

TWO-SIDED BELL-SHAPED SEQUENCES

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Abstract. A nonnegative real function f is *bell-shaped* if it converges to zero at $\pm\infty$ and the n th derivative of f changes sign n times for every $n = 0, 1, 2, \dots$. Similarly, a nonnegative sequence $(a(k) : k \in \mathbb{Z})$ is *bell-shaped* if it converges to zero at $\pm\infty$ and the n th iterated difference of $a(k)$ changes sign n times for every $n = 0, 1, 2, \dots$. A characterisation of bell-shaped functions was given by Thomas Simon and the first named author, and recently a similar result for one-sided bell-shaped sequences was found by the authors. In the present article we give a complete description of two-sided bell-shaped sequences. Our main result proves that bell-shaped sequences are convolutions of *Pólya frequency sequences* and what we call *absolutely monotone-then-completely monotone sequences*, and it provides an equivalent, and relatively easy to verify, condition in terms of holomorphic extensions of the generating function.

1. Introduction

1.1. History of bell-shape in a nutshell. The notion of a bell-shaped function has been present in mathematical literature since 1940s when it was introduced in the context of statistical games (see Section 6.11.C in [6]). A nonnegative, smooth real function f is *bell-shaped* if it converges to zero at $\pm\infty$ and its n th derivative $f^{(n)}$ changes sign exactly n times for every $n = 0, 1, 2, \dots$. Many common probability distributions have bell-shaped density functions, including the normal distribution $(2\pi)^{-1/2} \exp(-x^2/2)$, the Cauchy distribution $\pi^{-1}(1+x^2)^{-1}$, and the Lévy distribution $(2\pi)^{-1/2} \exp(1/(2x))\mathbf{1}_{(0,\infty)}(x)$. In fact, as proved in [7], all stable distributions have bell-shaped densities; see also [3, 12]. There are no compactly supported bell-shaped functions [4]. Density functions of hitting times of 1-D diffusion processes are examples of one-sided (that is, supported in a half-line) bell-shaped functions [5]. A complete characterisation of the class of bell-shaped functions was given in [8]. As a corollary, it follows that probability distributions with bell-shaped density functions are necessarily infinitely divisible.

At this point, a natural question arises: is there a discrete analogue of the theory of bell-shaped functions? In other words, can one prove similar results for appropriately defined bell-shaped sequences? This problem was tackled in the authors' previous work [9], where the following definition was introduced. A nonnegative two-sided sequence $(a(k) : k \in \mathbb{Z})$ is *bell-shaped* if it converges to zero at $\pm\infty$ and the sequence of its n th iterated differences $(\Delta^n a(k))$ changes sign exactly n times for every $n = 0, 1, 2, \dots$.

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A one-sided sequence $(a(k) : k \in \mathbb{N})$ (where $\mathbb{N} = \{0, 1, 2, \dots\}$) can be identified with the corresponding two-sided sequence satisfying $a(k) = 0$ for $k < 0$. The methods used in [9] only allowed to characterise one-sided bell-shaped sequences. Theorem 1.1 in [9] provides two equivalent conditions for a one-sided sequence to be bell-shaped, in terms of the holomorphic extension of the generating function, or in terms of Pólya frequency sequences and completely monotone sequences. More precisely, the former condition requires that the generating function is the exponential of a Pick function with appropriate boundary values. The latter one asserts that every one-sided bell-shaped sequence is the convolution of a summable Pólya frequency sequence and a completely monotone sequence which converges to zero. As a corollary, all discrete stable distributions (see [13]) have one-sided bell-shaped probability mass functions.

The above results are completely analogous to those available for one-sided bell-shaped functions. However, the theory of bell-shaped sequences is not entirely parallel to its continuous counterpart: while there are no compactly supported bell-shaped functions, many finitely supported sequences are bell-shaped. For example, binomial distributions have bell-shaped probability mass functions $\binom{n}{k} p^k (1-p)^{n-k} \mathbb{1}_{\{0,1,\dots,n\}}(k)$.

The main purpose of the present work is to extend the results of [9] and characterise all two-sided bell-shaped sequences. The major difference, and hence the main difficulty, lies in the fact that for one-sided bell-shaped sequences, the generating function is well defined and holomorphic in the unit disc in the complex plane, and standard inversion formulae apply. On the other hand, for two-sided sequences the generating function is only defined on the unit circle. Furthermore, generating functions of two-sided bell-shaped sequences are no longer exponentials of Pick functions. To overcome these difficulties, we need to develop a novel inversion formula, introduce and study a new class of holomorphic functions, and adjust the idea of the proof developed in [7, 8, 9].

1.2. Main results. Before we state our main theorem, we need auxiliary definitions.

Following [9], we say that a real function φ is *stepwise increasing*, if it is integer-valued and nondecreasing. A real function φ is called *increasing-after-rounding* if there exists a stepwise increasing function $\tilde{\varphi}$ such that $\tilde{\varphi} \leq \varphi \leq \tilde{\varphi} + 1$. *Stepwise decreasing* and *decreasing-after-rounding* functions are defined in an analogous way. It is straightforward to see that φ is increasing-after-rounding if $\lfloor \varphi \rfloor$ or $\lceil \varphi \rceil$ are stepwise increasing: in this case we may set $\tilde{\varphi} = \lfloor \varphi \rfloor$ or $\tilde{\varphi} = \lceil \varphi \rceil - 1$. The former condition is, however, slightly more general. We remark that in [7, 8], the term *level crossing condition* is used to describe increasing-after-rounding functions.

A (summable) two-sided *Pólya frequency sequence* is, up to multiplication by a constant, the probability mass function of the sum or difference of at most countably many independent Poissonian, geometric and Bernoulli random variables. A two-sided sequence $a(k)$ is said to be *absolutely monotone-then-completely monotone* if the one-sided sequences $a(k)$ and $a(-k)$ are completely monotone. We refer to Sections 2.3 and 2.4 for a detailed discussion

Theorem 1.1. *Suppose that $a(k)$ is a two-sided sequence. The following conditions are equivalent:*

- (a) $a(k)$ is a bell-shaped sequence;

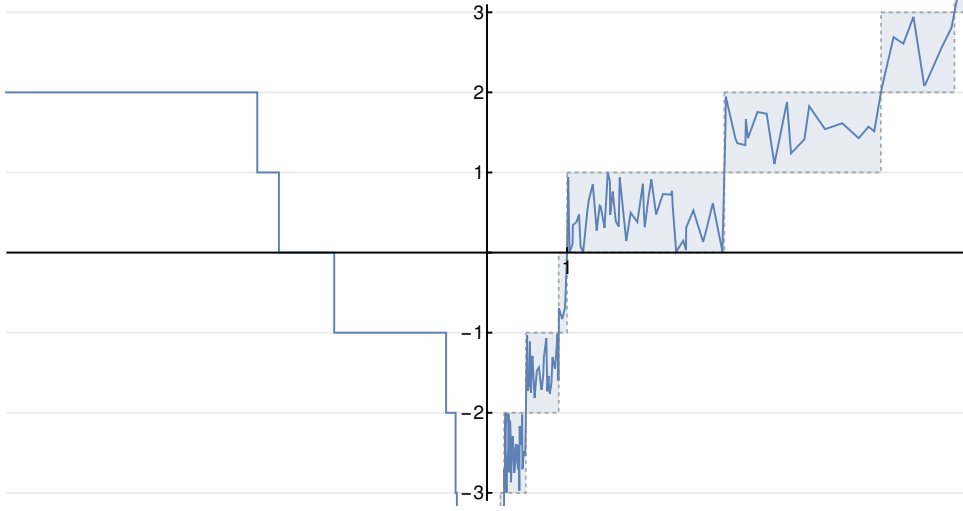


Figure 1. A sample function φ in Theorem 1.1(c).

- (b) $a(k)$ is the convolution of a summable Pólya frequency sequence $b(k)$ and an absolutely monotone-then-completely monotone sequence $c(k)$ which converges to zero as $k \rightarrow \pm\infty$;
- (c) The generating function of $a(k)$,

$$F(z) = \sum_{k=-\infty}^{\infty} a(k)z^k,$$

converges on the unit circle $|z| = 1$, except possibly $z = 1$, and F extends to a holomorphic function in the upper complex half plane, given by

$$F(z) = \exp\left(b^+z + \frac{b^-}{z} + c + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\varphi(s)ds\right) \quad (1.1)$$

when $\text{Im } z > 0$. Here $b^+, b^- \geq 0$, $c \in \mathbb{R}$, and φ is a Borel function on $\mathbb{R}$ such that (see Figure 1):

- (i) φ is stepwise decreasing on $(-\infty, 0)$;
- (ii) φ is increasing-after-rounding on $(0, \infty)$;
- (iii) $\varphi \leq 0$ on $(0, 1)$ and $\varphi \geq 0$ on $(1, \infty)$;
- (iv) φ satisfies the integrability condition

$$\int_{-\infty}^{\infty} \frac{|\varphi(s)|}{s^2+1} ds < \infty;$$

- (v) the function F defined by (1.1) satisfies

$$\lim_{t \rightarrow 0} (e^{it} - 1)F(e^{it}) = 0.$$

Additionally, every quadruple b^+, b^-, c, φ satisfying the conditions listed in item (c) corresponds to a unique bell-shaped sequence $a(k)$.

Remark 1.2. There seems to be no simple way to rewrite condition (v) in Theorem 1.1(c) directly in terms of the function φ . Nevertheless, it is a very natural condition, which, roughly speaking, corresponds to the fact that $\sum_{k=-\infty}^{\infty} \Delta a(k) = 0$. More precisely, we have the following two sufficient conditions for condition (v).

- If F is the generating function of a summable two-sided sequence $a(k)$, then F is continuous on the unit circle $|z| = 1$, and hence condition (v) is automatically satisfied.
- If $a(k)$ converges eventually monotonically to zero as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, then the generating function F of $a(k)$ is well-defined (as a conditionally convergent series) when $|z| = 1$, $z \neq 1$, and $(z-1)F(z)$ is the generating function of $\Delta a(k)$. Furthermore, in this case $\Delta a(k)$ is summable (it has eventually constant sign as $k \rightarrow \infty$ and as $k \rightarrow -\infty$) and it sums up to zero. Thus, $(z-1)F(z)$ extends to a continuous function on the unit circle $|z| = 1$, and it takes value 0 at $z = 1$. This is equivalent to say that condition (v) is satisfied.

Remark 1.3. If the generating function $F(z)$ of a given sequence $a(k)$ is given by an explicit formula when $|z| = 1$, $z \neq 1$, and this formula defines a holomorphic function in the upper complex half-plane $\text{Im } z > 0$ (denoted again by $F(z)$), then it is usually not very difficult to verify whether the conditions of Theorem 1.1(c) are satisfied. Indeed: it suffices to check that F is zero-free in the upper complex half-plane, and then study the continuous version of the argument of $F(z)$, which we denote by $\pi\Phi(z)$. One needs to show that Φ is the sum of two terms: the Poisson integral of an appropriate function φ (which is necessarily equal to the boundary values of Φ), and $b^+ \text{Im } z + b^- \text{Im } z^{-1}$. Here by saying that φ is an appropriate function we mean that conditions (i) through (iv) in Theorem 1.1(c) hold true.

Later in this section we apply the above procedure to two classes of sequences; see Examples 1.10 and 1.11.

Remark 1.4. Observe that

$$\frac{1}{s-z} - \frac{s}{s^2+1} = \frac{1}{s^2+1} \frac{1+sz}{s-z}.$$

Let us agree that $(1+sz)/(s-z) = z$ when $s = \infty$, so that $(1+sz)/(s-z)$ becomes a continuous function of s on $\mathbb{R} \cup \{\infty\}$, the one-point compactification of $\mathbb{R}$. Furthermore, let us denote

$$\sigma(ds) = b^+ \delta_\infty(ds) - b^- \delta_0(ds) + \frac{\varphi(s)}{s^2+1} ds.$$

Then formula (1.1) can be written as

$$F(z) = \exp\left(c + \int_{\mathbb{R} \cup \{\infty\}} \frac{1+sz}{s-z} \sigma(ds)\right). \quad (1.2)$$

Here σ is a finite signed measure on $\mathbb{R} \cup \{\infty\}$, with an appropriate density function on $\mathbb{R} \setminus \{0\}$.

Remark 1.5. Note that a sequence $a(k)$ is bell-shaped if and only if its mirror image $a(-k)$ is bell-shaped. If the generating function of $a(k)$ is given by the representation formula (1.1), then the generating function of $a(-k)$ is equal to $F(1/z)$. Furthermore, $F(1/z)$ is again given by the right-hand side of (1.1), with parameters $b^+, b^-, c, \varphi(s)$ replaced by $b^-, b^+, c, -\varphi(1/s)$.

One-sided bell-shaped sequences correspond to $b^- = 0$ and $\varphi(s) \geq 0$ in Theorem 1.1; see Theorem 1.1 in [9]. Combining this with the above observation, we find that bell-shaped sequences $a(k)$ such that $a(k) = 0$ for $k > 0$ correspond to $b^+ = 0$ and $\varphi(s) \leq 0$ in Theorem 1.1.

Our main result, Theorem 1.1, clearly resembles the corresponding statement for one-sided bell-shaped sequences (Theorem 1.1 in [9]), as well as the analogous result for bell-shaped functions (Theorems 1.1 and 1.3 in [8]). Proofs of all these theorems have a similar structure, developed mostly in [7, 8]. We stress, however, that the proof of Theorem 1.1 given below contains two essentially new elements, already mentioned in the introduction. First, we need a replacement for Post's inversion formula for the Laplace transform: its discrete variant employed in [9] does not apply to two-sided sequences. For this reason we prove in Proposition 2.4 an inversion formula that involves iterated differences of the Fourier-Laplace transform of the generating function. Next, unlike in the case of one-sided sequences, holomorphic functions defined by the right-hand side of (1.1) are no longer exponentials of Pick functions; instead, they are certain exponentials of differences of Pick functions. The pointwise limit of Pick functions is necessarily a Pick function, but this is no longer true for differences of Pick functions. We were unable to find a compactness result that would suit our needs in available literature, and so we needed to prove one. The property that we need in the proof of Theorem 1.1 is given in Lemma 2.5. Its proof turned out to be surprisingly long and technical.

Theorem 1.1 shows that, the convolution of bell-shaped sequences, whenever defined, corresponds to addition of the parameters b^+, b^-, c, φ in the representation formula (1.1) of the corresponding generating functions. Using this fact it is easy to construct bell-shaped sequences whose convolution is not bell-shaped (because the sum of corresponding functions φ is not increasing-after-rounding on $(0, \infty)$). We have, however, the following interesting result.

Corollary 1.6 (Wiener-Hopf factorisation of bell-shaped sequences). *Suppose that $b(k)$ is a one-sided bell-shaped sequence, and $c(k)$ is the mirror image of a one-sided bell-shaped sequence (that is, $c(k)$ is bell-shaped and $c(k) = 0$ for $k > 0$). Suppose furthermore that the convolution $a(k) = b * c(k)$ is well-defined, and that $a(k)$ converges to zero as $k \rightarrow \pm\infty$. Then $a(k)$ is bell-shaped. Conversely, every bell-shaped sequence can be factorised in the way described above.*

Proof. The direct part is a simple consequence of Theorem 1.1 and its one-sided version given in [9]. Indeed: suppose that φ_1 and φ_2 correspond to the functions φ in the representation (1.1) of the generating functions G and H of $b(k)$ and $c(k)$, respectively. The generating function of $a(k)$ is easily shown to be equal to $F(z) = G(z)H(z)$, and therefore F has representation (1.1) with $\varphi = \varphi_1 + \varphi_2$. However, by Remark 1.5, $\varphi_1(s) = 0$ for $s \in [0, 1]$ and $\varphi_2(s) = 0$ for $s \in [1, \infty)$. It is thus easy to see that φ has properties (i) through (iv). Furthermore, since $a(k)$ converges eventually monotonically to zero as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, condition (v) is satisfied by Remark 1.2. Consequently, F is the generating function of a bell-shaped sequence $a(k) = b * c(k)$.

The converse part of the corollary has an even simpler proof: it suffices to define $\varphi_1(s) = \max\{\varphi(s), 0\}$ and $\varphi_2(s) = \min\{\varphi(s), 0\}$, and observe that $\varphi(s) = \varphi_1(s) + \varphi_2(s)$, the parameters $b^+, 0, c, \varphi_1$ correspond to a one-sided bell-shaped sequence $b(k)$ (by Remark 1.5), while the parameters $0, b^-, 0, \varphi_2$ correspond to a sequence $c(k)$ which is the mirror image of a one-sided bell-shaped sequence (again by Remark 1.5). $\square$

By a very similar argument, we are able to describe when convolution powers or convolution roots of bell-shaped sequences are bell-shaped. We leave the details of

the proof to the interested reader, and we refer to Corollaries 1.4 and 1.5 in [8] for analogous results for bell-shaped functions.

Corollary 1.7 (infinitely divisible bell-shaped sequences). *Suppose that a probability mass function $a(k)$ is a two-sided bell-shaped sequence. Then $a(k)$ is the probability mass function of an infinitely divisible distribution if and only if the corresponding function φ defined in Theorem 1.1 is equal to zero on $(-\infty, 0)$. In this case the convolution roots of $a(k)$ are bell-shaped, too.*

Corollary 1.8 (convolution powers of bell-shaped sequences). *Suppose that $a(k)$ is a two-sided bell-shaped sequence, and the convolution powers of $a(k)$ are well-defined. Then the following two conditions are equivalent:*

- (a) *every convolution power of $a(k)$ is bell-shaped;*
- (b) *the corresponding function φ defined in Theorem 1.1 is nondecreasing on $(0, \infty)$.*

1.3. Examples. The following one-sided bell-shaped sequences (that is, sequences $a(k)$ equal to 0 for $k < 0$) have already been discussed in [9]:

- the delta sequence:

$$a(k) = \begin{cases} 1 & \text{for } k = 0; \\ 0 & \text{otherwise;} \end{cases}$$

- geometric sequences:

$$a(k) = q^k,$$

where $q \in (0, 1)$;

- probability mass functions of Poisson random variables:

$$a(k) = e^{-\lambda} \frac{\lambda^k}{k!},$$

where $\lambda > 0$;

- probability mass functions of Bernoulli and, more generally, binomial distributions:

$$a(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where $n = 1, 2, \dots$ and $p \in (0, 1)$;

- one-sided summable Pólya frequency sequences; that is, convolutions of a finite or countable infinite number of the above sequences;
- completely monotone sequences;
- probability mass functions of negative binomial distributions:

$$a(k) = \binom{\lambda + k - 1}{k} p^\lambda (1-p)^k,$$

where $\lambda \in (0, \infty)$ and $p \in (0, 1)$;

- probability mass functions of discrete stable distributions, with generating function

$$\sum_{k=0}^{\infty} a(k) z^k = \exp(-\lambda(1-z)^\nu),$$

where $\lambda \in (0, \infty)$ and $\nu \in (0, 1]$; see [13].

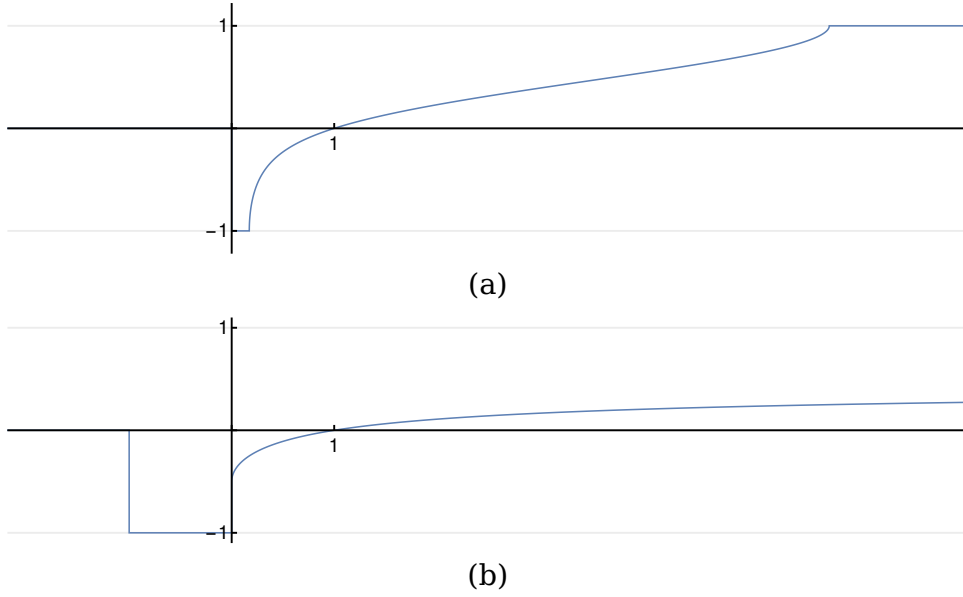


Figure 2. The function φ in: (a) Example 1.10; (b) Example 1.11.

The class of one-sided bell-shaped sequences is not closed under convolutions. However, the main result of [9] describes one-sided bell-shaped sequences as convolutions of completely monotone sequences which converge to zero and one-sided summable Pólya frequency sequences. This is in perfect analogy with our Theorem 1.1.

By Corollary 1.6, any convolution of a one-sided bell-shaped sequence with the mirror image of a one-sided bell-shaped sequence, whenever well-defined and convergent to zero at $\pm\infty$, is again bell-shaped. Thus, for example, two-sided discrete stable distributions are bell-shaped. Here by a two-sided discrete stable distribution we understand the convolution of a one-sided discrete stable distribution and the mirror image of a one-sided discrete stable distribution, both corresponding to the same parameter $\nu \in (0, 1]$.

The following beautiful result for one-sided sequences is due to Bondesson [1]. It is a discrete analogue of the similar theorem about hitting times of one-dimensional diffusions, given in [5].

Example 1.9 (hitting time of random walks). Consider a *random walk* X_n on $\mathbb{N} = \{0, 1, 2, \dots\}$: a discrete-time Markov chain with steps ± 1 as long as $X_n > 0$, and with an absorbing state 0. Suppose that $X_0 = x$, and let $N = \min\{n \in \mathbb{N} : X_n = 0\}$ denote the time to absorption. Then the probability mass function of $\frac{1}{2}(N - x)$ is a one-sided bell-shaped sequence. This follows immediately from Theorem 1 in [1] and Theorem 1.1 in [9]; see also Theorem 3 therein.

Motivated by Bondesson's result, below we give two examples of two-sided bell-shaped sequences, which illustrate a concept that will be significantly extended in a forthcoming work.

Example 1.10 (discrete Poisson kernel of a half-plane for a simple random walk). Consider a simple random walk (X_n, Y_n) on the square lattice $\mathbb{Z}^2$, performing steps $(\pm 1, 0)$ or $(0, \pm 1)$ with probabilities $\frac{1}{4}$. Suppose that $(X_0, Y_0) = (0, y)$, and let $N = \min\{n \in \mathbb{N} : Y_n = 0\}$ denote the hitting time of $\mathbb{Z} \times \{0\}$. Then X_N has a bell-shaped probability mass function.

In order to prove the above claim, we follow the approach described in Remark 1.3. By definition, the generating function F_y of the probability mass function of X_N satisfies

$$F_y(z) = \frac{zF_y(z) + z^{-1}F_y(z) + F_{y+1}(z) + F_{y-1}(z)}{4}$$

when $|z| = 1$ and $y \geq 1$. Additionally, we have $F_0(z) = 1$ and $|F_y(z)| \leq 1$ when $|z| = 1$ and $y \geq 1$. The solution of this second-order linear recurrence equation is given by

$$F_y(z) = \left(2 - \frac{z + z^{-1}}{2} - \sqrt{1 - \frac{z + z^{-1}}{2}} \sqrt{3 - \frac{z + z^{-1}}{2}}\right)^y$$

when $|z| = 1$; we omit the details. Observe that for every $y \geq 0$, the right-hand side extends to a holomorphic function in the upper complex half-plane $\text{Im } z > 0$. Indeed: when $\text{Im } z > 0$, then one easily finds that $1 - (z + z^{-1})/2 \in \mathbb{C} \setminus (-\infty, 0]$, and therefore $\sqrt{1 - (z + z^{-1})/2}$ and $\sqrt{3 - (z + z^{-1})/2}$ are well-defined and holomorphic in this region. In Appendix B we show that F_1 is given by (1.1), with $b^+ = b^- = 0$, with an appropriate constant c , and with

$$\varphi(s) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \text{Arg } F_1(s + it).$$

Additionally, we verify that (see Figure 2(a))

$$\varphi(s) = \begin{cases} 0 & \text{if } s < 0; \\ -1 & \text{if } 0 < s < 3 - 2\sqrt{2}; \\ -\frac{1}{\pi} \text{arccot} \frac{4 - s - s^{-1}}{\sqrt{s + s^{-1} - 2\sqrt{6 - s - s^{-1}}}} & \text{if } 3 - 2\sqrt{2} < s < 1; \\ \frac{1}{\pi} \text{arccot} \frac{4 - s - s^{-1}}{\sqrt{s + s^{-1} - 2\sqrt{6 - s - s^{-1}}}} & \text{if } 1 < s < 3 - \sqrt{2}; \\ 1 & \text{if } s > 3 + 2\sqrt{2}. \end{cases}$$

Since $F_y(z) = (F_1(z))^y$, we conclude that for every $y = 0, 1, 2, \dots$ the function F_y has the representation (1.1) with $b^+ = b^- = 0$, with constant c replaced by cy , and with φ replaced by $y\varphi$. Since φ is nondecreasing on $(0, \infty)$ and constant on $(-\infty, 0)$, it follows by Theorem 1.1 that F_y is the generating function of a two-sided bell-shaped sequence, that is, X_N has a bell-shaped probability mass function.

Example 1.11 (discrete Poisson kernel of a half-plane for two independent simple random walks). Consider a pair (X_n, Y_n) of independent simple random walks on $\mathbb{Z}$, so that (X_n, Y_n) is a random walk in $\mathbb{Z}^2$ which makes steps $(\pm 1, \pm 1)$ with probabilities $\frac{1}{4}$. In other words, (X_n, Y_n) is the simple random walk on the diagonal square lattice (the usual square lattice rotated by $\frac{\pi}{4}$). Suppose that $(X_0, Y_0) = (0, y)$, and let $N = \min\{n \in \mathbb{N} : Y_n = 0\}$ denote the hitting time of $\mathbb{Z} \times \{0\}$. Then $\frac{1}{2}(X_N + y)$ has a bell-shaped probability mass function.

As in the previous example, we follow the approach described in Remark 1.3. The generating function F_y of the probability mass function of $\frac{1}{2}(X_N + y)$ satisfies

$$F_y(z) = \frac{zF_{y+1}(z) + F_{y+1}(z) + F_{y-1}(z) + z^{-1}F_{y-1}(z)}{4}$$

when $|z| = 1$ and $y \geq 1$. Furthermore, $F_0(z) = 1$ and $|F_y(z)| \leq 1$ when $|z| = 1$ and $y \geq 1$. The solution of this second-order linear recurrence equation is given by

$$F_y(z) = \left(\frac{2 + i(z^{1/2} - z^{-1/2})}{z + 1} \right)^y$$

when $|z| = 1$ and $\operatorname{Im} z > 0$; again we omit the details. Clearly, the right-hand side defines a holomorphic function in the upper complex half-plane $\operatorname{Im} z > 0$. In Appendix B we prove that F_1 is given by (1.1), with $b^+ = b^- = 0$, with an appropriate constant c , and with

$$\varphi(s) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \operatorname{Arg} F_1(s + it).$$

Additionally, we verify that (see Figure 2(b))

$$\varphi(s) = \begin{cases} 0 & \text{if } s < -1; \\ -1 & \text{if } -1 < s < 0; \\ \frac{1}{\pi} \arctan \frac{s^{1/2} - s^{-1/2}}{2} & \text{if } s > 0. \end{cases}$$

Since $F_y(z) = (F_1(z))^y$, as in the previous example we conclude that for every $y = 0, 1, 2, \dots$ the function F_y has the representation (1.1) with $b^+ = b^- = 0$, with constant c replaced by cy , and with φ replaced by $y\varphi$. Additionally, φ is nondecreasing on $(0, \infty)$ and stepwise decreasing on $(-\infty, 0)$, and hence, by Theorem 1.1, F_y is the generating function of a two-sided bell-shaped sequence, as desired.

1.4. Organisation of the paper. The remaining part of this article is divided into two sections. In Preliminaries we recall the notions of Pick functions (Section 2.1), generating functions (Section 2.2), absolute and complete monotonicity (Section 2.3), and Pólya frequency sequences (Section 2.4). We also prove various auxiliary results, including an inversion formula (Section 2.5) and a compactness lemma (Section 2.6). In Section 3 of this paper we prove our main result, Theorem 1.1. We first discuss the easy parts (Section 3.1), before we prove the most difficult step, implication (a) $\Rightarrow$ (c) (Section 3.2). Detailed calculations for the proof of the inversion formula and for the examples discussed above are given in two appendices.

2. Preliminaries

Throughout the paper, we denote by $x^{\overline{n}}$ the *raising factorial*:

$$x^{\overline{n}} = x(x+1) \dots (x+n-1).$$

By $\log z$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ we denote the principal branch of the complex logarithm. We also set $\log(-s) = \log s + i\pi$, so that $\log$ becomes continuous on the closed upper complex half-plane with 0 removed. By $\operatorname{Arg} z = \operatorname{Im} \log z$ we denote the principal argument of $z \in \mathbb{C} \setminus \{0\}$.

2.1. Pick functions. A *Pick function*, also known under the names *Herglotz function*, *Nevanlinna function* or *Nevanlinna–Pick function*, is a holomorphic function f which maps the upper complex half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ to its closure; that is, $\operatorname{Im} f(z) \geq 0$ whenever $\operatorname{Im} z > 0$. By Theorem II.I in [2], every Pick function admits the *Stieltjes representation*

$$f(z) = bz + c + \int_{\mathbb{R}} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) \mu(ds), \quad (2.1)$$

where $b \geq 0$, $c \in \mathbb{R}$ and μ is a measure on $\mathbb{R}$ satisfying the integrability condition $\int_{\mathbb{R}} (s^2+1)^{-1} \mu(ds) < \infty$. The parameters b, c, μ are otherwise arbitrary, and they are uniquely determined by

$$b = \lim_{t \rightarrow \infty} \frac{\operatorname{Im} f(it)}{t}, \quad c = \operatorname{Re} f(i),$$

and, in the sense of vague convergence of measures,

$$\mu(ds) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \operatorname{Im} f(s+it) ds; \quad (2.2)$$

see Lemma II.1 in [2] or equation (3.10) in [10]. The measure μ is the *Stieltjes measure* of f .

As in Remark 1.4, formula (2.1) can be rewritten as

$$f(z) = c + \int_{\mathbb{R} \cup \{\infty\}} \frac{1+sz}{s-z} \sigma(ds), \quad (2.3)$$

where $c \in \mathbb{R}$ and σ is a finite measure on $\mathbb{R} \cup \{\infty\}$. More precisely, $\sigma(ds) = b\delta^\infty(ds) + (s^2+1)^{-1}\mu(ds)$, and in the above equation we agree that the integrand is equal to z when $s = \infty$.

It is easy to see that if f is a Pick function and f is not constant 0, then $\log f$ is another Pick function. Observe that $0 \leq \operatorname{Im} \log f(z) \leq \pi$ when $\operatorname{Im} z > 0$. By formula (2.2) (applied to $\log f$), the Stieltjes measure corresponding to $\log f$ necessarily has a density function with values in $[0, 1]$. Additionally, the corresponding coefficient b is equal to zero. This brings us to the *exponential representation* of the Pick function f :

$$f(z) = \exp \left(c + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) \varphi(s) ds \right), \quad (2.4)$$

where $c \in \mathbb{R}$ and φ is a Borel function on $\mathbb{R}$ with values in $[0, 1]$. Furthermore, $c = \log |f(i)|$, φ is determined uniquely up to equality almost everywhere, and

$$\varphi(s) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \operatorname{Arg} f(s+it) \quad (2.5)$$

for almost every $s \in \mathbb{R}$. For further discussion, we refer to equation II.6 in [2] and to Section 7.2 in [10].

2.2. Sequences and generating functions. A (real) *two-sided sequence* is a function $a : \mathbb{Z} \rightarrow \mathbb{R}$. For clarity, we often write $a(k)$ rather than a_k for the entries of the sequence, and we usually write ‘the sequence $a(k)$ ’ instead of the more formal ‘the sequence a ’. We identify a *one-sided sequence* with the corresponding two-sided sequence $a(k)$ such that $a(k) = 0$ when $k < 0$.

The *forward difference operator* is defined by

$$\Delta a(k) = a(k+1) - a(k),$$

and its powers Δ^n are defined in a straightforward way for $n = 0, 1, 2, \dots$. Clearly, if $a(k)$ is a summable sequence, then $\Delta a(k)$ is summable. Furthermore, if $a(k)$ converges eventually monotonically to zero as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, then $\Delta a(k)$ has an eventually constant sign as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, and hence $\Delta a(k)$ is a summable sequence, which sums up to zero.

For a two-sided sequence $a(k)$ we define its generating function as

$$F(z) = \sum_{k=-\infty}^{\infty} a(k)z^k$$

whenever the series converges (absolutely or conditionally). Clearly, the generating function of $\Delta a(k)$ is given by $(z-1)F(z)$ whenever $F(z)$ is defined.

If $a(k)$ is a bounded one-sided sequence, the generating function F is well defined and holomorphic in the unit disc $|z| < 1$ in the complex plane. If $a(k)$ is additionally summable, then $F(z)$ converges absolutely and it is continuous in the closed unit disc $|z| \leq 1$. If a one-sided sequence $a(k)$ converges eventually monotonically to zero as $k \rightarrow \infty$, then, by Dirichlet's test, the generating function $F(z)$ of $a(k)$ converges in the closed disc $|z| \leq 1$ except possibly $z = 1$. Furthermore, in this case the sequence of increments $\Delta a(k)$ is summable, its generating function is a continuous function on the closed unit disc $|z| \leq 1$, and for $z \neq 1$ it is equal to $(z-1)F(z)$. It follows that F is continuous in the closed disc $|z| \leq 1$, except possibly $z = 1$.

For a two-sided summable sequence $a(k)$, the generating function $F(z)$ converges absolutely on the unit circle $|z| = 1$. For general $a(k)$, the generating function $F(z)$ may diverge at every point z . However, if $a(k)$ converges eventually monotonically to zero as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, then, by the discussion in the preceding paragraph, $F(z)$ converges on the unit circle $|z| = 1$ except possibly $z = 1$, and it is a continuous function on this set. Furthermore, in this case F determines uniquely the sequence $a(k)$. Indeed: the continuous function $(z-1)F(z)$ on the unit circle is the generating function of the summable sequence $\Delta a(k)$, which is thus determined uniquely by F : it is the sequence of Fourier coefficients of $(z-1)F(z)$. Furthermore, since $a(k)$ converges to 0 as $k \rightarrow -\infty$, the sequence $\Delta a(k)$ uniquely determines the sequence $a(k)$.

The *convolution* of two-sided sequences $a(k)$ and $b(k)$ is defined in the usual way:

$$(a * b)(k) = \sum_{j=-\infty}^{\infty} a(j)b(k-j)$$

whenever the series converges (absolutely or conditionally) for every $k \in \mathbb{Z}$. Clearly, $\Delta(a * b) = \Delta a * b = a * \Delta b$ whenever $a * b$ is well-defined. Suppose that F and G are generating functions of two-sided sequences $a(k)$ and $b(k)$. If $a(k)$ and $b(k)$ are summable sequences, then the generating function of their convolution is equal to $F(z)G(z)$ when $|z| = 1$. More generally, if $a(k)$ is summable and $b(k)$ converges eventually monotonically to zero as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, then the convolution of these sequences converges absolutely, $(a * b)(k)$ converges to zero as $k \rightarrow \pm\infty$, and the generating function of the convolution is again given by $F(z)G(z)$ when $|z| = 1$, $z \neq 1$.

We recall the Abel's *summation by parts* formula:

$$\sum_{k=-\infty}^{\infty} a(k) \Delta b(k) = - \sum_{k=-\infty}^{\infty} \Delta a(k-1) b(k)$$

whenever either of the sums converges and $a(k)b(k)$ converges to zero as $k \rightarrow \pm\infty$. Iterating this identity n times, we find that

$$\sum_{k=-\infty}^{\infty} a(k) \Delta^n b(k) = (-1)^n \sum_{k=-\infty}^{\infty} \Delta^n a(k-n) b(k),$$

provided that either of the sums converges and for every $j = 0, 1, \dots, n-1$ the sequence $\Delta^j a(k-j) \Delta^{n-1-j} b(k)$ converges to zero as $k \rightarrow \pm\infty$.

A two-sided sequence $a(k)$ is said to change sign at least n times if there is a subsequence $a(k_0), a(k_1), \dots, a(k_n)$ of alternating signs, in the sense that $a(k_{j-1})a(k_j) < 0$ for $j = 1, 2, \dots, n$. If $a(k)$ changes sign at least n times, but not $n+1$ times, then we simply say that $a(k)$ *changes sign n times*.

By a discrete counterpart of Rolle's theorem, if $a(k)$ changes sign n times and $a(k)$ converges to zero as $k \rightarrow \pm\infty$, then $\Delta a(k)$ changes sign at least $n+1$ times: there is at least one sign change of $\Delta a(k)$ on each (finite or infinite) interval delimited by the locations of sign changes of $a(k)$.

2.3. Absolutely monotone-then-completely monotone sequences. Recall that a one-sided sequence $a(k)$, $k \geq 0$, is *completely monotone*, or $\mathcal{CM}$, if $(-1)^n \Delta^n a(k) \geq 0$ for every $n \geq 0$ and $k \geq 0$. A reversed one-sided sequence $a(k)$ defined for $k \leq 0$ is called *absolutely monotone*, or $\mathcal{AM}$, if the sequence $a(-k)$, $k \geq 0$, is completely monotone. Equivalently, $a(k)$ is absolutely monotone if $\Delta^n a(k) \geq 0$ for every $n \geq 0$ and $k \leq -n$.

The celebrated Hausdorff's theorem identifies completely monotone sequences with moment sequences of measures on $[0, 1]$: a sequence $a(k)$, $k \geq 0$, is completely monotone if and only if there exists a finite measure μ on $[0, 1]$ such that

$$a(k) = \int_{[0,1]} s^k \mu(ds).$$

Furthermore, by the dominated convergence theorem, $a(k)$ converges to zero if and only if $\mu(\{1\}) = 0$. By Fubini's theorem, the generating function of a $\mathcal{CM}$ sequence is given by

$$F(z) = \sum_{k=0}^{\infty} a(k) z^k = \int_{[0,1]} \frac{1}{1-sz} \mu(ds) \quad (2.6)$$

in the open unit disc $|z| < 1$. If $a(k)$ converges to zero as $k \rightarrow \infty$, then formula (2.6) holds also when $|z| = 1$ and $z \neq 1$. By Fubini's theorem, formula (2.6) also holds when $z = 1$, with both sides possibly infinite. We conclude that if $a(k)$ is $\mathcal{CM}$ and it converges to zero, then (2.6) holds in the closed unit disc $|z| \leq 1$. Observe that the right-hand side of (2.6) in fact defines a holomorphic function of $z \in \mathbb{C} \setminus [1, \infty)$.

Following the analogous definition of $\mathcal{AM}$ - $\mathcal{CM}$ functions in [7], in this section we introduce the class of $\mathcal{AM}$ - $\mathcal{CM}$ sequences.

Definition 2.1. A two-sided sequence $a(k)$ is said to be *absolutely monotone-then-completely monotone*, or $\mathcal{AM}\text{-}\mathcal{CM}$ in short, if it is not identically zero, the one-sided sequence $a(k)$, $k \geq 0$, is completely monotone, while the reversed one-sided sequence $a(k)$, $k \leq 0$, is absolutely monotone.

Equivalently, $a(k)$ is $\mathcal{AM}\text{-}\mathcal{CM}$ if and only if both one-sided sequences: $a(k)$, $k \geq 0$, and $a(-k)$, $k \geq 0$, are completely monotone. By Hausdorff's characterisation of completely monotone sequences, a two-sided sequence $a(k)$ is $\mathcal{AM}\text{-}\mathcal{CM}$ if and only if there are finite measures μ_+ , μ_- such that

$$\begin{aligned} a(k) &= \int_{[0,1]} s^k \mu_+(ds) && \text{for } k \geq 0, \\ a(k) &= \int_{[0,1]} s^{-k} \mu_-(ds) && \text{for } k \leq 0, \end{aligned}$$

and by comparing the two conflicting definitions of $a(0)$ we see that necessarily $\mu_+([0,1]) = \mu_-([0,1])$. Furthermore, $a(k)$ converges to zero as $k \rightarrow \pm\infty$ if and only if $\mu_+(\{1\}) = \mu_-(\{1\}) = 0$, and in this case, by (2.6), the generating function of $a(k)$ is given by

$$\begin{aligned} F(z) &= \sum_{k=-\infty}^{\infty} a(k) z^k = \sum_{k=0}^{\infty} a(k) z^k + \sum_{k=0}^{\infty} a(-k) z^{-k} - a(0) \\ &= \int_{[0,1]} \frac{1}{1-sz} \mu_+(ds) + \int_{[0,1]} \frac{1}{1-s/z} \mu_-(ds) - a(0) \end{aligned}$$

whenever $|z| = 1$, $z \neq 1$. Since $a(0) = \int_{[0,1]} \mu_+(ds)$, the above expression simplifies to

$$F(z) = \int_{(0,1)} \frac{sz}{1-sz} \mu_+(ds) + \int_{[0,1)} \frac{1}{1-s/z} \mu_-(ds). \quad (2.7)$$

Conversely, if μ_+ and μ_- are finite measures on $[0,1)$ such that $\mu_+([0,1)) = \mu_-([0,1))$, then the above formula defines the generating function F of a unique $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $a(k)$ which converges to zero as $k \rightarrow \pm\infty$.

Observe that the right-hand side of (2.7) defines a holomorphic function of $z \in \mathbb{C} \setminus [0, \infty)$, and we use the same symbol $F(z)$ to denote this holomorphic extension. The main result of this section describes the exponential representation of the function $F(z)$.

Lemma 2.2. *If F is the generating function of an $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $a(k)$ which converges to zero as $k \rightarrow \pm\infty$, then F extends to a holomorphic function on $\mathbb{C} \setminus [0, \infty)$, given by*

$$F(z) = \exp\left(c + \int_0^\infty \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \varphi(s) ds\right), \quad (2.8)$$

and we have

$$\lim_{t \rightarrow 0} (e^{it} - 1) F(e^{it}) = 0. \quad (2.9)$$

Here $c \in \mathbb{R}$ and φ is a Borel function on $(0, \infty)$ which takes values in $[-1, 0]$ over $(0, 1)$ and in $[0, 1]$ over $(1, \infty)$. If we define $\varphi(s) = 0$ for $s \leq 0$, then

$$\varphi(s) = \frac{1}{\pi} \lim_{t \rightarrow 0^+} \text{Arg } F(s + it) \quad (2.10)$$

for almost every $s \in \mathbb{R}$.

Conversely, if F is given by (2.8) and all conditions listed above are satisfied, then F is the generating function of a unique $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $a(k)$ which converges to zero as $k \rightarrow \pm\infty$.

We split the proof into two parts.

Proof of the direct part. Suppose that F is the generating function of an $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $a(k)$ which converges to zero as $k \rightarrow \pm\infty$. We divide the proof into three steps.

Step 1. Recall that $(z-1)F(z)$ is the generating function of the sequence $\Delta a(k)$, which is summable, and which sums up to zero. Thus, $(z-1)F(z)$ extends continuously to the unit circle $|z| = 1$, and it takes value zero at $z = 1$. As in Remark 1.2, this proves (2.9).

Step 2. We already know that F is given by (2.7). In order to derive (2.8), we first claim that $G(z) = (1 - 1/z)F(z)$, which is the generating function of the sequence $\Delta a(k-1)$, is a Pick function. By (2.7), we have

$$G(z) = \int_{(0,1)} \frac{sz-s}{1-sz} \mu_+(ds) + \int_{[0,1)} \frac{1-1/z}{1-s/z} \mu_-(ds) \quad (2.11)$$

Observe that

$$\frac{sz-s}{1-sz} = \frac{s(1-s)}{s^2+1} \frac{1+z/s}{1/s-z} - \frac{s(s+1)}{s^2+1}$$

and

$$\frac{1-1/z}{1-s/z} = \frac{1-s}{s^2+1} \frac{1+sz}{s-z} + \frac{s+1}{s^2+1}.$$

Thus, (2.11) can be rewritten as in (2.3):

$$G(z) = d + \int_{[0,\infty)} \frac{1+sz}{s-z} \sigma(ds), \quad (2.12)$$

where

$$\sigma(A) = \int_{(0,1)} \mathbb{1}_A(1/s) \frac{s(1-s)}{s^2+1} \mu_+(ds) + \int_{[0,1)} \mathbb{1}_A(s) \frac{1-s}{s^2+1} \mu_-(ds)$$

and

$$d = - \int_{(0,1)} \frac{s(s+1)}{s^2+1} \mu_+(ds) + \int_{[0,1)} \frac{s+1}{s^2+1} \mu_-(ds).$$

In particular, G is indeed a Pick function, and our claim is proved. We also note that $\sigma(\{1\}) = 0$.

Step 3. The exponential representation (2.4) of G reads

$$G(z) = \exp\left(\tilde{c} + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \tilde{\varphi}(s) ds\right) \quad (2.13)$$

when $\text{Im } z > 0$, where $\tilde{c} \in \mathbb{R}$ and $\tilde{\varphi}$ is a Borel function on $\mathbb{R}$ taking values in $[0, 1]$. Furthermore,

$$\tilde{\varphi}(s) = \frac{1}{\pi} \lim_{t \rightarrow 0^+} \text{Arg } G(s + it)$$

for almost every $s > 0$. By (2.7), we have $F(z) > 0$ when $z < 0$, and hence $G(z) = (1 - 1/z)F(z) > 0$ when $z < 0$. Since G is continuous on $\mathbb{C} \setminus [0, \infty)$, the above formula for $\tilde{\varphi}(s)$ implies that $\tilde{\varphi}(s) = 0$ for $s < 0$. Let us define

$$\varphi(s) = \tilde{\varphi}(s) - \mathbb{1}_{(0,1)}(s),$$

so that $\varphi(s) = 0$ for $s < 0$, $\varphi(s) \in [-1, 0]$ for $s \in (0, 1)$, and $\varphi(s) \in [0, 1]$ for $s \in (1, \infty)$. With this definition, we find that

$$G(z) = \exp\left(\tilde{c} + \int_0^1 \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) ds + \int_0^\infty \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \varphi(s) ds\right)$$

when $\text{Im } z > 0$. By a straightforward calculation, the first integral on the right-hand side is equal to $\log(1 - 1/z) - \frac{1}{2} \log 2$, and thus

$$G(z) = (1 - 1/z) \exp\left(c + \int_0^\infty \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \varphi(s) ds\right),$$

where $c = \tilde{c} - \frac{1}{2} \log 2$. This proves (2.8). Finally, we have

$$\begin{aligned} \frac{1}{\pi} \lim_{t \rightarrow 0^+} \text{Arg } F(s + it) &= \frac{1}{\pi} \lim_{t \rightarrow 0^+} \left(\text{Arg } G(s + it) - \text{Arg} \left(1 - \frac{1}{s + it}\right) \right) \\ &= \tilde{\varphi}(s) - \mathbb{1}_{(0,1)}(s) = \varphi(s) \end{aligned}$$

for almost every $s \in \mathbb{R}$. Formula 2.10 follows, and the proof is complete. $\square$

Proof of the converse part. Suppose that φ is a Borel function on $(0, \infty)$ which takes values in $[-1, 0]$ on $(0, 1)$ and in $[0, 1]$ on $(1, \infty)$, $c \in \mathbb{R}$, F is given by (2.8), and (2.9) holds. We essentially reverse the argument from the proof of the direct part.

Step 1. Observe that F is a holomorphic function in $\mathbb{C} \setminus [0, \infty)$, and $F(z) > 0$ for $z < 0$. Reversing Step 3 of the proof of the direct part, we find that $G(z) = (1 - 1/z)F(z)$ is a Pick function, with exponential representation (2.13), where $\tilde{\varphi}(s) = \varphi(s) + \mathbb{1}_{(0,1)}(s)$ and $\tilde{c} \in \mathbb{R}$.

Step 2. The Stieltjes representation (2.3) of the Pick function G reads

$$G(z) = d + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + sz}{s - z} \sigma(ds),$$

where $d \in \mathbb{R}$ and σ is a finite measure on $\mathbb{R} \cup \{\infty\}$. Since G is a holomorphic function in $\mathbb{C} \setminus [0, \infty)$, σ necessarily vanishes on $(-\infty, 0)$.

Recall that the integrand on the right-hand side is understood to be equal to z for $s = \infty$. For $s > 0$ and $z < -1$ we have

$$\left| \frac{1 + sz}{z(s - z)} \right| \leq \frac{1 + s|z|}{|z|(s + |z|)} \leq 1.$$

Hence, by the dominated convergence theorem, we have

$$\lim_{z \rightarrow -\infty} \frac{G(z)}{z} = \lim_{z \rightarrow -\infty} \int_{[0, \infty) \cup \{\infty\}} \frac{1 + sz}{z(s - z)} \sigma(ds) = \sigma(\{\infty\}).$$

Since $F(z) > 0$ for $z < 0$, we have $G(z) = (1 - 1/z)F(z) > 0$ for $z < 0$, and so the left-hand side is nonpositive. Therefore, $\sigma(\{\infty\}) = 0$. In other words, G has the representation (2.12).

Reversing the argument from Step 2 of the proof of the direct part, we find that instead of (2.11), we have

$$G(z) = \tilde{d} + \sigma(\{1\}) \frac{1+z}{1-z} + \int_{(0,1)} \frac{sz-s}{1-sz} \mu_+(ds) + \int_{[0,1)} \frac{1-1/z}{1-s/z} \mu_-(ds)$$

for some $\tilde{d} \in \mathbb{R}$ and some finite measures μ_+ on $(0, 1)$ and μ_- on $[0, 1)$. However, by (2.9), $G(e^{it}) = e^{-it}(e^{it} - 1)F(e^{it})$ converges to zero as $t \rightarrow 0$. Using the above expression for $G(e^{it})$, the dominated convergence theorem, and the estimates

$$\left| \frac{se^{it} - s}{1 - se^{it}} \right| = \frac{s|e^{it} - 1|}{|1 - se^{it}|} \leq \frac{st}{s \sin t} \leq \frac{\pi}{2}$$

for $s \in (0, 1)$ and $t \in (0, \frac{\pi}{2})$, and

$$\left| \frac{1 - e^{-it}}{1 - se^{-it}} \right| = \left| 1 - \frac{(1-s)e^{-it}}{1 - se^{-it}} \right| \leq 1 + \frac{1-s}{|1 - se^{-it}|} \leq 2$$

for $s \in [0, 1)$ and $t \in (0, \frac{\pi}{2})$, we find that

$$0 = \lim_{t \rightarrow 0^+} G(e^{it}) = \tilde{d} + \sigma(\{1\})i\infty.$$

Hence, $\tilde{d} = \sigma(\{1\}) = 0$. We conclude that in fact (2.11) holds with no modifications, and with $\sigma(\{1\}) = 0$. It follows that F is given by (2.7):

$$F(z) = \int_{(0,1)} \frac{sz}{1-sz} \mu_+(ds) + \int_{[0,1)} \frac{1}{1-s/z} \mu_-(ds).$$

Step 3. It remains to extend the definition of μ_+ at 0 so that $\mu_+([0, 1)) = \mu_-([0, 1))$. In other words, we let $\mu_+(\{0\}) = \mu_-([0, 1)) - \mu_+((0, 1))$. By the dominated convergence theorem,

$$\begin{aligned} \lim_{z \rightarrow -\infty} F(z) &= \lim_{z \rightarrow -\infty} \int_{(0,1)} \frac{sz}{1-sz} \mu_+(ds) + \lim_{z \rightarrow -\infty} \int_{[0,1)} \frac{1}{1-s/z} \mu_-(ds) \\ &= -\mu_+((0, 1)) + \mu_-([0, 1)) = \mu_+(\{0\}). \end{aligned}$$

On the other hand, $F(z) > 0$ when $z < 0$, and hence $\mu_+(\{0\}) \geq 0$. By the observation made after equation (2.7), $F(z)$ is the generating function of an $\mathcal{AM}$ - $\mathcal{CM}$ sequence $a(k)$ which converges to zero as $k \rightarrow \pm\infty$. $\square$

Lemma 2.2 is an extension of the results of Section 2.7 in [9], where one-sided completely monotone sequences were studied. It is closely related to the results of Section 3 in [7], where $\mathcal{AM}$ - $\mathcal{CM}$ functions were introduced.

2.4. Pólya frequency sequences. A two-sided summable sequence $a(k)$ is a *Pólya frequency sequence* if it is not identically zero, and the doubly infinite matrix $(a(k-l) : k, l \in \mathbb{Z})$ is totally positive, that is, all of its minors are nonnegative. Summable Pólya frequency sequences can be characterised by their generating functions, which are necessarily of the form

$$F(z) = \sum_{k=-\infty}^{\infty} a(k)z^k = z^m \exp\left(b^+z + \frac{b^-}{z} + c\right) \prod_{j=1}^{\infty} \frac{(1 + \gamma_j^+z)(1 + \gamma_j^-/z)}{(1 - \delta_j^+z)(1 - \delta_j^-/z)}, \quad (2.14)$$

where m is an integer, $b^+, b^- \in [0, \infty)$, $c \in \mathbb{R}$, and $\gamma_j^\pm, \delta_j^\pm$ are nonnegative summable sequences with $\gamma_j^\pm \leq 1$ and $\delta_j^\pm < 1$; see Theorem 8.9.5 in [6].

Another characterisation of summable two-sided Pólya frequency sequences involves the variation diminishing property: a summable two-sided sequence $a(k)$ is,

up to sign, a Pólya frequency sequence if and only if the convolution with $a(k)$ does not increase the number of sign changes; see Theorem 5.1.5 in [6]. We will need this result (in fact, only its easy direct half) in the proof the implication (b) $\Rightarrow$ (a) in Theorem 1.1.

We now derive the exponential representation of generating functions of summable two-sided Pólya frequency sequences. By (2.14), we have

$$F(z) = \exp\left(m \log z + b^+ z + \frac{b^-}{z} + c + \sum_{j=1}^{\infty} (\log(1 + \gamma_j^+ z) + \log(1 + \gamma_j^- / z) - \log(1 - \delta_j^+ z) - \log(1 - \delta_j^- / z))\right)$$

when $\text{Im } z > 0$. By a straightforward calculation,

$$\begin{aligned} \log z &= \int_{-\infty}^0 \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) ds, \\ \log(1 + \gamma_j^+ z) &= \int_{-\infty}^{-1/\gamma_j^+} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) ds + \log \sqrt{(\gamma_j^+)^2 + 1}, \\ \log(1 + \gamma_j^- / z) &= - \int_{-\gamma_j^-}^0 \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) ds + \log \sqrt{(\gamma_j^-)^2 + 1}, \\ \log(1 - \delta_j^+ z) &= - \int_{1/\delta_j^+}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) ds + \log \sqrt{(\delta_j^+)^2 + 1}, \\ \log(1 - \delta_j^- / z) &= \int_0^{\delta_j^-} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) ds + \log \sqrt{(\delta_j^-)^2 + 1}. \end{aligned}$$

It follows that

$$F(z) = \exp\left(b^+ z + \frac{b^-}{z} + d + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) \varphi(s) ds\right)$$

when $\text{Im } z > 0$, where

$$d = c + \sum_{j=1}^{\infty} \left(\log \sqrt{(\gamma_j^+)^2 + 1} + \log \sqrt{(\gamma_j^-)^2 + 1} - \log \sqrt{(\delta_j^+)^2 + 1} - \log \sqrt{(\delta_j^-)^2 + 1} \right)$$

(the series converges, because $\log \sqrt{s^2 + 1} \leq s$ for $s \geq 0$) and

$$\varphi(s) = m \mathbb{1}_{(-\infty, 0)}(s) + \sum_{j=1}^{\infty} \left(\mathbb{1}_{(-\infty, -1/\gamma_j^+)}(s) - \mathbb{1}_{(-\gamma_j^-, 0)}(s) - \mathbb{1}_{(0, \delta_j^-)}(s) + \mathbb{1}_{(1/\delta_j^+, \infty)}(s) \right).$$

In particular, φ is stepwise decreasing on $(-\infty, 0)$, stepwise increasing on $(0, \infty)$, and equal to zero near 1. Finally, we note the sequences $\gamma_j^{\pm}$ and $\delta_j^{\pm}$ are summable if and only if

$$\int_{-\infty}^{\infty} \frac{|\varphi(s)|}{s^2 + 1} ds < \infty.$$

The above reasoning can be clearly reversed, so that we obtain an equivalent form of the generating function of a summable two-sided Pólya frequency sequence. We state this result as a lemma.

Lemma 2.3. *If $F(z)$ is the generating function of a summable two-sided Pólya frequency sequence $a(k)$, then F extends to a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$, given by*

$$F(z) = \exp\left(b^+ z + \frac{b^-}{z} + d + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \varphi(s) ds\right). \quad (2.15)$$

Here $b^\pm \geq 0$, $d \in \mathbb{R}$, and φ is a stepwise decreasing function on $(-\infty, 0)$, a stepwise increasing function on $(0, \infty)$, $\varphi(s) = 0$ in a neighbourhood of 1, and

$$\int_{-\infty}^{\infty} \frac{|\varphi(s)|}{s^2+1} ds < \infty. \quad (2.16)$$

Conversely, if F is given by (2.15) and all conditions listed above are satisfied, then F is the generating function of a unique summable two-sided Pólya frequency sequence $a(k)$.

For a similar discussion of the one-sided case, see Section 2.8 in [9]. Pólya frequency functions in a similar context were discussed in Section 4 in [7].

2.5. Generating functions and inversion formulae. The classical Post's inversion formula tells us that if G is the Laplace transform of F :

$$G(x) = \int_0^{\infty} e^{-tx} F(t) dt,$$

then, under suitable assumptions on F , we have

$$F(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n (n/t)^{n+1}}{n!} G^{(n)}(n/t).$$

This identity, applied to the Fourier transform F of a bell-shaped function, played a crucial role in the analysis of bell-shaped functions in [8].

For one-sided bell-shaped sequences, the following discrete counterpart of Post's inversion formula was used in [9]. If G is the moment sequence of a function F :

$$G(k) = \int_0^1 t^k F(t) dt,$$

then, again under suitable assumptions on F , we have

$$F(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n (k_n + 1)^{\overline{n+1}}}{n!} \Delta^n G(k_n),$$

where Δ is the forward difference operator and

$$\lim_{n \rightarrow \infty} \frac{k_n}{n + k_n} = t.$$

In [9] this formula was applied to the generating function F of a one-sided bell-shaped sequence.

In our case, the generating function F of a two-sided bell-shaped sequence is only defined on the unit circle. Therefore, we need a different variant of Post's inversion formula. We were not able to find this result in the literature, so we provide a complete proof.

Proposition 2.4 (yet another Post's inversion formula). *Let F be an integrable function on the unit circle in the complex plane, and let*

$$G(x) = \int_0^{\infty} e^{itx} F(e^{it}) dt$$

when $\text{Im } x > 0$. Then for every $t \in (0, \pi)$ such that F is continuous at e^{it} we have

$$F(e^{it}) = \lim_{n \rightarrow \infty} \frac{(-1)^n (x_n)^{\overline{n+1}}}{in!} \Delta^n G(x_n), \quad (2.17)$$

where Δ is the forward difference operator and

$$x_n = \frac{n}{2} \left(i \cot \frac{t}{2} - 1 \right).$$

The same result holds true in the more general case when instead of integrability of F we assume that $(e^{it} - 1)^n F(e^{it})$ is integrable over $(0, 2\pi)$ for sufficiently large n , provided that in (2.17) we agree that

$$\Delta^n G(x) = \int_0^\infty e^{itx} (e^{it} - 1)^n F(e^{it}) dt$$

for n large enough.

Proof. We divide the argument into four steps.

Step 1. Suppose that G is integrable over the unit circle. The n th iterated difference of $x \mapsto e^{isx}$ is equal to $e^{isx} (e^{is} - 1)^n$. Thus,

$$\begin{aligned} \Delta^n G(x) &= \int_0^\infty e^{isx} (e^{is} - 1)^n F(e^{is}) ds \\ &= (2i)^n \int_0^\infty e^{is(x+n/2)} \left(\sin \frac{s}{2} \right)^n F(e^{is}) ds \end{aligned}$$

when $\text{Im } x > 0$. We fix $t \in (0, \pi)$ and we let $c = \frac{1}{2} \cot \frac{t}{2} \in (0, \infty)$, so that $x_n = -\frac{n}{2} + nic$. The above formula evaluated at $x = x_n$ leads to

$$\Delta^n G(x_n) = (2i)^n \int_0^\infty \left(e^{-cs} \sin \frac{s}{2} \right)^n F(e^{is}) ds. \quad (2.18)$$

Step 2. Suppose for the moment that F is constant 1. Then $G(x) = i/x$, and hence, by induction,

$$\Delta^n G(x) = \frac{(-1)^n in!}{x^{\overline{n+1}}}.$$

On the other hand, $\Delta^n G(x)$ is given by (2.18). It follows that

$$(2i)^n \int_0^\infty \left(e^{-cs} \sin \frac{s}{2} \right)^n ds = \frac{(-1)^n in!}{(x_n)^{\overline{n+1}}}.$$

In other words, if we write

$$M_n = \int_0^\infty \left(e^{-cs} \sin \frac{s}{2} \right)^n ds,$$

then

$$(2i)^n M_n = \frac{(-1)^n in!}{(x_n)^{\overline{n+1}}}.$$

Turning back to an arbitrary integrable function F , and combining the above expression with (2.18), we find that

$$\frac{(-1)^n (x_n)^{\overline{n+1}}}{in!} \Delta^n G(x_n) = \frac{1}{M_n} \int_0^\infty \left(e^{-cs} \sin \frac{s}{2} \right)^n F(e^{is}) ds.$$

Step 3. The function $|e^{-cs} \sin \frac{s}{2}|$, defined for $s \in (0, \infty)$, has a strict global maximum at $s = 2 \operatorname{arccot}(2c) = t$. Using a standard ‘approximation to the identity’ argument, one can show that if F is continuous at e^{it} , then

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \int_0^\infty \left(e^{-cs} \sin \frac{s}{2} \right)^n F(e^{is}) ds = F(e^{it}),$$

and the desired result follows. For completeness, we provide full details in Lemma A.1 in Appendix A.

Step 4. If we only assume that $(z - 1)^n F(z)$ is integrable over the unit circle for n large enough, then the first equality in Step 1 holds now by assumption when n is large enough. Otherwise, the proof is exactly the same. $\square$

2.6. Completeness of a class of holomorphic functions. Below we prove that the class of functions F given by (1.1) is closed under pointwise limits over the semi-circle $|z| = 1$, $\operatorname{Im} z > 0$. This is very similar to various analogous results for other classes of functions, but we failed to find the statement needed here in literature.

It is well-known that pointwise convergence of Pick functions on a sufficiently large subset of the upper complex half-plane is equivalent to locally uniform convergence, as well as to the vague convergence of the corresponding Stieltjes measures. A brief discussion and some references can be found in Section 2.6 in [9]. In that paper this property of Pick functions easily led to a similar completeness result for the class of generating functions of one-sided bell-shaped sequences; see Step 10 of the proof of Theorem 3.1 in [9]. However, unlike in the one-sided case, logarithms of generating functions of two-sided bell-shaped sequences are generally not Pick functions. Thus, a different argument is needed.

A similar completeness result for the class of Fourier transforms of two-sided bell-shaped functions is given in Lemma 3.1 in [8]. To some extent, the proof given below for two-sided bell-shaped sequences is similar to the argument used in [8]. Our case is, however, more complicated, as the representing measures σ_n change sign twice, and one of this sign changes occurs at an undetermined position.

Lemma 2.5. *Suppose that*

$$F_n(z) = \exp \left(b_n^+ z + \frac{b_n^-}{z} + c_n + \int_{-\infty}^\infty \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) \varphi_n(s) ds \right), \quad (2.19)$$

where $b_n^\pm, c_n, \varphi_n$ satisfy the conditions listed in Theorem 1.1(c), except possibly (v). Suppose, furthermore, that $F_n(e^{it})$ converges as $n \rightarrow \infty$ to a finite limit $F(e^{it})$ for every $t \in (0, \pi)$. Then either $F(e^{it}) = 0$ for every $t \in (0, \pi)$ or F extends to a holomorphic function in the upper complex half-plane, given by

$$F(z) = \exp \left(b^+ z + \frac{b^-}{z} + c + \int_{-\infty}^\infty \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) \varphi(s) ds \right), \quad (2.20)$$

where $b^\pm, c, \varphi$ satisfy the conditions listed in Theorem 1.1(c), except possibly (v). Furthermore, in the latter case, as $n \rightarrow \infty$, the sequence c_n converges to c , and the signed measures

$$\sigma_n(ds) = b_n^+ \delta_\infty(ds) - b_n^- \delta_0(ds) + \frac{\varphi_n(s)}{s^2 + 1} ds$$

(see (1.2)) converge vaguely on $\mathbb{R} \cup \{\infty\}$ to the corresponding signed measure

$$\sigma(ds) = b^+ \delta_\infty(ds) - b^- \delta_0(ds) + \frac{\varphi(s)}{s^2 + 1} ds.$$

Proof. Assume that $F(e^{it})$ is not identically equal to zero. We divide the argument into six steps.

Step 1. We denote $k_n = -\varphi_n(-1)$. Note that since φ_n only takes integer values on $(-\infty, 0)$, k_n is an integer. We consider the function

$$G_n(z) = z^{k_n} F_n(z).$$

Clearly, $|G_n(e^{it})| = |F_n(e^{it})|$ for $t \in (0, \pi)$. When $\text{Im } z > 0$, we have

$$\int_{-\infty}^0 \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) ds = \log z, \quad (2.21)$$

and therefore, by (2.19),

$$G_n(z) = \exp(k_n \log z) F_n(z) = \exp \left(b_n^+ z + \frac{b_n^-}{z} + c_n + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) \psi_n(s) ds \right),$$

where

$$\psi_n(s) = \varphi_n(s) + k_n \mathbf{1}_{(-\infty, 0)}(s) ds.$$

Since φ_n is stepwise decreasing on $(-\infty, 0)$ and $\psi_n(s) = \varphi_n(s) - \varphi_n(-1)$ for $s < 0$, we have $\psi_n \geq 0$ on $(-\infty, -1) \cup (1, \infty)$ and $\psi_n \leq 0$ on $(-1, 1)$. Define accordingly

$$\begin{aligned} \varrho_n(ds) &= b_n^+ \delta_\infty(ds) - b_n^- \delta_0(ds) + \frac{\psi_n(s)}{s^2+1} ds \\ &= \sigma_n(ds) + \frac{k_n}{s^2+1} \mathbf{1}_{(-\infty, 0)}(s) ds. \end{aligned}$$

Thus, ϱ_n is a finite signed measure, ϱ_n is nonnegative on $(\mathbb{R} \cup \{\infty\}) \setminus [-1, 1]$ and nonpositive on $[-1, 1]$, and $\varrho_n(\{-1, 1\}) = 0$. The number

$$\begin{aligned} M_n &= \int_{\mathbb{R} \cup \{\infty\}} \frac{s^2-1}{s^2+1} \varrho_n(ds) \\ &= b_n^+ + b_n^- + \int_{-\infty}^{\infty} \frac{s^2-1}{(s^2+1)^2} \psi_n(s) ds, \end{aligned} \quad (2.22)$$

with the former integrand extended continuously at $s = \infty$, is therefore nonnegative and finite.

Rewriting the expression for G_n as in (1.2), we obtain

$$G_n(z) = \exp \left(c_n + \int_{\mathbb{R} \cup \{\infty\}} \frac{1+sz}{s-z} \varrho_n(ds) \right). \quad (2.23)$$

In the next step we study the properties of $|G_n(z)|$ when $z = e^{it}$, $t \in (0, \pi)$: we prove a variant of uniform continuity of these functions.

Step 2. For $t \in (0, \pi)$ the expression for $G_n(e^{it})$ takes form

$$G_n(e^{it}) = \exp \left(c_n + \int_{\mathbb{R} \cup \{\infty\}} \frac{(s^2-1) \cos t + i(s^2+1) \sin t}{s^2 - 2s \cos t + 1} \varrho_n(ds) \right),$$

with the integrand extended continuously at $s = \infty$. In particular, by the dominated convergence theorem,

$$-\frac{d}{dt}(\log |G_n(e^{it})|) = \int_{\mathbb{R} \cup \{\infty\}} \frac{(s^2 - 1)(s^2 + 1) \sin t}{(s^2 - 2s \cos t + 1)^2} \varrho_n(ds).$$

Observe that

$$(1 - |\cos t|)(s^2 + 1) \leq s^2 - 2s \cos t + 1 \leq 2(s^2 + 1),$$

and hence

$$\frac{\sin t}{4} M_n \leq -\frac{d}{dt}(\log |G_n(e^{it})|) \leq \frac{\sin t}{(1 - |\cos t|)^2} M_n \quad (2.24)$$

(see (2.22)). Finally, $|G_n(e^{it})| = |F_n(e^{it})|$, so that (2.24) also holds with G_n replaced by F_n .

Step 3. We claim that M_n is a bounded sequence. By assumption, $F(e^{it_1}) \neq 0$ for some $t_1 \in (0, \pi)$. We choose an arbitrary $t_2 \in (0, t_1)$. By (2.24) (for F_n instead of G_n),

$$\begin{aligned} \log |F_n(e^{it_2})| - \log |F_n(e^{it_1})| &= - \int_{t_2}^{t_1} \frac{d}{dt}(\log |G_n(e^{it})|) dt \\ &\geq \int_{t_2}^{t_1} \frac{\sin t}{4} M_n dt = \frac{\cos t_2 - \cos t_1}{4} M_n. \end{aligned}$$

As $n \rightarrow \infty$, the expression on the left-hand side has a limit $\log |F(e^{it_2})| - \log |F(e^{it_1})| \in [-\infty, \infty)$. Thus, the sequence M_n is necessarily bounded. Our claim is proved.

Since ψ_n is increasing-after-rounding on $(0, \infty)$, for $r \in [1, 2]$ we have

$$0 \leq \psi_n(r) \leq \int_2^3 (1 + \psi_n(s)) ds \leq 1 + \frac{25}{2} \int_2^3 \frac{s^2 - 1}{(s^2 + 1)^2} \psi_n(s) ds \leq 1 + \frac{25M_n}{2}.$$

Similarly, for $r \in [\frac{1}{2}, 1]$,

$$0 \leq -\psi_n(r) \leq 2 \int_0^{1/2} (1 - \psi_n(s)) ds \leq 1 + \frac{25}{12} \int_0^{1/2} \frac{s^2 - 1}{(s^2 + 1)^2} \psi_n(r) dr \leq 1 + \frac{25M_n}{12}.$$

It follows that $|\psi_n(r)| \leq 1 + \frac{25}{2} M_n$ for $r \in [\frac{1}{2}, 2]$. Similarly, since ψ_n is stepwise decreasing on $(-\infty, 0)$, the same argument shows that $|\psi_n(r)| \leq \frac{25}{2} M_n$ for $r \in [-2, -\frac{1}{2}]$. Finally, for $s \in (\mathbb{R} \cup \{\infty\}) \setminus [-2, 2]$ we have

$$\frac{5}{3} \frac{s^2 - 1}{s^2 + 1} \geq 1,$$

and similarly for $s \in (-\frac{1}{2}, \frac{1}{2})$,

$$\frac{5}{3} \frac{s^2 - 1}{(s^2 + 1)^2} \leq -1.$$

By combining the above estimates, we find that the total variation norm of ϱ_n , denoted as $\|\varrho_n\|$, satisfies

$$\begin{aligned} \|\varrho_n\| &= \int_{\mathbb{R} \cup \{\infty\} \setminus [-2, 2]} \varrho_n(ds) + \int_{(-1/2, 1/2)} (-\varrho_n)(ds) + \int_{-2}^{-1/2} \frac{|\psi_n(s)|}{1+s^2} ds + \int_{1/2}^2 \frac{|\psi_n(s)|}{1+s^2} ds \\ &\leq \frac{5}{3} \int_{\mathbb{R} \cup \{\infty\} \setminus [-2, 2]} \frac{s^2-1}{s^2+1} \varrho_n(ds) + \frac{5}{3} \int_{(-1/2, 1/2)} \frac{s^2-1}{s^2+1} \varrho_n(ds) \\ &\quad + \int_{-2}^{-1/2} \frac{1 + \frac{25}{2}M_n}{1+s^2} ds + \int_{1/2}^2 \frac{1 + \frac{25}{2}M_n}{1+s^2} ds \\ &\leq \frac{5}{3} M_n + 3(1 + \frac{25}{2}M_n) \leq 3 + 40M_n. \end{aligned}$$

We conclude that $\|\varrho_n\|$ is a bounded sequence.

Step 4. We already know that the sequence M_n is bounded, and that $F(e^{it_1}) \neq 0$ for some $t_1 \in (0, \pi)$. Pick any $t_2 \in (0, \pi)$. By (2.24) and the mean value theorem, the ratio

$$\frac{|F_n(e^{it_1})|}{|F_n(e^{it_2})|} = \exp(\log |F_n(e^{it_1})| - \log |F_n(e^{it_2})|)$$

is bounded by a constant (which depends only on t_1, t_2 and the bound on M_n). Passing to the limit as $n \rightarrow \infty$, we find that $F(e^{it_2}) \neq 0$.

In particular, $|F(i)| \neq 0$. However, by (2.19),

$$|F(i)| = \lim_{n \rightarrow \infty} |F_n(i)| = \lim_{n \rightarrow \infty} e^{c_n}.$$

Therefore, c_n has a finite limit c .

Step 5. We have already proved that the sequence c_n converges to some $c \in \mathbb{R}$, and that the finite signed measures ϱ_n on $\mathbb{R} \cup \{\infty\}$ have bounded total variation norms. In particular, each subsequence of ϱ_n has a vaguely convergent further subsequence.

Suppose that ϱ is a partial limit of ϱ_n in the sense of vague convergence of measures. Passing to the limit along the corresponding subsequence n_j in the representation (2.23) of $G_n(e^{it})$, we find that

$$\lim_{j \rightarrow \infty} G_{n_j}(e^{it}) = \exp\left(c + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + se^{it}}{s - e^{it}} \varrho(ds)\right)$$

for $t \in (0, \pi)$. On the other hand,

$$\lim_{j \rightarrow \infty} \exp(-ik_{n_j}t) G_{n_j}(e^{it}) = \lim_{j \rightarrow \infty} F_{n_j}(e^{it}) = F(e^{it}).$$

Therefore, $\exp(-ik_{n_j}t)$ converges pointwise for every $t \in (0, \pi)$. A standard argument shows that k_{n_j} necessarily converges to a finite limit k , and we conclude that

$$\begin{aligned} F(e^{it}) &= e^{-ikt} \lim_{j \rightarrow \infty} G_{n_j}(e^{it}) \\ &= \exp\left(-ikt + c + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + se^{it}}{s - e^{it}} \varrho(ds)\right) \end{aligned}$$

for $t \in (0, \pi)$.

Clearly, $(1 + s^2)\mathbb{1}_{\mathbb{R}}(s)\varrho_{n_j}(ds)$ converges vaguely to $(1 + s^2)\mathbb{1}_{\mathbb{R}}(s)\varrho(ds)$ on $\mathbb{R}$ (but not necessarily on $\mathbb{R} \cup \{\infty\}$). Since $(1 + s^2)\mathbb{1}_{\mathbb{R}}(s)\varrho_{n_j}(ds)$ has a stepwise decreasing density function $\psi_n(s)$ on $(-\infty, 0)$, by Lemma 2.2 in [9], also the limiting measure

$(1 + s^2)\mathbb{1}_{\mathbb{R}}(s)\varrho(ds)$ has a stepwise decreasing density function on $(-\infty, 0)$. The same argument shows that $(1 + s^2)\mathbb{1}_{\mathbb{R}}(s)\varrho(ds)$ has an increasing-after-rounding density function on $(0, \infty)$. We denote the density function of $(1 + s^2)\mathbb{1}_{\mathbb{R}}(s)\varrho(ds)$ on $\mathbb{R} \setminus \{0\}$ by ψ . Additionally, we set $b^+ = \varrho(\{\infty\})$ and $b^- = -\varrho(\{0\})$. Undoing the transformation that led to (1.2) in Remark 1.4, we find that our representation of $F(e^{it})$ reads:

$$F(e^{it}) = \exp\left(-ikt + b^+e^{it} + \frac{b^-}{e^{it}} + c + \int_{-\infty}^{\infty} \left(\frac{1}{s - e^{it}} - \frac{s}{s^2 + 1}\right) \psi(s) ds\right)$$

for $t \in (0, \pi)$. If we write $\varphi(s) = \psi(s) - k\mathbb{1}_{(-\infty, 0)}(s)ds$, then, by (2.21), the above expression is equivalent to (2.20) with $z = e^{it}$, and clearly the right-hand side of (2.20) defines a holomorphic function of z in the upper complex half-plane $\text{Im } z > 0$. The first part of the lemma is thus proved.

Step 6. Above we have shown that there is a number $c \in \mathbb{R}$ and a finite signed measure σ on $\mathbb{R} \cup \{\infty\}$ such that

$$F(z) = \exp\left(c + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + sz}{s - z} \sigma(ds)\right) \quad (2.25)$$

when $|z| = 1$ and $\text{Im } z > 0$ (cf. (1.2)). Here c is the limit of c_n , and we have

$$\sigma(ds) = \varrho(ds) - \frac{k}{1 + s^2} \mathbb{1}_{(-\infty, 0)}(s)ds,$$

where ϱ is the vague limit of a subsequence of

$$\varrho_n(ds) = \sigma_n(ds) + \frac{k_n}{1 + s^2} \mathbb{1}_{(-\infty, 0)}(s)ds,$$

and k is the corresponding partial limit of k_n . It follows that σ is the vague limit of the corresponding subsequence of σ_n . Additionally, since every subsequence of ϱ_n has a vaguely convergent further subsequence, the sequence σ_n has the same property.

A standard argument shows that (2.25) determines the pair c, σ uniquely. Indeed: suppose that $\tilde{c}$ and $\tilde{\sigma}$ is another such pair. Then

$$1 = \exp\left((c - \tilde{c}) + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + sz}{s - z} (\sigma - \tilde{\sigma})(ds)\right),$$

and hence, for some integer m , we have

$$(c - \tilde{c}) + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + sz}{s - z} (\sigma - \tilde{\sigma})(ds) = 2m\pi i$$

when $|z| = 1$ and $\text{Im } z > 0$. The left-hand side defines a holomorphic function in the upper complex half-plane, and this function is necessarily constant. By uniqueness of the Cauchy–Stieltjes transform (see, for example, Theorem II.1 in [2]), $\sigma - \tilde{\sigma}$ is necessarily a zero measure, and $c - \tilde{c} = 2m\pi i$. However, $c - \tilde{c}$ is real, and we conclude that $c - \tilde{c} = 0$.

We have thus proved that every subsequence of σ_n has a vaguely convergent further subsequence, and the vague limit of this subsequence is necessarily the measure σ constructed above. This, however, means that σ_n converges vaguely to σ , and the proof is complete. $\square$

3. Proof of the main result

This section is devoted to the proof of Theorem 1.1. It is divided into three parts, which correspond to three implications in the theorem, respectively: (b) $\Rightarrow$ (a), (a) $\Rightarrow$ (c), and (c) $\Rightarrow$ (b). For clarify, below we state these implications as three separate theorems.

3.1. Convolutions of $\mathcal{AM}$ - $\mathcal{CM}$ and Pólya frequency sequences and their generating functions. We begin with the easy parts of Theorem 1.1. The following result covers implication (b) $\Rightarrow$ (a) in Theorem 1.1.

Theorem 3.1. *Suppose that $a(k)$ is the convolution of a summable two-sided Pólya frequency sequence $b(k)$ and an $\mathcal{AM}$ - $\mathcal{CM}$ sequence $c(k)$ which converges to zero as $k \rightarrow \pm\infty$. Then $a(k)$ is bell-shaped.*

Proof. The convolution $a(k) = (b * c)(k)$ is nonnegative, not identically zero, and it converges to zero as $k \rightarrow \pm\infty$ by the dominated convergence theorem. Thus, our goal is to prove that the sequence $\Delta^n a(k)$ changes its sign exactly n times for $n = 0, 1, 2, \dots$

By the discrete Rolle's theorem and induction, the sequence $\Delta^n a(k)$ changes sign at least n times. To prove the converse inequality, we observe that by the definition of an $\mathcal{AM}$ - $\mathcal{CM}$ sequence we have $(-1)^n \Delta^n c(k) \geq 0$ for $k \geq 0$ and $\Delta^n c(k) \geq 0$ for $n \leq -k$. Hence, $\Delta^n c(k)$ changes sign at most n times, at positions $\alpha_0 = -n + 1$, $\alpha_1 = -n + 2$, $\dots$, $\alpha_{n-2} = -1$ and $\alpha_{n-1} \geq 0$. The variation diminishing property of summable Pólya frequency sequences implies that also the convolution of $b(k)$ and $\Delta^n c(k)$ changes sign no more than n times. It remains to observe that $(b * \Delta^n c)(k) = \Delta^n (b * c)(k)$. $\square$

We now turn to implication (c) $\Rightarrow$ (b) in Theorem 1.1. Our result in fact proves equivalence of these conditions. First, however, we need an auxiliary lemma.

Lemma 3.2. *Let $p, q \geq 0$. Suppose that*

$$F(z) = \exp\left(b^+ z + \frac{b^-}{z} + c + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \varphi(s) ds\right)$$

when $|z| = 1$, $z \neq 1$, where $b^+, b^- \in [0, \infty)$, $c \in \mathbb{R}$ and φ is a Borel function on $\mathbb{R}$ such that $\varphi(s) \leq -p$ for $s \in (0, 1)$, $\varphi(s) \geq q$ for $s \in (1, \infty)$, and $\int_{-\infty}^{\infty} |\varphi(s)|/(s^2+1) ds < \infty$. If

$$\lim_{t \rightarrow 0^+} (e^{it} - 1)F(e^{it}) = 0,$$

then $p + q < 1$.

Proof. When $t \in (0, \frac{\pi}{2})$, we have

$$|F(e^{it})| = \exp\left((b^+ + b^-) \cos t + c + \int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{1}{s - e^{it}} - \frac{s}{s^2+1}\right) \varphi(s) ds\right).$$

Since

$$\operatorname{Re}\left(\frac{1}{s - e^{it}} - \frac{s}{s^2+1}\right) = \frac{(s^2 - 1) \cos t}{(s^2 + 1)(s^2 - 2s \cos t + 1)}$$

has the same sign as $\varphi(s)$ for $s \in (0, \infty)$, we have

$$\begin{aligned} |F(e^{it})| \geq & \exp \left((b^+ + b^-) \cos t + c + \int_{-\infty}^0 \operatorname{Re} \left(\frac{1}{s - e^{it}} - \frac{s}{s^2 + 1} \right) \varphi(s) ds \right. \\ & - p \int_0^1 \operatorname{Re} \left(\frac{1}{s - e^{it}} - \frac{s}{s^2 + 1} \right) ds \\ & \left. + q \int_1^\infty \operatorname{Re} \left(\frac{1}{s - e^{it}} - \frac{s}{s^2 + 1} \right) ds \right). \end{aligned}$$

By a straightforward calculation, we obtain

$$\begin{aligned} |F(e^{it})| \geq & 2^{p/2+q/2} |e^{it} - 1|^{-p-q} \times \\ & \times \exp \left((b^+ + b^-) \cos t + c + \int_{-\infty}^0 \operatorname{Re} \left(\frac{1}{s - e^{it}} - \frac{s}{s^2 + 1} \right) \varphi(s) ds \right). \end{aligned}$$

By the dominated convergence theorem, as $t \rightarrow 0^+$, the exponent in brackets has a finite limit

$$b^+ + b^- + c + \int_{-\infty}^0 \operatorname{Re} \left(\frac{1}{s - 1} - \frac{s}{s^2 + 1} \right) \varphi(s) ds.$$

Therefore, if $(e^{it} - 1)F(e^{it})$ converges to zero as $t \rightarrow 0^+$, then $|e^{it} - 1|^{1-p-q}$ also converges to zero, and thus $p + q < 1$. $\square$

Theorem 3.3. Suppose that $a(k)$ is the convolution of a summable two-sided Pólya frequency sequence $b(k)$ and an $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $c(k)$ which converges to zero as $k \rightarrow \pm\infty$. Then the generating function F of $a(k)$ is equal to

$$F(z) = \sum_{k=-\infty}^{\infty} a(k)z^k = \exp \left(b^+ z + \frac{b^-}{z} + c + \int_{-\infty}^{\infty} \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) \varphi(s) ds \right) \quad (3.1)$$

when $|z| = 1$, $z \neq 1$, where $b^+, b^- \in [0, \infty)$, $c \in \mathbb{R}$ and φ is a Borel function on $\mathbb{R}$ such that

- (i) φ is stepwise decreasing on $(-\infty, 0)$;
- (ii) φ is increasing-after-rounding on $(0, \infty)$;
- (iii) $\varphi \leq 0$ on $(0, 1)$ and $\varphi \geq 0$ on $(1, \infty)$;
- (iv) $\int_{-\infty}^{\infty} |\varphi(s)|/(s^2 + 1) ds < \infty$;
- (v) $(e^{it} - 1)F(e^{it})$ converges to 0 as $t \rightarrow 0$.

Conversely, if b^+, b^-, c, φ and the function F defined by (3.1) satisfy the above conditions, then F is the generating function of the convolution $a(k)$ of a summable two-sided Pólya frequency sequence $b(k)$ and an $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $c(k)$ which converges to zero as $k \rightarrow \pm\infty$.

Proof. By Lemma 2.2, the generating function H of an $\mathcal{AM}\text{-}\mathcal{CM}$ sequence $c(k)$ which converges to zero as $k \rightarrow \pm\infty$ has the exponential representation

$$H(z) = \exp \left(c_2 + \int_0^\infty \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) \varphi_2(s) ds \right)$$

for a constant $c_2 \in \mathbb{R}$ and a Borel function φ_2 on $\mathbb{R}$ which is equal to zero on $(-\infty, 0)$, takes values in $[-1, 0]$ on $(0, 1)$, and takes values in $[0, 1]$ on $(1, \infty)$. Similarly, by

Lemma 2.3, the generating function G of a summable two-sided Pólya frequency sequence $b(k)$ is given by

$$G(z) = \exp\left(b^+z + \frac{b^-}{z} + c_1 + \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) \varphi_1(s) ds\right)$$

for some constants $b^+, b^- \geq 0$ and $c_1 \in \mathbb{R}$, and a function φ_1 on $\mathbb{R}$ which is stepwise decreasing on $(-\infty, 0)$, is stepwise increasing on $(0, \infty)$, and satisfies $\varphi(s) = 0$ in a neighbourhood of 1. The generating function of the convolution $a(k) = (b * c)(k)$ is a function F satisfying $F(z) = G(z)H(z)$ when $|z| = 1$ and $z \neq 1$. We conclude that F is given by (3.1), with $c = c_1 + c_2 \in \mathbb{R}$ and $\varphi(s) = \varphi_1(s) + \varphi_2(s)$. It remains to note that $\varphi = \varphi_1$ is stepwise decreasing on $(-\infty, 0)$, $\varphi = \varphi_1 + \varphi_2$ is increasing-after-rounding on $(0, \infty)$, and φ is clearly nonpositive on $(0, 1)$ and nonnegative on $(1, \infty)$. Additionally, both $\varphi_1(s)/(s^2+1)$ and $\varphi_2(s)/(s^2+1)$ are integrable (the former by Lemma 2.3, the latter because φ_2 is bounded), and therefore $|\varphi(s)|/(s^2+1)$ is integrable over $\mathbb{R}$. Finally, G is continuous on the unit circle in the complex plane and $(e^{it} - 1)H(e^{it})$ converges to 0 as $t \rightarrow 0$, so clearly the limit of $(e^{it} - 1)F(e^{it})$ as $t \rightarrow 0$ is zero.

In order to prove the converse part of the theorem, we first show that every function φ with the properties listed in the statement of the theorem can be written as a sum $\varphi = \varphi_1 + \varphi_2$, where φ_1 and φ_2 have the properties discussed in the proof of the direct part of the theorem.

For $s < 0$, we simply define $\varphi_1(s) = \varphi(s)$ and $\varphi_2(s) = 0$. For $s > 0$ the definition is slightly more complicated. By definition, there is a stepwise increasing function $\tilde{\varphi}$ on $(0, \infty)$ such that $\tilde{\varphi}(s) \leq \varphi(s) \leq \tilde{\varphi}(s) + 1$ for $s > 0$. Since $\varphi(s) \leq 0$ for $s \in (0, 1)$ and $\varphi(s) \geq 0$ for $s \in (1, \infty)$, with no loss of generality we may assume that $\tilde{\varphi}(s) \leq -1$ for $s \in (0, 1)$ and $\tilde{\varphi}(s) \geq 0$ for $s \in [1, \infty)$ (otherwise we replace $\tilde{\varphi}$ by $\min\{\tilde{\varphi}, -1\}$ on $(0, 1)$ and by $\max\{\tilde{\varphi}, 0\}$ on $[1, \infty)$). We define $\varphi_1(s) = \tilde{\varphi}(s) + \mathbf{1}_{(0,1)}(s)$ and $\varphi_2(s) = \varphi(s) - \varphi_1(s)$. Clearly, φ_1 is stepwise increasing on $(0, \infty)$, $\varphi_2(s) \in [-1, 0]$ for $s \in (0, 1)$, and $\varphi_2(s) \in [0, 1]$ for $s \in (1, \infty)$. It remains to show that $\varphi_1(s) = 0$ in a neighbourhood of 1. Indeed: if $\varphi_1(s) \geq 1$ in some right neighbourhood of 1, then $\varphi(s) \geq 1$ for $s \in (1, \infty)$, which would contradict Lemma 3.2. Similarly, if $\varphi_1(s) \leq -1$ in some left neighbourhood of 1, then $\varphi(s) \leq -1$ for $s \in (0, 1)$, and again we would arrive at a contradiction with Lemma 3.2.

We return to the proof of the converse part of the theorem. Suppose that F is given by (3.1), define φ_1 and φ_2 as described above, and let G and H be defined as in the proof of the direct part (with, say, $c_1 = 0$ and $c_2 = c$). We observe that since $\varphi_1(s) = 0$ in a neighbourhood of 1, G is continuous on the unit circle in the complex plane and $G(1) \neq 0$. Hence, $(e^{it} - 1)H(e^{it}) = (e^{it} - 1)F(e^{it})/G(e^{it})$ converges to zero as $t \rightarrow 0$. Therefore, by Lemma 2.2, H is the generating function of an AM-CM sequence $c(k)$ which converges to zero as $k \rightarrow \pm\infty$. Furthermore, since φ_2 is bounded, $\varphi_1(s)/(s^2+1)$ is integrable over $\mathbb{R}$, and so Lemma 2.3 implies that G is the generating function of a summable Pólya frequency sequence $b(k)$. The convolution $a(k)$ of $b(k)$ and $c(k)$ has therefore generating function equal to $F(z)$ when $|z| = 1$, $z \neq 1$, and the proof is complete. $\square$

3.2. Generating functions of bell-shaped sequences. We now prove the difficult part of Theorem 1.1: implication (a) $\Rightarrow$ (c). Our argument follows the idea of

the proof in the one-sided case in [9], but the the generating function F is only defined on the unit circle, and so a different transform and a nonstandard inversion formula need to be employed. Note that despite these additional difficulties, our new approach also brings some simplifications: we no longer need Step 8 from the proof of Theorem 3.1 in [9].

Theorem 3.4. *Suppose that $a(k)$ is a two-sided bell-shaped sequence. Then the generating function F of $a(k)$ is given by (3.1), with b^+, b^-, c, φ satisfying conditions (i) through (v) listed in Theorem 3.3.*

Proof. The argument is divided into nine steps.

Step 1. The generating function of $a(k)$ is given by

$$F(z) = \sum_{k=-\infty}^{\infty} a(k)z^k$$

when $|z| = 1$, $z \neq 1$.

If $a(k)$ is summable, then F is continuous and bounded on the unit circle. In this case, when $\operatorname{Im} x > 0$, we define

$$G(x) = \int_0^{\infty} e^{itx} F(e^{it}) dt.$$

The inversion formula given in Proposition 2.4 reads

$$F(e^{it}) = \lim_{n \rightarrow \infty} \frac{(-1)^n (x_n)^{\overline{n+1}}}{in!} \Delta^n G(x_n), \quad (3.2)$$

where $t \in (0, \pi)$ and

$$x_n = \frac{n}{2} \left(i \cot \frac{t}{2} - 1 \right). \quad (3.3)$$

This will be the starting point for our reasoning.

In the general case, F may fail to be integrable over the unit circle in the complex plane, and we need the following minor modification. Since $a(k)$ converges to zero and it is eventually monotone as $k \rightarrow \infty$ and as $k \rightarrow -\infty$, the sequence $\Delta a(k)$ is summable, and its generating function is equal to $(z-1)F(z)$ when $|z| = 1$, $z \neq 1$. It follows that $(z-1)F(z)$ extends to a continuous functions on the unit circle in the complex plane, and hence we may apply the second part of Proposition 2.4 to get the same conclusion: (3.2) holds with x_n defined in (3.3).

Note that formula (3.2) is analogous to the result of Step 1 in the proof of Theorem 3.1, but the definition of G is essentially different there, and appropriately defined integers j_n are used there instead of the complex numbers x_n defined above.

Step 2. Suppose that $\operatorname{Im} x > 0$. If $a(k)$ is integrable, then, using the definition of F and Fubini's theorem, we obtain

$$\begin{aligned} G(x) &= \int_0^{\infty} e^{itx} \left(\sum_{k=-\infty}^{\infty} a(k) e^{ikt} \right) dt \\ &= \sum_{k=-\infty}^{\infty} a(k) \left(\int_0^{\infty} e^{it(x+k)} dt \right) \\ &= \sum_{k=-\infty}^{\infty} \frac{ia(k)}{x+k}. \end{aligned}$$

Evaluating the n th iterated difference of both sides with respect to x , we find that

$$\Delta^n G(x) = \sum_{k=-\infty}^{\infty} ia(k) \Delta_x^n \frac{1}{x+k} = \sum_{k=-\infty}^{\infty} ia(k) \Delta_k^n \frac{1}{x+k}, \quad (3.4)$$

where Δ_x and Δ_k denote the forward difference operators with respect to variables x and k , respectively. In the general case, by the definition of $\Delta^n G(x)$ (see Proposition 2.4) and Fubini's theorem, for $n \geq 1$ we have

$$\begin{aligned} \Delta^n G(x) &= \int_0^\infty e^{itx} (e^{it} - 1)^n \left(\sum_{k=-\infty}^{\infty} a(k) e^{ikt} \right) dt \\ &= \sum_{k=-\infty}^{\infty} a(k) \left(\int_0^\infty (e^{it} - 1)^n e^{it(x+k)} dt \right) \\ &= \sum_{k=-\infty}^{\infty} a(k) \left(\int_0^\infty \Delta_k^n e^{it(x+k)} dt \right) \\ &= \sum_{k=-\infty}^{\infty} a(k) \Delta_k^n \left(\int_0^\infty e^{it(x+k)} dt \right) \\ &= \sum_{k=-\infty}^{\infty} a(k) \Delta_k^n \frac{i}{x+k}, \end{aligned}$$

and we come to the same conclusion (3.4).

Let P be a polynomial of degree at most n . As in Step 2 in the proof of Theorem 3.1 in [9], we observe that $(P(k) - P(-x))/(x+k)$ is a polynomial in k of degree at most $n-1$, and hence

$$\Delta_k^n \frac{P(k)}{x+k} = P(-x) \Delta_k^n \frac{1}{x+k}.$$

Therefore,

$$P(-x) \Delta^n G(x) = \sum_{k=-\infty}^{\infty} ia(k) \Delta_k^n \frac{P(k)}{x+k}.$$

Applying summation by parts n times to the right-hand side, we conclude that

$$P(-x) \Delta^n G(x) = (-1)^n i \sum_{k=-\infty}^{\infty} \frac{P(k)}{x+k} \Delta^n a(k-n). \quad (3.5)$$

Note that the boundary terms vanish: for every $j = 0, 1, \dots, n-1$, the sequence $\Delta_k^{n-1-j}(P(x)/(x+k))$ is bounded by a constant times $(1+|k|)^j$, and the sequence $(1+|k|)^j \Delta^j a(k-j)$ converges to zero as $k \rightarrow \pm\infty$ by Lemma 2.3 in [9].

Step 3. Combining (3.2) and (3.5), we obtain

$$F(e^{it}) = \lim_{n \rightarrow \infty} \frac{(x_n)^{n+1}}{n! P_n(-x_n)} \sum_{k=-\infty}^{\infty} \frac{P_n(k) \Delta^n a(k-n)}{x_n + k}, \quad (3.6)$$

where $t \in (0, \pi)$, x_n is given by (3.3), and P_n is an arbitrary sequence of polynomials of degree at most n . We choose these polynomials in such a way that

$$P_n(k) \Delta^n a(k-n) \geq 0$$

for every $k \in \mathbb{Z}$: we set

$$P_n(k) = \prod_{j=0}^{n-1} (\alpha_{n,j} - k),$$

where $\alpha_{n,0}, \alpha_{n,1}, \dots, \alpha_{n,n-1}$ denote the locations of sign changes of the sequence $\Delta^n a(k-n)$. To be specific, we let $\alpha_{n,-1} = -\infty$, and we define inductively

$$\alpha_{n,j} = \min\{k > \alpha_{n,j-1} : (-1)^j \Delta^n a(k-n) > 0\}$$

for $j = 0, 1, 2, \dots, n-1$. With this choice of P_n , formula (3.6) can be rewritten as

$$F(e^{it}) = \lim_{n \rightarrow \infty} \left(\frac{x_n + n}{n!} \prod_{j=0}^{n-1} \frac{x_n + j}{x_n + \alpha_{n,j}} \right) \sum_{k=-\infty}^{\infty} \frac{P_n(k) \Delta^n a(k-n)}{x_n + k}. \quad (3.7)$$

This part is very similar to Step 3 of the proof of Theorem 3.1 in [9].

Step 4. Recall that in formula (3.7), $t \in (0, \pi)$ and x_n is given by (3.3). On the right-hand side of (3.7) only x_n depends on t , namely,

$$x_n = \frac{n}{2} \left(i \cot \frac{t}{2} - 1 \right) = \frac{n}{2} \left(-\frac{e^{it/2} + e^{-it/2}}{e^{it/2} - e^{-it/2}} - 1 \right) = \frac{ne^{it}}{1 - e^{it}}.$$

When $x \in \mathbb{C} \setminus \mathbb{Z}$, we denote by $G_n(x)$ the expression under the limit in (3.7), with x_n replaced by x :

$$G_n(x) = \left(\frac{x + n}{n!} \prod_{j=0}^{n-1} \frac{x + j}{x + \alpha_{n,j}} \right) \sum_{k=-\infty}^{\infty} \frac{P_n(k) \Delta^n a(k-n)}{x + k}. \quad (3.8)$$

Thus, $G_n(x_n)$ converges to $F(e^{it})$ as $n \rightarrow \infty$. This part is analogous to Step 4 of the proof of Theorem 3.1 in [9]. In the next two steps, we denote the two factors in the definition (3.8) of G_n by $G_n^\flat$ and $G_n^\sharp$, and we study each of them separately.

Step 5. For $x \in \mathbb{C} \setminus \mathbb{Z}$ we denote

$$G_n^\sharp(x) = \sum_{k=-\infty}^{\infty} \frac{P_n(k) \Delta^n a(k-n)}{x + k}. \quad (3.9)$$

Since $P_n(k) \Delta^n a(k-n) \geq 0$, we have $\operatorname{Im} G_n^\sharp(x) \leq 0$ when $\operatorname{Im} x > 0$. Since $G_n^\sharp$ is not identically zero, it follows that $1/G_n^\sharp$ is a Pick function. Note that $G_n^\sharp$ is real-valued and decreasing on each interval $(k, k+1)$, $k \in \mathbb{Z}$. Hence, the exponential representation of $1/G_n^\sharp$ is given by

$$\frac{1}{G_n^\sharp(x)} = \exp \left(c_n^\sharp + \int_{-\infty}^{\infty} \left(\frac{1}{s-x} - \frac{s}{s^2+1} \right) \psi_n^\sharp(s) ds \right) \quad (3.10)$$

when $\operatorname{Im} x > 0$, where $c_n^\sharp \in \mathbb{R}$ and

$$\psi_n^\sharp(s) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \operatorname{Arg} \frac{1}{G_n^\sharp(s+it)} = \begin{cases} 1 & \text{if } G_n^\sharp(s) < 0, \\ 0 & \text{if } G_n^\sharp(s) > 0. \end{cases}$$

Furthermore, for each $j \in \mathbb{Z}$ there is a number $\beta_{n,j} \in [-j-1, -j]$ such that

$$\{s \in (-j-1, -j) : G_n^\sharp(s) < 0\} = (-\beta_{n,j}, -j),$$

and hence

$$\psi_n^\sharp(s) = \sum_{j=-\infty}^{\infty} \mathbb{1}_{[-\beta_{n,j}, -j)}(s) \quad (3.11)$$

for almost every $s \in \mathbb{R}$. This part is essentially the same as Step 5 in the proof of Theorem 3.1 in [9].

Step 6. The other factor in the definition (3.8) of G_n reads

$$G_n^b(x) = \frac{x+n}{n!} \prod_{j=0}^{n-1} \frac{x+j}{x+\alpha_{n,j}}$$

for $x \in \mathbb{C} \setminus \mathbb{Z}$. Its exponential representation in the upper half-plane $\text{Im } x > 0$ follows from the elementary identities

$$\begin{aligned} \log(x+n) &= \frac{1}{2} \log(n^2+1) + \int_{-\infty}^{-n} \left(\frac{1}{s-x} - \frac{s}{s^2+1} \right) ds, \\ \log \frac{x+j}{x+\alpha_{n,j}} &= \frac{1}{2} \log \frac{j^2+1}{\alpha_{n,j}^2+1} + \int_{-\alpha_{n,j}}^{-j} \left(\frac{1}{s-x} - \frac{s}{s^2+1} \right) ds. \end{aligned}$$

By definition, we have

$$G_n^b(x) = \exp \left(c_n^b + \int_{-\infty}^{\infty} \left(\frac{1}{s-x} - \frac{s}{s^2+1} \right) \psi_n^b(s) ds \right), \quad (3.12)$$

where $c_n^b \in \mathbb{R}$ and

$$\psi_n^b(s) = \sum_{j=0}^n \mathbb{1}_{(-\infty, -j)}(s) - \sum_{j=0}^{n-1} \mathbb{1}_{(-\infty, -\alpha_{n,j})}(s). \quad (3.13)$$

This part is identical to Step 6 of the proof of Theorem 3.1 in [9].

Step 7. Combining (3.10) and (3.12) together, we obtain the exponential representation of the function G_n defined in (3.8), namely

$$G_n(x) = \exp \left(a_n + \int_{-\infty}^{\infty} \left(\frac{1}{s-x} - \frac{s}{s^2+1} \right) \psi_n(s) ds \right), \quad (3.14)$$

where $a_n = a_n^b - a_n^\sharp \in \mathbb{R}$ and $\psi_n = \psi_n^b - \psi_n^\sharp$. In this step we prove that ψ_n is stepwise decreasing on $(-n, 0)$, increasing-after-rounding and nonnegative on $(-\infty, -n)$, and increasing-after-rounding and nonpositive on $(0, \infty)$. This is the counterpart of Step 7 of the proof of Theorem 3.1 in [9], but the analysis requires more care in the present case.

To simplify the notation, in this step only we fix n , and we write $\alpha_{n,j} = \alpha_j$ and $\beta_{n,j} = \beta_j$. By (3.10) and (3.13), we have

$$\psi_n(s) = \sum_{j=0}^n \mathbb{1}_{(-\infty, -j)}(s) - \sum_{j=0}^{n-1} \mathbb{1}_{(-\infty, -\alpha_j)}(s) - \sum_{j=-\infty}^{\infty} \mathbb{1}_{[-\beta_j, -j)}(s).$$

Recall that $-j - 1 \leq -\beta_j \leq -j$. Elementary manipulations lead to

$$\begin{aligned}
\psi_n &= \sum_{j=0}^{n-1} (\mathbb{1}_{(-\infty, -j)} - \mathbb{1}_{(-\infty, -\alpha_j)}) + \left(\sum_{j=n}^{\infty} \mathbb{1}_{[-j-1, -j)} - \sum_{j=-\infty}^{\infty} \mathbb{1}_{[-\beta_j, -j)} \right) \\
&= \sum_{j=0}^{n-1} (\mathbb{1}_{(-\infty, -j)} - \mathbb{1}_{(-\infty, -\alpha_j)} - \mathbb{1}_{[-\beta_j, -j)}) \\
&\quad - \sum_{j=-\infty}^{-1} \mathbb{1}_{[-\beta_j, -j)} + \sum_{j=n}^{\infty} (\mathbb{1}_{[-j-1, -j)} - \mathbb{1}_{[-\beta_j, -j)}) \\
&= \sum_{j=0}^{n-1} (\mathbb{1}_{(-\infty, -\beta_j)} - \mathbb{1}_{(-\infty, -\alpha_j)}) - \sum_{j=-\infty}^{-1} \mathbb{1}_{[-\beta_j, -j)} + \sum_{j=n}^{\infty} \mathbb{1}_{[-j-1, -\beta_j)}.
\end{aligned}$$

The last two terms on the right-hand side define a function which takes values in $\{0, 1\}$ on $(-\infty, -n)$, which is equal to zero on $(-n, 0)$, and which takes values in $\{-1, 0\}$ on $(0, \infty)$. Let us now inspect the first term, that is,

$$\tilde{\psi}_n(s) = \sum_{j=0}^{n-1} (\mathbb{1}_{(-\infty, -\beta_j)}(s) - \mathbb{1}_{(-\infty, -\alpha_j)}(s)). \quad (3.15)$$

This function only takes integer values, it has n upward jumps at $-\alpha_j$, and it has n downward jumps at $-\beta_j$, where $j = 0, 1, \dots, n-1$. In particular, $\tilde{\psi}_n(s) = 0$ outside a finite interval. Since $-\beta_j \in [-n, 0]$ for every $j = 0, 1, \dots, n-1$, the function $\tilde{\psi}_n$ is stepwise increasing on $(-\infty, -n)$ and on $(0, \infty)$.

It follows that on $(-\infty, -n)$, the function ψ_n is a sum of a stepwise increasing nonnegative function $\tilde{\psi}_n$ and a function that takes values in $\{0, 1\}$. Thus, ψ_n is increasing-after-rounding and nonnegative on $(-\infty, n)$. Similarly, ψ_n is increasing-after-rounding and nonpositive on $(0, \infty)$. It remains to show that ψ_n is stepwise decreasing on $(-n, 0)$.

We already know that ψ_n takes integer values on $(-n, 0)$ and it has downward jumps at $-\beta_j \in [-j-1, -j]$ for $j = 0, 1, \dots, n-1$. It may have an upward jump at $-\alpha_j$ with $j = 0, 1, \dots, n-1$, as long as this number belongs to $(-n, 0)$. Suppose that $-\alpha_j \in (-n, 0)$, that is, $\alpha_j \in \{1, 2, \dots, n-1\}$, and write $m = \alpha_j$. By definition, $P_n(m) = 0$, and so the function $G_n^\sharp$ defined in (3.9) does not have a pole at $y = -m$. Therefore, $G_n^\sharp$ is decreasing on $(-m-1, -m+1)$.

If $G_n^\sharp(-m) < 0$, then $G_n^\sharp(y) < 0$ for $y \in (-m, -m+1)$, and so, by definition, $\beta_{m-1} = m$. In this case the upward jump of ψ_n at $-\alpha_j = -m$ is cancelled by a downward jump at $-\beta_{m-1} = -m$.

If $G_n^\sharp(-m) \geq 0$, then $G_n^\sharp(y) \geq 0$ for $y \in (-m-1, -m)$, and hence $\beta_m = m$. In this case the upward jump of ψ_n at $-\alpha_j = -m$ is cancelled by a downward jump at $-\beta_m = -m$.

The above argument shows that every downward jump of ψ_n on $(-n, 0)$ is cancelled by an appropriate upward jump, and so ψ_n is indeed stepwise decreasing on $(-n, 0)$, as desired.

Step 8. Recall that $x_n = ne^{it}/(1 - e^{it})$, and our ultimate goal is to express $F(e^{it})$ in terms of e^{it} . For this reason we substitute $x = nz/(1 - z)$ in the exponential

representation (3.14) of the function G_n . In other words, when $\text{Im } z > 0$ we define

$$F_n(z) = G_n\left(\frac{nz}{1-z}\right) = \exp\left(c_n + \int_{-\infty}^{\infty} \left(\frac{1-z}{s(1-z) - nz} - \frac{s}{s^2+1}\right) \psi_n(s) ds\right),$$

so that $F_n(e^{it}) = G_n(x_n)$ converges to $F(e^{it})$ for every $t \in (0, \pi)$. As in Step 9 of the proof of Theorem 3.1 in [9], we substitute $s = nr/(1-r)$ in the integral on the right-hand side. In order to do so, we denote

$$\varphi_n(r) = \psi_n\left(\frac{nr}{1-r}\right).$$

It is straightforward to see that φ_n is stepwise decreasing on $(-\infty, 0)$, increasing-after-rounding and nonpositive on $(0, 1)$, and increasing-after-rounding and non-negative on $(1, \infty)$. Substitution leads to

$$F_n(z) = \exp\left(d_n + \int_{-\infty}^{\infty} \left(\frac{1}{r-z} - \frac{r}{r^2+1}\right) \varphi_n(r) dr\right) \quad (3.16)$$

when $\text{Im } z > 0$ for an appropriate $d_n \in \mathbb{R}$; we omit the easy details, which are exactly the same as in the corresponding part of [9].

Step 9. We know that $F_n(z)$ is given by (3.16) when $\text{Im } z > 0$, and that $F_n(e^{it})$ converges to $F(e^{it})$ as $n \rightarrow \infty$ for every $t \in (0, \pi)$. The desired representation (3.1) of the limiting function F follows now from Lemma 2.5. Clearly, φ satisfies conditions (i) through (iv) of Theorem 3.3. In order to prove the remaining condition (v), observe, as in Remark 1.2, that the sequence $\Delta a(k)$ is summable and it sums up to 0, and hence its generating function is continuous on the unit circle in the complex plane, and it takes value 0 at $z = 1$. The generating function of $\Delta a(k)$ is, however, equal to $(z-1)F(z)$ when $|z| = 1$, $z \neq 1$, and hence the limit of $(e^{it}-1)F(e^{it})$ as $t \rightarrow 0$ is equal to zero. $\square$

Appendix A. Auxiliary ‘approximation to identity’ lemma

The following result is used in the proof of Proposition 2.4

Lemma A.1. *Suppose that φ is a continuous integrable real-valued function on $[0, \infty)$ such that $\varphi(s)$ converges to 0 as $s \rightarrow \infty$, $|\varphi|$ attains a strict global maximum at a point $t \in (0, \infty)$, and $\varphi(t) > 0$. Denote*

$$M_n = \int_0^\infty (\varphi(s))^n ds.$$

If F is an integrable function on the unit circle in the complex plane and F is continuous at e^{it} , then

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \int_0^\infty (\varphi(s))^n F(e^{is}) ds = F(e^{it}).$$

Instead of integrability of F it is sufficient to assume that $(\varphi(t))^n F(e^{it})$ is integrable over $(0, \infty)$ for n large enough.

Proof. Our goal is to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \int_0^\infty (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds = 0.$$

Fix $\varepsilon > 0$, and choose $\delta > 0$ small enough, so that $|F(e^{is}) - F(e^{it})| < \varepsilon$ and $\varphi(s) > 0$ whenever $|s - t| < \delta$. By assumptions, there is $\vartheta > 0$ such that $|\varphi(s)| \leq (1 - 2\vartheta)\varphi(t)$

when $s \geq 0$ and $|s - t| \geq \delta$. Furthermore, we can find $\eta > 0$ such that $\varphi(s) > (1 - \vartheta)\varphi(t)$ when $|s - t| < \eta$. On one hand, we have

$$\begin{aligned} & \left| \int_{[0, \infty) \setminus (t-\delta, t+\delta)} (\varphi(s))^n ds \right| \\ & \leq \int_{[0, \infty) \setminus (t-\delta, t+\delta)} |\varphi(s)|^n ds \\ & \leq ((1 - 2\vartheta)\varphi(t))^{n-1} \int_{[0, \infty) \setminus (t-\delta, t+\delta)} |\varphi(s)| ds \end{aligned} \quad (\text{A.1})$$

and, in the same vein,

$$\begin{aligned} & \left| \int_{[0, \infty) \setminus (t-\delta, t+\delta)} (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds \right| \\ & \leq \int_{[0, \infty) \setminus (t-\delta, t+\delta)} |\varphi(s)|^n |F(e^{is}) - F(e^{it})| ds \\ & \leq ((1 - 2\vartheta)\varphi(t))^{n-1} \int_{[0, \infty) \setminus (t-\delta, t+\delta)} |\varphi(s)| |F(e^{is}) - F(e^{it})| ds. \end{aligned} \quad (\text{A.2})$$

On the other one, we have

$$M_n \geq \int_{(t-\delta, t+\delta)} (\varphi(s))^n ds \geq \int_{(t-\eta, t+\eta)} (\varphi(s))^n ds \geq 2\eta((1 - \vartheta)\varphi(t))^n, \quad (\text{A.3})$$

and

$$\left| \int_{(t-\delta, t+\delta)} (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds \right| \leq \varepsilon \int_{(t-\delta, t+\delta)} (\varphi(s))^n ds \leq \varepsilon M_n. \quad (\text{A.4})$$

Since $(1 - \vartheta)\varphi(t) > (1 - 2\vartheta)\varphi(t)$, the right-hand sides of (A.1) and (A.2) are negligible compared to the right-hand side of (A.3) as $n \rightarrow \infty$. Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{M_n} \int_{[0, \infty) \setminus (t-\delta, t+\delta)} (\varphi(s))^n ds &= 0, \\ \lim_{n \rightarrow \infty} \frac{1}{M_n} \int_{(t-\delta, t+\delta)} (\varphi(s))^n ds &= 1, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \left| \int_{[0, \infty) \setminus (t-\delta, t+\delta)} (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds \right| = 0.$$

By (A.4), we also have

$$\limsup_{n \rightarrow \infty} \frac{1}{M_n} \left| \int_{(t-\delta, t+\delta)} (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds \right| \leq \varepsilon.$$

The last two formulae imply that

$$\limsup_{n \rightarrow \infty} \frac{1}{M_n} \left| \int_0^\infty (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds \right| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the desired result follows.

If we relax the integrability assumption on F to integrability of $(\varphi(t))^m F(e^{it})$ over $(0, \infty)$ for some m , then we only need to replace the bound (A.2) by

$$\begin{aligned} & \left| \int_{[0, \infty) \setminus (t-\delta, t+\delta)} (\varphi(s))^n (F(e^{is}) - F(e^{it})) ds \right| \\ & \leq ((1 - 2\vartheta)\varphi(t))^{n-m} \int_{[0, \infty) \setminus (t-\delta, t+\delta)} |\varphi(s)|^m |F(e^{is}) - F(e^{it})| ds. \end{aligned}$$

Otherwise, the proof is exactly the same. $\square$

Appendix B. Holomorphic character of two functions

Below we give a detailed derivation of the representation formula for two holomorphic functions which appear in Examples 1.10 and 1.11.

B.1. Independent simple random walks. We begin with the function F_1 defined in Example 1.11. For simplicity, we drop the index 1 from the notation, that is, we consider the function

$$F(z) = \frac{2 + i(z^{1/2} - z^{-1/2})}{z + 1}.$$

Our goal is to prove that F has the form given in (1.1), so that Theorem 1.1 applies. We have already observed that F is a holomorphic function in the upper complex half-plane $\text{Im } z > 0$. Observe that in this region

$$2 + i(z^{1/2} - z^{-1/2}) = 2 - (-z)^{1/2} - (-z)^{-1/2} = (i(-z)^{1/4} - i(-z)^{-1/4})^2.$$

Furthermore, $\text{Re}(i(-z)^{1/4} - i(-z)^{-1/4}) > 0$, and hence

$$\text{Arg}(2 + i(z^{1/2} - z^{-1/2})) = 2 \text{Arg}(i(-z)^{1/4} - i(-z)^{-1/4})$$

is well-defined and takes values in $(-\pi, \pi)$. Clearly, $\text{Arg}(z + 1)$ is well-defined and takes values in $(0, \pi)$ when $\text{Im } z > 0$. It follows that the argument of F has a continuous version in the upper complex half-plane $\text{Im } z > 0$, which we denote by $\pi\Phi$, and

$$\Phi(z) = \frac{1}{\pi} \text{Arg}(2 + i(z^{1/2} - z^{-1/2})) - \frac{1}{\pi} \text{Arg}(z + 1).$$

Furthermore, $\Phi(z) \in [-2, 1]$ for every z , and hence Φ is a bounded harmonic function in the upper complex half-plane. By Poisson's representation formula,

$$\Phi(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im} \frac{1}{s - z} \varphi(s) ds,$$

where $\varphi(s) = \lim_{t \rightarrow 0^+} \Phi(s + it)$ is the boundary limit of Φ , defined for almost every $s \in \mathbb{R}$. Recall that $\pi\Phi$ is the imaginary part of the continuous logarithm of F . It follows that for some constant $c \in \mathbb{R}$,

$$F(z) = \exp \left(c + \int_{-\infty}^{\infty} \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) \varphi(s) ds \right),$$

that is, F is indeed given by (1.1). Furthermore, for $s \neq -1$, we have

$$\lim_{t \rightarrow 0^+} \text{Arg}(s + it + 1) = \pi \mathbf{1}_{(-\infty, -1)}(s).$$

When $s > 0$, clearly

$$\begin{aligned} \lim_{t \rightarrow 0^+} \operatorname{Arg}(2 + i((s + it)^{1/2} - (s + it)^{-1/2})) &= \operatorname{Arg}(2 + i(s^{1/2} - s^{-1/2})) \\ &= \arctan \frac{s^{1/2} - s^{-1/2}}{2}. \end{aligned}$$

Finally, when $s < 0$ and $s \neq -1$, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \operatorname{Arg}(2 + i((s + it)^{1/2} - (s + it)^{-1/2})) \\ &= \lim_{t \rightarrow 0^+} \operatorname{Arg}(2 - (-s)^{1/2} - (-s)^{-1/2} + \tfrac{1}{2}i(-s)^{-1/2}t - \tfrac{1}{2}i(-s)^{-3/2}t + O(t^2)) \\ &= \pi \mathbf{1}_{(-\infty, -1)}(s) - \pi \mathbf{1}_{(-1, 0)}(s). \end{aligned}$$

It follows that

$$\begin{aligned} \pi\varphi(s) &= \lim_{t \rightarrow 0^+} \pi\Phi(s + it) \\ &= \lim_{t \rightarrow 0^+} \operatorname{Arg}(2 + i((s + it)^{1/2} - (s + it)^{-1/2})) - \lim_{t \rightarrow 0^+} \operatorname{Arg}(s + it + 1) \end{aligned}$$

has the desired form, namely,

$$\varphi(s) = \begin{cases} 0 & \text{if } s < -1; \\ -1 & \text{if } -1 < s < 0; \\ \frac{1}{\pi} \arctan(\frac{1}{2}(s^{1/2} - s^{-1/2})) & \text{if } s > 0, \end{cases}$$

as claimed in Example 1.11. We note that in fact we have $\varphi(s) \in [-1, 1]$ for almost every $s \in \mathbb{R}$, and thus $\Phi(z) \in (-1, 1)$ for every z in the upper complex half-plane $\operatorname{Im} z > 0$, that is, Φ is in fact the principal branch of the argument of $F(z)$.

B.2. Two-dimensional simple random walk. We turn to the properties of the function F_1 introduced in Example 1.10. Again we drop the index 1 from the notation, that is, we consider the function

$$F(z) = 2 - \frac{z + z^{-1}}{2} - \sqrt{1 - \frac{z + z^{-1}}{2}} \sqrt{3 - \frac{z + z^{-1}}{2}}.$$

In Example 1.10 we argued that F is a holomorphic function in the upper complex half-plane $\operatorname{Im} z > 0$, and in this region $1 - \frac{1}{2}(z + z^{-1}) \in \mathbb{C} \setminus (-\infty, 0]$. If we write $w = 1 - \frac{1}{2}(z + z^{-1})$, then $w \in \mathbb{C} \setminus (-\infty, 0]$, and

$$F(z) = 1 + w - \sqrt{w}\sqrt{w+2} = \frac{(1+w)^2 - w(w+2)}{1+w+\sqrt{w}\sqrt{w+2}} = \frac{1}{1+w+\sqrt{w}\sqrt{w+2}}.$$

If $w > 0$, then $1 + w + \sqrt{w}\sqrt{w+2} > 0$. Suppose that $\operatorname{Im} w > 0$. Then $0 < \operatorname{Arg}(w+2) < \operatorname{Arg} w < \pi$, and hence $0 < \operatorname{Arg}(\sqrt{w}\sqrt{w+2}) < \pi$. Therefore, $\operatorname{Im}(1 + w + \sqrt{w}\sqrt{w+2}) > 0$. Similarly, if $\operatorname{Im} w < 0$, then $\operatorname{Im}(1 + w + \sqrt{w}\sqrt{w+2}) < 0$. It follows that $F(z) \in \mathbb{C} \setminus (-\infty, 0]$. Consequently, $\Phi(z) = \frac{1}{\pi} \operatorname{Arg} F(z)$ is well-defined in the upper complex half-plane with the principal branch of the logarithm, and $\Phi(z) \in [-1, 1]$. As in the previous section, we use Poisson's representation formula for the bounded harmonic function Φ to find that F is indeed given by (1.1), with

$$\pi\varphi(s) = \lim_{t \rightarrow 0^+} \operatorname{Arg} F(s + it).$$

In order to evaluate the above limit, we observe that if $z = s + it$ and, as before, $w = 1 - \frac{1}{2}(z + z^{-1})$, then

$$w = 1 - \frac{1}{2}(s + s^{-1}) - \frac{1}{2}it + \frac{1}{2}is^{-2}t + O(t^2)$$

as $t \rightarrow 0^+$. Thus, as $t \rightarrow 0^+$, w converges to $1 - \frac{1}{2}(s + s^{-1})$. Additionally, $\text{Im } w$ is negative for small $t > 0$ when $|s| > 1$ and positive for small $t > 0$ when $|s| < 1$.

In terms of variable w introduced above, we have

$$\lim_{t \rightarrow 0^+} \text{Arg } F(s + it) = - \lim_{t \rightarrow 0^+} \text{Arg}(1 + w + \sqrt{w}\sqrt{w+2}).$$

When $s < 0$, the limit of w is positive, and therefore $F(s + it)$ converges to a positive limit as $t \rightarrow 0^+$. That is,

$$\lim_{t \rightarrow 0^+} \text{Arg } F(s + it) = 0.$$

When $0 < s < 3 - 2\sqrt{2}$, then w converges to a number in $(-\infty, -2)$, so that $F(z)$ converges to a negative number. Furthermore, the imaginary part of w is positive for t small enough. Therefore,

$$\lim_{t \rightarrow 0^+} \text{Arg } F(s + it) = - \lim_{t \rightarrow 0^+} \text{Arg}(1 + w + \sqrt{w}\sqrt{w+2}) = -\pi.$$

Similarly, when $s > 3 + 2\sqrt{2}$, then again w converges to a number in $(-\infty, -2)$ and $F(z)$ converges to a negative number, but the imaginary part of w is negative for t small enough. It follows that

$$\lim_{t \rightarrow 0^+} \text{Arg } F(s + it) = - \lim_{t \rightarrow 0^+} \text{Arg}(1 + w + \sqrt{w}\sqrt{w+2}) = \pi.$$

Finally, when $3 - 2\sqrt{2} < s < 3 + 2\sqrt{2}$ and $s \neq 1$, then the limit of w lies in $(-2, 0)$, and hence

$$\begin{aligned} \lim_{t \rightarrow 0^+} \text{Arg } F(s + it) &= - \lim_{t \rightarrow 0^+} \text{Arg}(1 + w + i\sqrt{-w}\sqrt{w+2}) \\ &= - \text{Arg}\left(2 - \frac{1}{2}(s + s^{-1}) + i\sqrt{\frac{1}{2}(s^{-1} + s) - 1}\sqrt{3 - \frac{1}{2}(s^{-1} + s)}\right). \end{aligned}$$

The above expression belongs to $(-\pi, 0)$ when $s < 1$ and to $(0, \pi)$ when $s > 1$. We conclude that φ indeed has the desired form:

$$\varphi(s) = \begin{cases} 0 & \text{if } s < 0; \\ -1 & \text{if } 0 < s < 3 - 2\sqrt{2}; \\ -\frac{1}{\pi} \text{arccot} \frac{4 - s - s^{-1}}{\sqrt{s + s^{-1} - 2}\sqrt{6 - s - s^{-1}}} & \text{if } 3 - 2\sqrt{2} < s < 1; \\ \frac{1}{\pi} \text{arccot} \frac{4 - s - s^{-1}}{\sqrt{s + s^{-1} - 2}\sqrt{6 - s - s^{-1}}} & \text{if } 1 < s < 3 - \sqrt{2}; \\ 1 & \text{if } s > 3 + 2\sqrt{2}. \end{cases}$$

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