right)=q^{-\operatorname{val}(a)}: T^{\prime} \rightarrow q^{\mathbb{Z}} \quad\left(a \in F^{*}\right) .
$$

As $G=B G^{\prime}$ and $G / B$ is compact, the restriction map $\left.f \mapsto f\right|_{G^{\prime}}$ gives isomorphisms

$$
\begin{equation*}
\left.\left(\operatorname{ind}_{B}^{G}(\sigma)\right)\right|_{G^{\prime}} \rightarrow \operatorname{ind}_{B^{\prime}}^{G^{\prime}}\left(\left.\sigma\right|_{T^{\prime}}\right),\left.\quad\left(i_{B}^{G}(\sigma)\right)\right|_{G^{\prime}} \rightarrow i_{B^{\prime}}^{G^{\prime}}\left(\left.\sigma\right|_{T^{\prime}}\right) . \tag{4.14}
\end{equation*}
$$

4.3.2. Cuspidal representations of $G L_{2}(F)$.

When $\chi$ is a smooth $R$-character of $T$, $\operatorname{ind}_{B}^{G}(\chi)$ is called a principal series of $G$. An irreducible smooth $R$-representation of $G$ which is not a subquotient of a principal series, is called supercuspidal. It is called cuspidal when its $U$-coinvariants are 0 . A supercuspidal representation is cuspidal (the converse is true only when $q+1 \neq 0$ in $R$ ). The principal series and the cuspidal $R$-representations are infinite dimensional. Similarly for $G^{\prime}$.

Let $\Pi$ be an irreducible smooth $R$-representation of $G$ and $\pi \in L(\Pi)$. Then,

$$
\begin{equation*}
\Pi \text { is cuspidal if and only if } \pi \text { is cuspidal. } \tag{4.15}
\end{equation*}
$$

Indeed, $L(\Pi)$ is the $B$-orbit of $\pi$, the $U$-coinvariant functor is exact and commutes with the restriction to $G^{\prime}$. We say that $L(\Pi)$ is cuspidal if $\Pi$ is. By the formula (4.14), $L(\Pi)$ is supercuspidal if $\Pi$ is.

Let $\Pi$ be a cuspidal $R$-representation of $G$. It is the compact induction of an extended maximal simple type $(J, \Lambda)$ (Bushnell-Kutzko94] Bushnell-Henniart02] when $R=$ $\mathbb{C}$, Vignéras96, III.3-4] for general $R$ )

$$
\Pi=\operatorname{ind}_{J}^{G}(\Lambda)
$$

The group $J$ contains $Z$ and a unique maximal open compact subgroup $J^{0}$. Let $J^{1}$ be the pro-p radical of $J^{0}$. The representation $\left.\Lambda\right|_{J^{0}}$ is irreducible, equal to $\lambda=\kappa \otimes \bar{\sigma}$ where $\left.\kappa\right|_{J^{1}}$ is irreducible and $\bar{\sigma}$ is inflated from an irreducible $R$-representation $\sigma$ of $J^{0} / J^{1}$. The type ( $J, \Lambda$ ) is unique modulo $G$-conjugacy ([Bushnell-Henniart02, 15.5 Induction theorem] when $R=\mathbb{C}$, Vignéras96, III.5.3] for general $R$.

The open normal subgroup $J G^{\prime}$ of $G$ has index $\left|F^{*} / \operatorname{det}(J)\right|$, and by Mackey theory

$$
\begin{equation*}
\left.\Pi\right|_{J G^{\prime}}=\oplus_{g \in G / J G^{\prime}} \operatorname{ind}_{J g}^{J G^{\prime}} \lambda^{g} \tag{4.16}
\end{equation*}
$$

Denote $J^{\prime}, J^{0^{\prime}}, J^{1^{\prime}}$ the intersections of $J, J^{0}, J^{1}$ with $G^{\prime}$. We have $J^{\prime}=\left(J^{0}\right)^{\prime}$ and the length of

$$
\left.\left(\operatorname{ind}_{J g}^{J G^{\prime}} \lambda^{g}\right)\right|_{G^{\prime}} \simeq \operatorname{ind}_{J^{\prime g}}^{G^{\prime}}\left(\left.\lambda^{g}\right|_{J^{\prime g}}\right)
$$

is independent of $g$. By transitivity of the restriction $\left.\Pi\right|_{G^{\prime}}=\oplus_{g \in G / J G^{\prime}} \operatorname{ind}_{J^{\prime} g}^{G^{\prime}}\left(\left.\lambda^{g}\right|_{J^{\prime g}}\right)$, and

$$
\begin{equation*}
|L(\Pi)|=\left|F^{*} / \operatorname{det}(J)\right| \lg \left(\operatorname{ind}_{J^{\prime}}^{G^{\prime}}\left(\left.\lambda\right|_{J^{\prime}}\right)\right) . \tag{4.17}
\end{equation*}
$$

It follows from Lemma 2.3 3), remark 2.4 and the formula (4.16) that:
Lemma 4.6. If $\left|F^{*} / \operatorname{det}(J)\right|=2$ then $\operatorname{det}\left(G_{\Pi}\right) \subset \operatorname{det}(J)$.

[^2]Remark 4.7. We have $\operatorname{det}\left(G_{\Pi}\right)=\operatorname{det}(J) \Leftrightarrow G_{\Pi}=J G^{\prime}$. If $\left|F^{*} / \operatorname{det}(J)\right|=2$, the group $J$ determines a quadratic separable extension $E / F$ such that $\operatorname{det}(J)=N_{E / F}\left(E^{*}\right)$. The representation $\operatorname{ind}_{J^{\prime}}^{G^{\prime}}\left(\left.\lambda\right|_{J^{\prime}}\right)$ is irreducible if and only if $|L(\Pi)|=\left|F^{*} / \operatorname{det}(J)\right|$.

There is a smooth $R$-character $\chi$ of $F^{*}$ such that $\Lambda \simeq \Lambda_{0} \otimes(\chi \circ \operatorname{det})$ and $\left(J, \Lambda_{0}\right)$ is either of level 0 or of positive level. We say that the $L$-packet $L(\Pi)$ and its elements are of level 0 or of positive level accordingly.

Level 0. $J=Z G L_{2}\left(O_{F}\right), J^{0}=G L_{2}\left(O_{F}\right), J^{0} / J^{1} \simeq G L_{2}\left(k_{F}\right), \kappa=1, \sigma$ is a cuspidal $R$-representation of $G L_{2}\left(k_{F}\right), \lambda=\bar{\sigma}$. We have $\operatorname{det} J=\operatorname{val}^{-1}(2 \mathbb{Z})$, and by (4.17):

$$
\begin{equation*}
|L(\Pi)|=2 \lg \left(\left.\lambda\right|_{J^{\prime}}\right)=2 \lg \left(\left.\sigma\right|_{S L_{2}\left(k_{F}\right)}\right), \tag{4.18}
\end{equation*}
$$

because $\left.\lambda\right|_{J^{\prime}}$ is semisimple with length $\lg \left(\left.\sigma\right|_{S L_{2}\left(\mathbb{F}_{q}\right)}\right)$, and for any irreducible component $\left.\lambda^{\prime} \subset \lambda\right|_{J^{\prime}}$, the compact induction $\operatorname{ind}_{J^{\prime}}^{G^{\prime}}\left(\lambda^{\prime}\right)$ is irreducible (Henniart-Vignéras22 Corollary 4.29).

The cardinality of the cuspidal $L$-packet $L(\Pi)$ of level 0 can be computed via (4.17), (4.18), and Remark 7.4 b ) given in the appendix on the classification of the irreducible $R$-representations of $G L_{2}(k)$ and of $S L_{2}(k)$ for a finite field $k$ with char ${ }_{k} \neq \operatorname{char}_{R}$. We have two cases:
(i) $\left|F^{*} / \operatorname{det}\left(G_{\Pi}\right)\right|=2$ and $E_{\Pi} / F$ is the unramified quadratic extension.
(ii) $p$ is odd, $\operatorname{det}\left(G_{\Pi}\right)=\left(F^{*}\right)^{2}$ and $E_{\Pi} / F$ is the unique biquadratic extension. This case occurs for a unique packet $L(\Pi)$.

Proposition 4.8. When $p=2$, each level 0 cuspidal L-packet has size 2.
When $p$ is odd, there is a unique level 0 cuspidal L-packet of size 4 , the other level 0 cuspidal L-packets have size 2.

Positive Level. $J=E^{*} J^{0}$ for a quadratic separable extension $E / F, J^{0}=O_{E}^{*} J^{1}$, $J^{0} / J^{1} \simeq k_{E}^{*}, \sigma$ is an $R$-character of $k_{E}^{*}, \lambda=\kappa \otimes \sigma$ and $\left.\lambda\right|_{J^{\prime}}$ is irreducible. The representation $\lambda_{1}=\left.\lambda\right|_{J^{1}}$ is irreducible of $G$-intertwining equal to $J$, because $J$ normalizes $\lambda_{1}$ and the $G$ intertwining of $\sigma$ is already $J$ Bushnell-Henniart06, 15.1]. We have $N_{E / F}\left(E^{*}\right) \subset \operatorname{det}(J)$. If the quadratic extension $E / F$ is tamely ramified, then $\operatorname{det}(J)=N_{E / F}\left(E^{*}\right)$, because $J=E^{*} J^{1}, J^{1}=\left(1+P_{F}\right) J^{1}$ and $1+P_{F} \subset \operatorname{det}\left(E^{*}\right)=N_{E / F}\left(E^{*}\right)$.

If $p=2$ a tamely ramified quadratic extension of $F$ is unramified, and $E / F$ is unramified if and only if $\operatorname{det}(J)=\operatorname{Ker}\left((-1)^{v a l}\right)$.

If $p$ is odd, each quadratic extension of $F$ is tamely ramified.
Proposition 4.9. If $p$ is odd, each positive level cuspidal L-packet $L(\Pi)$ has size 2 and the extension $E_{\Pi}$ of $F$ is isomorphic to $E$.
Proof. ${ }^{5}$ The central subgroup $1+P_{F}$ of $J^{1}=\left(1+P_{F}\right) J^{\prime 1}$ acts by scalars, the representation $\lambda_{1}^{\prime}=\left.\lambda\right|_{J^{\prime}}$ is still irreducible of $G$-intertwining $J$, so its $G^{\prime}$-intertwining is $J^{\prime}$. The isotypic

[^3]component of $\left.\Pi\right|_{J^{1}}$ of type $\lambda_{1}$ is the space of $\lambda$, so the isotypic component of $\left.\Pi\right|_{J^{11}}$ of type $\lambda_{1}^{\prime}$ is still the space of $\lambda$. As in the proof of Henniart-Vignéras22, Corollary 4.29], we deduce that $\operatorname{ind}_{J^{\prime}}^{G^{\prime}}\left(\left.\lambda\right|_{J^{\prime}}\right)$ is irreducible. Apply Lemma 4.6.

Remark 4.10. When $p=2$ and $E / F$ is ramified, then $J^{0} \cap G^{\prime}$ is a pro-2-group. Indeed, the determinant induces a morphism $J^{0} / J^{1} \rightarrow k_{F}^{*}$ equal via the natural isomorphism $J^{0} / J^{1} \rightarrow k_{E}=k_{F}^{*}$ to the automorphism $x \rightarrow x^{2}$ on $k_{F}^{*}$. Hence $\left(J^{0}\right)^{\prime}=\left(J^{1}\right)^{\prime}$ is a pro-2group. Note also that $\Lambda$ is a character [Bushnell-Henniart02, §15].
Corollary 4.11. (Propositions $4.8,4.9$ ) When $p$ is odd, there is a unique cuspidal L-packet of size 4, and it is of level 0 . The other cuspidal L-packets have size 1 or 2.
4.3.3. Principal series of $G L_{2}(F)$. We recall the description of the normalized principal series $i_{B}^{G}(\chi)$ of $G$ for a smooth $R$-character $\chi$ of $T$.

Denote by $\chi_{1}, \chi_{2}$ the smooth $R$-characters of $F^{*}$ such that

$$
\begin{equation*}
\chi(\operatorname{diag}(a, d))=\chi_{1}(a) \chi_{2}(d), \quad\left(a, d \in F^{*}\right) \tag{4.19}
\end{equation*}
$$

and by $\chi^{w}$ the character $\chi^{w}(\operatorname{diag}(a, d))=\chi(\operatorname{diag}(d, a))$ of $T$. In particular in (4.13), $\nu^{w}=\nu^{-1}$ and $\nu / \nu^{w}=\delta$.
Proposition 4.12. (i) For two smooth $R$-characters $\chi, \chi^{\prime}$ of $T$, $\left[i_{B}^{G}(\chi)\right]$ and $\left[i_{B}^{G}\left(\chi^{\prime}\right)\right]$ are disjoint or equal, with equality if and only if $\chi^{\prime}=\chi$ or $\chi^{w}$.
(ii) The smooth dual of $i_{B^{\prime}}^{G^{\prime}}(\chi)$ is $i_{B^{\prime}}^{G^{\prime}}\left(\chi^{-1}\right)$.
(iii) $\left(i_{B}^{G}(\chi)\right)_{U}$ has dimension 2 , contains $\chi^{w}$ and has quotient $\chi$.
(iv) $\operatorname{dim} W_{Y}\left(i_{B}^{G}(\chi)\right)=1$ when $Y \neq 0$ (Vignéras96 III.5.10).
(v) $i_{B}^{G}(\chi)$ is reducible if and only if $\chi_{1} \chi_{2}^{-1}=q^{ \pm \mathrm{val}}$.
(vi) $\operatorname{ind}_{B}^{G}(1)=i_{B}^{G}\left(\nu^{-1}\right)$ contains the trivial representation 1 and

- if $q+1 \neq 0$ in $R, \lg \left(\operatorname{ind}_{B}^{G}(1)\right)=2$, in particular $\mathrm{St}=\left(\operatorname{ind}_{B}^{G} 1\right) / 1$ is irreducible (the Steinberg $R$-representation). The representation $\operatorname{ind}_{B}^{G} 1$ is semi-simple if and only if $q=1$ in $R$ (and $\operatorname{char}_{R} \neq 2$ ).
- if $q+1=0$ in $R, \lg \left(\operatorname{ind}_{B}^{G}(1)\right)=3$, $\operatorname{ind}_{B}^{G}(1)$ is indecomposable of quotient $(-1)^{\mathrm{val}} \circ$ det, and $\operatorname{ind}_{B}^{G}(1) / 1$ contains a cuspidal representation

$$
\Pi_{0}=\operatorname{ind}_{Z G L_{2}\left(O_{F}\right)}^{G} \tilde{\sigma}_{0}
$$

where $\tilde{\sigma}_{0}$ is the inflation to $Z G L\left(2, O_{F}\right)$ of the cuspidal subquotient $\sigma_{0}$ of $\operatorname{ind}_{B\left(k_{F}\right)}^{G L\left(2, k_{F}\right)} 1$ (appendix).

This is Vignéras89, Theorem3] but the proof of (i) is incomplete. What is missing is the proof that $\Pi_{0}$ occurs only in $i_{B}^{G}(\nu)$ and $i_{B}^{G}\left(\nu^{-1}\right)$ when $q+1=0$ in $R$. This is equivalent to $X_{\Pi_{0}}=\left\{1,(-1)^{\mathrm{val}} \circ \operatorname{det}\right\}$ with the notation (4.10). This follows from Remark 7.4 a) given in the appendix.

Remark 4.13. 1) The Steinberg representation St is infinite dimensional and not cuspidal.
2) When $\operatorname{char}_{R} \neq 2$, the principal series $\left[i_{B}^{G}(\chi)\right]$ are multiplicity free.

When $\operatorname{char}_{R}=2$, then $q$ is odd, $\operatorname{ind}_{B}^{G}(1)$ has length 3 , of subquotients $\Pi_{0}$ and the trivial representation 1 as a subrepresentation and a quotient.

Corollary 4.14. The non-supercuspidal irreducible smooth $R$-representations of $G$ are:

- the characters $\chi \circ$ det for the smooth $R$-characters $\chi$ of $F^{*}$,
- the principal series $i_{B}^{G}(\chi)$ for the smooth $R$-characters $\chi$ of $T$ with $\chi_{1} \chi_{2}^{-1} \neq q^{ \pm \text {val }}$.
- if $q+1 \neq 0$ in $R$, the twists $(\chi \circ \operatorname{det}) \otimes$ St of the Steinberg representation by the smooth $R$-characters $\chi$ of $F^{*}$.
- if $q+1=0$ in $R$, the twists ( $\chi \circ \operatorname{det}) \otimes \Pi_{0}$ of the cuspidal non-supercuspidal representation $\Pi_{0}$ by the smooth $R$-characters $\chi$ of $F^{*}$.

The only isomorphisms between those representations are $i_{B}^{G}(\chi) \simeq i_{B}^{G}\left(\chi^{w}\right)$ for the irreducible principal series and $(\chi \circ \operatorname{det}) \otimes \Pi_{0} \simeq\left((-1)^{\text {val }} \chi \circ\right.$ det $) \otimes \Pi_{0}$.
4.3.4. Let $\ell$ be a prime number different from $p$. An irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representation $\tau$ of $G$ or $G^{\prime}$ is integral if it preserves a lattice. It then gives by reduction modulo $\ell$ and semi-simplification a finite length semi-simple smooth $\mathbb{F}_{\ell}^{a c}$-representation, of isomorphism class (not depending of the lattice) which we write $r_{\ell}(\tau)$. The restriction from $G$ to $G^{\prime}$ from irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representations $\tilde{\Pi}$ of $G$ to finite length semi-simple smooth $\mathbb{Q}_{\ell}^{a c}$-representations of $G^{\prime}$ respects integrality and commutes with the reduction modulo $\ell$. When $\tilde{\Pi}$ is integral, then any irreducible representation $\left.\tilde{\pi} \subset \tilde{\Pi}\right|_{G^{\prime}}$ is integral, the length of the reduction $r_{\ell}(\tilde{\pi})$ modulo $\ell$ of $\tilde{\pi}$ does not depend on the choice of $\tilde{\pi}$. If $\Pi=r_{\ell}(\tilde{\Pi})$ is irreducible, we have

$$
\begin{equation*}
|L(\Pi)|=|L((\tilde{\Pi}))| \lg \left(r_{\ell}(\tilde{\pi})\right) \tag{4.20}
\end{equation*}
$$

and by formula (4.11):

$$
\begin{equation*}
\lg \left(r_{\ell}(\tilde{\pi})\right)=\left|X_{\Pi} / X_{\tilde{\Pi}}\right| \text { when } \operatorname{char}_{R} \neq 2 \tag{4.21}
\end{equation*}
$$

Proposition 4.15. Each irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\Pi$ of $G$ is the reduction modulo $\ell$ of some integral irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\Pi}$ of $G$.
Proof. Corollary 4.14 when $\Pi$ is not-cuspidal, Vignéras01 when $\Pi$ is cuspidal.
A supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\Pi}=\operatorname{ind}_{J}^{G} \tilde{\Lambda}$ of $G$ is integral if and only if $\tilde{\Lambda}$ is integral. Then, its reduction modulo $\ell$ is irreducible Vignéras89, equal to $\Pi=\operatorname{ind}_{J}^{G} \Lambda$ where $\Lambda=r_{\ell}(\tilde{\Lambda})$. The reduction modulo $\ell$ of the $L$-packet $L(\tilde{\Pi})$ is $L(\Pi)$. The reduction modulo $\ell$ respects level 0 and positive level. Conversely, any cuspidal $\mathbb{F}_{\ell}^{a c}$-representation $\Pi=\operatorname{ind}_{J}^{G} \Lambda$ of $G$ is the reduction modulo $\ell$ of an integral cuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\Pi}=\operatorname{ind}_{J}^{G} \tilde{\Lambda}$ of $G$ where $\Lambda=r_{\ell}(\tilde{\Lambda})$ Vignéras01. By the unicity of the extended maximal simple type $(J, \Lambda)$ modulo $G$ (see $\S 4.3 .2$ ), two supercuspidal integral $\mathbb{Q}_{\ell}^{a c}$-representations have isomorphic reduction modulo $\ell$ if and only if the reduction modulo $\ell$ of their extended maximal simple types are $G$-conjugate.

Any supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $G^{\prime}$ is integral, as $\tilde{\pi} \in L(\tilde{\Pi})$ where $\tilde{\Pi}$ is a supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation of $G$, and some twist of $\tilde{\Pi}$ by a character is integral. From formulas (4.18) and (4.21),

$$
\begin{equation*}
\lg \left(r_{\ell}(\tilde{\pi})\right)=|L(\sigma)| /|L(\tilde{\sigma})| \text { when } \tilde{\Pi} \text { has level } 0 \tag{4.22}
\end{equation*}
$$

Proposition 4.16. When $\tilde{\pi}$ is supercuspidal of level 0 , the length of $r_{\ell}(\tilde{\pi})$ is $\leq 2$.

When $p$ is odd and $\tilde{\pi}$ is supercuspidal, the representation $r_{\ell}(\tilde{\pi})$ is irreducible if the level of $\tilde{\pi}$ is positive or if $\ell=2$.

Any cuspidal $\mathbb{F}_{\ell}^{a c}$-representation $\pi$ of $G^{\prime}$ is the reduction modulo $\ell$ of a supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation of $G^{\prime}$, except may be when $p=2$ and $\pi$ is of positive level and $E / F$ is unramified.
Proof. For $\tilde{\Pi}$ of level 0 , one computes in the appendix the integer $\lg \left(\left.\sigma\right|_{\left.S L_{2}\left(k_{F}\right)\right)} / \lg \left(\left.\tilde{\sigma}\right|_{S L_{2}\left(k_{F}\right)}\right)\right.$, and one sees that it is equal to 1 or 2 and that there exists $\tilde{\sigma}$ such that it is 1 .

For $p$ odd, if the level of $\tilde{\pi}$ is positive we have $\lg \left(\left.\Pi\right|_{G^{\prime}}\right)=\lg \left(\left.\tilde{\Pi}\right|_{G^{\prime}}\right)$ by Proposition 4.9, hence $r_{\ell}(\tilde{\pi})$ is irreducible.

For $\ell=2$ (so $p$ is odd), if the level of $\tilde{\pi}$ is 0 , then $r_{\ell}(\tilde{\pi})$ is also irreducible by the formula (4.22) and Lemma 7.3 in the appendix.

Assume now $p=2$ (so $\ell$ is odd), $\pi$ is in a cuspidal $L$-packet $L(\Pi)$ of positive level and $E / F$ is ramified. Let $\tilde{\Pi}$ a $\mathbb{Q}_{\ell}^{a c}$-lift of $\Pi$. The reduction modulo $\ell$ from $X_{\tilde{\Pi}}$ onto $X_{\Pi}$ is injective.
Lemma 4.17. The reduction modulo $\ell$ from $X_{\tilde{\Pi}}$ onto $X_{\Pi}$ is a bijection.
Proof. Le $\chi \in X_{\Pi}, \chi \neq 1$, and $\tilde{\chi}$ the unique $\mathbb{Q}_{\ell}^{a c}$ lift of $\chi$ of order 2 . We have $\tilde{\Pi}=\operatorname{ind}_{J}^{G} \tilde{\Lambda}$ where $\tilde{\Lambda}$ is a character (Remark (4.10). We have $\Pi=\operatorname{ind}_{J}^{G} \Lambda$ where $\Lambda=r_{\ell}(\tilde{\Lambda})$ and $(J, \chi \Lambda)=$ $\left(J,{ }^{g} \Lambda\right)$ for $g \in G$ normalizing $J$. So $\tilde{\chi} \tilde{\Lambda}=\epsilon^{g} \tilde{\Lambda}$ for a $\mathbb{Q}_{\ell}^{a c}$-character $\epsilon$ of $J$ of order a power of $\ell$. So, $\left.\epsilon\right|_{J_{1}}=1$ and $\left.\epsilon\right|_{Z}=1$. As $E / F$ is ramified, the index of $Z J^{1}$ in $J$ is 2 hence $\epsilon=1$ and $\tilde{\chi} \in X_{\tilde{\Pi}}$.

The last case of the proposition follows.
We shall show that the proposition remains true for all $L$-packets. When $\operatorname{char}_{F} \neq 2$ and $\operatorname{char}_{R} \neq 2$, compare with Proposition 6.7 in [Cui-Lanard-Lu24].

### 4.4. Local Langlands $R$-correspondence for $G L_{2}(F)$.

4.4.1. By local class field theory, the smooth $R$-characters $\chi$ of $F^{*}$ identify with the smooth $R$-characters $\chi \circ \alpha_{F}$ of $W_{F}$ where $\alpha_{F}: W_{F} \rightarrow F^{*}$ is the Artin reciprocity map sending a arithmetic Frobenius Fr to $p_{F}^{-1}$ ([Bushnell-Henniart02] §29). This is the local Langlands $R$-correspondence for $G L_{1}(F)$.

A two-dimensional Deligne $R$-representation of the Weil group $W_{F}$ is a pair $(\sigma, N)$ where $\sigma$ is a two dimensional semi-simple smooth $R$-representation of the Weil group $W_{F}$ and $N$ a nilpotent $R$-endomorphism of the space of $\sigma$ with the usual requirement:

$$
\begin{equation*}
\sigma(w) N=N\left|\alpha_{F}(w)\right|_{F} \sigma(w) \text { for } w \in W_{F} \tag{4.23}
\end{equation*}
$$

Two two-dimensional Deligne $R$-representations $(\sigma, N)$ and ( $\sigma^{\prime}, N^{\prime}$ ) of $W_{F}$ are isomorphic if there exists a linear isomorphism $f: V \rightarrow V^{\prime}$ from the space $V$ of $\sigma$ to the space $V^{\prime}$ of $\sigma^{\prime}$ such that $\sigma^{\prime}(w) \circ f=f \circ \sigma(w)$ for $w \in W_{F}$ and $N^{\prime} \circ f=f \circ N$.

For a smooth $R$-character $\chi$ of $F^{*}$, the twist $(\sigma, N) \otimes\left(\chi \circ \alpha_{F}\right)$ of $(\sigma, N)$ by $\chi \circ \alpha_{F}$ is $\left(\sigma \otimes\left(\chi \circ \alpha_{F}\right), N\right)$.

When $R=\mathbb{Q}_{\ell}^{a c},(\sigma, N)$ is called integral if $\sigma$ is integral.

Remark 4.18. When $\sigma$ is irreducible we have $N=0$.
When $\sigma=\left(\chi_{1} \oplus \chi_{2}\right) \circ \alpha_{F}$, if $\chi_{1} \chi_{2}^{-1} \neq q^{ \pm \mathrm{val}}$ then $N=0$, When $N \neq 0$, we have $\left\{\chi_{1}, \chi_{2}\right\}=\left\{\chi_{i}, q^{- \text {val }} \chi_{i}\right\}$ for some $i$ and $N$ sends the $\chi_{i} \circ \alpha_{F^{-}}$-eigenspace to the $q^{- \text {val }} \chi_{i} \circ \alpha_{F^{-}}$ eigenspace or 0 . Therefore when $\chi_{1} \chi_{2}^{-1}=q^{\text {val }}$,

If $q-1 \neq 0$ and $q+1 \neq 0$ in $R$, then $N=0$ or the kernel of $N$ is the ( $\chi_{2} \circ \alpha_{F}$ )-eigenline.
If $q-1 \neq 0$ and $q+1=0$ in $R$, then $N=0$, or the kernel of $N$ is the ( $\chi_{2} \circ \alpha_{F}$ )-eigenline, or the kernel of $N$ is the $\left(\chi_{1} \circ \alpha_{F}\right)$-eigenline.

If $q-1=0$, then $N$ is any nilpotent.
The local Langlands $R$-correspondence for $G=G L_{2}(F)$ is a canonical bijection

$$
\begin{equation*}
L L_{R}: \Pi \mapsto\left(\sigma_{\Pi}, N_{\Pi}\right) \tag{4.24}
\end{equation*}
$$

from the isomorphism classes of the irreducible smooth $R$-representations $\Pi$ of $G$ onto the equivalence classes of the two-dimensional Weil-Deligne $R$-representations of $W_{F}{ }^{6}$. It identifies supercuspidal $R$-representations of $G$ and irreducible two-dimensional $R$-representations of $W_{F}$, commutes with the automorphisms of $R$ respecting a chosen square root of $q$, with the twist by smooth $R$-characters $\chi$ of $F^{*}$ :

$$
\begin{equation*}
L L_{R}(\Pi \otimes(\chi \circ \operatorname{det}))=L L_{R}(\Pi) \otimes\left(\chi \circ \alpha_{F}\right) \tag{4.25}
\end{equation*}
$$

The local Langlands complex correspondence was proved by Kutzko Bushnell-Henniart02, $\S 33]$. An isomorphism $\mathbb{C} \simeq \mathbb{Q}_{\ell}^{a c}$ and the choice of a square root of $q$ in $\mathbb{Q}_{\ell}^{a c}$ transfers $L L_{\mathbb{C}}$ to a local Langlands $\mathbb{Q}_{\ell}^{a c}$-correspondence $L L_{\mathbb{Q}_{\ell}^{a c}}$ respecting integrality. Any irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\Pi$ of $G$ lifts to a $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\Pi}$ of $G$ (Proposition 4.15) and $L L_{\mathbb{Q}_{\ell}^{a c}}$ descends to a local Langlands $\mathbb{F}_{\ell}^{a c}$-correspondence $L L_{\mathbb{F}_{\ell}^{a c}}$ compatible with reduction modulo $\ell$ in the sense of Vignéras01, §1.8.5]. The nilpotent part $N_{\Pi}$ is subtle but the semi-simple part $\sigma_{\Pi}$ is simply the reduction modulo $\ell$ of $\sigma_{\tilde{\Pi}}$,

$$
\begin{equation*}
\sigma_{\Pi}=r_{\ell}\left(\sigma_{\tilde{\Pi}}\right) \tag{4.26}
\end{equation*}
$$

The local Langlands correspondence $L L_{R}$ of $G$ over $R$ is deduced from $L L_{\mathbb{Q}_{\ell}^{a c}}$ when $\operatorname{char}_{R}=$ 0 and from $L L_{\mathbb{F}_{\ell}^{a c}}$ when $\operatorname{char}_{R}=\ell$ Vignéras97, §3.3], Vignéras01, §I.7-8]. We recall from loc.cit. a representative $\left(\sigma_{\Pi}, N_{\Pi}\right)$ of $L L_{R}(\Pi)$ for an irreducible smooth $R$-representation $\Pi$ of $G$.

Proposition 4.19. A) Let $\Pi$ be an irreducible subquotient of the un-normalized $R$-principal series $\operatorname{ind}_{B}^{G}(1)$ of $G$. Then, $\sigma_{\Pi}=\left(\left(q^{1 / 2}\right)^{-\mathrm{val}} \oplus\left(q^{1 / 2}\right)^{\mathrm{val}}\right) \circ \alpha_{F}$. We have $N_{\Pi}=0$ if
$\Pi=1$ the trivial character if $q+1 \neq 0$ in $R, \Pi=\Pi_{0}$ cuspidal if $q+1=0$ in $R$.
Otherwise $N_{\Pi} \neq 0$. When $q-1 \neq 0$ in $R$, the kernel of $N_{\Pi}$ is the
$\left(q^{1 / 2}\right)^{-\mathrm{val}} \circ \alpha_{F}$-eigenline if $q+1=0$ in $R$ and $\Pi=1$,
$\left(q^{1 / 2}\right)^{\text {val }} \circ \alpha_{F}$-eigenline if $q+1=0$ in $R$ and $\Pi=q^{v a l} \circ$ det,
$\left(q^{1 / 2}\right)^{-\mathrm{val}} \circ \alpha_{F}$-eigenline if $q+1 \neq 0$ in $R$ and $\Pi=$ St the Steinberg representation.
B) Let $\Pi$ be the irreducible normalized principal series $i_{B}^{G}(\eta)$, i.e. $\eta \neq q^{ \pm \mathrm{val}}$, with the notation of (4.29). Then $\sigma_{\Pi}=(\eta \oplus 1) \circ \alpha_{F}$ and $N_{\Pi}=0$.

[^4]C) Let $\Pi$ be a supercuspidal $R$-representation of $G$. Then $\sigma_{\Pi}$ is irreducible and $N_{\Pi}=0$.
4.4.2. For a two dimensional semi-simple smooth $R$-representation $\sigma$ of $W_{F}$, put
$$
X_{\sigma}=\left\{\text { smooth } R \text {-characters } \chi \text { of } F^{*} \text { such that }\left(\chi \circ \alpha_{F}\right) \otimes \sigma \simeq \sigma\right\}
$$

The square of each $\chi \in X_{\sigma}$ is trivial because $\operatorname{dim}_{R} \sigma=2$. We shall compute $X_{\sigma}$ when $\operatorname{char}_{R} \neq 2$. When $\operatorname{char}_{R}=2, X_{\sigma}=\{1\}$.

To a pair $(E, \xi)$ where $E$ is a quadratic separable extension of $F$ and $\xi$ is a smooth $R$-character of $E^{*}$ different from its conjugate $\xi^{\tau}$ by a generator $\tau$ of $\operatorname{Gal}(E / F)$ (i.e. $\xi$ is not trivial on $\operatorname{Ker} N_{E / F}=\left\{x / x^{\tau} \mid x \in E^{*}\right\}$ ), is associated a 2-dimensional irreducible smooth $R$-representation of $W_{F}$

$$
\sigma(E, \xi)=\operatorname{ind}_{W_{E}}^{W_{F}}\left(\xi \circ \alpha_{E}\right)
$$

The character $\xi$ is unique modulo $\operatorname{Gal}(E / F)$-conjugation.
When $\operatorname{char}_{R} \neq 2$, let $\sigma$ be a two dimensional irreducible smooth $R$-representation of $W_{F}$ and $E / F$ a quadratic separable extension. By Clifford's theory Bushnell-Henniart06, §10, 41.3 Lemma], with the notation (4.4)

$$
\eta_{E} \in X_{\sigma} \Leftrightarrow \sigma \simeq \sigma(E, \xi) \text { for some } \xi
$$

Proposition 4.20. When $\operatorname{char}_{R} \neq 2$,

$$
X_{\sigma(E, \xi)}= \begin{cases}\left\{1, \eta_{E}\right\} & \text { if }\left(\xi / \xi^{\tau}\right)^{2} \neq 1 \\ \left\{1, \eta_{E}, \eta_{E^{\prime}}, \eta_{E} \eta_{E^{\prime}}\right\} & \text { if }\left(\xi / \xi^{\tau}\right)^{2}=1, \xi / \xi^{\tau}=\eta_{E^{\prime}} \circ N_{E / F}\end{cases}
$$

For each biquadratic separable extension $K / F$, there exists a two dimensional irreducible smooth $R$-representation $\sigma$ of $W_{F}$, unique modulo twist by a character, with $X_{\sigma}=\left\{1, \eta_{E}, \eta_{E^{\prime}}, \eta_{E^{\prime \prime}}\right\}$ for the three quadratic extensions $E, E^{\prime}, E^{\prime \prime}$ of $F$ contained in $K$.

Proof. $\chi \in X_{\sigma(E, \xi)} \Leftrightarrow\left(\chi \circ \alpha_{F}\right) \otimes \operatorname{ind}_{W_{E}}^{W_{F}}\left(\xi \circ \alpha_{E}\right) \simeq \operatorname{ind}_{W_{E}}^{W_{F}}\left(\xi \circ \alpha_{E}\right) \Leftrightarrow \xi\left(\chi \circ N_{E / F}\right)=\xi$ or $\xi^{\tau}$. $\xi\left(\chi \circ N_{E / F}\right)=\xi \Leftrightarrow \chi$ is trivial on $N_{E / F}\left(E^{*}\right)$, so $\chi=1$ or $\eta_{E}$.
$\xi\left(\chi \circ N_{E / F}\right)=\xi^{\tau} \Leftrightarrow \chi=\eta_{E^{\prime}}$ for a quadratic separable extension $E^{\prime} \neq E$ of $F$, as $\chi^{2}=1$.
In the latter case, the order of $\xi^{\tau} / \xi$ is $2, \xi^{\tau} / \xi$ is fixed by $\tau$ and determines $\chi$ up to multiplication by $\eta_{E}$. Let $K / F$ be the biquadratic extension generated by $E$ and $E^{\prime}$ and $E^{\prime \prime} / F$ the third quadratic extension contained in $K / F$. We have $\eta_{E} \eta_{E^{\prime}}=\eta_{E^{\prime \prime}}$.

The unicity in the second assertion follows from the fact that for two smooth $R$-characters $\xi_{1}, \xi_{2}$ of $E^{*}, \xi_{1}^{\tau} / \xi_{1}=\xi_{2}^{\tau} / \xi_{2} \Leftrightarrow \xi_{1}=\xi_{2}\left(\chi \circ N_{E / F}\right)$ for a smooth $R$-character $\chi$ of $F^{*}$. The existence when $p$ is odd follows. When $E / F$ is unramified, the character $\xi$ of $E^{*}$ trivial on $1+p_{F} O_{E}, \xi\left(p_{F}\right)=-1$ and $\chi(x)=x^{(q+1) / 2}$ if $x^{q^{2}-1}=1$, satisfies $\xi^{\tau} / \xi \neq 1$ and $\left(\xi^{\tau} / \xi\right)^{2}=1$ hence $\xi^{\tau} / \xi=\eta_{E^{\prime}} \circ N_{E / F}=\eta_{E} \eta_{E^{\prime}} \circ N_{E / F}$ for $E^{\prime} / F$ ramified. If $p$ is odd, there is a unique biquadratic extension $K / F$ of $F$. When $p=2$, given a quadratic separable extension $E^{\prime} / F$ different from $E / F$, there exists a smooth $R$-character $\xi$ of $F$ such that
$\xi^{\tau} / \xi=\eta_{E^{\prime}} \circ N_{E / F}=\eta_{E} \eta_{E^{\prime}} \circ N_{E / F}$, because $\operatorname{char}_{R} \neq 2$, and this is known when $R=\mathbb{C}$ (Bushnell-Henniart06, §41] when $p \neq 2$, but the proof does not use $p \neq 2)^{7}$. 8

Remark 4.21. Let $\Pi$ be a supercuspidal $R$-representation of $G$. Then $\Pi$ has level 0 (resp. $L(\Pi)$ has level 0 ), if and only if $\sigma_{\Pi}=\operatorname{ind}_{W_{E}}^{W_{F}}\left(\xi \circ \alpha_{E}\right)$ where $E / F$ is quadratic unramified and $\xi$ is a tame character of $E^{*}$ (resp. $\xi^{\tau} / \xi$ is a tame character of $E^{*}$ where $\tau$ is the non-trivial element of $\operatorname{Gal}(E / F))$.

Remark 4.22. Assume $\operatorname{char}_{R} \neq 2$. Let $\sigma=\chi_{1} \circ \alpha_{F} \oplus \chi_{2} \circ \alpha_{F}$ be a reducible two dimensional semi-simple smooth $R$-representation of $W_{F}$. Then $\chi \circ \alpha_{F} \in X_{\sigma} \Leftrightarrow\left\{\chi \chi_{1}, \chi \chi_{2}\right\}=$ $\left\{\chi_{1}, \chi_{2}\right\} \Leftrightarrow \chi=1$ or $\chi \chi_{1}=\chi_{2}, \chi \chi_{2}=\chi_{1} \Leftrightarrow \chi=1$ or $\chi=\chi_{2} \chi_{1}^{-1}, \chi^{2}=1$. If $\chi_{1} \chi_{2}^{-1}=\eta_{E}$ for a quadratic separable extension $E / F$, then $X_{\sigma}=\left\{1, \eta_{E}\right\}$. Otherwise, $X_{\sigma}=\{1\}$.
4.4.3. Application to the cuspidal $L$-packets.

For a two dimensional Weil-Deligne $R$-representation $(\sigma, N)$ of $W_{F}$, put $X_{(\sigma, N)}$ for the group of $\chi \in X_{\sigma}$ such that there exists an isomorphism of $\chi \otimes \sigma$ onto $\sigma$ preserving $N$. For any irreducible $R$-representation $\Pi$ of $G$, applying the formulas (4.24), (4.25) and (4.11) we obtain :

$$
\begin{equation*}
X_{\Pi}=\left\{\chi \circ \operatorname{det} \mid \chi \in X_{\left(\sigma_{\Pi}, N_{\Pi}\right)}\right\} \tag{4.27}
\end{equation*}
$$

When $\operatorname{char}_{R} \neq 2$, the cardinality of the $L$-packet $L(\Pi)$ is $\left|X_{\sigma_{\Pi}}\right|$.
Proposition 4.23. 1) When $\operatorname{char}_{R} \neq 2$, we have:

- The cardinality of a cuspidal L-packet is 1,2 or 4 .
- The map $L(\Pi) \mapsto E_{\Pi}$ is a bijection from the cuspidal L-packets of size 4 to the biquadratic separable extensions of $F$.

2) There is a bijection from the cuspidal L-packets of size 4 to the biquadratic separable extensions of $F$, sending the unique cuspidal L-packet of size 4 to the unique biquadratic separable extension of $F$ when char $_{R}=2$, and equal to the map $L(\Pi) \mapsto E_{\Pi}$ of Proposition 4.23 when char $_{R} \neq 2$.

Proof. a) Assume char ${ }_{R} \neq 2$. If $\Pi$ is cuspidal and $X_{\Pi} \neq\{1\}$ then $\eta_{E} \in X_{\Pi}$ for some quadratic separable extension $E / F, \sigma_{\Pi}=\sigma(E, \xi)$ for some $\xi$ and $\left|X_{\sigma(E, \xi)}\right|=2$ or 4 by Proposition 4.20. When $p=2$ then the map is a bijection by Proposition 4.20 via the local Langlands correspondence.
b) Assume $p$ is odd $\left(\operatorname{and} \operatorname{char}_{R} \neq p\right)$. There is an unique biquadratic separable extension of $F$ and an unique cuspidal $L$-packet of size 4 (Corollary 4.11).
c) As $p$ is odd when $\operatorname{char}_{R}=2$, the proposition follows from a) and b).

When $R=\mathbb{F}_{\ell}^{a c}$ and $\ell \neq p$, it is well known that an irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\sigma$ of $W_{F}$ lifts to an integral irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\sigma}$ of $W_{F}$ of dimension 2

[^5]9. The order of $X_{\tilde{\sigma}}$ is smaller or equal to the order of $X_{\sigma}$. We give now all the cases where the orders are different.

Theorem 4.24. Assume $\ell \neq 2$.

1) Let $\tilde{\sigma}$ be a lift to $\mathbb{Q}_{\ell}^{a c}$ of a two-dimensional irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\sigma$ of $W_{F}$. The cardinalities of $X_{\sigma}$ and of $X_{\tilde{\sigma}}$ are different if and only if $\left|X_{\sigma}\right|=4,\left|X_{\tilde{\sigma}}\right|=2$, and this happens if and only if

$$
p=2, \ell \text { divides } q+1, \tilde{\sigma}=\operatorname{ind}_{W_{E}}^{W_{F}}\left(\tilde{\xi} \circ \alpha_{E}\right)
$$

where $E / F$ is a quadratic unramified extension, $\tilde{\xi}$ a smooth $\mathbb{Q}_{\ell}^{a c}$-character of $E^{*}$ such that
(i) the order of $\tilde{\xi}^{\tau} / \tilde{\xi}$ on $1+P_{E}$ is 2 where $\operatorname{Gal}(E / F)=\{1, \tau\}$.
(ii) $\tilde{\xi}(b) \neq 1, \tilde{\xi}(b)^{\ell^{s}}=1$ for a root of unity $b \in E^{*}$ of order $q+1$, and $s$ is a positive integer such that $\ell^{s}$ divides $q+1$.
2) Each irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\sigma$ of $W_{F}$ of dimension 2 admits a lift $\tilde{\sigma}$ to $\mathbb{Q}_{\ell}^{a c}$ such that $\left|X_{\tilde{\sigma}}\right|=\left|X_{\sigma}\right|$.
Proof. 1) Let $\Pi$ be the supercuspidal smooth $\mathbb{F}_{\ell}^{a c}$-representation of $G$ and $\tilde{\Pi}$ the integral supercuspidal smooth $\mathbb{Q}_{\ell}^{a c}$-representation of $G$ lifting $\Pi$ such that $\sigma=\sigma_{\Pi}, \tilde{\sigma}=\sigma_{\tilde{\Pi}}$ by the Langlands correspondence (4.24). We have $\left|X_{\Pi}\right|=\left|X_{\sigma}\right|,\left|X_{\tilde{\Pi}}\right|=\left|X_{\tilde{\sigma}}\right|$ (4.27). By Proposition 4.16, $\left|X_{\sigma}\right|=\left|X_{\tilde{\sigma}}\right|$ or $2\left|X_{\tilde{\sigma}}\right|$, except may be when $p=2$ and $\tilde{\Pi}$ has positive level. In this exceptional case, $\eta_{E} \in X_{\tilde{\Pi}}$. By Proposition 4.22, $\left|X_{\sigma}\right|$ and $\left|X_{\tilde{\sigma}}\right|$ are equal to 1,2 or 4 . Therefore, $\left|X_{\sigma}\right| \neq\left|X_{\tilde{\sigma}}\right|$ is equivalent to $\left|X_{\sigma}\right|=4$ and $\left|X_{\tilde{\sigma}}\right|=2$.

When $\left|X_{\sigma}\right|=4$ and $\left|X_{\tilde{\sigma}}\right|=2, \sigma=\operatorname{ind}_{W_{E}}^{W_{F}} \xi, \tilde{\sigma}=\operatorname{ind}_{W_{E}}^{W_{F}} \tilde{\xi}$ for a quadratic unramified extension $E / F$, an integral smooth $\mathbb{Q}_{\ell}^{a c}$-character $\tilde{\xi}$ of $E^{*}$, of reduction $\xi$ modulo $\ell$, with $\xi / \xi^{\tau} \neq 1$ where $\tau$ is the generator $\tau$ of $\operatorname{Gal}(E / F)$, and $\left(\xi / \xi^{\tau}\right)^{2}=1$. This implies $\left(\tilde{\xi} / \tilde{\xi}^{\tau}\right)^{2}=1$ on $p_{F}^{\mathbb{Z}}\left(1+P_{E}\right)$ because $\ell \neq p$. We have $E^{*}=p_{F}^{\mathbb{Z}}\left(1+P_{E}\right) \mu_{E}$ where $\mu_{E}=\left\{x \in E^{*} \mid x^{q^{2}-1}=1\right\}$. We have $\tau(x)=x^{q}$ if $x \in \mu_{E}$. The group $\left\{x^{q-1} \mid x \in \mu_{E}\right\}$ is generated by an arbitrary root of unity $b \in E^{*}$ of order $q+1$. So

$$
\left(\tilde{\xi} / \tilde{\xi}^{\tau}\right)^{2}=1 \Leftrightarrow \tilde{\xi}(b)^{2}=1 \Leftrightarrow\left|X_{\tilde{\sigma}}\right|=4, \quad\left(\tilde{\xi} / \tilde{\xi}^{\tau}\right)^{2} \neq 1 \Leftrightarrow \tilde{\xi}(b)^{2} \neq 1 \Leftrightarrow\left|X_{\tilde{\sigma}}\right|=2 .
$$

In the exceptional case, $p=2$ hence $\ell$ is odd and $\xi(b)^{2}=1$ implies $\xi(b)=1$ (and conversely), or equivalently, the order of $\tilde{\xi}(b)$ is a power of $\ell$ dividing $q+1$. There exists a lift $\tilde{\xi}$ of $\xi$ such that $\tilde{\xi}(b) \neq 1$ if and only if $\ell$ divides $q+1$.
2) Given a positive integer $s$, each element $x \in\left(\mathbb{F}_{\ell}^{a c}\right)^{*}, x \neq 1$, is the reduction modulo $\ell$ of an element $\tilde{x} \in\left(\mathbb{Z}_{\ell}^{a c}\right)^{*}$ such that $\tilde{x}^{\ell^{s}} \neq 1$.

Corollary 4.25. 1) The reduction modulo $\ell$ of a supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $G^{\prime}$ has length $\leq 2$. It has length 2 if and only if

$$
p=2, \ell \text { divides } q+1, \sigma_{\Pi}=\operatorname{ind}_{W_{E}}^{W_{F}}\left(\tilde{\xi} \circ \alpha_{E}\right)
$$

where $\tilde{\pi} \in L(\tilde{\Pi}), E / F$ is unramified, and $\tilde{\xi}$ is a smooth $\mathbb{Q}_{\ell}^{a c}$-character of $E^{*}$ such that

[^6](i) the order of $\tilde{\xi}^{\tau} / \tilde{\xi}$ on $1+P_{E}$ is 2 where $\operatorname{Gal}(E / F)=\{1, \tau\}$.
(ii) $\tilde{\xi}(b) \neq 1, \tilde{\xi}(b)^{\ell^{s}}=1$ for a root of unity $b \in E^{*}$ of order $q+1$, and $\ell^{s}$ divides $q+1$.
2) Each cuspidal $\mathbb{F}_{\ell}^{a c}$-representation $\pi$ of $G^{\prime}$ is the reduction modulo $\ell$ of an integral supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $G^{\prime}$.

Proof. When $\ell \neq 2$, the corollary follows from Theorem 4.24 and the formula (4.21) via the local Langlands correspondence.

When $\ell=2$ so $p$ is odd, the reduction modulo 2 of any supercuspidal $\mathbb{Q}_{2}^{a c}$-representation of $G^{\prime}$ is irreducible (Proposition 4.16), and the corollary holds also.

Remark 4.26. Assume $p \neq 2$. A pair $(E, \theta)$ where $E / F$ is a quadratic extension of $F$ and $\theta$ is a smooth $R$-character of $E^{*}$, is called admissible (Bushnell-Henniart06 §18.2) if :

1) $\theta$ does not factorize through $N_{E / F}$ (equivalently is regular with respect to $\operatorname{Gal}(E / F)$ )
2) If $\left.\theta\right|_{1+P_{E}}$ does factorize through $N_{E / F}$ (equivalently is invariant under $\operatorname{Gal}(E / F)$ ), then $E / F$ is unramified.

To an admissible pair $(E, \theta)$ is associated the two-dimensional irreducible $R$-representation $\sigma(E, \theta)=\operatorname{ind}_{W_{E}}^{W_{F}}\left(\theta \circ \alpha_{E}\right)$ of $W_{F}$, and when $R=\mathbb{C}$ an explicitly constructed supercuspidal representation $\pi(E, \theta)$ of $G$ (loc.cit. §19). Isomorphism classes of supercuspidal complex representations of $G$, are parametrized by isomorphism classes of admissible pairs $(E, \theta)$ (loc.cit. $\S 20.2$ ). The Langlands local correspondence sends $\pi(E, \theta)$ to $\sigma(E, \mu \theta)$ where the explicit "rectifyer" $\mu$ is a tame character of $E^{*}$ depending only on $\left.\theta\right|_{1+P_{E}}$. As the Langlands correspondence is compatible with automorphisms of $\mathbb{C}$ preserving $\sqrt{q}$, the previous classification in terms of admissible pairs transfers to $R$-representations where $R$ is an algebraically closed field of characteristic 0 (given a choice of square root of $q$ in $R$ ). The classification and correspondence for $R=\mathbb{Q}_{\ell}^{a c}$ reduce modulo $\ell \neq p$ (the integrality property for a pair $(E, \theta)$ is that $\theta$ takes integral values) to get a similar classification of supercuspidal $\mathbb{F}_{\ell}^{a c}$-representations in terms of admissible pairs. The integral admissible pairs over $\mathbb{Q}_{\ell}^{a c}$ that do not reduce to admissible pairs over $\mathbb{F}_{\ell}^{a c}$, yield under reduction to cuspidal but not supercuspidal $\mathbb{F}_{\ell}^{a c}$-representations.
4.5. Principal series. Notations of $\S 4$. We identify a smooth $R$-character $\eta$ of $T^{\prime}$ with a $R$-character of $F^{*}$ and of $T$ by:

$$
\begin{equation*}
\eta(\operatorname{diag}(a, d))=\eta\left(\operatorname{diag}\left(a, a^{-1}\right)\right)=\eta(a) \quad\left(a, d \in F^{*}\right) \tag{4.29}
\end{equation*}
$$

Proposition 4.12 describes $i_{B}^{G}(\eta)$. The transfer of the properties (i) to (iv) to

$$
i_{B^{\prime}}^{G^{\prime}}(\eta)=\left.\left(i_{B}^{G}(\eta)\right)\right|_{G^{\prime}}
$$

is easy and gives:
(i) For smooth $R$-characters $\eta, \eta^{\prime}$ of $F^{*},\left[i_{B^{\prime}}^{G^{\prime}}(\eta)\right]$ and $\left[i_{B^{\prime}}^{G^{\prime}}\left(\eta^{\prime}\right)\right]$ are disjoint if $\eta^{\prime} \neq \eta^{ \pm 1}$, and equal if $\eta^{\prime}=\eta^{ \pm 1}$.
(ii) The smooth dual of $i_{B^{\prime}}^{G^{\prime}}(\eta)$ is $i_{B^{\prime}}^{G^{\prime}}\left(\eta^{-1}\right)$.
(iii) $\left(i_{B^{\prime}}^{G^{\prime}}(\eta)\right)_{U}$ has dimension 2, contains $\eta^{-1}$ and $\eta$ is a quotient.
(iv) $\operatorname{dim} W_{Y}\left(i_{B^{\prime}}^{G^{\prime}}(\eta)\right)=1$ for all $Y \neq 0$.

The transfer of the properties (v) and (vi) is harder.

Proposition 4.27.
(i) $i_{B^{\prime}}^{G^{\prime}}(\eta)$ is reducible if and only if $\eta=q^{ \pm \mathrm{val}}$, or $\eta \neq 1$ and $\eta^{2}=1$.
(ii) When $\operatorname{char}_{R} \neq 2$, $i_{B^{\prime}}^{G^{\prime}}\left(\eta_{E}\right)$ is semi-simple of length 2 , when $E / F$ is a quadratic separable extension, which is ramified if $q+1=0$ in $R$.
(iii) When $\operatorname{char}_{R}=2$, the only reducible principal series is $i_{B^{\prime}}^{G^{\prime}}(1)=\operatorname{ind}_{B^{\prime}}^{G^{\prime}}(1)$.
(iv) The length of $i_{B^{\prime}}^{G^{\prime}}\left(q^{-\mathrm{val}}\right)$ and of $i_{B^{\prime}}^{G^{\prime}}\left(q^{\mathrm{val}}\right)=\operatorname{ind}_{B^{\prime}}^{G^{\prime}}(1)$ is

$$
\lg \left(\operatorname{ind}_{B^{\prime}}^{G^{\prime}} 1\right)= \begin{cases}2 & \text { if } q+1 \neq 0 \text { in } R \\ 4 & \text { if } q+1=0 \text { in } R \text { and } \operatorname{char}_{R} \neq 2 \\ 6 & \text { if } \operatorname{char}_{R}=2\end{cases}
$$

Note that $\operatorname{char}_{R}=2$ implies $q+1=0$ in $R$.
Proof. We show (i) (ii) and (iii).
If $i_{B}^{G}(\eta)$ is reducible, then its restriction $i_{B^{\prime}}^{G^{\prime}}(\eta)$ to $G^{\prime}$ is reducible. By Proposition 4.12, $i_{B}^{G}(\eta)$ is reducible if and only if $\eta=q^{ \pm \text {val }}$.

Assume $i_{B}^{G}(\eta)$ irreducible, i.e. $\eta \neq q^{ \pm \mathrm{val}}$. If $\operatorname{char}_{R} \neq 2$, we have $X_{i_{B}^{G}(\eta)}=2$ if and only $\eta \neq 1$ and $\eta^{2}=1$ by the Langlands correspondence and Remark 4.2210. We have $\eta \neq 1, \eta^{2}=1$ if and only if $\eta=\eta_{E}$ for a quadratic separable extension $E / F$, which is ramified if $q+1=0$ in $R$ (notation 4.4) as $\eta \neq q^{ \pm \mathrm{val}}$. If $\operatorname{char}_{R}=2$, then $p$ is odd, $\eta \neq 1$, and $i_{B^{\prime}}^{G^{\prime}}(\eta)$ is irreducible. Indeed, the irreducible components of $i_{B^{\prime}}^{G^{\prime}}(\eta)$ are $B$-conjugate ( $\S 6.2 .1)$. They give a partition of the set of irreducible components of $\left.\left(i_{B^{\prime}}^{G^{\prime}}(\eta)\right)\right|_{B^{\prime}}$. The character $\eta$ appears with multiplicity 1 as $\eta \neq \eta^{-1}$, but as it is fixed by $B$, the partition is trivial, i.e. $i_{B^{\prime}}^{G^{\prime}}(\eta)$ is irreducible.

We show (iv) ${ }^{11}$ When $q+1 \neq 0$ in $R$, the restriction to $G^{\prime}$ of the Steinberg representation St of $G$ is irreducible, otherwise it would contain a cuspidal representation as $\operatorname{dim}_{R} \mathrm{St}_{U}=1$ which is impossible by (4.15). When $q+1=0$ in $R$, the cuspidal $R$-representation $\Pi_{0}$ (see Proposition(4.12) is induced from the inflation to $Z G L_{2}\left(O_{F}\right)$ of a cuspidal $R$-representation $\sigma_{0}$ of $G L_{2}\left(k_{F}\right)$. By (4.18), $\lg \left(\left.\Pi_{0}\right|_{G^{\prime}}\right)=2 \lg \left(\left.\sigma_{0}\right|_{S L_{2}\left(k_{F}\right)}\right)$. The representation $\left.\sigma_{0}\right|_{S L_{2}\left(k_{F}\right)}$ is irreducible if $\operatorname{char}_{R} \neq 2$, and has length 2 if $\operatorname{char}_{R}=2$ (Appendix).

Corollary 4.28. The non-supercuspidal smooth $R$-representations of $G^{\prime}$ are:
The trivial character.
If $q+1 \neq 0$ in $R$, the Steinberg $R$-representation st $=\left.\mathrm{St}\right|_{G^{\prime}}$.
The principal series $i_{B^{\prime}}^{G^{\prime}}(\eta)$ for the smooth $R$-characters $\eta$ of $F^{*}$ with $\eta \neq q^{ \pm \text {val }}$ and $\eta \neq \eta_{E}$ for any quadratic separable extension $E / F$.

If $\operatorname{char}_{R} \neq 2$, the two irreducible components $\pi_{E}^{ \pm}$of $i_{B^{\prime}}^{G^{\prime}}\left(\eta_{E}\right)$ for a quadratic separable extension $E / F$, which is ramified if $q+1=0$ in $R$.

If $\operatorname{char}_{R} \neq 2$ and $q+1=0$ in $R$, the two irreducible components of $\left.\Pi_{0}\right|_{G^{\prime}}$.
If $\operatorname{char}_{R}=2$, the four irreducible components of $\left.\Pi_{0}\right|_{G^{\prime}}$.

[^7]The only isomorphisms between those representations are $i_{B^{\prime}}^{G^{\prime}}(\eta) \simeq i_{B^{\prime}}^{G^{\prime}}\left(\eta^{-1}\right)$ for the irreducible principal series.

We get for non supercuspidal $L$-packets:
Proposition 4.29. When $q+1=0$ in $R$, there is a unique cuspidal non-supercuspidal L-packet. Its size is $\left\{\begin{array}{ll}2 & \text { if } \operatorname{char}_{R} \neq 2 \\ 4 & \text { if } \operatorname{char}_{R}=2\end{array}\right.$.

When $\operatorname{char}_{R}=2$, every non-cuspidal L-packet is a singleton.
When $\operatorname{char}_{R} \neq 2$, the non-cuspidal L-packets are singletons or of size 2 . Those of size 2 are in bijection with the isomorphism classes of the quadratic separable extensions of $F$.

This proposition and Corollary 4.11 imply :
Corollary 4.30. The L-packets of size 4 are cuspidal.
We consider now the reduction modulo a prime number $\ell \neq p$. A non-cuspidal irreducible $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $G^{\prime}$ is integral except when $\tilde{\pi} \simeq i_{B^{\prime}}^{G^{\prime}}(\tilde{\eta})$ for a non-integral smooth $\mathbb{Q}_{\ell}^{a c}$-character $\tilde{\eta}$ of $F^{*}$. When $\tilde{\pi}$ is integral, we deduce from Corollary 4.28 the length of the reduction $r_{\ell}(\tilde{\pi})$ modulo $\ell$ of $\tilde{\pi}$.

Proposition 4.31. 1) The reduction $r_{\ell}(\tilde{\pi})$ modulo $\ell$ of $\tilde{\pi}$ irreducible non-cuspidal and integral is irreducible with the exceptions:

If $\ell=2, \lg \left(r_{\ell}(\tilde{s} t)\right)=5, \lg \left(r_{\ell}\left(\tilde{\pi}_{E}^{ \pm}\right)\right)=3, \lg \left(r_{\ell}\left(i_{B^{\prime}}^{G^{\prime}}(\tilde{\eta})\right)\right)=6$ for $\tilde{\eta}$ of order a finite power of $\ell$.

If $\ell \neq 2$ and $\ell$ divides $q+1, \lg \left(r_{\ell}(\tilde{s} t)\right)=3, \lg \left(r_{\ell}\left(i_{B^{\prime}}^{G^{\prime}}(\tilde{\eta})\right)\right)=4$ for $\tilde{\eta}$ of order a finite power of $\ell, \lg \left(r_{\ell}\left(i_{\tilde{B}} G^{\prime}(\tilde{\eta})\right)\right)=2$ if $\tilde{\eta}=\tilde{\eta}_{E} \tilde{\xi}$, for a ramified quadratic separable extension $E / F$ and a character $\tilde{\xi}$ of order a power of $\ell$.
2) Each non-cuspidal irreducible $\mathbb{F}_{\ell}^{a c}$-representation of $G^{\prime}$ is the reduction modulo $\ell$ of an integral non-cuspidal irreducible $\mathbb{Q}_{\ell}^{a c}$-representation of $G^{\prime}$.

## 5. Local Langlands $R$-correspondence for $S L_{2}(F)$

5.0.1. If $(\sigma, N)$ is a two-dimensional Deligne $R$-representation of the Weil group $W_{F}$ (\$4.4.1), a choice of a basis of the space of $\sigma$ gives a Deligne morphism of $W_{F}$ into $G L_{2}(R)$ 12. In this way equivalence classes of two-dimensional Deligne $R$-representation of $W_{F}$ identify with Deligne morphism of $W_{F}$ into $G L_{2}(R)$, up to $G L_{2}(R)$-conjugacy.

A Deligne morphism of $W_{F}$ into $P G L_{2}(R)$ is a pair $(\sigma, N)$ where $\sigma: W_{F} \rightarrow P G L_{2}(R)$ is a smooth morphism, semisimple in the sense that if $\sigma\left(W_{F}\right)$ is in a parabolic subgroup $P$ then it is in a Levi of $P$, and $N$ is a nilpotent 13 element in $\operatorname{Lie}\left(P G L_{2}(R)\right)$ with the usual requirement (4.23). We say that $(\sigma, N)$ is irreducible if $\sigma\left(W_{F}\right)$ is not contained in a proper parabolic subgroup (that means that $N=0$ and the inverse image of $\sigma\left(W_{F}\right)$ in $G L_{2}(R)$

[^8]acts irreducibly on $R^{2}$ ). The question arises whether a Deligne morphism $(\sigma, N)$ of $W_{F}$ into $P G L_{2}(R)$ lifts to a two-dimensional Weil-Deligne $R$-representation.

When $(\sigma, N)$ is reducible, we may assume that $\sigma$ takes value in the diagonal torus of $P G L_{2}(R)$, and that $N$ is upper triangular. The map $x \rightarrow \operatorname{diag}(x, 1)$ modulo scalars is an isomorphism from $R^{*}$ to this torus, so $\sigma$ comes from an $R$-character $\chi$ of $W_{F}$, and $\sigma$ lifts to the two-dimensional $\chi \oplus 1$. That deals with the case where $N=0$. When $N \neq 0$, then $(\sigma, N)$ lifts to $\left(q^{-\mathrm{val}} \oplus 1, N\right)$.

The following lemma answers the question, more generally for irreducible Deligne morphisms of $W_{F}$ into $P G L_{n}(R)$ for integers $n \geq 2$ (the definitions in the first alinea for $n=2$ generalize to $n>2$ ).
Lemma 5.1. Any irreducible smooth morphism $\rho: W_{F} \rightarrow P G L_{n}(R)$ has finite image and its natural extension to $\mathrm{Gal}_{F}$ lifts to an ireducible smooth $R$-representation of $\mathrm{Gal}_{F}$ of dimension $n$.

Proof. Because the inertia group $I_{F}$ of $W_{F}$ is profinite and $\rho$ is smooth, $\rho\left(I_{F}\right)$ is finite. Let $\phi$ be a Frobenius element in $W_{F}$. If the order of $\rho(\phi)$ is finite, then $\rho\left(W_{F}\right)$ is finite, so $\rho$ extends by continuity to a smooth $R$-representation $\rho^{\prime}$ of $\mathrm{Gal}_{F}$. The proof of Tate's theorem (Serre77] §6.5) applies with $R$ instead of $\mathbb{C}$ and that shows that $\rho^{\prime}$ lifts to a smooth $R$-representation of $\mathrm{Gal}_{F}$. Let us show that $\rho(\phi)$ has finite order. Since $\rho(\phi)$ acts by conjugation on $\rho\left(I_{F}\right)$ which is finite, a power $\rho\left(\phi^{d}\right)$ for some positive $d$ acts trivially on $\rho\left(I_{F}\right)$. But it also acts trivially on $\rho(\phi)$, hence on all of $\rho\left(W_{F}\right)$. Let $A \in G L_{n}(R)$ be a lift of $\rho\left(\phi^{d}\right)$. For $B \in G L_{n}(R)$, the commutator $(A, B)$ depends only on the image of $B$ in $P G L_{n}(R)$, and if $B$ has image $\rho(i)$ for $i \in I_{F}$, then $(A, B)$ is a scalar $\mu(i)$. If $B^{\prime} \in G L_{n}(R)$ has image $\rho\left(i^{\prime}\right)$ for $i^{\prime} \in I_{F}$, then $A\left(B B^{\prime}\right) A^{-1}=A B A^{-1} A B^{\prime} A^{-1}$, giving $\mu\left(i i^{\prime}\right)=\mu(i) \mu\left(i^{\prime}\right)$, so conjugation by $A$ induces a morphism $\mu: I_{F} \rightarrow R^{*}$. Since $\rho\left(I_{F}\right)$ is finite, a power $A^{e}$ for some positive $e$ commutes with the inverse image $J$ in $G L_{n}(R)$ of $\rho\left(W_{F}\right)$. Let $V$ be an eigenspace of $A^{e}$. It is stable under $J$. If $V \neq R^{n}$, that yields a proper parabolic subgroup $P$ (the image in $P G L_{n}(R)$ of the stabilizer of $V$ ) of $P G L_{n}(R)$ which contains $\rho\left(W_{F}\right)$, contrary to the hypothesis. So $A^{e}$ is scalar, which implies that $\rho(\phi)$ has finite order dividing $d e$.

Two 2-dimensional Deligne $R$-representations of $W_{F}$ in $G L_{2}(R)$ are twists of each other by a smooth $R$-character of $W_{F}$ if and only if they give the same Deligne morphism of $W_{F}$ in $P G L_{2}(R)$ if and only if the two corresponding irreducible smooth $R$-representations $\Pi, \Pi^{\prime}$ of $G$ are twists of each other by a smooth $R$-character of $G(4.25)$ if and only if $\Pi$ and $\Pi^{\prime}$ define the same $L$-packet $L(\Pi)=L\left(\Pi^{\prime}\right)$ of irreducible smooth $R$-representations of $G^{\prime}$ (4.4).
5.0.2. From the above the local Langlands correspondence for $G$ induces a bijection between $L$-packets of irreducible smooth $R$-representations of $G^{\prime}$ and Deligne morphisms of $W_{F}$ in $P G L_{2}(R)$ up to $P G L_{2}(R)$-conjugacy. We would like to understand the internal structure of a given packet in terms of an associated Deligne morphism $W_{F} \rightarrow P G L_{2}(R)$ (called its $L$-parameter).

Let $\Pi$ be an irreducible smooth $R$-representation of $G$. The $L$-packet $L(\Pi)$ is principal homogeneous space of $G / G_{\Pi}$. The packet containing the trivial representation of $G^{\prime}$ is a singleton, so the parametrization is trivial. When $L(\Pi)$ is a packet of infinitedimensional representations of $G^{\prime}$ we take as a base point in $L(\Pi)$ the element with nonzero Whittaker model with respect to the character $\psi$ of $F$ (that is, $\theta_{0}$ of $U$ ) fixed in §4.1). Let $C_{\Pi}$ denote the centralizer of the image in $P G L_{2}(R)$ of a Deligne morphism $\left(\sigma_{\Pi}, N_{\Pi}\right)$ of $W_{F}$ in $G L_{2}(R)$ associated to $\Pi$, and $S_{\Pi}$ the component group of $C_{\Pi}$. We shall compute $C_{\Pi}$ and $S_{\Pi}$, and when $\operatorname{char}_{R} \neq 2$ we shall construct a canonical isomorphism from $G / G_{\pi}$ onto the $R$-characters of $S_{\Pi}$. In this way we get an enhanced local Langlands correspondence for $S L_{2}(F)$ in the sense of Aubert-Baum-Plymen-Solleveld16, Aubert-Mendes-Plymen-Solleveld17 if $\operatorname{char}_{R} \neq 2$ but not if $\operatorname{char}_{R}=2$. J.-F. Dat tells us that our results for $\operatorname{char}_{R}=2$ should still be compatible with the stacky approach of Fargues and Scholze to the semisimple Langlands correspondence. For example, for a supercuspidal $R$-representation $\Pi$ of $G$, the two components of $\left.\Pi\right|_{G^{\prime}}$ should be indexed by the two irreducible $R$-representations of the group scheme $\mu_{2}$.

Assume that $\operatorname{char}_{R} \neq 2$. The group of $R$-characters of $G / G_{\Pi}$ is $X_{\Pi}$, and $X_{\Pi}=\{\chi \circ$ $\left.\operatorname{det} \mid \chi \in X_{\left(\sigma_{\Pi}, N_{\Pi}\right)}\right\}$ (4.27). We now construct an homomorphism $\varphi: X_{\left(\sigma_{\Pi}, N_{\Pi}\right)} \rightarrow S_{\Pi}$. Let $\chi \in X_{\left(\sigma_{\Pi}, N_{\Pi}\right)}$. By definition, there exists $A \in G L_{2}(R)$ such that $A N_{\Pi}=N_{\Pi}$ and $A \sigma_{\pi}(w) A^{-1}=\chi(w) \sigma_{\pi}(w)$ for $w \in W_{F}$. The image $\bar{A}$ of $A$ in $P G L_{2}(R)$ belongs to $C_{\Pi}$ and we shall show that its image $\varphi(\chi)$ in $S_{\Pi}$ does not depend on the choice of $A$.

Proposition 5.2. Assume that $\operatorname{char}_{R} \neq 2$. The map $\varphi: X_{\left(\sigma_{\Pi}, N_{\Pi}\right)} \rightarrow S_{\Pi}$ is a group isomorphism.

Theorem 5.3. When $\operatorname{char}_{R} \neq 2$, the map $\varphi: X_{\left(\sigma_{\Pi}, N_{\Pi}\right)} \rightarrow S_{\Pi}$ is a group isomorphism, and $S_{\Pi}=\{1\}, \mathbb{Z} / 2 \mathbb{Z}$ or $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}$.

When $\operatorname{char}_{R}=2, S_{\Pi}=\{1\}$ for each $\Pi$, but the length of $\left.\Pi\right|_{G^{\prime}}$ is
1 if $\Pi$ is not cuspidal,
2 if $\Pi$ is supercuspidal,
4 if $\Pi$ is cuspidal not supercuspidal.
Proof. A) Let $\Pi$ be a supercuspidal $R$-representation of $G$. Then $\sigma_{\Pi}$ is irreducible and $N_{\Pi}=0$ (Proposition 4.19).

When $\operatorname{char}_{R} \neq 2$, in [Cui-Lanard-Lu24, Proposition 6.4], an isomorphism $\phi: X_{\sigma_{\Pi}} \rightarrow C_{\Pi}$ is constructed when $\operatorname{char}_{F} \neq 2$, but the proof does not use this hypothesis. This implies $C_{\Pi}=S_{\Pi}$. One checks that $\varphi(\chi)=\phi(\chi)$ for $\chi \in X_{\sigma_{\Pi}}$. an isomorphism.

When $\operatorname{char}_{R}=2, p$ is odd, the cardinality of $L(\Pi)$ is 2 or 4 (Propositions 4.8, 4.9), $\sigma_{\Pi}=\operatorname{ind}_{W_{E}}^{W_{F}}(\theta)$ where $E / F$ is a quadratic separable extension and $\theta$ a smooth $R$-character of $W_{E}$ (or equivalently of $E^{*}$ ) different from its conjugate $\theta^{\tau}$ by a generator $\tau$ of $\operatorname{Gal}(E / F)$. The character $\theta^{\tau} / \theta$ has finite odd order, say $m$, and $\sigma_{\Pi}\left(W_{F}\right) \subset G L_{2}(R)$ is a dihedral group of order $2 m$, generated by a matrix $\left(\begin{array}{cc}a & 0 \\ 0 & a^{-1}\end{array}\right)$ of order $m$ and $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ modulo conjugation in $G L_{2}(R)$. So $C_{\Pi}=\{1\}$ and there is no enhanced correspondence.
B) Let $\Pi=i_{B}^{G}(\eta)$ be an irreducible normalized principal series with the notation of (4.29), with $\eta \neq q^{ \pm \text {val }}$. The cardinality of $L(\Pi)$ is 2 if $\eta \neq 1, \eta^{2}=1$, and $L(\Pi)$ is a singleton otherwise. We have $\sigma_{\Pi}=(\eta \oplus 1) \circ \alpha_{F}, N_{\Pi}=0$ (Proposition 4.19) and we easily see that $C_{\Pi}$ is:
$P G L_{2}(R)$ when $\eta=1$, so $S_{\Pi}=\{1\}$.
The diagonal torus when $\eta \neq 1, \eta^{2} \neq 1, S_{\Pi}=\{1\}$.
The normalizer of the trivial torus when $\eta \neq 1, \eta^{2}=1$, so $\operatorname{char}_{R} \neq 2$ and $S_{\Pi}=\mathbb{Z} / 2 \mathbb{Z}$. We have $X_{\Pi}=\{1, \eta \circ \operatorname{det}\}$ (Remark 4.22) and $\varphi(\eta)$ is not trivial, so $\varphi: X_{\Pi} \rightarrow S_{\Pi}$ is an isomorphism.
C) Let $\Pi$ be an irreducible subquotient of $\operatorname{ind}_{B}^{G} 1$. The length of $\left.\Pi\right|_{G^{\prime}}$ is (Corollary 4.28, Proposition 4.29):

1 when $\Pi=1, q^{v a l} \circ \operatorname{det}$ or St,
2 when $\Pi=\Pi_{0}$ if $\operatorname{char}_{R} \neq 2$ and $q+1=0$ in $R$,
4 when $\Pi=\Pi_{0}$ if $\operatorname{char}_{R}=2$.
We have $\sigma_{\Pi}=\left(\left(q^{1 / 2}\right)^{\mathrm{val}} \oplus\left(q^{-1 / 2}\right)^{\mathrm{val}}\right) \circ \alpha_{F}$ (formula (4.24), Proposition4.19). The centralizer $C_{\Pi}^{\prime}$ of the image of $\sigma_{\Pi}\left(W_{F}\right)$ in $P G L_{2}(R)$ is the image in $P G L_{2}(R)$ of

$$
\begin{gathered}
\left\{A \in G L_{2}(R) \mid A \operatorname{diag}(q, 1) A^{-1} \in R^{*} \operatorname{diag}(q, 1)\right\}= \\
\left\{A=\left(\begin{array}{cc}
x & y \\
z & t
\end{array}\right) \in G L_{2}(R) \left\lvert\,\left(\begin{array}{cc}
x q & y \\
z q & t
\end{array}\right)=u\left(\begin{array}{cc}
x q & y q \\
z & t
\end{array}\right)\right. \text { for some } u \in R^{*}\right\} .
\end{gathered}
$$

If $x \neq 0$ or $t \neq 0$ then $u=1$, and if $y \neq 0$ then $q u=1$. If $z \neq 0$ then $u=q$. So, $C_{\Pi}^{\prime}$ is:
$P G L_{2}(R)$ if $q-1=0$ in $R$.
The diagonal torus when $q-1 \neq 0$ in $R$ and $q+1 \neq 0$ in $R$.
The centralizer of the diagonal torus if $q-1 \neq 0$ in $R$ and $q+1=0$ in $R$.
We have $N_{\Pi}=0$, hence $C_{\Pi}=C_{\Pi}^{\prime}$, when:
$\Pi=1$ when $q+1 \neq 0$ in $R$, hence $C_{1}=P G L_{2}(R)$ if $q+1 \neq 0, q-1=0$ in $R$ (so $\operatorname{char}_{R} \neq 2$ ) and $C_{1}$ is the diagonal torus if $q+1 \neq 0, q-1 \neq 0$ in $R$. In both cases $S_{1}=\{1\}$.
$\Pi=\Pi_{0}$ cuspidal when $q+1=0$ in $R$. Recalling Proposition 4.29, when $\operatorname{char}_{R} \neq 2$, $\lg \left(\left.\Pi_{0}\right|_{G^{\prime}}\right)=2$ and $C_{\Pi_{0}}$ is the normalizer of the diagonal torus and $S_{\Pi}=\mathbb{Z} / 2 \mathbb{Z}$. We have $X_{\sigma_{\Pi_{0}}}=\left\{1,(-1)^{v a l}\right\}$ (Corollary 4.14). As in B), $\varphi\left((-1)^{v a l}\right)$ is not trivial, so $\varphi: X_{\Pi} \rightarrow S_{\Pi}$ is an isomorphism.

But when $\operatorname{char}_{R}=2$, then $q-1=0$ in $R$ and $C_{\Pi_{0}}=P G L_{2}(R)$. As $S_{\Pi_{0}}=\{1\}$ and $\lg \left(\left.\Pi_{0}\right|_{G^{\prime}}\right)=4$, there is no enhanced correspondence.

We suppose now $N_{\Pi} \neq 0$. Then (Proposition 4.19) $\Pi=$ St when $q+1 \neq 0$ in $R$, $\Pi$ is a character when $q+1=0$ in $R$. In both cases $\left.\Pi\right|_{G^{\prime}}$ is irreducible (Corollary 4.28). We can suppose that $N_{\Pi}$ is a non-trivial upper triangular matrix. A similar analysis gives that $C_{\Pi}$ is
the diagonal torus if if $q-1 \neq 0$ in $R$,
the upper triangular subgroup if $q-1=0$ in $R$.
In both cases $S_{\Pi}=\{1\}$.
Remark 5.4. We computed the centralizer $C_{\Pi} \subset P G L_{2}(R)$ :
$C_{\Pi}$ is finite if and only if $\Pi$ is supercuspidal.

When $C_{\Pi}$ is connected, it is isomorphic to $P G L_{2}(R)$, the upper triangular subgroup, the diagonal subgroup, or $\{1\}$.

When $C_{\Pi}$ has two connected components it is isomorphic to the normalizer of the diagonal subgroup or to $\mathbb{Z} / 2 \mathbb{Z}$.

When $C_{\Pi}$ has four connected components, it is isomorphic to the Klein group $\mathbb{Z} / 2 \mathbb{Z} \times$ $\mathbb{Z} / 2 \mathbb{Z}$.
5.0.3. Assume $\operatorname{char}_{R}=2$. A kind of lifting has been introduced by Treumann-Venkatesh16 and generalized in Feng23. They consider a (connected) split reductive $F$-group $\underline{H}$, equipped with an involution $\iota$ such that the group of fixed points $\underline{H}^{\iota}$ is (connected) split reductive. They set up a correspondence, called linkage, between $\iota$-invariant irreducible smooth $R$-representations $\Pi$ of $H=\underline{H}(F)$ and irreducible smooth $R$-representations of $H^{\iota}=\underline{H}^{\iota}(F)$. More precisely they show that there is a unique isomorphism $\iota_{\Pi}$ from $\Pi$ to its conjugate $\Pi^{\iota}$ by $\iota$, which has trivial square. They say that an irreducible smooth $R$ representation $\pi$ of $H^{\iota}$ is linked with $\Pi$ if the Frobenius twist of $\pi$ occurs as a subquotient of the representation $T(\Pi)=\operatorname{Ker}\left(1+\iota_{\Pi}\right) / \operatorname{Im}\left(1+\iota_{\Pi}\right)$ of $H^{\iota}$. They ask for an interpretation of linkage in terms of dual groups.

Let us consider the special case where $\underline{H}=G L_{2}$ and $\iota(g)=g / \operatorname{det}(g){ }^{14}$. Then $\underline{H}^{\iota}=S L_{2}$, so $H=G, H^{\iota}=G^{\prime}$. Let $\Pi$ be an irreducible smooth $R$-representation of $G$ of central character $\omega_{\Pi}$. It is invariant under $\iota$ if and only if $\Pi \simeq \Pi \otimes\left(\omega_{\Pi} \circ \operatorname{det}\right)$. This implies that $\omega_{\Pi}$ has trivial square, so is trivial because $\operatorname{char}_{R}=2$. In other words, $\Pi$ is $\iota$-invariant if and only if $\Pi$ factors to a representation of $P G L_{2}(F)$. It follows that then $\iota_{\Pi}$ is identity, and $T(\Pi)$ is simply the restriction of $\Pi$ to $G^{\prime}$, which we have throroughly investigated. In particular $T(\Pi)$ has finite length, as expected. The dual group of $\underline{H}$ over $R$ is $G L_{2}(R)$, that of $\underline{H}^{\iota}$ is $P G L_{2}(R)$. They ask for an interpretation of linkage in terms of a natural homomorphism from $P G L_{2}(R)$ to $G L_{2}(R)$.

Let $\sigma_{\Pi}: W_{F} \rightarrow G L_{2}(R)$ be the semi-simple $L$-parameter of $\Pi$. The map $\varphi^{-1}\left(\sigma_{\Pi}\right)$ : $W_{F} \rightarrow G L_{2}(R)$ followed by the quotient map $G L_{2}(R) \rightarrow P G L_{2}(R)$, is the semi-simple $L$-parameter $\rho_{\Pi}: W_{F} \rightarrow P G L_{2}(R)$ of the Frobenius twist of any constituent $\pi$ of $\left.\Pi\right|_{G^{\prime}}$.

The map $\Psi(g)=\varphi(g) / \operatorname{det}(g)$ for $g \in G L_{2}(R)$ and the Frobenius map $\varphi: x \rightarrow x^{2}$ of $R$, is trivial on scalar matrices, hence factors through an homomorphism $\Psi: P G L_{2}(R) \rightarrow$ $G L_{2}(R)$. The homomorphism $\Psi$ is injective of image $S L_{2}(R)$. Now if $\Pi$ is $\iota$-invariant, the determinant of $\sigma_{\Pi}$ is trivial so $\sigma_{\Pi}=\Psi \circ \rho_{\Pi}$ and the conjectures of Treumann-Venkatesh16, $\S 6.3]$ are indeed true in our special case.

## 6. Representations of $S L_{2}(F)$ near the identity

6.1. Assume $\operatorname{char}_{F}=0$ and $R=\mathbb{C}$. Let $H$ be the group of $F$-points of a connected reductive group over $F$. We denote by $C_{c}^{\infty}(X ; \mathbb{C})$ the space of smooth complex functions with compact support on a locally profinite space $X$. The exponential map exp from Lie $(H)$ to $H$ induces an $H$-equivariant bijection between a neighbourhood of 0 in $\operatorname{Lie}(H)$ and a neighbourhood of 1 in $H$. So a function $f \in C_{c}^{\infty}(H ; \mathbb{C})$ with support small enough

[^9]around 1 gives a smooth function $f \circ \exp$ around 0 in $C_{c}^{\infty}(\operatorname{Lie}(H) ; \mathbb{C})$. Also there are only finitely many nilpotent orbits of $H$ in $\operatorname{Lie}(H)$, for the adjoint action. For each such orbit $\mathfrak{O}$, there is an $H$-invariant measure on $\mathfrak{O}$, and a function $\varphi \in C_{c}^{\infty}(\operatorname{Lie}(H) ; \mathbb{C})$ can be integrated along $\mathfrak{O}$ with respect to that measure, yielding an orbital integral $I_{\mathfrak{V}}(\varphi)$. Choosing a nondegenerate invariant bilinear form on $\operatorname{Lie}(H)$, a non-trivial character of $\operatorname{Lie}(H)$ and a Haar measure on $\operatorname{Lie}(H)$ yields a Fourier transform $\hat{\varphi}$ for a function $\varphi \in C_{c}^{\infty}(\operatorname{Lie}(H) ; \mathbb{C})$. Fix also a Haar measure $d h$ on $H$.

Theorem 6.1. Let $\Pi$ be a smooth complex representation of $H$ with finite length. Then there is an open neighbourhood $V(\Pi)$ of 1 in $H$ and for each nilpotent orbit $\mathfrak{O}$ a unique complex number $c_{\mathfrak{V}}=c_{\mathfrak{V}}(\Pi)$ such that if $f \in C_{c}^{\infty}(H ; \mathbb{C})$ has compact support in $V(\Pi)$ then the trace $\operatorname{tr}_{\Pi}(f)$ of the linear endomorphism $\int_{H} f(h) \Pi(h) d h$ is equal to

$$
\begin{equation*}
\operatorname{tr}_{\Pi}(f)=\sum_{\mathfrak{O}} c_{\mathfrak{O}}(\Pi) I_{\mathfrak{O}}(\hat{\varphi}) \quad \text { where } \quad \varphi=f \circ \exp \tag{6.1}
\end{equation*}
$$

This was first proved by Roger Howe when $H=G L_{n}(F)$, and the general case is due to Harish-Chandra.

As is usual, we say that a nilpotent orbit $\mathfrak{O}^{\prime}$ is smaller than a nilpotent orbit $\mathfrak{O}$ if $\mathfrak{O}^{\prime}$ is contained in the closure of $\mathfrak{O}$. With the normalizations as in Varma14 we have:

Theorem 6.2. Let $\Pi$ be a smooth complex representation of $H$ with finite length. When $\mathfrak{O}$ is maximal among the orbits with $c_{\mathfrak{V}}(\Pi) \neq 0$, then $c_{\mathfrak{V}}(\Pi)$ is equal to the dimension of generalized Whittaker spaces for $\Pi$ attached to $\mathfrak{O}$.

The result when $p$ is odd due to Moeglin-Waldspurger87 is extended to $p=2$ in [Varma14] in general). When $\mathfrak{O}$ is a regular nilpotent orbit, the generalized Whittaker model is the usual one, and the result then goes back to Rodier [Rodier74]. Varma actually proves that with that normalization all coefficients $c_{\mathfrak{O}}(\Pi)$ are rational Varma14.
6.2. Assume $R=\mathbb{C}$. For any $F$, when $H$ is an open normal subgroup of $G L_{r}(D)$ where $D$ is a finite dimensional central division $F$-algebra, Theorem 6.1 still holds, with the exponential map replaced by the map $X \mapsto 1+X$ Lemaire04]. In the special case where $H=G L_{r}(D)$, Theorem 6.2 also holds, at least for the natural generalized Whittaker space attached to each nilpotent orbit Henniart-Vignéras23.
6.2.1. We use the notations and definitions introduced in $\S 4.1$. Any irreducible smooth $R$-representation $\pi$ of $G^{\prime}=S L_{2}(F)$ extends to an open normal subgroup $H$ of $G=G L_{2}(F)$ by Theorem [3.2. The group $H$ contains $Z G^{\prime}$ and $G / H \simeq F^{*} / \operatorname{det}(H)$ is a finite power of 2. Only when $\operatorname{char}_{F} \neq 2$ we can take $H=Z G^{\prime}$. Put

$$
\begin{equation*}
V_{H}=F^{*} / \operatorname{det}(H), \quad \operatorname{dim}_{\mathbb{F}_{2}} V_{H}=d, \quad|G / H|=2^{d} \tag{6.2}
\end{equation*}
$$

Remark 6.3. We have $d=0$ if and only if $H=G$, and $d=1$ if and only if $\operatorname{det} H=N_{E / F}\left(E^{*}\right)$ for a quadratic separable extension $E / F$. When $p$ is odd then $d \leq 2$ and $d=2$ if and only if $H=Z G^{\prime}$. If $p=2$ and $\operatorname{char}_{F}=0$, by formula (4.1) $d \leq e+2$ with equality if and only if $H=Z G^{\prime}$. If char ${ }_{F}=2$, then $d$ can be any non-negative integer.

A nilpotent matrix can be conjugated in a lower triangular nilpotent matrix $Y$ by an element of $G^{\prime}$. Two such matrices $Y$ and $Y^{\prime}$ are $H$-conjugate if and only if their bottom left coefficients differ by multiplication by an element of $\operatorname{det}(H)$.
(6.3) The number of $H$-orbits in the nilpotent matrices in $M_{2}(F)$ is $1+2^{d}$.

The 0-matrix forms the smallest nilpotent $H$-orbit (the "trivial" one). The non trivial nilpotent $H$-orbits are maximal, and parametrized by $V_{H}$ via their bottom left coefficient.

With the same arguments as those given for $Z G^{\prime}$ in $\S 4.1$, any irreducible smooth $R$ representation $\pi$ of $H$ appears in the restriction to $H$ of an irreducible smooth representation $\Pi$ of $G$, unique modulo torsion by a smooth $R$-character of $G$. The irreducible components $\pi$ of $\left.\Pi\right|_{H}$ are $G$-conjugate (even $B$-conjugate) and the $G$-stabilizer of $\pi$ does not depend on the choice of $\pi$ in $\left.\Pi\right|_{H}$, and denoted by $G_{\left.\Pi\right|_{H}}$. The representation $\left.\Pi\right|_{H}$ is semi-simple of multiplicity 1 with length

$$
\begin{equation*}
\lg \left(\left.\Pi\right|_{H}\right)=\left|G / G_{\left.\Pi\right|_{H}}\right| \text { dividing } \lg \left(\left.\Pi\right|_{Z G^{\prime}}\right)=\left|G / G_{\Pi}\right|=|L(\pi)|, \tag{6.4}
\end{equation*}
$$

hence equal to 1,2 or 4 by Theorem 1.1. The representation $\left.\pi\right|_{G^{\prime}}$ is semi-simple of multiplicity 1 with length $\lg \left(\left.\pi\right|_{G^{\prime}}\right)=\lg \left(\left.\Pi\right|_{G^{\prime}}\right) / \lg \left(\left.\Pi\right|_{H}\right)=\left|G_{\left.\Pi\right|_{H}} / G_{\Pi}\right|$.

For a lower triangular matrix $Y \neq 0$, we have:

$$
\sum_{\left.\pi \subset \Pi\right|_{H}} \operatorname{dim}_{R} W_{Y}(\pi)=\operatorname{dim}_{R} W_{Y}(\Pi)=1
$$

There is a single irreducible $\pi$ in $\left.\Pi\right|_{H}$ with $W_{Y}(\pi) \neq 0$, and $\operatorname{dim}_{R} W_{Y}(\pi) \neq 0 \Leftrightarrow \operatorname{dim}_{R} W_{Y}(\pi)=$ 1. If $W_{Y}(\pi) \neq 0$ then $W_{Y^{\prime}}(\pi) \neq 0$ when $Y^{\prime}$ and $Y$ are $H$-conjugate. We consider $\operatorname{dim}_{R} W_{Y}(\pi)$ as a function $m_{\pi}$ on $V_{H}$. Because $\pi$ extends to $G_{\left.\Pi\right|_{H}}, m_{\pi}$ is invariant under translations by

$$
W_{\left.\Pi\right|_{H}}=\operatorname{det}\left(G_{\left.\Pi\right|_{H}}\right) / \operatorname{det}(H) .
$$

It follows that $m_{\pi}$ is the characteristic function of an affine subspace $A_{\pi}$ of $V_{H}$ with direction $W_{\left.\Pi\right|_{H}}$, each such affine subspace being obtained exactly for one $\left.\pi \subset \Pi\right|_{H}$. For $g \in G$ we denote $\pi^{g}(x)=\pi\left(g x g^{-1}\right)$ for $g \in G, x \in H$, so $\pi^{g h}=\left(\pi^{g}\right)^{h}$ for $g, h \in G$. We have $A_{\pi^{g}}=\operatorname{det}(g) A_{\pi}$. We have a disjoint union (the Whittaker decomposition):

$$
\begin{equation*}
V_{H}=\sqcup_{\left.\pi \subset \Pi\right|_{H}} A_{\pi} . \tag{6.5}
\end{equation*}
$$

If $\lg \left(\left.\Pi\right|_{H}\right)=1, m_{\pi}$ is the constant function on $V_{H}$ with value 1. If $\lg \left(\left.\Pi\right|_{H}\right)=2$, the two irreducible components of $\left.\Pi\right|_{H}$ yield the characteristic functions of two affine hyperplanes of $V_{H}$ with the same direction. Finally for $\lg \left(\left.\Pi\right|_{H}\right)=4$, we get the characteristic functions of four affine subspaces of codimension 2 in $V_{H}$ with the same direction. In particular when $p$ is odd and $\lg \left(\left.\Pi\right|_{H}\right)=4$, then $H=Z G^{\prime}$ and $m_{\pi}$ is a non-zero delta function on $V_{H}=F^{*} /\left(F^{*}\right)^{2}$.

Let $C\left(V_{H} ; \mathbb{Z}\right)$ denote the $\mathbb{Z}$-module of functions $f: V_{H} \mapsto \mathbb{Z}$. For an integer $0 \leq r<d$, let $I_{r}$ denote the $\mathbb{Z}$-submodule of $C\left(V_{H} ; \mathbb{Z}\right)$ generated by the characteristic functions of the $r$-dimensional affine subspaces of $V_{H}$. We have $I_{0}=C\left(V_{H} ; \mathbb{Z}\right)$.
Lemma 6.4. When $0<r<d, 2 I_{r-1}$ is included in $I_{r}$ and the exponent of $I_{0} / I_{r}$ is $2^{r}$.

Proof. Let $W$ be a $r$ - 1 -dimensional vector subspace of $V_{H}$ and $\{0, e, f, e+f\}$ a supplementary plane. For an affine subspace $A$ of $V_{H}$ of direction $W$, the affine subspaces $A_{e}=A \cup A+e, A_{f}=A \cup A+f$ and $B=A+e \cup A+f$ of $V_{H}$ are $r$-dimensional, and taking their charactersitic functions $\chi$, we get $\chi_{A_{e}}+\chi_{A_{f}}-\chi_{B}=2 \chi_{A}$. Thus $2 I_{r-1} \subset I_{r}$. By induction $2^{r} I_{0} \subset I_{r}$. The map $s_{r}: C\left(V_{H} ; \mathbb{Z}\right) \mapsto \mathbb{Z} / 2^{r} \mathbb{Z}$ given by the sum of coordinates is surjective and vanishes on $I_{r}$ but not on $I_{r-1}$. So the exponent of $I_{0} / I_{r}$ is $2^{r}$.
6.2.2. Let us precise Theorem 6.1 for an open normal subgroup $H$ of $G=G L_{2}(F)$ as in §6.2.1.

Notation 6.5. On $G$ (hence on $H$ ) we put a Haar measure $d g$, and on Lie $G=\operatorname{Lie} H=$ $M_{2}(F)$ we put the Haar measure $d X$ such that $X \mapsto 1+X$ preserves measures near 0 . The invariant bilinear map $\left(X, X^{\prime}\right) \mapsto \operatorname{tr}\left(X X^{\prime}\right)$ on $\operatorname{Lie}(H)$ is non-degenerate. The Fourier transform $\varphi \mapsto \hat{\varphi}$ on $C_{c}^{\infty}(\operatorname{Lie}(H) ; \mathbb{C})$. is taken with respect to the non-trivial character $\psi \circ \operatorname{tr}$ on $\operatorname{Lie}(H)$. For each nilpotent $H$-orbit $\mathfrak{O}$ in $\operatorname{Lie}(H)$, we normalize the nilpotent orbital integral $I_{\mathfrak{V}}(\hat{\varphi})$ in the same way as (Varma14 §3); that normalization is valid even when $\operatorname{char}_{F}>0$. By Remark 2 of loc.cit., for large enough $i, K_{i}=1+M_{2}\left(P_{F}^{i}\right)$ and a lower triangular nilpotent matrix $Y$, the measure of $\operatorname{Ad}\left(K_{i}\right)(Y)$ is 0 if $Y=0$ and $q^{-2 i}$ otherwise. In particular $I_{0}(\hat{\varphi})=\varphi(0)$ for the nilpotent trivial orbit $0 \in$ Lie $H$.

Theorem 6.6. Let $\pi$ be a smooth complex representation of $H$ with finite length. There is an open neighbourhood $V(\pi)$ of 1 in $H$ and for each nilpotent $H$-orbit $\mathfrak{O}$ a unique complex number $c_{\mathfrak{Q}}=c_{\mathfrak{O}}(\pi)$ such that if $f \in C_{c}^{\infty}(H ; \mathbb{C})$ has compact support in $V(\pi)$ then

$$
\begin{equation*}
\operatorname{tr}_{\pi}(f)=c_{0}(\pi) f(1)+\sum_{\mathfrak{O} \neq 0} c_{\mathfrak{O}}(\pi) I_{\mathfrak{O}}(\hat{\varphi}) \quad \text { where } \varphi(X)=f(1+X) \text { for } 1+X \in V(\pi) \tag{6.6}
\end{equation*}
$$

We call (6.6) the germ expansion and $c_{0}(\pi)$ the constant coefficient of the trace of $\pi$ around 1. A character twist of $\pi$ does not change $c_{0}(\pi)$. For $\pi$ irreducible, $c_{\mathfrak{O}}(\pi)=0$ for all $\mathfrak{O} \neq 0$ if and only if $\pi$ is degenerate (by Theorem 6.2) if and only if $\operatorname{dim}_{\mathbb{C}} \pi=1$. In this case $c_{0}(\pi)=1$.

We can determine that constant coefficient $c_{0}(\pi)$ for any irreducible smooth representation $\pi$ of $H$ from the case of $G$, because $\pi$ appears in the restriction to $H$ of an irreducible smooth complex representation $\Pi$ of $G$. The irreducible components of $\left.\Pi\right|_{H}$ being $G$ conjugate to $\pi$ have the same constant coefficient, and

$$
\begin{equation*}
c_{0}(\Pi)=\lg \left(\left.\Pi\right|_{H}\right) c_{0}(\pi) \tag{6.7}
\end{equation*}
$$

We have Henniart-Vignéras23 :
$c_{0}\left(1_{G}\right)=1$.
When $\Pi$ is parabolically induced, for example when $\Pi$ is tempered and not a discrete series,

$$
c_{0}(\Pi)=0 .
$$

When $\Pi$ is a discrete series representation of formal degree $d(\Pi)$,

$$
c_{0}(\Pi)=-d(\Pi) / d(\mathrm{St})
$$

When $\Pi$ is a supercuspidal complex smooth representation of $G$ of minimal level $f_{\Pi}$ (the minimal leve $\sqrt{15}$ of the character twists of $\Pi$ ),

$$
c_{0}(\Pi)=\left\{\begin{array}{l}
-2 q^{f_{\Pi}} \text { if } f_{\Pi} \text { is an integer }  \tag{6.8}\\
-(q+1) q^{f_{\Pi}-1 / 2} \text { if } f_{\Pi} \text { is a half-integer (not an integer) }
\end{array} .\right.
$$

When $f_{\Pi}$ is a half integer (not an integer), $\Pi$ has positive level ( $\S 4.3 .2$ ), $\Pi=\operatorname{ind}_{J}^{G} \Lambda$ where $J=E^{*}\left(1+Q^{f_{\Pi}+1 / 2}\right)$, where $E / F$ is ramified, $Q$ is the Jacobson radical of an Iwahori order in $M_{2}(F)$, and $\Lambda$ is trivial on $1+Q^{2 f_{\Pi}+1}$ Bushnell-Henniart06, $\left.\S 15\right]$. Let $\chi \in X_{\Pi} \backslash\{1\}$. Then $\chi$ is ramified [Bushnell-Henniart06, 20.3 Lemma]. The level $r_{\chi}$ of $\chi$ is the largest positive integer $r$ such that $\chi$ is non-trivial on $1+P_{F}^{r}$ when $\chi$ is ramified. We have

$$
\begin{equation*}
1 \leq r_{\chi}<f_{\Pi} \tag{6.9}
\end{equation*}
$$

Indeed, if $r_{\chi}>f_{\Pi}$ then $\chi \circ$ det is non-trivial on $1+Q^{2 r_{\chi}}\left(\right.$ as $\left.\operatorname{det}\left(1+Q_{\chi}^{2 r}\right)=1+P_{F}^{r_{\chi}}\right)$, and $(\chi \circ \operatorname{det}) \otimes \Lambda$ would be non trivial on $1+Q^{2 r_{\chi}}$ implying that the level of $(\chi \circ \operatorname{det}) \otimes \Pi$ is at least $r_{\chi}$ by (loc.cit.(15.8.1)), contrary to the assumption that $\chi \in X_{\Pi}$. So $f_{\Pi}<r_{\chi}$ as $r_{\chi}$ is an integer but not $f_{\Pi}$.
Lemma 6.7. If $f_{\Pi}=1 / 2$ then $X_{\Pi}=\{1\}$. If $q=2$ and $f_{\Pi}=3 / 2$ then $X_{\Pi}$ cannot have 4 elements.
Proof. If $f_{\Pi}=1 / 2$, then $X_{\Pi}$ is trivial by the formula (6.9). If $f_{\Pi}=3 / 2$, then $r_{\chi}=1$, and if $q=2$ there are only 2 quadratic characters of level 1 . That implies that $X_{\Pi}$ cannot have 4 elements.
Proposition 6.8. Let $\Pi$ be an irreducible complex smooth representation of $G$ and $\pi$ an irreducible representation of $H$ contained in $\left.\Pi\right|_{H}$. Then
$c_{0}(\pi)=-1 / 2$ if $p$ is odd, $\Pi$ is cuspidal of minimal level 0 and $L(\Pi)$ has 4 elements.
$c_{0}(\pi)$ is an integer otherwise, and $c_{0}(\pi)<0$ if $\pi$ is cuspidal.
Proof. By formulas (6.4), (6.7), (6.8), we have
$c_{0}\left(1_{G}\right)=1$, so $c_{0}\left(1_{H}\right)=1$.
$c_{0}(\mathrm{St})=-1$ so $c_{0}\left(\mathrm{st}_{H}\right)=-1$, since the restriction $\mathrm{st}_{H}$ of St to $H$ is irreducible as st $=\left.\mathrm{St}\right|_{G^{\prime}}$ is irreducible.
$c_{0}(\Pi)=0$ so $c_{0}(\pi)=0$, when $\Pi$ is an irreducible principal series.
$c_{0}(\Pi)<0$ so $c_{0}(\pi)<0$, when $\Pi$ supercuspidal of level $f_{\Pi}$ (the minimal level). If $p$ is odd, then $c_{0}(\Pi)$ is an even integer by (6.8), so that $c_{0}(\pi)$ is an integer if $L(\Pi)$ has 1 or 2 elements by (6.7); if $L(\Pi)$ has 4 elements, then $f_{\Pi}=0$ by Proposition 4.9 and $c_{0}(\Pi)=-2$, so $c_{0}(\pi)=-1 / 2$. If $p=2$, then $c_{0}(\Pi)$ is a multiple of 4 (so $c_{0}(\pi)$ is an integer) by (6.8) except when:
(i) $f_{\Pi}=0$, where $c_{0}(\Pi)=-2$. But $L(\Pi)$ has size 2 by Proposition 4.8, so $c_{0}(\pi)=-1$.
(ii) $f_{\Pi}=1 / 2$, where $c_{0}(\Pi)=-(q+1)$. But $L(\Pi)$ has size 1 by Lemma 6.7, so $c_{0}(\pi)=$ $-(q+1)$.

[^10](iii) $f_{\Pi}=3 / 2$ and $q=2$, where $c_{0}(\Pi)=-6$. But $L(\Pi)$ has size 1 or 2 by Lemma 6.7, so $c_{0}(\pi)=-6$ or -3.
Theorem 6.9. Let $\pi$ be a finite length complex representation of $H, Y \neq 0$ a lower triangular matrix in $M_{2}(F)$ and $\mathfrak{O}$ its $H$-orbit. Then $c_{\mathfrak{Q}}(\pi)=\operatorname{dim}_{\mathbb{C}} W_{Y}(\pi)$.

Proof. The proof uses the same idea as Rodier74. Remarking that the lower triangular group $B^{-}$of $G$ acts transtively on the lower triangular nilpotent matrices $Y$, and that for $g \in B^{-}$we have $c_{\mathfrak{O}}(\pi)=c_{\mathfrak{O} g}\left(\pi^{g}\right), \operatorname{dim}_{\mathbb{C}}\left(W_{Y}(\pi)\right)=\operatorname{dim}_{\mathbb{C}}\left(W_{Y^{g}}\left(\pi^{g}\right)\right)$, it is enough to consider the case where $Y=\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right)$. We stick to that $Y$ (so $\theta_{Y}=\theta$ with the notation 4.1).

For each positive integer $i$, we define a character $\chi_{i}$ of the pro- $p$ group $K_{i}=1+M_{2}\left(P_{F}^{i}\right)$, by the formula

$$
\chi_{i}(1+X)=\psi \circ \operatorname{tr}\left(p_{F}^{-2 i} Y X\right)=\psi\left(p_{F}^{-2 i} X_{1,2}\right), \quad X=\left(\begin{array}{ll}
X_{1,1} & X_{1,2}  \tag{6.10}\\
X_{2,1} & X_{2,2}
\end{array}\right) \in M_{2}\left(P_{F}^{i}\right)
$$

The character $\chi_{i}$ is trivial on $K_{2 i}$. Conjugating by the diagonal matrix $d_{i}=\operatorname{diag}\left(p_{F}^{i}, p_{F}^{-i}\right)$ we get a character $\theta_{i}$ on $H_{i}=d_{i}^{-1} K_{i} d_{i}=1+\left(\begin{array}{cc}P_{F}^{i} & P_{F}^{-i} \\ P_{F}^{3 i} & P_{F}^{i}\end{array}\right)$ such that $\theta_{i}(1+X)=\psi\left(X_{1,2}\right)$. The limit of the groups $H_{i}$ as $i \rightarrow \infty$ is the group $U$. We will prove that the $\theta_{i}$ approximate the character $\theta_{Y}$ of $U$ in the sense that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \operatorname{dim}_{\mathbb{C}} \operatorname{Hom}_{H_{i}}\left(\theta_{i}, \pi\right)=\operatorname{dim}_{\mathbb{C}} W_{Y}(\pi) \tag{6.11}
\end{equation*}
$$

On the other hand we will also prove in §6.2.3, following Varma14, that

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} \operatorname{Hom}_{K_{i}}\left(\chi_{i}, \pi\right)=c_{\mathfrak{V}}(\pi) \text { for large } i \tag{6.12}
\end{equation*}
$$

Since $\operatorname{dim}_{\mathbb{C}} \operatorname{Hom}_{H_{i}}\left(\theta_{i}, \pi\right)=\operatorname{dim}_{\mathbb{C}} \operatorname{Hom}_{K_{i}}\left(\chi_{i}, \pi\right)$, we shall get the result.
6.2.3. Let us proceed to the proof of the formulas (6.11) and (6.12), through a sequence of lemmas, rather easy compared to the analogous statements in the more general cases treated by Rodier74], Moeglin-Waldspurger87] and Varma14 when $\operatorname{char}_{F}=0$, in Henniart-Vignéras23 for arbitrary char ${ }_{F}$.

For $X \in M_{2}(F)$, put $\delta_{i}(X)=\chi_{i}^{-1}(1+X)$ if $X \in M_{2}\left(P_{F}^{i}\right)$ and $\delta_{i}(X)=0$ outside. With the notation 6.5, the Fourier transform $\hat{\delta}_{i}$ of $\delta_{i}$ is

$$
\hat{\delta}_{i}(X)= \begin{cases}q^{-4 i} \operatorname{vol}\left(M_{2}\left(O_{F}\right), d X\right) & \text { if } X \in p_{F}^{-2 i} Y+M_{2}\left(P_{F}^{-i}\right)  \tag{6.13}\\ 0 & \text { otherwise }\end{cases}
$$

Lemma 6.10. The $K_{1}$-normalizer of $\chi_{i}$ is $\left(Z U^{-} \cap K_{1}\right) K_{i}$.
Proof. For a positive integer $j \leq i$, we prove that the $K_{1}$-normalizer of the restriction of $\chi_{i}$ to $K_{2 i-j}$ is $\left(Z U^{-} \cap K_{1}\right) K_{j}$ by induction on $j$. This is clear for $j=1$ and the case $j=i$ gives what we want. Assume that the claim is true for $j<i$ and let us prove it for $j+1$. Let $g \in K_{1}$ normalizing the restriction of $\chi_{i}$ to $K_{2 i-j-1}$. By induction $g \in\left(Z U^{-} \cap K_{1}\right) K_{j}$ and
we may assume $g \in K_{j}$. Write $g=1+X$ with $X \in M_{2}\left(P_{F}^{j}\right)$. Then $g^{-1} Y g \equiv Y+Y X-X Y$ modulo $M_{2}\left(P_{F}^{j+1}\right)$ and the hypothesis on $g$ means that $Y X-X Y \equiv 0$ modulo $M_{2}\left(P_{F}^{j+1}\right)$, which gives that $p_{F}^{-j} X$ commutes with $Y$ modulo $P_{F}$. But the commutant of $Y$ modulo $P_{F}$ in $M_{2}\left(k_{F}\right)$ is made out of lower triangular matrices with the same diagonal elements. Consequently $g \in\left(Z U^{-} \cap K_{1}\right) K_{j+1}$ as claimed.
Lemma 6.11. The $K_{i}$-orbit of $Y$ is the set of nilpotent matrices in $Y+M_{2}\left(P_{F}^{i}\right)$
Proof. It is clear that $g Y g^{-1}$ is a nilpotent element in $Y+M_{2}\left(P_{F}^{i}\right)$ for $g \in K_{i}$. Conversely let $Y+p_{F}^{i} Z$ nilpotent (hence of trace 0) with $Z \in M_{2}\left(O_{F}\right)$. If $g=1+p_{F}^{i} X$ with $X \in M_{2}\left(O_{F}\right)$, then $g\left(Y+p_{F}^{i} Z\right) g^{-1} \equiv Y+p_{F}^{i}(Y X-X Y+Z)$ modulo $M_{2}\left(P_{F}^{i+1}\right)$. We choose $X$, as we can, so that $Y X-X Y+Z \equiv 0$ modulo $P_{F}$. So $g\left(Y+p_{F}^{i} Z\right) g^{-1} \in Y+M_{2}\left(P_{F}^{i+1}\right)$. The $K_{i}$-orbit of $Y$ is closed in $M_{2}(F)$. We finish the proof by successive approximations.

Let $\pi$ be a smooth representation of $H$ on a complex vector space $V$, and $\phi: V \rightarrow V_{\theta}$ be the quotient map from $V$ to the $\theta$-coinvariants $V_{\theta}$ of $V$. For large enough $i$ such that $H_{i} \subset H$ let $V_{i}$ be the $\theta_{i}$-isotypic component of $V$.

Lemma 6.12. For large enough $i, \phi\left(V_{i}\right)=V_{\theta}$.
Proof. It is the same as that of Lemma 8.7 in Henniart-Vignéras23.
We have

$$
H_{i+1}=\left(H_{i+1} \cap H_{i}\right)\left(H_{i+1} \cap U\right),\left[H_{i+1}:\left(H_{i+1} \cap H_{i}\right)\right]=\left[\left(H_{i+1} \cap U\right):\left(H_{i} \cap U\right)\right]=q^{-1}
$$

and $\theta_{i+1}=\theta_{i}$ on $H_{i+1} \cap H_{i}$. Let $e_{i}=f_{i} d g$ where $d g$ is the Haar measure on $H$ giving the volume 1 to $H_{i}$ and $f_{i}$ is the function on $G$ with support $H_{i}$ and value $\theta_{i}^{-1}$ on $H_{i}$.

Lemma 6.13. We have $e_{i} e_{i+1} e_{i}=q^{-1} e_{i}$ when $i>1$ and $H_{i} \subset H$. In particular, the map $v \rightarrow \pi\left(e_{i+1}\right) v: V_{i} \rightarrow V_{i+1}$ is injective.

Proof. The lemma is equivalent to $\pi\left(e_{i} e_{i+1} e_{i}\right) v=q^{-1} \pi\left(e_{i}\right) v$ for all $v \in V$ and $(\pi, V)$ as above. The projector $V \rightarrow V_{i}$ is $\pi\left(e_{i}\right)$ and

$$
\pi\left(e_{i} e_{i+1} e_{i}\right) v=q^{-1} \sum_{u \in\left(H_{i+1} \cap U\right) /\left(H_{i} \cap U\right)} \pi\left(e_{i} \theta_{i+1}(u)^{-1} u e_{i}\right) v
$$

If $\pi\left(e_{i} u e_{i}\right) v \neq 0$ for $u \in H_{i+1} \cap U$, then $u$ intertwines $\theta_{i}$. To interpret that condition we conjugate $\theta_{i}$ back to $\chi_{i}$. Then $H_{i}$ is sent to $K_{i}$ and $H_{i+1}$ is sent to $d_{1}^{-1} K_{i+1} d_{1}$ which, we remark, is contained in $K_{i-1}$. By Lemma 6.10, $u \in H_{i+1} \cap U$ conjugates to an element in $\left(Z U^{-} \cap K_{1}\right) K_{i}$, so that $u \in H_{i} \cap U$. We deduce that $\pi\left(e_{i} e_{i+1} e_{i}\right) v=q^{-1} \pi\left(e_{i}\right) v$ as claimed.

Proof of the formula (6.11)
Fix a large integer $i$ such that the lemmas apply. The projector $\pi\left(e_{i}\right): V \rightarrow V_{i}$ can be obtained by first projecting onto $V^{H_{i} \cap B^{-}}$, and then applying the projector $\pi\left(e_{i, U}\right)$ where $e_{i, U}=\left.f_{i}\right|_{H_{i} \cap U} d u$ for the Haar measure on $H \cap U$ giving the volume 1 to $H_{i} \cap U$. As $V_{i} \subset V^{H_{i+1} \cap B^{-}}$, we have $\pi\left(e_{i+1}\right)=\pi\left(e_{i+1, U}\right)$ on $V_{i}$. It follows that for $v \in V_{i}$ and $v_{1}=$ $\pi\left(e_{i+1}\right) v=\pi\left(e_{i+1, U}\right) v$ have the same image $\phi\left(v_{1}\right)=\phi(v)$ in $V_{\theta}$. Iterating the process we get for positive integers $k$, vectors $v_{k}=\pi\left(e_{j+k}\right) v_{k-1}=\pi\left(e_{j+k, U}\right) v_{k-1}$ with $\phi\left(v_{k}\right)=\phi(v)$. As
$e_{i+1, U} e_{i, U}=e_{i+1, U}$ we have $v_{k}=\pi\left(e_{i+k, U}\right) v$. But $\phi(v)=0$ is equivalent to $\pi\left(e_{i+k, U}\right) v=0$ for large $k$. As $v_{k}=0$ implies $v_{k-1}=0$ by Lemma 6.13, we get that $\phi$ is injective on $V_{i}$. Since it is also surjective by Lemma 6.12, we deduce that it gives an isomorphism $V_{i} \simeq V_{\theta}$. This ends the proof of (6.11).

Proof of the formula (6.12)
Fix an integer $i$ such that $K_{i} \subset H$. We have $\operatorname{dim}_{\mathbb{C}}\left(\operatorname{Hom}_{K_{i}} \chi_{i}, \pi\right)=\operatorname{tr} \pi\left(e_{i}^{\prime}\right)$ where $e_{i}^{\prime}=f_{i}^{\prime} d g$ where $d g$ is the Haar measure on $H$ giving the volume 1 to $K_{i}$ and $f_{i}^{\prime}$ is the function on $G$ with support $K_{i}$ and value $\chi_{i}^{-1}$ on $K_{i}$. We have, $f_{i}^{\prime}(1+X)=\delta_{i}(X)$. To prove (6.12), it suffices to apply the germ expansion (6.6) to $\operatorname{tr}_{\pi}$ and to show that for large $i$, $I_{\mathfrak{O}}\left(\hat{\delta}_{i}\right)=1$ whereas $I_{\mathfrak{V}^{\prime}}\left(\hat{\delta}_{i}\right)=0$ for any nilpotent orbit $\mathfrak{O}^{\prime} \neq \mathfrak{O}$. From the formula (6.13), $\hat{\delta}_{i}$ is a multiple of the characteristic function of $-p_{F}^{-2 i} Y+M_{2}\left(P_{F}^{-i}\right)$ and from Lemma 6.11 the nilpotent elements there form the $K_{i}$-orbit of $p_{F}^{-2 i} Y$. It follows that $I_{\mathfrak{V}^{\prime}}\left(\hat{\delta}_{i}\right)=0$ if $\mathfrak{O}^{\prime} \neq \mathfrak{O}$. That $I_{\mathfrak{O}}\left(\hat{\delta}_{i}\right)=1$ is proved exactly as in the proof of Lemma 7 in Varma14.
6.2.4. For a locally profinite space $X, x \in X$, and a field $C$, two linear forms $f, f^{\prime}$ on $C_{c}^{\infty}(V ; C)$ for some open neighbourhood $V$ of $x$ in $X$, are called equivalent if their restrictions to $C_{c}^{\infty}(W ; C)$ for some open neighbourhood $W$ of $x$ contained in $V$ are equal. The equivalence class of $f$ is called its germ $\tilde{f}$ at $x$. Denote $\mathfrak{G}_{x}(X)$ the space of the germs at $x$.

For a locally profinite space $X^{\prime}$, an open subset $W$ in $X$ and an open subset $W^{\prime}$ in $X^{\prime}$, an homeomorphism $j: W \rightarrow W^{\prime}$ gives by functoriality an isomorphism $C_{c}^{\infty}\left(W^{\prime} ; C\right) \rightarrow$ $C_{c}^{\infty}(W ; C)$ and an isomorphism $\mathfrak{G}_{j(x)}\left(X^{\prime}\right) \rightarrow \mathfrak{G}_{x}(X)$ from the space of the germs of $X^{\prime}$ at $j(x)$ to the space of the germs of $X$ at $x \in W$.

The nilpotent orbital integrals $\mathcal{F}_{\mathfrak{O}}: \varphi \mapsto I_{\mathfrak{O}}(\hat{\varphi})$ for $\varphi \in C_{c}^{\infty}($ Lie $H ; \mathbb{C})$, and the nilpotent $H$-orbits $\mathfrak{O}$ in $\operatorname{Lie}(H)$, are linearly independent $H$-equivariant linear forms on $C_{c}^{\infty}(\operatorname{Lie} H ; \mathbb{C})$. They form a basis of a $\mathbb{Z}$-module $I_{H}$ with rank $1+2^{d}$ (6.3). For each $H$-equivariant open neighborhood $V$ of 0 in Lie $H$, the $\mathcal{F}_{\mathfrak{G}}$ remain independent as linear forms on $C_{c}^{\infty}(V ; \mathbb{C})$. The germs $\tilde{\mathcal{F}}_{\mathfrak{O}}$ form a basis of the $\mathbb{Z}$-module $\tilde{I}_{H}$ of germs of elements of $I_{H}$. Denote by $I_{H}^{W h}$ the $\mathbb{Z}$-submodule of $I_{H}$ of basis $\mathcal{F}_{\mathfrak{G}}$ for $\mathfrak{O} \neq 0$.

Theorems 6.6 and 6.9 say that the germ at 1 of the trace of an irreducible complex smooth representation $\pi$ of $H$ identifies via the map $X \rightarrow 1+X$ with the germ at 0 of a unique element $T_{\pi}=c_{0}(\pi) \mathcal{F}_{0}+T_{\pi}^{W h}$ where $c_{0}(\pi) \in \mathbb{Q}$, and $T_{\pi}^{W h} \in I_{H}^{W h}$ is determined by the non-degenerate Whittaker models of $\pi$. Note that $T_{\pi}^{W h}=0$ if and only if $\operatorname{dim}_{\mathbb{C}} \pi=1$.

Denote by $T_{H}^{W h}$ the $\mathbb{Z}$-submodule of $I_{H}^{W h}$ generated by respectively the $T_{\pi}^{W h}$, for all irreducible complex smooth representations $\pi$ of $H$. Write $\tilde{I}_{H}^{W h}, \tilde{T}_{H}^{W h}$ for the space of germs at 0 of $I_{H}^{W h}, T_{H}^{W h}$.
Theorem 6.14. We have $\tilde{T}_{H}=\tilde{I}_{H}$ when $d=0,1$.
The $\mathbb{Z}$-submodule $\tilde{T}_{H}^{W h}$ is a submodule of $\tilde{I}_{H}^{W h}$ of finite index. The exponent of $\tilde{I}_{H}^{W h} / \tilde{T}_{H}^{W h}$ is $2^{d-2}$ when $d \geq 2$.

Proof. When $d=0$, then $I_{H}$ has $\mathbb{Z}$-rank 2, and the germs of the traces of the trivial representation 1 and of the Steinberg representation $\mathrm{st}_{H}$ form a $\mathbb{Z}$-basis $\left\{\tilde{\mathrm{tr}}_{1}, \tilde{\mathrm{tr}}_{\mathrm{st}_{H}}\right\}$ of $\tilde{I}_{H}$.

When $d=1$, then $I_{H}$ has $\mathbb{Z}$-rank 3 , $\operatorname{det} H=N_{E / F}\left(E^{*}\right)$ for a quadratic separable extension $E / F$ (Remark 6.3), the principal series $\left.\left(i_{B}^{G} \eta_{E}\right)\right|_{H}$ is semi-simple of length 2 and multiplicity free (Lemma 2.3 and footnote in the proof of Proposition 4.27), and the germs of the traces of the trivial representation 1 and of the two components $\pi_{E}^{+}, \pi_{E}^{-}$of $\left.\left(i_{B}^{G} \eta_{E}\right)\right|_{H}$ form a $\mathbb{Z}$-basis $\left\{\tilde{\operatorname{tr}}_{1}, \tilde{\operatorname{tr}}_{\pi_{E}^{+}}, \tilde{\operatorname{tr}}_{\pi_{E}^{-}}\right\}$of $\tilde{I}_{H}$.

When $d \geq 2$, the theorem follows from Lemma 6.4.
Theorem 6.14 can be equally well expressed in terms of the Grothendieck group $\operatorname{Gr}_{R}(H)$. This is under this form that the theorem extends to $R$-representations. For an open compact subgroup $K$ of $H$, and $\pi$ a finite length smooth complex representation $\pi$ of $H$, $\left.\pi\right|_{K}$ is semi-simple wifh finite multiplicities, and is determined by the restriction of the trace of $\pi$ to $C_{c}^{\infty}(K, \mathbb{C})$.

Corollary 6.15. There are $2^{d}$ virtual finite length smooth complex representations $\pi_{1}, \ldots, \pi_{2^{d}}$ of $H$ with the following property: for any finite length smooth complex representation $\pi$ of $H$, there are unique integers $a_{0}(\pi), a_{1}(\pi), \ldots, a_{2^{d}}(\pi)$, such that on some compact open subgroup $K=K(\pi)$ of $H$,

$$
\pi \simeq a_{0}(\pi) 1+\sum_{i=1}^{2^{d}} a_{i}(\pi) \pi_{i}
$$

Proof. By Theorem [6.14, the $\mathbb{Z}$-module $\tilde{T}_{H}^{W h}$ has a basis $\left\{\tilde{T}_{\pi_{1}}^{W h}, \ldots, \tilde{T}_{\pi_{2 d}}^{W h}\right\}$ where $\pi_{1}, \ldots, \pi_{2^{d}}$ are virtual finite length smooth representations of $H$. By Theorem 6.6, for any finite length smooth representation $\pi$ of $H$ there exist a unique rational number $a_{0}(\pi)$ and unique integers $a_{1}(\pi), \ldots, a_{2^{d}}(\pi)$, such that

$$
\operatorname{tr}_{\pi}=a_{0}(\pi) \operatorname{tr}_{1}+\sum_{i=1}^{2^{d}} a_{i}(\pi) \operatorname{tr}_{\pi_{i}}
$$

on restriction to $C_{c}^{\infty}(K(\pi), \mathbb{C})$ for some compact open subgroup $K(\pi)$ of $H$. As $a_{0}(\pi)=$ $\operatorname{dim}_{\mathbb{C}} \pi^{K(\pi)}-\sum_{i=1}^{2^{d}} a_{i}(\pi) \operatorname{dim}_{\mathbb{C}} \pi_{i}^{K(\pi)}$, we see that $a_{0}(\pi)$ is an integer. Equivalently, on restriction to $K(\pi)$,

$$
\pi \simeq a_{0}(\pi) 1+\sum_{i=1}^{2^{d}} a_{i}(\pi) \pi_{i}
$$

6.2.5. This has consequences for the representations of $G^{\prime}$.

An irreducible complex representation of $G^{\prime}$ extends to $Z G^{\prime}$, and we can apply Theorem 6.6 to $H=Z G^{\prime}$ when $\operatorname{char}_{F} \neq 2$. When $p$ is odd, there is an unique $L$-packet $\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}$ of $G^{\prime}$ with 4 elements (Proposition 4.23). One can enumerate the 4 non-trivial nilpotent $G^{\prime}$-orbits $\mathfrak{O}_{1}, \ldots, \mathfrak{O}_{4}$ such that $c_{\mathfrak{D}_{i}}\left(\tau_{j}\right)=\left\{\begin{array}{l}1 \text { if } i=j \\ 0 \text { if } i \neq j\end{array}\right.$. For $i=1, \ldots, 4$ we choose a lower triangular element $Y_{i} \in \mathfrak{O}_{i}$.

Theorem 6.16. ( $p$ odd, $R=\mathbb{C}$ ) Let $\pi$ be a finite length smooth complex representation of $G^{\prime}$. On restriction to a small enough compact open subgroup $K(\pi)$ of $G^{\prime}$, we have

$$
\begin{equation*}
\pi \simeq a_{0}(\pi) 1+\sum_{i=1}^{4} c_{\mathfrak{刃}_{i}}(\pi) \tau_{i}, \quad c_{\mathfrak{D}_{i}}(\pi)=\operatorname{dim}_{\mathbb{C}} W_{Y_{i}}(\pi) \tag{6.14}
\end{equation*}
$$

where $a_{0}(\pi)=\operatorname{dim}_{\mathbb{C}} \pi^{K(\pi)}-\sum_{i=1}^{4} c_{\mathfrak{D}_{i}}(\pi) \operatorname{dim}_{\mathbb{C}} \tau_{i}^{K(\pi)}$. The constant term in Theorem (6.6) is

$$
c_{0}(\pi)=a_{0}(\pi)-\left(\sum_{i=1}^{4} c_{\mathfrak{Q}_{i}}(\pi)\right) / 2
$$

The constant term $c_{0}(\pi)$ can be computed using (6.7) and (6.8).
Remark 6.17. When char ${ }_{F}=0, p$ odd and $R=\mathbb{C}$, the theorem was known (Assem94) and the last section of (Nevins23).
6.2.6. For any $p$, let $\pi$ be an irreducible smooth complex representation of $G^{\prime}$ and $r$ the cardinality of the $L$-packet of $\pi$.

For any $L$-packet $\left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$ of size 4 , there exist integers $a_{0}, b_{0}$ such that on a small enough compact open subgroup $K$ of $G^{\prime}$ we have

$$
\begin{equation*}
\operatorname{ind}_{B^{\prime}}^{G^{\prime}} 1 \simeq b_{0} T_{1}+\sum_{i=1}^{4} \tau_{i}, \quad \text { and if } r=1, \quad \pi \simeq a_{0} T_{1}+\sum_{i=1}^{4} \tau_{i} . \tag{6.15}
\end{equation*}
$$

If $r=2$, then $\operatorname{det}\left(G_{\pi}\right)=N_{E / F}\left(E^{*} / F\right)$ for a quadratic separable extension $E / F$. Choose a bi-quadratic separable extension of $F$ containing $E$. There exist $\tau_{1}$ and $\tau_{2}$ in the associated $L$-packet of size 4 (Proposition4.23) and an integer $a_{0}$ such that on a small enough compact open subgroup $K$ of $G^{\prime}$ we have

$$
\begin{equation*}
\pi \simeq a_{0} T_{1}+\sum_{i=1}^{2} \tau_{i} \tag{6.16}
\end{equation*}
$$

Therefore, when $R=\mathbb{C}$ we have:
Theorem 6.18. Let $\pi$ be an irreducible smooth $R$-representation of $G^{\prime}$. There are an integer $a_{0}$ and irreducible smooth $R$-representations $\left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$ of $G^{\prime}$ forming an $L$ packet, such that on a small enough compact open subgroup $K$ of $G^{\prime}$ we have

$$
\pi \simeq a_{0} 1+\sum_{i=1}^{4 / r} \tau_{i}
$$

where $r$ is the cardinality of the L-packet containing $\pi$.
6.2.7. Let us prove Theorem 6.18 for any $R$.

Put $R_{c}=\mathbb{Q}^{a c}$ when $\operatorname{char}_{R}=0$ and $R_{c}=\mathbb{F}_{\ell}^{a c}$ when $\operatorname{char}_{R}=\ell>0$.
a) We show first that Theorem 1.8 for $R_{c}$ extends to $R$. A cuspidal $R$-representation of $G^{\prime}$ is the scalar extension $\pi_{R}=R \otimes_{R_{c}} \pi$ to $R$ of a cuspidal $R_{c}$-representation $\pi$ of $G^{\prime}$ Vignéras96 and the $L$-packets of size 4 are cuspidal. The scalar extension from $R_{c}$ to $R$ respects irreducibility, identifies the $L$-packets of size 4 over $R_{c}$ with those over $R$
and sends the $L$-packets of size $r$ over $R_{c}$ to $L$-packets of size $r$ over $R$. Theorem 1.8 for $R_{c}$-representations imply Theorem 1.8 extends for $R$-representations which are scalar extensions of $R_{c}$-representations:

$$
\pi \simeq a_{0} 1+\sum_{i=1}^{4 / r} \tau_{i} \text { implies by scalar extension } \pi_{R} \simeq a_{0} 1+\sum_{i=1}^{4 / r} \tau_{i, R}
$$

The only irreducible smooth $R$-representations of $G^{\prime}$ which are not scalar extensions of $R_{c^{\prime}}$-representations, are principal series $i_{B^{\prime}}^{G^{\prime}}(\eta)$. But

$$
\begin{equation*}
i_{B^{\prime}}^{G^{\prime}}(\eta) \simeq \operatorname{ind}_{B^{\prime}}^{G^{\prime}}(1) \text { on some small open compact subgroup } K \text { of } G^{\prime} \tag{6.17}
\end{equation*}
$$

and we have (6.15) for the $R_{c}$-representation $\operatorname{ind}_{B^{\prime}}^{G^{\prime}}(1)$. Therefore, for any $L$-packet $\left\{\tau_{1, R}, \tau_{2, R}, \tau_{3, R}, \tau_{4, R}\right\}$ of size 4 , there is an integer $a_{0}$ such that

$$
\operatorname{ind}_{B^{\prime}}^{G^{\prime}}(1) \simeq a_{0} 1+\sum_{i=1}^{4} \tau_{i, R} \quad \text { on some small open compact subgroup } K \text { of } G^{\prime}
$$

b) Theorem 6.18 for $\mathbb{C}$ extends to $\mathbb{Q}^{a c}$ because the scalar extension from $\mathbb{Q}^{a c}$ to $\mathbb{C}$ respects irreducibility, representations in an $L$-packet of size 4 are cuspidal, and cuspidal representations complex representations of $G^{\prime}$ are defined over $\mathbb{Q}^{a c}$.
c) Via an isomorphism $\mathbb{C} \simeq \mathbb{Q}_{\ell}^{a c}$, Theorem 1.8 for $\mathbb{C}$ extends to $\mathbb{Q}_{\ell}^{a c}$. Theorem 6.18 for $\mathbb{Q}_{\ell}^{a c}$ extends to $\mathbb{F}_{\ell}^{a c}$-representations. Indeed, from Proposition 4.31 an irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representation $\pi$ of $G^{\prime}$ in an $L$-packet of size $r$, lifts to an integral irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\pi}$ of $G^{\prime}$ in an $L$-packet of size $r$ (Proposition 1.6). From Theorem 6.18 for $\mathbb{Q}_{\ell}^{a c}$, there is an $L$-packet $\left\{\tilde{\tau}_{1}, \tilde{\tau}_{2}, \tilde{\tau}_{3}, \tilde{\tau}_{4}\right\}$ of irreducible smooth $\mathbb{Q}_{\ell}^{a c}$-representations of $G^{\prime}$ and an integer $a_{0}$, such that on a small enough compact open subgroup $K$ of $G^{\prime}$ we have

$$
\tilde{\pi} \simeq a_{0} 1+\sum_{i=1}^{4 / r} \tilde{\tau}_{i} . \text { By reduction modulo } \ell \pi \simeq a_{0} 1+\sum_{i=1}^{4 / r} \tau_{i}
$$

where the reduction $\left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$ modulo $\ell$ of $\left\{\tilde{\tau}_{1}, \tilde{\tau}_{2}, \tilde{\tau}_{3}, \tilde{\tau}_{4}\right\}$ forms an $L$-packet of irreducible smooth $\mathbb{F}_{\ell}^{a c}$-representations of $G^{\prime}$. This ends the proof of Theorem 1.8.
Remark 6.19. From the proof, the recipes giving (6.15) and (6.16) remain valid for $R$.
6.2.8. Assume $p$ odd and $R=\mathbb{C}$. Let $x$ be a vertex of the Bruhat-Tits building of $G^{\prime}$. In Nevins23, are defined admissible complex representations $\tau_{x, 1}, \ldots \tau_{x, 5}$ of the maximal open compact subgroup $G_{x}^{\prime}$ fixing $x$ such that the following is true. Let $\pi$ be an irreducible smooth complex representation of $G^{\prime}$ of depth $r_{\pi}$ in the sense of Moy-Prasad. Then, there are integers $a_{\pi, 1}, \ldots, a_{\pi, 5}$ such that on restriction to $G_{x, r_{\pi+}}^{\prime}$,

$$
\pi \simeq \sum_{i=1}^{5} a_{\pi, i} \tau_{x, i}
$$

Now allow any $R$ with char ${ }_{R} \neq p$ and $p$ odd. The representations $\tau_{x, i}$ of Nevins transfered to $\mathbb{Q}_{\ell}^{a c}$ are integral, defined over $\mathbb{Q}^{a c}$ and can be transfered to $R$-representations $\tau_{x, i, R}$. The proof in $\$ 6.2 .7$ applies and shows that the above result is also valid over $R$ with $\tau_{x, 1, R}, \ldots \tau_{x, 5, R}$.

## 7. Appendix - The finite group $S L_{2}\left(\mathbb{F}_{q}\right)$

Let $k$ be a finite field of characteristic $p$ with $q$ elements. In this appendix we classify irreducible representations of $G=G L_{2}(k)$ and of $G^{\prime}=S L_{2}(k)$ over an algebraically closed field $R$ of characteristic 0 or $\ell>0, \ell \neq p$. We could use [Bonnafé11] for char ${ }_{R} \neq 2$ and Kleshchev-Tiep09] for any $R$, but we prefer using the same methods as in the main text.

Note that the irreducible $R$-representations of the finite groups $G$ and $G^{\prime}$ are defined over the algebraic closure $R_{c}$ of the prime field, and we can freely pass from $R$ to any other algebraically closed field of the same characteristic. Thus it is enough to consider the cases where $R=\mathbb{C}$ or $R=\mathbb{F}_{\ell}^{a c}$. We aim also to prove the following theorem.

Theorem 7.1. Any irreducible $\mathbb{F}_{\ell}^{a c}$ representation $\sigma$ of $G L_{2}(k)$ is the reduction modulo $\ell$ of a $\mathbb{Q}_{\ell}^{a c}$-representation $\tilde{\sigma}$ of $G L_{2}(k)$ such that $\left.\tilde{\sigma}\right|_{S L_{2}(k)}$ and $\left.\sigma\right|_{S L_{2}(k)}$ have the same length.

Any irreducible $\mathbb{F}_{\ell}^{a c}$-representation of $S L_{2}(k)$ is the reduction modulo $\ell$ of a $\mathbb{Q}_{\ell}^{a c}$-representation of $S L_{2}(k)$.

Write $Z$ for the centre of $G, B$ for the upper triangular subgroup of $G$, and $U$ for its unipotent radical. Let us first recall the known classification of the $R$-representations of $G$ (Bushnell-Henniart02] for $R=\mathbb{C}$ and Vignéras88 for $R=\mathbb{F}_{\ell}^{a c}$ ).

The parabolically induced representation $\operatorname{ind}_{B}^{G}(1)$ realised by the space of constant functions on $B \backslash G$, contains the trivial character. It also has the trivial character as a quotient, given by the functional $\lambda$ which sums the values of functions on $B \backslash G$. The map from the trivial subrepresentation to the trivial quotient is multiplication by $q+1$, so is an isomorphism if $\ell$ does not divide $q+1$, and is 0 otherwise. In the first case the quotient St $=\operatorname{ind}_{B}^{G}(1) / 1$ is irreducible, in the second case $\operatorname{Ker}(\lambda) / 1$ is a cuspidal but not supercuspidal representation $\sigma_{0}$ of $G$.

The irreducible (classes of) $R$-representations $\sigma$ of $G$ are :

1) The characters $\chi \circ$ det where $\chi$ is an $R$-character of $k^{*}$.
2) When $q+1 \neq 0$ in $R$, the twists $(\chi \circ \operatorname{det}) \otimes$ St of St by the $R$-characters $\chi \circ \operatorname{det}$ of $G$.
$2^{\prime}$ ) When $q+1=0$ in $R$, the twists $(\chi \circ \operatorname{det}) \otimes \sigma_{0}$ of $\sigma_{0}$ by the $R$-characters $\chi \circ \operatorname{det}$ of $G$.
3) The irreducible principal series $\operatorname{ind}_{B}^{G}\left(\chi_{1} \otimes \chi_{2}\right)$, where $\chi_{1}$ and $\chi_{2}$ are two distinct $R$-characters of $k^{*}$.
4) The supercuspidal representations $\sigma(\theta)$, where $\theta$ is an $R$-character of $k_{2}^{*}, \theta \neq \theta^{q}$, where $k_{2} / k$ is a quadratic extension.

The only isomorphisms between those representations are given by exchanging $\chi_{1}$ and $\chi_{2}$ in 3), $\theta$ and $\theta^{q}$ in 4).

Twisting by an $R$-character $\chi \circ \operatorname{det}$ of $G$ has the obvious effect, for example sending $\theta$ to $(\chi \circ N) \theta$ where $N(x)=x^{q+1}$ for $x \in k_{2}^{*}$ in 4$)$.

Any irreducible $R$-representation $\tau$ of $G^{\prime}$ is contained in the restriction $\left.\sigma\right|_{G^{\prime}}$ to $G^{\prime}$ of an irreducible $R$-representation $\sigma$ of $G$. The representation $\left.\sigma\right|_{G^{\prime}}$ is semi-simple of multiplicity 1 and its irreducible components are $G$-conjugate. The stabilizer of $\tau$ contains $Z G^{\prime}$ and $G / Z G^{\prime}$ is isomorphic to $k^{*} /\left(k^{*}\right)^{2}$. We have $\left|k^{*} /\left(k^{*}\right)^{2}\right|=1$ when $p=2$ and $\left|k^{*} /\left(k^{*}\right)^{2}\right|=2$ when $p$ is odd. Therefore $\left.\sigma\right|_{G^{\prime}}$ is irreducible when $p=2$ and $\left.\sigma\right|_{G^{\prime}}$ has length 1 or 2 when $p$ is odd.

When $\operatorname{char}_{R} \neq 2$, the length $\lg \left(\left.\sigma\right|_{G^{\prime}}\right)$ of $\left.\sigma\right|_{G^{\prime}}$ is the number of $R$-characters $\chi$ of $k^{*}$ such that $(\chi \circ \operatorname{det}) \otimes \sigma \simeq \sigma$, so

$$
\lg \left(\left.\sigma\right|_{G^{\prime}}\right)= \begin{cases}2 & \text { in case } \left.3) \text { if }\left(\chi_{1} / \chi_{2}\right)^{2}=1 \text { and in case } 4\right) \text { if }\left(\theta^{q-1}\right)^{2}=1  \tag{7.1}\\ 1 & \text { otherwise }\end{cases}
$$

The restrictions $\left.\sigma_{1}\right|_{G^{\prime}},\left.\sigma_{2}\right|_{G^{\prime}}$ of two irreducible representations $\sigma_{1}, \sigma_{2}$ of $G$ are isomorphic if and only $\sigma_{1}, \sigma_{2}$ are twists of each other by an $R$-character of $G$. Otherwise $\left.\sigma\right|_{G^{\prime}},\left.\sigma_{2}\right|_{G^{\prime}}$ are disjoint. So, we have a classification of the (isomorphism classes of) irreducible representations of $G^{\prime}$ when $\operatorname{char}_{R} \neq 2$.

Remark 7.2. The restriction to $B$ of a cuspidal representation of $G$ is the Kirillov representation $\kappa$ of $B$ (the irreducible $R$-representation of B induced by any non-trivial $R$-character of $U$ ). The restriction of $\kappa$ to $U$ is the direct sum of all non-trivial $R$-characters of $U$. The group $B$ acts transitively on such characters, whereas $B^{\prime}=B \cap G^{\prime}$ acts with two orbits. It follows that the restriction of $\kappa$ to $B^{\prime}$ has two inequivalent irreductible components. Consequently a cuspidal representation of $G$ restricts to $G^{\prime}$ with length 1 or 2.

Let $\ell$ be an odd prime number different from $p$. Let us consider the reduction modulo $\ell$ of the previous irreducibles $\sigma$ over $\mathbb{Q}_{\ell}^{a c}$ (since $G$ is finite they are integral). For an integral $\mathbb{Q}_{\ell}^{a c}$-character $\chi$ (with values in $\mathbb{Z}_{\ell}^{a c}$ ) let $\bar{\chi}$ denote its reduction modulo $\ell$. Reduction modulo $\ell$ is compatible with twisting by a $\mathbb{Q}_{\ell}^{a c}$-character $\chi \circ$ det in the sense that the reduction of $(\chi \circ \operatorname{det}) \otimes \sigma$ is the twist by $\bar{\chi} \circ \operatorname{det}$ of the reduction of $\sigma$.

1) The trivial $\mathbb{Q}_{\ell}^{a c}$-character of $G$ reduces to the trivial $\mathbb{F}_{\ell}^{a c}$-character.
2) When $\ell$ does not divide $q+1$, the Steinberg $\mathbb{Q}_{\ell}^{a c}$-representation reduces to the Steinberg $\mathbb{F}_{\ell}^{a c}$-representation.

2') When $\ell$ divides $q+1$, the Steinberg $\mathbb{Q}_{\ell}^{a c}$-representation reduces to a length 2 representation with subrepresentation $\sigma_{0}$ and trivial quotient (for the natural integral structure).
3) The irreducible principal series $\operatorname{ind}_{B}^{G}\left(\chi_{1} \otimes \chi_{2}\right)$ reduces to the irreducible principal series $\operatorname{ind}_{B}^{G}\left(\bar{\chi}_{1} \otimes \bar{\chi}_{2}\right)$ when $\bar{\chi}_{1} \neq \bar{\chi}_{2}$, and to $\left(\bar{\chi}_{1} \circ \operatorname{det}\right) \otimes \operatorname{ind}_{B}^{G}(1)$ (of length 2 when $\ell$ does not divide $q+1$, and length 3 otherwise) when $\bar{\chi}_{1}=\bar{\chi}_{2}$ (for the natural integral structure).
4) The supercuspidal $\mathbb{Q}_{\ell}^{a c}$-representation $\sigma(\theta)$, reduces to the supercuspidal $\mathbb{F}_{\ell}^{a c}$-representation $\sigma(\bar{\theta})$ if $\bar{\theta} \neq(\bar{\theta})^{q}=\overline{\theta^{q}}$, and otherwise (which can happen only if $\ell$ divides $q+1$ ) to ( $\eta \circ \operatorname{det}$ ) $\otimes \sigma_{0}$ where $\eta$ is the $\mathbb{F}_{\ell}^{a c}$-character of $\mathbb{F}_{q}^{*}$ such that $\eta \circ N=\bar{\theta}$.

A given $\mathbb{F}_{\ell}^{a c}$-character of $k^{*}$ or $k_{2}^{*}$ has a unique lift to a $\mathbb{Z}_{\ell}^{a c}$-character of the same order, and from the above it is clear that any irreducible $\mathbb{F}_{\ell}^{a c}$-representation $\sigma$ of $G$ lifts to a $\mathbb{Q}_{\ell}^{a c}$ representation. Moreover, one can choose a lift of $\sigma$ with the same length on restriction to $G^{\prime}$, thus proving the theorem when $\ell$ is odd.

Let us finally assume char ${ }_{R}=2$. Then $p$ is odd and $q+1=0$ in $R$. Write $q-1=2^{s} m$ with a positive integer $s$ and an odd integer $m$. The number of irreducible $R$-representations of $G$ (resp. $Z G^{\prime}$ ) is the number of conjugacy classes in $G$ (resp. $Z G^{\prime}$ ) of elements of odd order. Let $g \in G$ of odd order. Then $\operatorname{det}(g) \in k^{*}$ has odd order so $\operatorname{det}(g) \in\left(k^{*}\right)^{2}$ and $g \in Z G^{\prime}$. The $G$-conjugacy class of $g$ is equal to its $Z G^{\prime}$-conjugacy class unless the $G$-centralizer of $g$ is entirely in $Z G^{\prime}$. In that exceptional case, the $G$-equivalence class of $g$ is the union of
two $Z G^{\prime}$-equivalence classes. This happens only when $g=z u$ where $z \in Z$ (of odd order) and $u \neq 1$ is unipotent. That shows that $m$ is the number of $Z G^{\prime}$-conjugacy classes of elements of odd order minus the number of $G$-conjugacy of such elements. Consequently $m$ is the number of irreducible $R$-representations of $Z G^{\prime}$ minus the number of irreducible $R$-representations of $G$.

Consider first $\sigma(\theta)$ for a $\mathbb{Q}_{2}^{a c}$-character $\theta$ of $k_{2}^{*}$ of order $2^{s+1}$. Certainly $\bar{\theta}$ is trivial so that the reduction of $\sigma(\theta)$ modulo 2 is $\sigma_{0}$. But $\ell\left(\left.\sigma(\theta)\right|_{G^{\prime}}\right)=2$ by (7.1), from which it follows that $\ell\left(\left.\sigma_{0}\right|_{G^{\prime}}\right) \geq 2$. We have seen however that $\ell\left(\left.\sigma_{0}\right|_{G^{\prime}}\right) \leq 2$ (Remark [7.2), so $\ell\left(\left.\sigma_{0}\right|_{G^{\prime}}\right)=2$, and each irreducible component of $\left.\sigma_{0}\right|_{G^{\prime}}$ lifts to an irreducible component of $\left.\sigma(\theta)\right|_{G^{\prime}}$. The $\mathbb{F}_{2}^{a c}$-characters $\chi$ of $k^{*}$ have an odd order, their number is $m$, and the representations $(\chi \circ \operatorname{det}) \otimes \sigma_{0}$ are not equivalent (the order of $\chi$ is odd). We deduce:

Lemma 7.3. All irreducible $\mathbb{F}_{2}^{a c}$-representations of $G$ restrict irreducibly to $G^{\prime}$ except the twists of $\sigma_{0}$ by characters.

The reduction modulo 2 of any supercuspidal $\mathbb{Q}_{2}^{a c}$-representation of $G^{\prime}$ is irreducible.
We deduce the classification of irreducible $R$-representations of $G^{\prime}$ when $\operatorname{char}_{R}=2$ and Theorem 7.1] when $\ell=2$.

Remark 7.4. For use in the main text we summarize:
a) When $q+1=0$ in $R$, then $\left.\sigma_{0}\right|_{S L_{2}(k)}$ is irreducible if $\operatorname{char}_{R} \neq 2$, and has length 2 if $\operatorname{char}_{R}=2$.
b) In 4) let $b \in k_{2}$ be an element of order $q+1$. We have $\theta \neq \theta^{q} \Leftrightarrow \theta(b) \neq 1$ and $\left.\sigma(\theta)\right|_{S L_{2}(k)}$ is irreducible if $\theta^{2}(b) \neq 1$, and has length 2 if $\theta^{2}(b)=1$.

When $\operatorname{char}_{R}=2$, or when $p=2$ hence $(2, q+1)=1$, we have $\theta(b) \neq 1 \Leftrightarrow \theta\left(b^{2}\right) \neq 1$ hence $\left.\sigma(\theta)\right|_{S L_{2}(k)}$ is irreducible for all $\theta \neq \theta^{q}$.

When $\operatorname{char}_{R} \neq 2$ and $p$ is odd, there exists $\theta$ such that $\theta(b) \neq 1, \theta(b)^{2}=1$, unique modulo the twist by a character $\chi$ such that $\chi(b)=1$. The corresponding representations $\sigma(\theta)$ of $G$ are twists of each other by a character of $G$. Their restrictions to $S L_{2}(k)$ are isomorphic and reducible of length 2.

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[^0]:    ${ }^{1}$ This proof was suggested by Peyi Cui, and replaces a more complicated argument of ours

[^1]:    ${ }^{2}$ the map $f$ will also appear in $\$ 5.0 .3$

[^2]:    $3_{\text {in }}$ loc.cit. it is proved only that $\left(J^{0}, \lambda\right)$ is unique modulo $G$-conjugacy, but $J$ is the normalizer of $\left(J^{0}, \lambda\right)$ and $\Lambda$ is $\lambda$-isotypic part of $\Pi$

[^3]:    ${ }^{4}$ When $\operatorname{char}_{F}=2$ the quadratic extension appearing in the construction Bushnell-Henniart06 is not necessarily separable. It is generated by an element $x \in G$, determined up to some open subgroup of $G$, so that modifying $x$ slightly yields a separable extension
    ${ }^{5}$ This can also be obtained using Cui20

[^4]:    ${ }^{6}\left(\sigma_{\Pi}, N_{\Pi}\right)$ is called the $L$-parameter of $\Pi$

[^5]:    ${ }^{7}$ We gave a direct proof when $p$ is odd, this was unnecessary
    ${ }^{8}$ When $p$ is odd and $\operatorname{char}_{R}=2$, there no $\xi$ such that $\sigma(E, \xi)$ is induced from a character of $W_{E^{\prime}}$ for a quadratic extension $E^{\prime} / F$ distinct from $E / F$.

[^6]:    ${ }^{9} \sigma$ extends to a $\mathbb{F}_{\ell}^{a c}$-representation of the Galois group $\mathrm{Gal}_{F}$. As $\mathrm{Gal}_{F}$ is solvable this representation lifts to a $\mathbb{Q}_{\ell}^{a c}$-representation of $\mathrm{Gal}_{F}$ that one restricts to $W_{F}$ to get $\tilde{\sigma}$

[^7]:    ${ }^{10}$ or directly because for a smooth $R$-character $\chi$ of $F^{*}$, the property (i) in Proposition 4.12 implies $(\chi \circ \operatorname{det}) \otimes i_{B}^{G}(\eta) \simeq i_{B}^{G}(\eta) \Leftrightarrow \chi \eta=\eta$ or $\eta^{-1} \Leftrightarrow \chi=1$ or $\chi=\eta$ and $\eta^{2}=1$.
    ${ }^{11}$ See Cui20 Example 3.11 Method 2.

[^8]:    ${ }^{12}$ We use the same notation $(\sigma, N)$ for the Deligne morphism of $W_{F}$ into $G L_{2}(R)$
    ${ }^{13} N$ is nilpotent in $\operatorname{Lie}\left(P G L_{2}(R)\right)$ if the Zariski closure of the $P G L_{2}(R)$-orbit of $N$ contains 0

[^9]:    ${ }^{14} \iota(g)$ is conjugate to the transpose of the inverse of $g$

[^10]:    ${ }^{15}$ The level is the normalized level of Bushnell-Henniart06 §12.6 and the depth in the sense of MoyPrasad.

