On the relaxation to equilibrium of a quantum oscillator interacting with a radiation field

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Abstract

In this article we investigate from the point of view of spectral theory the problem of relaxation to thermodynamical equilibrium of a quantum harmonic oscillator interacting with a radiation field. Our starting point is a system of infinitely many Pauli master equations governing the time evolution of the occupation probabilities of the available quantum states. The system we consider is derived from the evolution equation for the reduced density operator obtained after the initial interaction of the oscillator with the radiation field, the latter acting as a heat bath. We provide a complete spectral analysis of the infinitesimal generator of the equations, showing thereby that it generates an infinite-dimensional dynamical system of hyperbolic type. This implies that every global solution to the equations converges exponentially rapidly toward the corresponding Gibbs equilibrium state. We also provide a complete spectral analysis of a linear pencil naturally associated with the Pauli equations. All of our considerations revolve around the notion of compactness, more specifically around the existence of a new compact embedding result involving a space of sequences of Sobolev type into a weighted space of square-integrable summable sequences.

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1 Introduction and outline

In this article we investigate the relaxation to thermodynamical equilibrium of a one-dimensional quantum harmonic oscillator interacting with a radiation field, for instance a quantized electromagnetic field in a vacuum or lattice vibrations in a crystal, acting as a heat bath. More specifically, we are primarily interested in the long time behavior of the occupation probabilities associated with each quantum energy state of the oscillator after its initial interaction with the bath. The system governing the evolution of those probabilities consists of infinitely many master equations, that is, one such equation for each quantum state. The equations we consider are typical gain-loss equations whose structure is already apparent in [18], *albeit* in a very different context. They represent a Markovian approximation to a more general evolution system that takes the form of an integro-differential equation with a memory term for the reduced density operator under consideration. Accordingly, we organize our article in the following way: In Section 2 we define the class of master equations to be analyzed and prove the self-adjointness of their generator on a dense domain of a suitable Hilbert space. We proceed by proving that the generator in question has a compact resolvent and thereby a discrete spectrum, which requires a concrete realization of its domain as a weighted Sobolev space of sequences compactly embedded in the given Hilbert space. This allows us to prove that the system generates a dynamical system of hyperbolic type, and from there we obtain the spectral resolution of the corresponding evolution semi-group, which is holomorphic and contracting. Knowing this, we then prove that any initial probability distribution remains a probability distribution for all times, which eventually converges exponentially rapidly, as time becomes large, to the Gibbs equilibrium distribution characterized by the temperature of the bath. In Section 3 we determine the spectrum of a linear pencil naturally associated with the system of equations analyzed in Section 2. We conclude the article with an appendix, in which we prove a generalization of the compactness result used in Sections 2 and 3 along with some consequences. Throughout this work we make a point of giving detailed proofs of all statements, with appropriate references whenever necessary.

Needless to say, much is known regarding the properties of a quantum harmonic oscillator in a heat bath and their physical consequences (see, e.g. [5], [12] or [26] and some of the references therein). Thus, the focus of our approach to the problem of relaxation by means of spectral arguments is not so much about the results they imply as it is about the strategy to obtain them. As such, our method may then be considered as complementary to the existing ones.

For other applications of master equations in non-equilibrium statistical mechanics, we refer the reader for instance to [3, 11, 16, 22] and their references.

2 A spectral theorem for a class of infinitely many Pauli master equations

Our starting point to implement the above program is the system of Pauli master equations given by

$$\begin{aligned} \frac{dp_{\mathsf{m}}(t)}{dt} &= \sum_{\mathsf{n}=0}^{+\infty} \left(r_{\mathsf{m},\mathsf{n}} p_{\mathsf{n}}(t) - r_{\mathsf{n},\mathsf{m}} p_{\mathsf{m}}(t) \right), \quad t \in (0, +\infty) \\ p_{\mathsf{m}}(0) &= p_{\mathsf{m}}^{*} \end{aligned} \tag{1}$$

for every $\mathsf{m} \in \mathbb{N}$, where (p^*_{m}) stands for any sequence of initial conditions satisfying

$$p^*_{\mathsf{m}} \geq 0, \quad \sum_{\mathsf{m}=0}^{+\infty} p^*_{\mathsf{m}} = 1, \tag{2}$$

and where the transition rates from level n to level m are

$$r_{\mathsf{m},\mathsf{n}} = \begin{cases} -(\rho\mathsf{m} + \sigma(\mathsf{m}+1)) & \text{for} \quad \mathsf{m} = \mathsf{n}, \\ \\ \rho\mathsf{n}\delta_{\mathsf{m},\mathsf{n}-1} + \sigma(\mathsf{n}+1)\delta_{\mathsf{m},\mathsf{n}+1} & \text{for } \mathsf{m} \neq \mathsf{n} \end{cases} \tag{3}$$

with $\rho, \sigma > 0$. It will soon be reminded that the functions $t \mapsto p_{\mathsf{m}}(t)$ are the occupation probabilities of level m at time t of the harmonic oscillator after its initial interaction with the heat bath.

While system (1) takes the form

$$\begin{aligned} \frac{dp_{\mathsf{m}}(t)}{dt} &= \rho(\mathsf{m}+1)p_{\mathsf{m}+1}(t) + \sigma\mathsf{m}p_{\mathsf{m}-1}(t) - (\rho\mathsf{m} + \sigma(\mathsf{m}+1))\, p_{\mathsf{m}}(t), \\ p_{\mathsf{m}}(0) &= p^*_{\mathsf{m}}, \end{aligned} \tag{4}$$

as a consequence of (3), we note that the right-hand side of the preceding equation is ill-defined for $\mathsf{m} = 0$ because of the presence of $p_{\mathsf{m}-1}(t)$. Strictly speaking, it would indeed be more accurate to require (4) for $\mathsf{m} \geqslant 1$ while imposing

$$\begin{aligned} \frac{dp_0(t)}{dt} &= \rho p_1(t) - \sigma p_0(t), \\ p_0(0) &= p^*_0 \end{aligned}$$

for $\mathsf{m} = 0$. However, since the coefficient of $p_{\mathsf{m}-1}(t)$ vanishes when $\mathsf{m} = 0$ we will keep using (4) for the sake of convenience and simplicity of the notation. A similar remark applies to the second line of (3).

Before we make a first contact with statistical mechanics, we have to determine the unique stationary solution to (4):

Lemma 1. *Aside from the positivity of σ and ρ, let us assume that $\sigma < \rho$. Then system (4) possesses exactly one stationary solution $\mathsf{p_S}$ given by*

$$p_{\mathsf{S},\mathsf{m}} = \left(\frac{\sigma}{\rho}\right)^{\mathsf{m}} \left(1 - \frac{\sigma}{\rho}\right) \tag{5}$$

for every $\mathsf{m} \in \mathbb{N}$.

Proof. Setting the right-hand side of (4) equal to zero we get the recurrence relation

$$p_{\mathsf{S},\mathsf{m}+1} = \frac{(\rho\mathsf{m} + \sigma(\mathsf{m}+1))\, p_{\mathsf{S},\mathsf{m}} - \sigma\mathsf{m}p_{\mathsf{S},\mathsf{m}-1}}{\rho(\mathsf{m}+1)},$$

so that we obtain

$$p_{\mathsf{S},\mathsf{m}} = \left(\frac{\sigma}{\rho}\right)^{\mathsf{m}} p_{\mathsf{S},0}$$

for every $\mathsf{m} \in \mathbb{N}$ from an easy induction argument. Since $\sigma < \rho$, the result then follows from the normalization condition (2). ■

From now on we identify the energy spectrum of the one-dimensional quantum harmonic oscillator with $\lambda_{\mathsf{m}} = \mathsf{m} \in \mathbb{N}$, which only requires one to rescale and shift the corresponding Hamiltonian operator in an irrelevant way. When the oscillator is in equilibrium with the heat bath at inverse temperature $\beta = (k_{\mathsf{B}}T)^{-1} > 0$, where k_{B} stands for Boltzmann's constant, the corresponding Gibbs probability distribution $\mathsf{p}_{\beta,\mathsf{Gibbs}}$ is then

$$p_{\beta,\mathsf{Gibbs},\mathsf{m}} = Z_{\beta}^{-1} \exp\left[-\beta \mathsf{m}\right] \tag{6}$$

where

$$Z_{\beta} = \sum_{\mathsf{m}=0}^{+\infty} \exp\left[-\beta \mathsf{m}\right] = \left(1 - \exp\left[-\beta\right]\right)^{-1} \tag{7}$$

is the associated partition function. We can easily arrange for $\mathsf{p}_{\beta,\mathsf{Gibbs}}$ to be a trivial solution to (4) when the initial condition is chosen as $p_{\mathsf{m}}^{*} = p_{\beta,\mathsf{Gibbs},\mathsf{m}}$ for every $\mathsf{m} \in \mathbb{N}$. Indeed the following result holds:

Lemma 2. *The rates (3) satisfy the so-called detailed balance conditions*

$$r_{\mathsf{m},\mathsf{n}} p_{\beta,\mathsf{Gibbs},\mathsf{n}} = r_{\mathsf{n},\mathsf{m}} p_{\beta,\mathsf{Gibbs},\mathsf{m}} \tag{8}$$

with respect to (6) for all $\mathsf{m}, \mathsf{n} \in \mathbb{N}$ *if, and only if,*

$$\frac{\sigma}{\rho} = \exp\left[-\beta\right]. \tag{9}$$

In this case we have

$$\mathsf{p}_{\mathsf{S}} = \mathsf{p}_{\beta,\mathsf{Gibbs}}. \tag{10}$$

Proof. It is sufficient to assume $\mathsf{m} \neq \mathsf{n}$, in which case (8) reads

$$\begin{aligned} & \left(\rho \mathsf{n} \delta_{\mathsf{m},\mathsf{n}-1} + \sigma(\mathsf{n}+1)\delta_{\mathsf{m},\mathsf{n}+1}\right) \exp\left[-\beta \mathsf{n}\right] \\ = \quad & \left(\rho \mathsf{m} \delta_{\mathsf{n},\mathsf{m}-1} + \sigma(\mathsf{m}+1)\delta_{\mathsf{n},\mathsf{m}+1}\right) \exp\left[-\beta \mathsf{m}\right], \end{aligned}$$

from which (9) follows when $\mathsf{m} = \mathsf{n} - 1$ or when $\mathsf{m} = \mathsf{n} + 1$. The converse is equally trivial and the substitution of (9) into (5) then gives (10). ■

The detailed balance conditions (8) thus imply that the parameters σ and ρ are not independent, and as a consequence the unique stationary solution to (4) is an equilibrium solution in the thermodynamical sense. Before proceeding, we note that (3) becomes

$$r_{\mathsf{m},\mathsf{n}} = \begin{cases} -\rho(\mathsf{m} + \exp[-\beta](\mathsf{m}+1)) \quad \text{for} \quad \mathsf{m} = \mathsf{n}, \\ \\ \rho\left(\mathsf{n}\delta_{\mathsf{m},\mathsf{n}-1} + \exp[-\beta](\mathsf{n}+1)\delta_{\mathsf{m},\mathsf{n}+1}\ \right) \quad \text{for } \mathsf{m} \neq \mathsf{n} \end{cases} \tag{11}$$

for every m as a consequence of (9), so that we may rewrite the equation in (4) as

$$\begin{aligned} & \frac{dp_{\mathsf{m}}(t)}{dt} \\ = \quad & \rho\left\{(\mathsf{m}+1)p_{\mathsf{m}+1}(t) + \exp[-\beta]\,\mathsf{m}p_{\mathsf{m}-1}(t) - (\mathsf{m} + \exp[-\beta](\mathsf{m}+1))\,p_{\mathsf{m}}(t)\right\}. \end{aligned} \tag{12}$$

Regarding the question of how the dynamics of the coupled system consisting of the harmonic oscillator and the radiation field acting as a heat bath leads to (12), we refer the reader for instance to [12, 17, 26-28]. In a nutshell, as in [12] and in Sections 3 and 4 of Chapter XVII in [26], the bath is modelled by an infinite system of quantum harmonic oscillators whose frequencies satisfy certain conditions. The time evolution of the density operator describing that coupling is then governed by the laws of Quantum Mechanics. Subsequently, various methods are used to eliminate the bath variables to eventually obtain the evolution equation of the so-called reduced density operator, that is, the density operator of the given oscillator after its initial interaction with the bath. Among others, one of those methods is the celebrated Projection Technique devised in [17, 27, 28]. As a consequence and within a Markovian approximation, equation (12) emerges naturally with the functions $t \to p_{\mathsf{m}}(t)$ being defined as the diagonal matrix elements of the reduced density operator with respect to the m^{th} Hermite function. We refer the reader for instance to [5-8] for a rigorous implementation of some of those formal developments.

We devote the remaining part of this section to proving that $\mathsf{p}_{\beta,\mathsf{Gibbs}}$ is in fact the global exponential attractor to (12), more specifically that the dynamical system generated by (12) is hyperbolic, which implies that its solution converges exponentially rapidly to $\mathsf{p}_{\beta,\mathsf{Gibbs}}$ as $\tau \to +\infty$ in a suitable topology. We first introduce a suitable functional space in order to make a self-adjoint realization out of the formal expression on the right-hand side of (12). Let $w_\beta := (w_{\beta,\mathsf{m}})$ be the sequence of weights given by $w_{\beta,\mathsf{m}} = \exp[\beta\mathsf{m}]$ for each $\mathsf{m} \in \mathbb{N}$, and let us consider the separable Hilbert space $l^2_{\mathbb{C},w_\beta}$ consisting of all complex sequences $\mathsf{p} = (p_{\mathsf{m}})$ satisfying

$$\|\mathsf{p}\|^2_{2,w_\beta} := \sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}}\,|p_{\mathsf{m}}|^2 < +\infty, \tag{13}$$

endowed with the usual algebraic operations and the sesquilinear form

$$(\mathsf{p},\mathsf{q})_{2,w_\beta} := \sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}} p_{\mathsf{m}} \bar{q}_{\mathsf{m}}. \tag{14}$$

Let us consider the sequence $(\mathsf{f_m})$ of elements in $l^2_{\mathbb{C},w_\beta}$ defined by $\mathsf{f_m} = w_{\beta,\mathsf{m}}^{-\frac{1}{2}} \mathsf{e_m}$ where $(\mathsf{e_m})_\mathsf{n} = \delta_{\mathsf{m,n}}$ for all $\mathsf{m,n} \in \mathbb{N}$. It is easily seen that $(\mathsf{f_m})$ provides an orthonormal basis of that space. Moreover we have

$$\sum_{\mathsf{m}=0}^{+\infty} r_{\mathsf{m,n}} = 0 \tag{15}$$

for every $\mathsf{n} \in \mathbb{N}$. Then the following result holds:

Proposition 1. *Let us define the mapping R by*

$$(R\mathsf{p})_\mathsf{m} = \sum_{\mathsf{n}=0}^{+\infty} r_{\mathsf{m,n}} p_\mathsf{n}. \tag{16}$$

Then R is an unbounded symmetric operator on the dense domain $D(R) \subset l^2_{\mathbb{C},w_\beta}$ consisting of all finite linear combinations of the $\mathsf{f_m}$.

Proof. By linearity it is sufficient to prove that

$$(R\mathsf{f_m}, \mathsf{f_n})_{2,w_\beta} = (\mathsf{f_m}, R\mathsf{f_n})_{2,w_\beta}$$

for all $\mathsf{m,n} \in \mathbb{N}$. Noting that $(\mathsf{f_m})_\mathsf{k} = w_{\beta,\mathsf{m}}^{-\frac{1}{2}} \delta_{\mathsf{m,k}}$ and $(R\mathsf{f_n})_\mathsf{k} = w_{\beta,\mathsf{n}}^{-\frac{1}{2}} r_{\mathsf{k,n}}$ we have

$$(R\mathsf{f_m}, \mathsf{f_n})_{2,w_\beta} = w_{\beta,\mathsf{m}}^{-\frac{1}{2}} w_{\beta,\mathsf{n}}^{\frac{1}{2}} r_{\mathsf{n,m}}$$

and

$$(\mathsf{f_m}, R\mathsf{f_n})_{2,w_\beta} = w_{\beta,\mathsf{m}}^{\frac{1}{2}} w_{\beta,\mathsf{n}}^{-\frac{1}{2}} r_{\mathsf{m,n}}$$

respectively, so that both expressions are equal by virtue of the detailed balance conditions. Furthermore we have

$$\sum_{\mathsf{k}=0}^{+\infty} w_{\beta,\mathsf{k}} \left| (R\mathsf{f_m})_\mathsf{k} \right|^2 = w_{\beta,\mathsf{m}}^{-1} \sum_{\mathsf{k}=0}^{+\infty} w_{\beta,\mathsf{k}} r_{\mathsf{k,m}}^2,$$

which, by taking (11) and the definition of the weights into account, leads to

$$\begin{aligned} \|R\mathsf{f_m}\|^2_{2,w_\beta} &= \rho^2 \left\{ (\mathsf{m} + \exp[-\beta] (\mathsf{m}+1)^2 + \exp[-\beta] \left(2\mathsf{m}^2 + 2\mathsf{m} + 1 \right) \right\} \\ &\geq \rho^2 \exp[-\beta] \, \mathsf{m}^2 \to +\infty \end{aligned}$$

as $\mathsf{m} \to +\infty$, so that R is indeed unbounded. $\blacksquare$

Remark. It is essential to have a functional space endowed with an appropriately weighted norm in order to make a symmetric operator out of (16) by means of the detailed balance conditions (8) (see, e.g., Section 7 in Chapter V of [26] for an informal discussion of this idea in a general context, which has been around for some time). Had we chosen the usual unweighted space $l^2_\mathbb{C}$

consisting of all square-integrable complex sequences instead, the operator R would not have been symmetric, the spectral analysis of its possible non self-adjoint extensions being thereby definitely more complicated. Such a situation was thoroughly investigated in [2], *albeit* with a different operator whose choice was motivated by the works on stochastic thermodynamics of chemical reaction systems set forth in [23] and [24].

It is straightforward to check that $\mathsf{p}_{\beta,\mathsf{Gibbs}} \in l^2_{\mathbb{C},w_\beta}$, but unfortunately $\mathsf{p}_{\beta,\mathsf{Gibbs}} \notin D(R)$. In fact, the Fourier coefficients of $\mathsf{p}_{\beta,\mathsf{Gibbs}}$ along the basis (f_m) are

$$(\mathsf{p}_{\beta,\mathsf{Gibbs}}, \mathsf{f}_\mathsf{m})_{2,w_\beta} = Z_\beta^{-1} w_{\beta,\mathsf{m}}^{-\frac{1}{2}} > 0$$

for every $\mathsf{m} \in \mathbb{N}$, so that

$$\mathsf{p}_{\beta,\mathsf{Gibbs}} = Z_\beta^{-1} \sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}}^{-\frac{1}{2}} \mathsf{f}_\mathsf{m}$$

is not a finite linear combination of the f_m. However, let R^* be the adjoint of R. As is well known, the operator R^* provides a closed extension of R, namely, $R \subseteq R^*$, so that its domain $D(R^*)$ is dense in $l^2_{\mathbb{C},w_\beta}$ as well. Moreover, recall that $\mathsf{q} \in D(R^*)$ means there exists a unique $\mathsf{q}^* \in l^2_{\mathbb{C},w_\beta}$ such that

$$(R\mathsf{p}, \mathsf{q})_{2,w_\beta} = (\mathsf{p}, \mathsf{q}^*)_{2,w_\beta}$$

for every $\mathsf{p} \in D(R)$, with $\mathsf{q}^* = R^*\mathsf{q}$ (for all these notions and various relations among them see, e.g., [4]). Then the following statement holds:

Lemma 3. *We have $\mathsf{p}_{\beta,\mathsf{Gibbs}} \in D(R^*)$, hence R^* is a strict extension of R. Moreover*

$$R^* \mathsf{p}_{\beta,\mathsf{Gibbs}} = 0, \tag{17}$$

so that $\nu = 0$ is an eigenvalue of R^*.

Proof. We have

$$(R\mathsf{f}_\mathsf{m}, \mathsf{p}_{\beta,\mathsf{Gibbs}})_{2,w_\beta} = Z_\beta^{-1} w_{\beta,\mathsf{m}}^{-\frac{1}{2}} \sum_{\mathsf{k}=0}^{+\infty} r_{\mathsf{k},\mathsf{m}} = 0$$

for every $\mathsf{m} \in \mathbb{N}$ by virtue of (15), so that by linearity

$$(R\mathsf{p}, \mathsf{p}_{\beta,\mathsf{Gibbs}})_{2,w_\beta} = 0$$

for every $\mathsf{p} \in D(R)$, which may be written as

$$(R\mathsf{p}, \mathsf{p}_{\beta,\mathsf{Gibbs}})_{2,w_\beta} = \left(\mathsf{p}, \mathsf{p}^*_{\beta,\mathsf{Gibbs}}\right)_{2,w_\beta}$$

with $\mathsf{p}^*_{\beta,\mathsf{Gibbs}} = 0$. Hence (17) holds and the first part of the statement follows from the fact that $\mathsf{p}_{\beta,\mathsf{Gibbs}} \notin D(R)$. ■

Let us now determine the action of R^* on elements of $D(R^*)$. Not surprisingly we obtain:

Lemma 4. *For every* $\mathsf{q} \in D(R^*)$ *we have*

$$(R^*\mathsf{q})_{\mathsf{m}} = \rho \left\{ (\mathsf{m}+1)\, q_{\mathsf{m}+1} + \exp\left[-\beta\right] \mathsf{m} q_{\mathsf{m}-1} - (\mathsf{m} + \exp\left[-\beta\right](\mathsf{m}+1)) q_{\mathsf{m}} \right\} \tag{18}$$

for every $\mathsf{m} \in \mathbb{N}$, *with*

$$\sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}} \left|(R^*\mathsf{q})_{\mathsf{m}}\right|^2 < +\infty. \tag{19}$$

Proof. In order to determine $\mathsf{q}^* = R^*\mathsf{q}$ we must have in particular

$$(R\mathsf{f}_{\mathsf{m}}, \mathsf{q})_{2,w_\beta} = (\mathsf{f}_{\mathsf{m}}, \mathsf{q}^*)_{2,w_\beta} \tag{20}$$

for every $\mathsf{m} \in \mathbb{N}$. But using (11) and the definition of the weights once again we get

$$\begin{aligned} & (R\mathsf{f}_{\mathsf{m}}, \mathsf{q})_{2,w_\beta} \\ = \;& \rho \exp\left[\frac{\beta \mathsf{m}}{2}\right] \left\{ (\mathsf{m}+1)\, \bar{q}_{\mathsf{m}+1} + \exp\left[-\beta\right] \mathsf{m} \bar{q}_{\mathsf{m}-1} - (\mathsf{m} + \exp\left[-\beta\right](\mathsf{m}+1)) \bar{q}_{\mathsf{m}} \right\}, \end{aligned} \tag{21}$$

while on the other hand we have

$$(\mathsf{f}_{\mathsf{m}}, \mathsf{q}^*)_{2,w_\beta} = \exp\left[\frac{\beta \mathsf{m}}{2}\right] \bar{q}^*_{\mathsf{m}}.$$

Therefore, solving (20) for q^*_{m} gives the desired result while (19) expresses the requirement $R^*\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$. ■

The operator R^* turns out to be the desired self-adjoint realization we need to deal with the right-hand side of (12):

Proposition 2. *The unbounded operator* R^* *is self-adjoint on* $D(R^*)$. *In other words, the operator* R *is essentially self-adjoint on* $D(R)$.

Proof. Since $R^{**} := (R^*)^*$ is the smallest closed extension of R we have $R \subseteq R^{**} \subseteq R^*$, so that it is sufficient to prove the symmetry of R^* on $D(R^*)$ for then

$$R \subseteq R^{**} \subseteq R^* \subseteq R^{**},$$

which implies $R \subset R^* = R^{**}$, the strict inclusion following from the information we already have. But from (18) we get

$$\begin{aligned} & (R^*\mathsf{p}, \mathsf{q})_{2,w_\beta} \\ = \;& \rho \sum_{\mathsf{m}=0}^{+\infty} \left(\exp\left[\beta \mathsf{m}\right](\mathsf{m}+1) p_{\mathsf{m}+1} \bar{q}_{\mathsf{m}} + \exp\left[\beta(\mathsf{m}-1)\right] \mathsf{m} p_{\mathsf{m}-1} \bar{q}_{\mathsf{m}} \right) \\ & -\rho \sum_{\mathsf{m}=0}^{+\infty} \exp\left[\beta \mathsf{m}\right] (\mathsf{m} + \exp\left[-\beta\right](\mathsf{m}+1))\, p_{\mathsf{m}} \bar{q}_{\mathsf{m}} \end{aligned} \tag{22}$$

for all $\mathsf{p},\mathsf{q}\in D(R^{*})$, while

$$\begin{aligned}
&\left(\mathsf{p},R^{*}\mathsf{q}\right)_{2,w_{\beta}}\\
=\;&\rho\sum_{\mathsf{m}=0}^{+\infty}\left(\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)p_{\mathsf{m}}\bar{q}_{\mathsf{m}+1}+\exp\left[\beta(\mathsf{m}-1)\right]\mathsf{m}p_{\mathsf{m}}\bar{q}_{\mathsf{m}-1}\right) \qquad (23)\\
&-\rho\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right]\left(\mathsf{m}+\exp\left[-\beta\right](\mathsf{m}+1)\right)p_{\mathsf{m}}\bar{q}_{\mathsf{m}},
\end{aligned}$$

from which we easily see that

$$\left(R^{*}\mathsf{p},\mathsf{q}\right)_{2,w_{\beta}}=\left(\mathsf{p},R^{*}\mathsf{q}\right)_{2,w_{\beta}}$$

by means of suitable shifts in the summation indices wherever necessary. ■

Next, we prove a series of results which will eventually lead to the determination of the spectrum of R^{*}. We begin with the following statement, which makes the information already provided by (17) more precise:

Proposition 3. *Let $\mathsf{q}\in D(R^{*})$ be such that $R^{*}\mathsf{q}=0$. Then we have*

$$\mathsf{q}=q_{0}Z_{\beta}\mathsf{p}_{\beta,\mathsf{Gibbs}} \qquad (24)$$

for some $q_{0}\in\mathbb{C}$, where Z_{β} is the partition function defined in (6). Thus, $\nu=0$ is a simple eigenvalue of R^{*} whose corresponding eigenspace is generated by $\mathsf{p}_{\beta,\mathsf{Gibbs}}$.

Proof. Requiring $\left(R^{*}\mathsf{q}\right)_{\mathsf{m}}=0$ for every m from (18) gives

$$q_{\mathsf{m}+1}=\frac{\left(\mathsf{m}+\exp\left[-\beta\right](\mathsf{m}+1)\right)q_{\mathsf{m}}-\exp\left[-\beta\right]\mathsf{m}q_{\mathsf{m}-1}}{\mathsf{m}+1}, \qquad (25)$$

hence $q_{1}=\exp\left[-\beta\right]q_{0}$ for some $q_{0}\in\mathbb{C}$ when $\mathsf{m}=0$. Therefore, if we assume $q_{\mathsf{m}}=\exp\left[-\beta\mathsf{m}\right]q_{0}$ for some m then (25) implies

$$\begin{aligned}
q_{\mathsf{m}+1}&=\frac{\left(\mathsf{m}+\exp\left[-\beta\right](\mathsf{m}+1)\right)\exp\left[-\beta\mathsf{m}\right]q_{0}-\mathsf{m}\exp\left[-\beta\mathsf{m}\right]q_{0}}{\mathsf{m}+1}\\
&=\exp\left[-\beta\left(\mathsf{m}+1\right)\right]q_{0},
\end{aligned}$$

so that

$$q_{\mathsf{m}}=\exp\left[-\beta\mathsf{m}\right]q_{0}$$

holds for every $\mathsf{m}\in\mathbb{N}$, which gives (24) according to (6). ■

Since R^{*} is self-adjoint, its entire spectrum is a subset of the real axis. The following statement will provide additional information about its localization:

Lemma 5. *The operator R^* is negative semi-definite. More specifically, for each $\mathsf{q} \in D(R^*)$ we have*

$$(R^*\mathsf{q},\mathsf{q})_{2,w_\beta} < 0 \tag{26}$$

unless q is of the form (24). In particular, the strict inequality (26) holds for every non-zero $\mathsf{q} \in D(R^*)$ in the orthogonal complement of the one-dimensional subspace generated by $\mathsf{p}_{\beta,\mathsf{Gibbs}}$. Moreover, if $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > 0$ then

$$\|(R^* - \nu)\,\mathsf{q}\|_{2,w_\beta} \geq |\nu|\,\|\mathsf{q}\|_{2,w_\beta}\,. \tag{27}$$

Proof. Since both sums on the right-hand side of the equality in (22) are real when $\mathsf{p} = \mathsf{q}$, the proof of (26) amounts to proving that

$$\begin{aligned}
&\sum_{\mathsf{m}=0}^{+\infty}\left(\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)q_{\mathsf{m}+1}\bar{q}_{\mathsf{m}} + \exp\left[\beta\left(\mathsf{m}-1\right)\right]\mathsf{m}q_{\mathsf{m}-1}\bar{q}_{\mathsf{m}}\right)\\
<\ &\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right]\left(\mathsf{m}+\exp\left[-\beta\right](\mathsf{m}+1)\right)\left|q_{\mathsf{m}}\right|^2
\end{aligned} \tag{28}$$

for every $\mathsf{q} \in D(R^*)$, unless q is of the form (24). In order to achieve that we proceed by using Cauchy's inequality with epsilon to deal with the left-hand side of the preceding inequality. We begin by writing

$$\begin{aligned}
&\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)\left|q_{\mathsf{m}+1}\bar{q}_{\mathsf{m}}\right|\\
=\ &\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\frac{\beta}{2}\mathsf{m}\right](\mathsf{m}+1)^{\frac{1}{2}}\left|q_{\mathsf{m}+1}\right| \times \exp\left[\frac{\beta}{2}\mathsf{m}\right](\mathsf{m}+1)^{\frac{1}{2}}\left|q_{\mathsf{m}}\right|.
\end{aligned}$$

Furthermore, for each $\varepsilon > 0$ and every m we have

$$\left(\sqrt{\varepsilon}\exp\left[\frac{\beta}{2}\mathsf{m}\right](\mathsf{m}+1)^{\frac{1}{2}}\left|q_{\mathsf{m}+1}\right| - \frac{\exp\left[\frac{\beta}{2}\mathsf{m}\right](\mathsf{m}+1)^{\frac{1}{2}}\left|q_{\mathsf{m}}\right|}{\sqrt{\varepsilon}}\right)^2 > 0 \tag{29}$$

unless

$$\left|q_{\mathsf{m}+1}\right| = \frac{\left|q_{\mathsf{m}}\right|}{\varepsilon}, \tag{30}$$

so that with the exception of this particular case we get

$$\begin{aligned}
&\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)\left|q_{\mathsf{m}+1}\bar{q}_{\mathsf{m}}\right|\\
<\ &\frac{1}{2}\left(\varepsilon\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)\left|q_{\mathsf{m}+1}\right|^2 + \frac{1}{\varepsilon}\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)\left|q_{\mathsf{m}}\right|^2\right)\\
=\ &\frac{1}{2}\left(\varepsilon\exp\left[-\beta\right]\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right]\mathsf{m}\left|q_{\mathsf{m}}\right|^2 + \frac{1}{\varepsilon}\sum_{\mathsf{m}=0}^{+\infty}\exp\left[\beta\mathsf{m}\right](\mathsf{m}+1)\left|q_{\mathsf{m}}\right|^2\right)
\end{aligned}$$

by expanding (29). Choosing then $\varepsilon = \exp[\beta]$ and regrouping terms we obtain

$$\begin{aligned}
&\sum_{\mathsf{m}=0}^{+\infty} \exp[\beta \mathsf{m}] (\mathsf{m}+1) |q_{\mathsf{m}+1} \bar{q}_{\mathsf{m}}| \\
<\ & \frac{1}{2} \left(\sum_{\mathsf{m}=0}^{+\infty} \exp[\beta \mathsf{m}]\, \mathsf{m}\, |q_{\mathsf{m}}|^2 + \exp[-\beta] \sum_{\mathsf{m}=0}^{+\infty} \exp[\beta \mathsf{m}] (\mathsf{m}+1) |q_{\mathsf{m}}|^2 \right) \qquad (31) \\
=\ & \frac{1}{2} \sum_{\mathsf{m}=0}^{+\infty} \exp[\beta \mathsf{m}] (\mathsf{m} + \exp[-\beta] (\mathsf{m}+1)) |q_{\mathsf{m}}|^2 ,
\end{aligned}$$

which is half of the right-hand side of (28). Similarly, from (31) we have

$$\begin{aligned}
&\sum_{\mathsf{m}=0}^{+\infty} \exp[\beta (\mathsf{m}-1)]\, \mathsf{m}\, |q_{\mathsf{m}-1} \bar{q}_{\mathsf{m}}| \\
=\ & \sum_{\mathsf{m}=0}^{+\infty} \exp[\beta \mathsf{m}] (\mathsf{m}+1) |q_{\mathsf{m}+1} \bar{q}_{\mathsf{m}}| \qquad (32) \\
<\ & \frac{1}{2} \sum_{\mathsf{m}=0}^{+\infty} \exp[\beta \mathsf{m}] (\mathsf{m} + \exp[-\beta] (\mathsf{m}+1)) |q_{\mathsf{m}}|^2
\end{aligned}$$

unless (30) holds, so that the combination of (31) and (32) indeed leads to (28) up to the exceptional case we alluded to.

Thus, it remains to examine the consequences of the recurrence relation (30). We first consider the case where $\mathsf{q} \in D(R^*)$ is real. With our choice of ε this leads to

$$q_{\mathsf{m}} = \exp[-\mathsf{m}\beta]\, q_0$$

for every $\mathsf{m} \in \mathbb{N}$, which gives (24) as in Proposition 3 but with some $q_0 \in \mathbb{R}$. The general case for $\mathsf{q} = \mathsf{r} + i\mathsf{s}$ with $\mathsf{r}, \mathsf{s} \in D(R^*)$ both real then follows from complexification. Indeed we have

$$(R^*\mathsf{q}, \mathsf{q})_{2,w_\beta} = (R^*\mathsf{r}, \mathsf{r})_{2,w_\beta} + (R^*\mathsf{s}, \mathsf{s})_{2,w_\beta} + i \left\{ (R^*\mathsf{s}, \mathsf{r})_{2,w_\beta} - (R^*\mathsf{r}, \mathsf{s})_{2,w_\beta} \right\}$$

where all the inner products are real, a consequence of (14) and (18) for those in the bracket on the right-hand side. We then necessarily have

$$(R^*\mathsf{q}, \mathsf{q})_{2,w_\beta} = (R^*\mathsf{r}, \mathsf{r})_{2,w_\beta} + (R^*\mathsf{s}, \mathsf{s})_{2,w_\beta} ,$$

so that according to the first part of the proof we have $(R^*\mathsf{q}, \mathsf{q})_{2,w_\beta} < 0$ unless both r and s are of the form

$$\mathsf{r} = r_0 Z_\beta \mathsf{p}_{\beta,\mathsf{Gibbs}}$$

and

$$\mathsf{s} = s_0 Z_\beta \mathsf{p}_{\beta,\mathsf{Gibbs}}$$

with $r_0, s_0 \in \mathbb{R}$, respectively. The only exceptional case for which $(R^* \mathsf{q}, \mathsf{q})_{2,w_\beta} = 0$ is therefore $\mathsf{q} = (r_0 + i s_0)\, Z_\beta \mathsf{p}_{\beta,\mathsf{Gibbs}}$.

As for the proof of (27) we have

$$\begin{aligned} & \left\| (R^* - \nu)\, \mathsf{q} \right\|_{2,w_\beta}^2 \\ = \quad & \left\| R^* \mathsf{q} \right\|_{2,w_\beta}^2 - 2\, (\operatorname{Re} \nu)\, (R^* \mathsf{q}, \mathsf{q})_{2,w_\beta} + |\nu|^2 \left\| \mathsf{q} \right\|_{2,w_\beta}^2 \qquad (33) \\ \geq \quad & |\nu|^2 \left\| \mathsf{q} \right\|_{2,w_\beta}^2 \qquad (34) \end{aligned}$$

since $\operatorname{Re} \nu > 0$ and $(R^* \mathsf{q}, \mathsf{q})_{2,w_\beta} \leqslant 0$ for every $\mathsf{q} \in D(R^*)$, which is the desired estimate. ■

The following result and the information already available show that the entire spectrum of R^* lies in the interval $(-\infty, 0]$:

Proposition 4. *For each $\nu \in \mathbb{C}$ such that $\operatorname{Re} \nu > 0$ the inverse operator $(R^* - \nu)^{-1}$ exists on $l^2_{\mathbb{C},w_\beta}$ and we have*

$$\left\| (R^* - \nu)^{-1} \mathsf{p} \right\|_{2,w_\beta} \leqslant |\nu|^{-1} \left\| \mathsf{p} \right\|_{2,w_\beta} \qquad (35)$$

for every $\mathsf{p} \in l^2_{\mathbb{C},w_\beta}$. In other words, each such ν belongs to the resolvent set of R^*.

Proof. It follows from (27) that there exists at most one solution to the equation

$$(R^* - \nu)\, \mathsf{q} = \mathsf{p}$$

for any $\mathsf{p} \in l^2_{\mathbb{C},w_\beta}$, namely, that the operator $R^* - \nu$ is injective.

Next, we prove that $R^* - \nu$ is surjective, namely, that

$$\operatorname{Ran}(R^* - \nu) = l^2_{\mathbb{C},w_\beta} \qquad (36)$$

where $\operatorname{Ran}(R^* - \nu)$ stands for the range of the operator. This is already true if $\operatorname{Im} \nu \neq 0$ so that it remains to consider the case $\nu > 0$. Since $R^* - \nu$ is then self-adjoint and injective, standard arguments first imply that its range is everywhere dense in $l^2_{\mathbb{C},w_\beta}$ (see, e.g., Chapter X in [4]), hence for every $\mathsf{p} \in l^2_{\mathbb{C},w_\beta}$ there exists a sequence $(\mathsf{q}_\mathsf{n}) \subset D(R^*)$ such that $(R^* - \nu)\, \mathsf{q}_\mathsf{n} \to \mathsf{p}$ strongly in $l^2_{\mathbb{C},w_\beta}$ as $\mathsf{n} \to +\infty$. Consequently, (27) implies that

$$\left\| \mathsf{q}_\mathsf{m} - \mathsf{q}_\mathsf{n} \right\|_{2,w_\beta} \leq \nu^{-1} \left\| (R^* - \nu)\, (\mathsf{q}_\mathsf{m} - \mathsf{q}_\mathsf{n}) \right\|_{2,w_\beta} \to 0$$

as $\mathsf{m}, \mathsf{n} \to +\infty$, hence that $\mathsf{q}_\mathsf{n} \to \mathsf{q}$ strongly in $l^2_{\mathbb{C},w_\beta}$ as $\mathsf{n} \to +\infty$ for some q. Therefore, $\mathsf{q} \in D(R^*)$ and

$$(R^* - \nu)\, \mathsf{q} = \mathsf{p}$$

since $R^* - \nu$ is closed, which gives (36). The boundedness of $(R^* - \nu)^{-1}$ for $\nu > 0$ then follows from the inverse mapping theorem, while (35) is a consequence of (27). ■

In order to unveil the exact nature of the spectrum of R^*, it is now essential to provide more concrete information about the domain $D(R^*)$. To this end, we consider the vector space $h^1_{\mathbb{C},w_\beta}$consisting of all complex sequences $\mathsf{p} = (p_\mathsf{m})$ satisfying

$$\|\mathsf{p}\|^2_{1,2,w_\beta} := \sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}} \left(1 + \mathsf{m}^2\right) |p_\mathsf{m}|^2 < +\infty. \tag{37}$$

We remark that $h^1_{\mathbb{C},w_\beta}$becomes a separable Hilbert space in its own right when endowded with the sesquilinear form

$$(\mathsf{p}, \mathsf{q})_{1,2,w_\beta} := \sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}} \left(1 + \mathsf{m}^2\right) p_\mathsf{m} \bar{q}_\mathsf{m} \tag{38}$$

and the norm (37), but for the time being we consider it as a vector subspace of $l^2_{\mathbb{C},w_\beta}$independently of its intrinsic topology as we wish to show that $D(R^*)$ and $h^1_{\mathbb{C},w_\beta}$ coincide. We begin with the easy part:

Lemma 6. *We have*

$$D(R) \subset h^1_{\mathbb{C},w_\beta} \subseteq D(R^*), \tag{39}$$

so that $h^1_{\mathbb{C},w_\beta}$ is everywhere dense in $l^2_{\mathbb{C},w_\beta}$. Moreover we have

$$R^* = \bar{R}^*_{\mathsf{rest}} \tag{40}$$

where $\bar{R}^*_{\mathsf{rest}}$ stands for the closure of the restriction R^*_{rest} of R^* to $h^1_{\mathbb{C},w_\beta}$.

Proof. We have $\mathsf{f}_\mathsf{m} \in h^1_{\mathbb{C},w_\beta}$ for each m and therefore $D(R) \subseteq h^1_{\mathbb{C},w_\beta}$. Moreover $\mathsf{p}_{\beta,\mathsf{Gibbs}} \in h^1_{\mathbb{C},w_\beta}$, so that the strict inclusion in (39) holds since $\mathsf{p}_{\beta,\mathsf{Gibbs}} \notin D(R)$. Furthermore, $h^1_{\mathbb{C},w_\beta}$ is everywhere dense in $l^2_{\mathbb{C},w_\beta}$ since $D(R)$ is. Now let $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ be arbitrary and let us define q^* by

$$q^*_\mathsf{m} := \rho \left\{ (\mathsf{m} + 1)\, q_{\mathsf{m}+1} + \exp\left[-\beta\right] \mathsf{m} q_{\mathsf{m}-1} - (\mathsf{m} + \exp\left[-\beta\right] (\mathsf{m} + 1)) q_\mathsf{m} \right\}$$

for every $\mathsf{m} \in \mathbb{N}$, having (18) in mind. From this and (37) we easily obtain the estimate

$$\|\mathsf{q}^*\|_{2,w_\beta} \leq c_{\beta,\rho} \|\mathsf{q}\|_{1,2,w_\beta} < +\infty \tag{41}$$

where $c_{\beta,\rho} > 0$ is a positive constant depending only on β and ρ, which means that $\mathsf{q}^* \in l^2_{\mathbb{C},w_\beta}$. The fact that

$$(R\mathsf{p}, \mathsf{q})_{2,w_\beta} = (\mathsf{p}, \mathsf{q}^*)_{2,w_\beta}$$

for every $\mathsf{p} \in D(R)$ then follows from a computation similar to that carried out in the proof of Lemma 4, which proves the second inclusion in (39).

Finally, writing R^*_{rest} for the restriction of R^* to $h^1_{\mathbb{C},w_\beta}$, which is symmetric and closable according the the second inclusion in (39), we have

$$R \subseteq R^*_{\mathsf{rest}} \subseteq R^*$$

and consequently

$$R^* \supseteq \left(R^*_{\mathsf{rest}}\right)^* \supseteq R^*$$

since R^* is self-adjoint, which means that $R^* = \left(R^*_{\mathsf{rest}}\right)^*$ and thereby

$$R^* = R^{**} = \left(R^*_{\mathsf{rest}}\right)^{**} = \bar{R}^*_{\mathsf{rest}}$$

where $\bar{R}^*_{\mathsf{rest}}$ stands for the closure of R^*_{rest}. ■

Let us now decompose R^*_{rest} into a sum of two operators, namely,

$$R^*_{\mathsf{rest}} = S + T \tag{42}$$

where

$$\begin{aligned}(S\mathsf{q})_{\mathsf{m}} \quad &: \quad = -\rho(\mathsf{m} + \exp\left[-\beta\right](\mathsf{m}+1))q_{\mathsf{m}}, \\ (T\mathsf{q})_{\mathsf{m}} \quad &: \quad = \rho\left\{(\mathsf{m}+1)\, q_{\mathsf{m}+1} + \exp\left[-\beta\right] \mathsf{m} q_{\mathsf{m}-1}\right\}\end{aligned} \tag{43}$$

for every $\mathsf{m} \in \mathbb{N}$, both defined on $h^1_{\mathbb{C},w_\beta}$. It is easily verified that S is a diagonal operator with respect to the basis $(\mathsf{f}_{\mathsf{m}})$, while T is the sum of an upper and a lower triangular operator. The following result holds:

Lemma 7. *Let $\nu > 0$ and let us define $S_\nu := S - \nu$. Then S_ν is self-adjoint on $h^1_{\mathbb{C},w_\beta}$ and its inverse S_ν^{-1} exists and is compact in $l^2_{\mathbb{C},w_\beta}$. Moreover, the operator TS_ν^{-1} remains bounded.*

Proof. We first prove that S is self-adjoint on $h^1_{\mathbb{C},w_\beta}$, namely, that

$$h^1_{\mathbb{C},w_\beta} \subseteq D(S^*) \subseteq h^1_{\mathbb{C},w_\beta}. \tag{44}$$

Indeed S is obviously symmetric with respect to (14), with

$$(S^*\mathsf{q})_{\mathsf{m}} = -\rho(\mathsf{m} + \exp\left[-\beta\right](\mathsf{m}+1))q_{\mathsf{m}}$$

for every m, so that the first inclusion in (44) holds. In order to prove the second inclusion, we remark that if $\mathsf{q} \in D(S^*)$ then from the very definition of this domain we have

$$\sum_{\mathsf{m}=0}^{+\infty} \exp\left[\beta \mathsf{m}\right] (\mathsf{m} + \exp\left[-\beta\right](\mathsf{m}+1))^2 \left|q_{\mathsf{m}}\right|^2 < +\infty,$$

from which the estimate

$$\|\mathsf{q}\|^2_{1,2,w_\beta} \leq \tilde{c}_{\beta,\rho} \|S^*\mathsf{q}\|^2_{2,w_\beta} < +\infty$$

follows with some $\tilde{c}_{\beta,\rho} > 0$ so that $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$. Therefore, S is self-adjoint on $h^1_{\mathbb{C},w_\beta}$ and so is S_ν.

Regarding the existence and the compactness of S_ν^{-1} we first note that

$$\left(S_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}} = a_{\mathsf{m}} q_{\mathsf{m}} \tag{45}$$

for each $\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$ and every m, where

$$a_{\mathsf{m}} := -\left(\rho(\mathsf{m} + \nu_\rho + \exp\left[-\beta\right](\mathsf{m}+1))\right)^{-1} \tag{46}$$

with $\nu_\rho := \rho^{-1}\nu > 0$. Then $a_{\mathsf{m}} \to 0$ as $\mathsf{m} \to +\infty$, so that the desired compactness follows from the fact that S_ν^{-1} may be viewed as the uniform limit of the sequence of finite-rank operators A_{N} given by

$$\left(A_{\mathsf{N}}\mathsf{q}\right)_{\mathsf{m}} = \begin{cases} a_{\mathsf{m}} q_{\mathsf{m}} & \text{for } 0 \leq \mathsf{m} \leq \mathsf{N}, \\ \\ 0 & \text{for } \mathsf{m} \geq \mathsf{N}+1. \end{cases}$$

As for the last statement of the lemma, we claim that

$$\left\|TS_\nu^{-1}\mathsf{q}\right\|_{2,w_\beta} \leq c_\beta \left\|\mathsf{q}\right\|_{2,w_\beta} \tag{47}$$

for every $\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$ and some $c_\beta > 0$ depending solely on β. Indeed we have

$$\left(TS_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}} = \rho\left\{(\mathsf{m}+1)\left(S_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}+1} + \exp\left[-\beta\right]\mathsf{m}\left(S_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}-1}\right\} \tag{48}$$

for every m from the definition of T in (43), along with the estimates

$$\left|\left(S_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}+1}\right| = \left|a_{\mathsf{m}+1}q_{\mathsf{m}+1}\right| \leq \frac{\left|q_{\mathsf{m}+1}\right|}{\rho\left(\mathsf{m}+1\right)} \tag{49}$$

and

$$\left|\left(S_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}-1}\right| = \left|a_{\mathsf{m}-1}q_{\mathsf{m}-1}\right| \leq \frac{\exp\left[\beta\right]\left|q_{\mathsf{m}-1}\right|}{\rho\mathsf{m}}$$

from (46), with $\mathsf{m} \neq 0$ in the last inequality. Therefore we obtain

$$\left|\left(TS_\nu^{-1}\mathsf{q}\right)_{\mathsf{m}}\right|^2 \leq 2\left(\left|q_{\mathsf{m}+1}\right|^2 + \left|q_{\mathsf{m}-1}\right|^2\right)$$

for every $\mathsf{m} \neq 0$, while for $\mathsf{m} = 0$ we have

$$\left|\left(TS_\nu^{-1}\mathsf{q}\right)_0\right|^2 \leq \left|q_1\right|^2$$

from (48) and (49). Lumping all things together using (13) we then get

$$\left\|TS_\nu^{-1}\mathsf{q}\right\|^2_{2,w_\beta} \leq \tilde{c}_\beta \left\|\mathsf{q}\right\|^2_{2,w_\beta}$$

for some $\tilde{c}_\beta > 0$, which gives the desired estimate. ■

Remark. Relation (47) becomes

$$\|T\mathsf{p}\|_{2,w_\beta} \leq c_\beta \|S_\nu \mathsf{p}\|_{2,w_\beta}$$

with $\mathsf{p} = S_\nu^{-1}\mathsf{q}$, which shows that the symmetric operator T is relatively bounded with respect to S_ν. However, the above bound is not one that allows us to apply directly the Kato-Rellich theory to prove the self-adjointness of $S+T$ on $h^1_{\mathbb{C},w_\beta}$ (see, e.g., Section 4 in Chapter V of [13]).

In order to determine the ultimate structure of the spectrum of R^* we still need, therefore, the following auxiliary result:

Proposition 5. *We have $D(R^*) = h^1_{\mathbb{C},w_\beta}$, and on this domain the operator R^* has a compact resolvent.*

Proof. For every $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ we may write

$$(R^* - \nu)\,\mathsf{q} = (S_\nu + T)\,\mathsf{q} = \left(I + TS_\nu^{-1}\right) S_\nu \mathsf{q} \tag{50}$$

from the decomposition (42) since $R^* = R^*_{\mathsf{rest}}$ on that subspace, where I stands for the identity operator and $\nu > 0$. Equivalently we have

$$(R^* - \nu)\, S_\nu^{-1}\mathsf{p} = \left(I + TS_\nu^{-1}\right) \mathsf{p}$$

whenever $\mathsf{p} = S_\nu \mathsf{q}$, which implies the equality of the two operators

$$(R^* - \nu)\, S_\nu^{-1} = I + TS_\nu^{-1} \tag{51}$$

on $\operatorname{Ran} S_\nu = D\left(S_\nu^{-1}\right) = l^2_{\mathbb{C},w_\beta}$.

We now proceed by showing that the operator on the right-hand side of (51) has a bounded inverse. On the one hand, the equation

$$(R^* - \nu)\, S_\nu^{-1}\mathsf{q} = \mathsf{0}$$

with $\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$ implies $S_\nu^{-1}\mathsf{q} = \mathsf{0}$ as a consequence of (27) and thereby $\mathsf{q} = \mathsf{0}$ from (45), so that the operator in (51) is injective. On the other hand, for any given $\mathsf{p} \in l^2_{\mathbb{C},w_\beta}$ the equation

$$(R^* - \nu)\, S_\nu^{-1}\mathsf{q} = \mathsf{p}$$

implies

$$S_\nu^{-1}\mathsf{q} = (R^* - \nu)^{-1}\,\mathsf{p} \in h^1_{\mathbb{C},w_\beta}$$

and so has the unique solution

$$\mathsf{q} = \mathsf{S}_\nu \left(\mathsf{R}^* - \nu\right)^{-1}\mathsf{p},$$

so that the operator in (51) is also surjective. Therefore, from the inverse mapping theorem we infer that the operator on the right-hand side of (51) has a bounded inverse in $l^2_{\mathbb{C},w_\beta}$, as desired.

Since $\left(I + TS_\nu^{-1}\right)^{-1}$ is bounded, we now easily infer that the operator

$$R^*_{\mathsf{rest}} - \nu = \left(I + TS_\nu^{-1}\right) S_\nu$$

is closed since both operators S_ν and $I + TS_\nu^{-1}$ are closed (see, e.g., Section 5 in Chapter III of [13]). Consequently R^*_{rest} is closed as well so that

$$R^* = \bar{R}^*_{\mathsf{rest}} = R^*_{\mathsf{rest}}$$

from (40), which proves the desired equality $D(R^*) = h^1_{\mathbb{C},w_\beta}$.

We complete the proof of the proposition by showing that the operator R^* has a compact resolvent. From (50) we first infer that the operator equality

$$(R^* - \nu)^{-1} = S_\nu^{-1}\left(I + TS_\nu^{-1}\right)^{-1}$$

holds on $h^1_{\mathbb{C},w_\beta}$, and hence on $l^2_{\mathbb{C},w_\beta}$ since $h^1_{\mathbb{C},w_\beta}$ is everywhere dense therein. Therefore $(R^* - \nu)^{-1}$ is compact as the product of a compact operator with a bounded operator. ■

Remarks. (1) We can easily verify that the compactness of the resolvent of R^* implies the compactness of the embedding

$$h^1_{\mathbb{C},w_\beta} \hookrightarrow l^2_{\mathbb{C},w_\beta} \tag{52}$$

when we consider $h^1_{\mathbb{C},w_\beta}$ as endowed with the sesquilinear form (38) and the induced norm (37). Indeed, for each $\nu > 0$ and every $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ we have the inequality

$$\left\|(R^* - \nu)\,\mathsf{q}\right\|_{2,w_\beta} \leq c_{\beta,\rho,\nu}\left\|\mathsf{q}\right\|_{1,2,w_\beta}$$

for some $c_{\beta,\rho,\nu} > 0$ as a consequence of (37) and (41). Therefore $R^* - \nu$ is a linear bounded operator from $h^1_{\mathbb{C},w_\beta}$ into $l^2_{\mathbb{C},w_\beta}$, which implies that

$$i_c := (R^* - \nu)^{-1}(R^* - \nu)$$

is compact since $(R^* - \nu)^{-1}$ is compact. But i_c is the restriction of the identity operator I to $h^1_{\mathbb{C},w_\beta}$, that is, the embedding map. In the appendix we shall derive the compactness of (52) independently from a more general embedding result which involves an intermediary space between $h^1_{\mathbb{C},w_\beta}$ and $l^2_{\mathbb{C},w_\beta}$.

(2) The compactness of (52) also allows us to prove the compactness of the operator S_ν^{-1} in a different manner, by showing that the set

$$D_\theta = \left\{\mathsf{q} \in h^1_{\mathbb{C},w_\beta} : \left\|\mathsf{q}\right\|_{2,w_\beta} \leq 1, \left\|S_\nu \mathsf{q}\right\|_{2,w_\beta} \leq \theta\right\} \tag{53}$$

where $\theta \in [0, +\infty)$ is compact in $l^2_{\mathbb{C},w_\beta}$. For the sake of completeness we shall provide the details in the appendix.

We can now state and prove the main result of this section:

Theorem. *The following statements hold:*

(a) The operator R^* has a purely discrete spectrum. More specifically, there exists an orthonormal basis $(\hat{\mathsf{p}}_{\mathsf{k}})_{\mathsf{k}\in\mathbb{N}}$ of $l^2_{\mathbb{C},w_\beta}$ consisting exclusively of eigenvectors of R^*, with $\hat{\mathsf{p}}_{\mathsf{k}} \in h^1_{\mathbb{C},w_\beta}$ satisfying

$$R^*\hat{\mathsf{p}}_{\mathsf{k}} = \nu_{\mathsf{k}}\hat{\mathsf{p}}_{\mathsf{k}} \tag{54}$$

for each k, the corresponding eigenvalues being such that

$$\dim_{\mathbb{C}} \mathrm{Ker}\,(R^* - \nu_{\mathsf{k}}) = 1 \tag{55}$$

and which we order as

$$\nu_{\mathsf{k}+1} < \nu_{\mathsf{k}} < \nu_0 := 0 \tag{56}$$

for every $\mathsf{k} \in \{1, 2, ...\}$. Moreover, $\nu_{\mathsf{k}} \to -\infty$ as $\mathsf{k} \to +\infty$.

(b) The operator R^*generates a holomorphic semigroup of contractions in $l^2_{\mathbb{C},w_\beta}$, written $\exp[tR^*]$ for $t \in [0, +\infty)$, and the norm-convergent spectral resolution

$$\exp[tR^*]\,\mathsf{p} = \sum_{\mathsf{k}=0}^{+\infty} (\mathsf{p}, \hat{\mathsf{p}}_{\mathsf{k}})_{2,w_\beta} \exp[t\nu_{\mathsf{k}}]\,\hat{\mathsf{p}}_{\mathsf{k}} \tag{57}$$

holds for each $\mathsf{p} \in l^2_{\mathbb{C},w_\beta}$ and every $t \in [0, +\infty)$. Moreover, this semigroup is self-adjoint in $l^2_{\mathbb{C},w_\beta}$.

Proof. The first part as well as the very last part of Statement (a) are an immediate consequence of Lemma 5 and Proposition 5 (see, e.g., Section 3 in Chapter IX of [9] on the relation between operators with compact resolvent and spectra). As for the non-degeneracy of the eigenvalues, we note that the equation

$$(R^* - \nu_{\mathsf{k}})\,\hat{\mathsf{p}}_{\mathsf{k}} = 0$$

is equivalent to the recurrence relation

$$\hat{p}_{\mathsf{k},\mathsf{m}+1} = \frac{(\mathsf{m} + \nu_{\mathsf{k},\rho} + \exp[-\beta](\mathsf{m}+1))\,\hat{p}_{\mathsf{k},\mathsf{m}} - \exp[-\beta]\,\mathsf{m}\hat{p}_{\mathsf{k},\mathsf{m}-1}}{\mathsf{m}+1}$$

for the components of the eigenvectors, where $\nu_{\mathsf{k},\rho} := \rho^{-1}\nu_{\mathsf{k}}$. Therefore

$$\begin{aligned}
\hat{p}_{\mathsf{k},1} &= (\nu_{\mathsf{k},\rho} + \exp[-\beta])\,\hat{p}_{\mathsf{k},0}, \\
\hat{p}_{\mathsf{k},2} &= \frac{(1 + \nu_{\mathsf{k},\rho} + 2\exp[-\beta])\,\hat{p}_{\mathsf{k},1} - \exp[-\beta]\,\hat{p}_{\mathsf{k},0}}{2}
\end{aligned}$$

and so on, each component being ultimately of the form

$$\hat{p}_{\mathsf{k},\mathsf{m}} = q_{\mathsf{m},\beta,\nu_{\mathsf{k},\rho}}\hat{p}_{\mathsf{k},0}$$

where the constants $q_{\mathsf{m},\beta,\nu_{\mathsf{k},\rho}} \in \mathbb{C}$ are fixed and depend solely on m, β and $\nu_{\mathsf{k},\rho}$, whereas $\hat{p}_{\mathsf{k},0} \in \mathbb{C}$ is a free parameter. Therefore Relation (55) holds.

As for the proof of statement (b), Proposition 4 implies that the hypotheses of the Hille-Yosida theorem hold true, so that R^* indeed generates a semigroup of contractions in $l^2_{\mathbb{C},w_\beta}$which we denote by $\exp[tR^*]$ for $t \in [0, +\infty)$. Furthermore, for a fixed $\varepsilon > 0$ let us define the auxiliary semigroup

$$S(t) := \exp[-\varepsilon t] \exp[tR^*] \tag{58}$$

whose infinitesimal generator is $R^*_\varepsilon := R^* - \varepsilon$. We then infer from Proposition 4 that $(R^* - \varepsilon)^{-1}$ exists as a linear bounded operator on $l^2_{\mathbb{C},w_\beta}$, hence that $\nu = 0$ belongs to the resolvent set of R^*_ε. Moreover, for every $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > 0$, $\operatorname{Im}\nu \neq 0$, and taking the inequality $(R^*_\varepsilon \mathsf{q}, \mathsf{q})_{2,w_\beta} \leq 0$ into account, an estimate similar to (27) gives

$$\|(R^*_\varepsilon - \nu)\, \mathsf{q}\|_{2,w_\beta} \geq |\nu|\, \|\mathsf{q}\|_{2,w_\beta} \geq |\operatorname{Im}\nu|\, \|\mathsf{q}\|_{2,w_\beta}\,,$$

so that the supremum norm of $(R^*_\varepsilon - \nu)^{-1}$ satisfies

$$\left\|(R^*_\varepsilon - \nu)^{-1}\right\|_\infty \leq |\operatorname{Im}\nu|^{-1}.$$

Therefore, the semigroup generated by R^*_ε is holomorphic (see, e.g., Condition (b) of Theorem 5.2 in [19]), and so is $\exp[tR^*]$ according to (58). Now, for any $\mathsf{p} \in l^2_{\mathbb{C},w_\beta}$ we have the norm-convergent expansion

$$\mathsf{p} = \sum_{\mathsf{k}=0}^{+\infty} (\mathsf{p}, \hat{\mathsf{p}}_\mathsf{k})_{2,w_\beta}\, \hat{\mathsf{p}}_\mathsf{k},$$

so that (57) follows from the standard spectral theory of semigroups. Finally, from (57) we obtain

$$(\exp[tR^*]\,\mathsf{p}, \mathsf{q})_{2,w_\beta} = (\mathsf{p}, \exp[tR^*]\,\mathsf{q})_{2,w_\beta}$$

for all $\mathsf{p}, \mathsf{q} \in l^2_{\mathbb{C},w_\beta}$ at once. ■

An immediate consequence of the preceding result is the following:

Corollary 1. *Let $\mathsf{p}^* \in l^2_{\mathbb{C},w_\beta}$ satisfy the two conditions in (2), and let us define*

$$\mathsf{p}(t) := \exp[tR^*]\, \mathsf{p}^* \tag{59}$$

for every $t \in [0, +\infty)$. Assuming that the eigenvalues of $R^*$are ordered as in (56) we then have

$$\|\mathsf{p}(t) - \mathsf{p}_{\beta,\mathsf{Gibbs}}\|_{2,w_\beta} \leq \exp[-t\,|\nu_1|]\, \|\mathsf{p}^*\|_{2,w_\beta} \tag{60}$$

for each $t \in [0,+\infty)$. Moreover, the function $t \to \mathsf{p}(t) = (p_{\mathsf{m}}(t))$ is differentiable for every $t \in (0,+\infty)$ and we have

$$\frac{d\mathsf{p}(t)}{dt} = R^* \mathsf{p}(t) \tag{61}$$

with

$$\left\| \frac{d\mathsf{p}(t)}{dt} \right\|_{2,w_\beta} \leq c_{\beta,\rho} t^{-1} \left\| \mathsf{p}^* \right\|_{2,w_\beta} \tag{62}$$

for some constant $c_{\beta,\rho} > 0$. Finally,

$$p_{\mathsf{m}}(t) \geq 0, \qquad \sum_{\mathsf{m}=0}^{+\infty} p_{\mathsf{m}}(t) = 1 \tag{63}$$

for every $t \in [0,+\infty)$.

Proof. From (57) we have

$$\begin{aligned} & \left\| \exp[tR^*]\, \mathsf{p}^* - (\mathsf{p}^*, \hat{\mathsf{p}}_0)_{2,w_\beta} \hat{\mathsf{p}}_0 \right\|_{2,w_\beta}^2 \\ = & \sum_{\mathsf{k}=1}^{+\infty} \left| (\mathsf{p}^*, \hat{\mathsf{p}}_{\mathsf{k}})_{2,w_\beta} \right|^2 \exp[-2t\,|\nu_{\mathsf{k}}|] \leq \exp[-2t\,|\nu_1|] \left\| \mathsf{p}^* \right\|_{2,w_\beta}^2 \end{aligned}$$

according to (56), so that (60) will follow if we can prove the relation

$$(\mathsf{p}^*, \hat{\mathsf{p}}_0)_{2,w_\beta}\, \hat{\mathsf{p}}_0 = \mathsf{p}_{\beta,\mathsf{Gibbs}}. \tag{64}$$

As $\hat{\mathsf{p}}_0$ is an eigenvector of R^* corresponding to the eigenvalue $\nu_0 = 0$ and satisfying $\|\hat{\mathsf{p}}_0\|_{2,w_\beta} = 1$, we may choose $\hat{\mathsf{p}}_0 = Z_\beta^{\frac{1}{2}} \mathsf{p}_{\beta,\mathsf{Gibbs}}$ without restricting the generality where Z_β is given by (7). Therefore we obtain

$$(\mathsf{p}^*, \hat{\mathsf{p}}_0)_{2,w_\beta} = Z_\beta^{-\frac{1}{2}}$$

as a consequence of the second relation in (2), which leads to (64).

Next, as a holomorphic semigroup, $\exp[tR^*]_{t\in[0,+\infty)}$ is *a fortiori* differentiable and equation (61) holds. Furthermore we have

$$\left\| R^* \exp[tR^*] \right\|_\infty \leq c_{\beta,\rho} t^{-1}$$

for some $c_{\beta,\rho} > 0$ and every $t \in (0,+\infty)$ for the uniform norm of the operator as a characteristic property of holomorphic semigroups (see, e.g, Section 2.5 in Chapter 2 of [19]), which proves (62).

As for the proof of the first relation in (63), let us define the family of orthogonal projections $(Q_{\mathsf{m}})_{\mathsf{m}\in\mathbb{N}}$ in $l^2_{\mathbb{C},w_\beta}$ by $(Q_{\mathsf{m}}\mathsf{q})_{\mathsf{k}} = q_{\mathsf{m}}\delta_{\mathsf{m},\mathsf{k}}$ for each $\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$ and every $\mathsf{k} \in \mathbb{N}$. Then we have

$$p_{\mathsf{m}}(t) = Q_{\mathsf{m}} \exp[tR^*]\, \mathsf{p}^* \tag{65}$$

for every $t \in [0, +\infty)$, and from the semigroup property we see that it is sufficient to have $p_{\mathsf{m}}(t) \geq 0$ for all sufficiently small $t \geq 0$ in order that the inequality be valid for every t. Now, the function given in (65) is continuous for $t > 0$ and right-continuous at $t = 0$ for every m. Therefore, it is impossible to have an $\mathsf{m}^* \in \mathbb{N}$ and a sufficiently small $t^* > 0$ such that $p_{\mathsf{m}^*}(t^*) < 0$, for by hypothesis either $p_{\mathsf{m}^*}(0) > 0$ or $p_{\mathsf{m}^*}(0) = 0$. In the first case, $p_{\mathsf{m}^*}(t^*) < 0$ would contradict the right-continuity of (65) for $\mathsf{m} = \mathsf{m}^*$ at $t = 0$. In the second case, equation (12) for $\mathsf{m} = \mathsf{m}^*$ gives

$$\lim_{t \searrow 0_+} \frac{dp_0(t)}{dt} = \rho p_1(0) \geq 0$$

if $\mathsf{m}^* = 0$, and

$$\lim_{t \searrow 0_+} \frac{dp_{\mathsf{m}^*}(t)}{dt} = \rho \left\{ (\mathsf{m}^* + 1) p_{\mathsf{m}^*+1}(0) + \exp\left[-\beta\right] \mathsf{m}^* p_{\mathsf{m}^*-1}(0) \right\} \geq 0$$

for $\mathsf{m}^* \neq 0$, again contradicting $p_{\mathsf{m}^*}(t^*) < 0$. Consequently we have $p_{\mathsf{m}}(t) \geq 0$ for every m and all sufficiently small $t \geq 0$.

It remains to prove the second relation in (63). We first note that the inequality

$$|p_{\mathsf{m}}(t) - p_{\beta,\mathsf{Gibbs},\mathsf{m}}| \leq \exp\left[-\frac{\beta \mathsf{m}}{2}\right] \|\mathsf{p}^*\|_{2,w_\beta} \tag{66}$$

holds for each $\mathsf{m} \in \mathbb{N}$ uniformly in $t \in [0, +\infty)$ as a consequence of (60). Then, taking (6) into account we obtain

$$|p_{\mathsf{m}}(t)| \leq \exp\left[-\frac{\beta \mathsf{m}}{2}\right] \|\mathsf{p}^*\|_{2,w_\beta} + Z_\beta^{-1} \exp\left[-\beta \mathsf{m}\right]$$

uniformly in t for each m, so that the series

$$\sum_{\mathsf{m}=0}^{+\infty} p_{\mathsf{m}}(t) \leq Z_{\frac{\beta}{2}} \|\mathsf{p}^*\|_{2,w_\beta} + 1 < +\infty \tag{67}$$

converges absolutely and uniformly on $(0, +\infty)$. Now let $t \in \mathsf{K} \subset (0, +\infty)$ where K stands for any compact subset of the interval. From (62) we then have, with $c_{\beta,\rho,\mathsf{K}} > 0$ depending solely on β, ρ and K,

$$\left|\frac{dp_{\mathsf{m}}(t)}{dt}\right| \leq c_{\beta,\rho} \exp\left[-\frac{\beta \mathsf{m}}{2}\right] t^{-1} \|\mathsf{p}^*\|_{2,w_\beta} \leq c_{\beta,\rho,\mathsf{K}} \exp\left[-\frac{\beta \mathsf{m}}{2}\right] \|\mathsf{p}^*\|_{2,w_\beta}$$

uniformly in t since this variable is bounded below by the positive distance between K and the origin. Consequently, the series

$$\sum_{\mathsf{m}=0}^{+\infty} \left|\frac{dp_{\mathsf{m}}(t)}{dt}\right| \leq c_{\beta,\rho,\mathsf{K}} Z_{\frac{\beta}{2}} \|\mathsf{p}^*\|_{2,w_\beta} < +\infty \tag{68}$$

converges uniformly on every compact subset $\mathsf{K} \subset (0, +\infty)$. Relations (67) and (68) then imply that the function $t \to \sum_{\mathsf{m}=0}^{+\infty} p_{\mathsf{m}}(t)$ is differentiable on $(0, +\infty)$, and that

$$\frac{d}{dt} \sum_{\mathsf{m}=0}^{+\infty} p_{\mathsf{m}}(t) = \sum_{\mathsf{m}=0}^{+\infty} \frac{dp_{\mathsf{m}}(t)}{dt} = \sum_{\mathsf{m}=0}^{+\infty} \left(R^{*}\mathsf{p}(t)\right)_{\mathsf{m}} = 0,$$

where the last two equalities follow directly from (12). This proves the second relation in (63) since $\sum_{\mathsf{m}=0}^{+\infty} p_{\mathsf{m}}(0) = 1$ by hypothesis. ■

Remarks. (1) To sum up, the oscillator initially steered away from thermodynamical equilibrium due to its interaction with the bath will eventually return there exponentially rapidly, as its initial probability distribution evolves according to (12) as a probability distribution for all times converging toward the Gibbs equilibrium distribution. Once thermodynamical equilibrium sets in, and owing to the physical significance of the parameter ρ in the above equations, it is possible to retrieve several known physical results, among which Planck's black body radiation law in case the radiation field is a quantized electromagnetic field (see, e.g., Chapter VI of [26]).

(2) Whereas the semigroup structure set forth in the preceding considerations is a manifestation of the Markovian nature of the evolution, it is well known that the approach to equilibrium in statistical mechanics is not Markovian in general, as was stressed in the introduction. We refer the reader for example to [10] and [20] for very nice examples of this fact.

In the next section we establish an elementary connection between the spectral properties of the operator R^{*} and those of a related linear pencil (for an introduction to the spectral theory of general pencils we refer the reader to [15]).

3 On the spectral properties of a linear pencil associated with R^{*}

We begin with the following:

Proposition 6. *There exist linear bounded operators $U, V : l^2_{\mathbb{C},w_\beta} \to h^1_{\mathbb{C},w_\beta}$ such that*

$$(R\mathsf{p}, \mathsf{q})_{2,w_\beta} = (\mathsf{p}, U\mathsf{q})_{1,2,w_\beta} \tag{69}$$

and

$$(\mathsf{p}, \mathsf{q})_{2,w_\beta} = (\mathsf{p}, V\mathsf{q})_{1,2,w_\beta} \tag{70}$$

for each $\mathsf{p} \in D(R)$ and every $\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$, respectively, and V^{-1} exists as an unbounded operator from $\operatorname{Ran} V$ *onto $l^2_{\mathbb{C},w_\beta}$. Furthermore, the restrictions $\tilde{U}, \tilde{V}$ of U, V to $h^1_{\mathbb{C},w_\beta}$ are compact and self-adjoint. Finally, we have the representation*

$$R^{*} = V^{-1}\tilde{U} \tag{71}$$

of the operator R^*.

Proof. We could argue by means of the Riesz-Fréchet representation theorem to prove (69) and (70), but we prefer an alternative way in order to get explicit expressions for the operators involved. On the one hand, it is sufficient to prove those equalities for $\mathsf{p} = \mathsf{f_m}$ and every m, that is,

$$(R\mathsf{f_m}, \mathsf{q})_{2,w_\beta} = (\mathsf{f_m}, U\mathsf{q})_{1,2,w_\beta} \tag{72}$$

and

$$(\mathsf{f_m}, \mathsf{q})_{2,w_\beta} = (\mathsf{f_m}, V\mathsf{q})_{1,2,w_\beta},$$

respectively. From (38) and the definition of the weights we first have

$$(\mathsf{f_m}, \mathsf{u})_{1,2,w_\beta} = \exp\left[\frac{\beta}{2}\mathsf{m}\right]\left(1+\mathsf{m}^2\right)\bar{u}_\mathsf{m} \tag{73}$$

for any $\mathsf{u} \in h^1_{\mathbb{C},w_\beta}$. On the other hand, from (21) we have

$$\begin{aligned} & (R\mathsf{f_m}, \mathsf{q})_{2,w_\beta} \\ = \;& \rho\exp\left[\frac{\beta}{2}\mathsf{m}\right]\left\{(\mathsf{m}+1)\,\bar{q}_{\mathsf{m}+1} + \exp[-\beta]\,\mathsf{m}\bar{q}_{\mathsf{m}-1} - (\mathsf{m}+\exp[-\beta]\,(\mathsf{m}+1))\,\bar{q}_\mathsf{m}\right\} \end{aligned}$$

for every $\mathsf{q} \in l^2_{\mathbb{C},w_\beta}$. Requiring equality between the preceding expression and (73) then allows us to solve for u_m and get

$$u_\mathsf{m} = \rho\left(\frac{\mathsf{m}+1}{\mathsf{m}^2+1}q_{\mathsf{m}+1} + \frac{\exp[-\beta]\,\mathsf{m}}{\mathsf{m}^2+1}q_{\mathsf{m}-1} - \frac{\mathsf{m}+\exp[-\beta]\,(\mathsf{m}+1)}{\mathsf{m}^2+1}q_\mathsf{m}\right) \tag{74}$$

for each m, from which we have

$$\begin{aligned} & \left(1+\mathsf{m}^2\right)|u_\mathsf{m}|^2 \\ \leq \;& c_\rho\left(\frac{(\mathsf{m}+1)^2}{\mathsf{m}^2+1}|q_{\mathsf{m}+1}|^2 + \frac{\exp[-2\beta]\,\mathsf{m}^2}{\mathsf{m}^2+1}|q_{\mathsf{m}-1}|^2 + \frac{(\mathsf{m}+\exp[-\beta]\,(\mathsf{m}+1))^2}{\mathsf{m}^2+1}|q_\mathsf{m}|^2\right) \end{aligned}$$

for some irrelevant constant $c_\rho > 0$, where the coefficients of each term on the right-hand side are uniformly bounded in m. We therefore obtain

$$\|\mathsf{u}\|_{1,2,w_\beta} \leq c_{\beta,\rho}\|\mathsf{q}\|_{2,w_\beta}$$

for some $c_{\beta,\rho} > 0$, so that

$$U : \mathsf{q} \to U\mathsf{q} := \mathsf{u}$$

is indeed a linear bounded operator from $l^2_{\mathbb{C},w_\beta}$ into $h^1_{\mathbb{C},w_\beta}$. The same conclusion holds for

$$V : \mathsf{q} \to V\mathsf{q} := \mathsf{v}$$

where

$$v_\mathsf{m} = \frac{q_\mathsf{m}}{\mathsf{m}^2+1}, \tag{75}$$

and (75) implies $\ker V = \{0\}$ so that V^{-1} exists as an unbounded operator from $\operatorname{Ran} V$ onto $l^2_{\mathbb{C},w_\beta}$. Furthermore, the restrictions $\tilde{U}, \tilde{V}$ of U, V to $h^1_{\mathbb{C},w_\beta}$ are compact as a consequence of (80) and we omit the elementary proofs of their self-adjointness. Finally, it is easily verified from (18), (74) and (75) that if $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ then $U\mathsf{q} = \tilde{U}\mathsf{q} \in \operatorname{Ran} V$ with

$$(R^*\mathsf{q})_{\mathsf{m}} = \left(V^{-1}\tilde{U}\mathsf{q}\right)_{\mathsf{m}} \tag{76}$$

for every $\mathsf{m} \in \mathbb{N}$, hence $R^* = V^{-1}\tilde{U}$ on $h^1_{\mathbb{C},w_\beta}$. ■

Let us now consider the linear pencil

$$L(\nu) := \tilde{U} - \nu\tilde{V}$$

on $h^1_{\mathbb{C},w_\beta}$, where $\nu \in \mathbb{C}$. Following Section 11.2 in [15] or Section 1 in [25], we define the spectrum of L, $\sigma(L)$, as the set of all $\nu \in \mathbb{C}$ for which $L(\nu)$ is not bijective, its resolvent set as $\mathbb{C} \setminus \sigma(L)$ and its point spectrum, $\sigma_{\mathsf{p}}(L)$, as the set of all $\nu \in \mathbb{C}$ for which $L(\nu)$ is not injective. Then we have:

Corollary 2. *The following statements hold:*
(a) We have $\sigma(L) = \mathbb{C}$, that is, the resolvent set of L is empty.
(b) The point spectrum consists exclusively of the values that appear in (56), that is,

$$\sigma_{\mathsf{p}}(L) = \{\nu \in \mathbb{C} : \nu = \nu_{\mathsf{k}},\, \mathsf{k} \in \mathbb{N}\},$$

which satisfy (55) and correspond to the basis eigenvectors $\hat{\mathsf{p}}_{\mathsf{k}}$ in (54). Moreover we have

$$L(\nu_{\mathsf{k}})\,\hat{\mathsf{p}}_{\mathsf{k}} = 0 \tag{77}$$

for every $\mathsf{k} \in \mathbb{N}$.

Proof. In order to prove statement (a), we first remark that for each $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ and every $\nu \in \mathbb{C}$ we have $L(\nu)\mathsf{q} \in \operatorname{Ran} V$ as a consequence of the argument given at the very end of the proof of Proposition 6. Furthermore, the inclusion $\operatorname{Ran} V \subset h^1_{\mathbb{C},w_\beta}$ is proper. For instance, the sequence p defined by

$$p_{\mathsf{m}} = \frac{\exp\left[-\frac{\beta}{2}\mathsf{m}\right]}{\mathsf{m}^2 + 1}$$

for every m is easily seen to belong to $h^1_{\mathbb{C},w_\beta}$, but not to $\operatorname{Ran} V$ according to (75). Therefore, given such a p there exists no $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ satisfying $L(\nu)\mathsf{q} = \mathsf{p}$ so that $L(\nu)$ is not surjective.

As for Statement (b), it is plain that (54) implies (77) as a consequence of the representation (76). Therefore we have the inclusion

$$\{\nu \in \mathbb{C} : \nu = \nu_{\mathsf{k}},\, \mathsf{k} \in \mathbb{N}\} \subseteq \sigma_{\mathsf{p}}(L).$$

Now let $\hat{\nu} \in \sigma_{\mathsf{p}}(L)$, so that there exists a $\hat{\mathsf{p}} \in h^1_{\mathbb{C},w_\beta}$, $\hat{\mathsf{p}} \neq 0$, satisfying

$$L(\hat{\nu})\,\hat{\mathsf{p}} = \tilde{U}\hat{\mathsf{p}} - \hat{\nu}\tilde{V}\hat{\mathsf{p}} = 0.$$

Multiplying the preceding expression on the left by V^{-1} gives

$$(R^* - \hat{\nu})\,\hat{\mathsf{p}} = 0$$

according to (76), hence the converse inclusion

$$\sigma_{\mathsf{p}}(L) \subseteq \{\nu \in \mathbb{C} : \nu = \nu_{\mathsf{k}},\ \mathsf{k} \in \mathbb{N}\ \}$$

also holds by virtue of the spectral properties of R^* proved in the preceding section. ■

We conclude this article with the following:

Appendix. On the compactness of the embedding $h^1_{\mathbb{C},w_\beta} \hookrightarrow l^2_{\mathbb{C},w_\beta}$

In this appendix we derive (52) as the consequence of a more general embedding result that involves an intermediary space between $h^1_{\mathbb{C},w_\beta}$ and $l^2_{\mathbb{C},w_\beta}$, and also prove the compactness of the set D_θ defined in (53). Let $l^1_{\mathbb{C},w_{\frac{\beta}{2}}}$ be the Banach space consisting of all complex sequences $\mathsf{p} = (p_{\mathsf{m}})$ satisfying

$$\|\mathsf{p}\|_{1,w_{\frac{\beta}{2}}} := \sum_{\mathsf{m}=0}^{+\infty} w_{\frac{\beta}{2},\mathsf{m}} |p_{\mathsf{m}}| < +\infty. \tag{78}$$

Then the following result holds:

Proposition A.1. *We have the embeddings*

$$h^1_{\mathbb{C},w_\beta} \hookrightarrow l^1_{\mathbb{C},w_{\frac{\beta}{2}}} \to l^2_{\mathbb{C},w_\beta} \tag{79}$$

where the first one is compact and the second one continuous. In particular, the embedding

$$h^1_{\mathbb{C},w_\beta} \hookrightarrow l^2_{\mathbb{C},w_\beta} \tag{80}$$

is compact.

Proof. Remembering that $w_{\beta,\mathsf{m}} = \exp[\beta\mathsf{m}]$ for every m we first have

$$\begin{aligned}
&\sum_{\mathsf{m}=1}^{+\infty} w_{\frac{\beta}{2},\mathsf{m}} |p_{\mathsf{m}}| \\
= &\sum_{\mathsf{m}=1}^{+\infty} \frac{1}{\mathsf{m}} w_{\frac{\beta}{2},\mathsf{m}} \mathsf{m} |p_{\mathsf{m}}| \leq \left(\sum_{\mathsf{m}=1}^{+\infty} \frac{1}{\mathsf{m}^2}\right)^{\frac{1}{2}} \left(\sum_{\mathsf{m}=1}^{+\infty} w_{\beta,\mathsf{m}} \mathsf{m}^2 |p_{\mathsf{m}}|^2\right)^{\frac{1}{2}} \qquad (81)\\
= &\frac{\pi\sqrt{6}}{6} \left(\sum_{\mathsf{m}=1}^{+\infty} w_{\beta,\mathsf{m}} \mathsf{m}^2 |p_{\mathsf{m}}|^2\right)^{\frac{1}{2}} \leq \frac{\pi\sqrt{6}}{6} \left(\sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}} \left(1+\mathsf{m}^2\right) |p_{\mathsf{m}}|^2\right)^{\frac{1}{2}}
\end{aligned}$$

by Schwarz inequality and summing the first series on the right-hand side. Furthermore we have

$$\left|p_{\mathsf{0}}\right| \leq \left(\sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}}\left(1+\mathsf{m}^{2}\right)\left|p_{\mathsf{m}}\right|^{2}\right)^{\frac{1}{2}}$$

and consequently

$$\|\mathsf{p}\|_{1,w_{\frac{\beta}{2}}} \leq \left(1+\frac{\pi\sqrt{6}}{6}\right)\|\mathsf{p}\|_{1,2,w_{\beta}}$$

from (37), (78) and (81), so that the first embedding in (79) is continuous.

Now let $\mathcal{K}$ be a bounded set in $h^{1}_{\mathbb{C},w_{\beta}}$, $\bar{\mathcal{K}}$ its closure in $l^{1}_{\mathbb{C},w_{\frac{\beta}{2}}}$ and let $\varepsilon>0$. Moreover, let $\kappa>0$ be the radius of a ball centered at zero containing $\mathcal{K}$ in $h^{1}_{\mathbb{C},w_{\beta}}$. In order to prove that $\bar{\mathcal{K}}$ is compact in $l^{1}_{\mathbb{C},w_{\frac{\beta}{2}}}$, it is then necessary and sufficient to show that there exists a finite-dimensional subspace $\mathcal{S}_{\varepsilon,\kappa}\subset l^{1}_{\mathbb{C},w_{\frac{\beta}{2}}}$ such that the distance of every $\mathsf{p}\in\bar{\mathcal{K}}$ to $\mathcal{S}_{\varepsilon,\kappa}$ satisfies

$$\operatorname{dist}\left(\mathsf{p},\mathcal{S}_{\varepsilon,\kappa}\right):=\inf_{\mathsf{q}\in\mathcal{S}_{\varepsilon,\kappa}}\|\mathsf{p}-\mathsf{q}\|_{1,w_{\frac{\beta}{2}}}\leq\varepsilon,$$

a property equivalent to the total boundedness of $\bar{\mathcal{K}}$ (see, e.g., [1] and [14]). From an estimate similar to (81), we first infer that there exists an integer $\mathsf{N}_{\varepsilon,\kappa}$ such that the inequality

$$\sum_{\mathsf{m}=\mathsf{N}}^{+\infty} w_{\frac{\beta}{2},\mathsf{m}}\left|p_{\mathsf{m}}\right| \leq \kappa\left(\sum_{\mathsf{m}=\mathsf{N}}^{+\infty}\frac{1}{\mathsf{m}^{2}}\right)^{\frac{1}{2}} \leq \kappa\frac{\varepsilon}{2\kappa}=\frac{\varepsilon}{2} \tag{82}$$

holds for every $\mathsf{N}\geq\mathsf{N}_{\varepsilon,\kappa}$. Then fix such an N and let

$$\mathcal{S}_{\varepsilon,\kappa}:=\operatorname{span}\left\{\mathsf{e}_{0},...,\mathsf{e}_{\mathsf{N}}\right\}$$

be the finite-dimensional subspace generated by the canonical sequences defined before the statement of Proposition 1. Let us write $\mathsf{p}=\sum_{\mathsf{m}=0}^{+\infty}p_{\mathsf{m}}\mathsf{e}_{\mathsf{m}}$ for any given $\mathsf{p}\in\mathcal{K}$ and let $\mathsf{p}_{\mathsf{N}}:=\sum_{\mathsf{m}=0}^{\mathsf{N}-1}p_{\mathsf{m}}\mathsf{e}_{\mathsf{m}}\in\mathcal{S}_{\varepsilon,\kappa}$. Since

$$\operatorname{dist}\left(\mathsf{p},\mathcal{S}_{\varepsilon,\kappa}\right)\leq\|\mathsf{p}-\mathsf{q}\|_{1,w_{\frac{\beta}{2}}}$$

for every $\mathsf{q}\in\mathcal{S}_{\varepsilon,\kappa}$, we may choose in particular $\mathsf{q}=\mathsf{p}_{\mathsf{N}}$ and therefore obtain

$$\operatorname{dist}\left(\mathsf{p},\mathcal{S}_{\varepsilon,\kappa}\right)\leq\|\mathsf{p}-\mathsf{p}_{\mathsf{N}}\|_{1,w_{\frac{\beta}{2}}}=\sum_{\mathsf{m}=\mathsf{N}}^{+\infty} w_{\frac{\beta}{2},\mathsf{m}}\left|p_{\mathsf{m}}\right|\leq\frac{\varepsilon}{2} \tag{83}$$

according to (82). Finally, for any limit point $\mathsf{p} \in \bar{\mathcal{K}}$ there exists a sequence $(\mathsf{p}_\mathsf{n}) \subset \mathcal{K}$ such that $\|\mathsf{p}-\mathsf{p}_\mathsf{n}\|_{1,w_{\frac{\beta}{2}}} \leq \frac{\varepsilon}{2}$ for each $\mathsf{n} \geq \mathsf{n}_\varepsilon$ and some integer n_ε, so that fixing such an n we get

$$\operatorname{dist}(\mathsf{p},\mathcal{S}_{\varepsilon,\kappa}) \leq \frac{\varepsilon}{2} + \inf_{\mathsf{q}\in\mathcal{S}_{\varepsilon,\kappa}} \|\mathsf{p}_\mathsf{n}-\mathsf{q}\|_{1,w_{\frac{\beta}{2}}} \leq \varepsilon$$

according to (83), which proves the compactness of $\bar{\mathcal{K}}$ and thereby that of the first embedding in (79).

As for the continuity of the second embedding, we simply note that

$$\|\mathsf{p}\|^2_{2,w_\beta} = \sum_{\mathsf{m}=0}^{+\infty} w_{\beta,\mathsf{m}} |p_\mathsf{m}|^2 \leq \left(\sum_{\mathsf{m}=0}^{+\infty} w_{\frac{\beta}{2},\mathsf{m}} |p_\mathsf{m}|\right)^2 = \|\mathsf{p}\|^2_{1,w_{\frac{\beta}{2}}}.$$

Consequently, the compactness of (80) is an immediate consequence of (79). ■

As a natural consequence of the preceding result we have:

Proposition A.2. *The set*

$$D_\theta = \left\{\mathsf{q} \in h^1_{\mathbb{C},w_\beta} : \|\mathsf{q}\|_{2,w_\beta} \leq 1, \|S_\nu \mathsf{q}\|_{2,w_\beta} \leq \theta\right\} \tag{84}$$

with $\theta \in [0,+\infty)$ is compact in $l^2_{\mathbb{C},w_\beta}$.

Proof. Let (q_N) be any sequence in D_θ. Then (q_N) is bounded with respect to the graph-norm

$$\|\mathsf{q}\|_{\mathcal{G},w_\beta} := \left(\|\mathsf{q}\|^2_{2,w_\beta} + \|S_\nu \mathsf{q}\|^2_{2,w_\beta}\right)^{\frac{1}{2}} \tag{85}$$

which we may introduce in $D(S_\nu) = h^1_{\mathbb{C},w_\beta}$, and which is easily seen to be equivalent to (37). In particular, since

$$(S_\nu \mathsf{q})_\mathsf{m} = -\rho(\mathsf{m} + \nu_\rho + \exp[-\beta](\mathsf{m}+1)) q_\mathsf{m} \tag{86}$$

for each m, we infer from (37) and (86) that

$$\|\mathsf{q}\|_{1,2,w_\beta} \leq c_{\beta,\rho} \|\mathsf{q}\|_{\mathcal{G},w_\beta}$$

for every $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$ for some $c_{\beta,\rho} > 0$ depending solely on β and ρ. Therefore, the sequence (q_N) is necessarily bounded with respect to the norm in $h^1_{\mathbb{C},w_\beta}$, and so there exist a subsequence $(\mathsf{q}_{\mathsf{N}'}) \subset (\mathsf{q}_\mathsf{N})$ and a $\mathsf{q}^* \in h^1_{\mathbb{C},w_\beta}$ such that $\mathsf{q}_{\mathsf{N}'} \rightharpoonup \mathsf{q}^*$ weakly in $h^1_{\mathbb{C},w_\beta}$ as $\mathsf{N}' \to +\infty$. Since (80) is compact, we may then conclude that $\mathsf{q}_{\mathsf{N}'} \to \mathsf{q}^*$ strongly in $l^2_{\mathbb{C},w_\beta}$. As a consequence we now claim that (84) is indeed compact in $l^2_{\mathbb{C},w_\beta}$, which amounts to showing that $\mathsf{q}^* \in D_\theta$. We already have $\|\mathsf{q}^*\|_{2,w_\beta} \leq 1$ since $\|\mathsf{q}_{\mathsf{N}'}\|_{2,w_\beta} \leq 1$ for every N', so that it remains to prove the inequality

$$\|S_\nu \mathsf{q}^*\|_{2,w_\beta} \leq \theta. \tag{87}$$

Since

$$\|S_\nu \mathsf{q}_{\mathsf{N}'}\|_{2,w_\beta} \leq \theta \tag{88}$$

for every $\mathsf{N}' \in \mathbb{N}$, we have *a fortiori*

$$\sum_{\mathsf{m}=0}^{\mathsf{M}} \exp[\beta \mathsf{m}] \left(\mathsf{m} + \nu_\rho + \exp[-\beta](\mathsf{m}+1)\right)^2 \left|(\mathsf{q}_{\mathsf{N}'})_{\mathsf{m}}\right|^2 \leq \left(\frac{\theta}{\rho}\right)^2$$

for each $\mathsf{M} \in \mathbb{N}$ from the very definition of S_ν. Moreover, the strong convergence $\mathsf{q}_{\mathsf{N}'} \to \mathsf{q}^*$ in $l^2_{\mathbb{C},w_\beta}$ implies that $(\mathsf{q}_{\mathsf{N}'})_{\mathsf{m}} \to q^*_{\mathsf{m}}$ in $\mathbb{C}$ for every m. Fixing M and letting $\mathsf{N}' \to +\infty$ in the preceding inequality we then obtain

$$\sum_{\mathsf{m}=0}^{\mathsf{M}} \exp[\beta \mathsf{m}] \left(\mathsf{m} + \nu_\rho + \exp[-\beta](\mathsf{m}+1)\right)^2 \left|q^*_{\mathsf{m}}\right|^2 \leq \left(\frac{\theta}{\rho}\right)^2,$$

which shows that the sum on the left-hand side is uniformly bounded and monotone increasing in M. Letting $\mathsf{M} \to +\infty$ in this last expression then gives (87). ■

Since S_ν is self-adjoint and satisfies

$$(S_\nu \mathsf{q}, \mathsf{q})_{2,w_\beta} \leq 0$$

for every $\mathsf{q} \in h^1_{\mathbb{C},w_\beta}$, we may invoke Theorem XIII.64 in [21] to conclude that S_ν has a compact resolvent, hence that $S_\nu^{-1} = (S_\nu - 0)^{-1}$ is compact since $\nu = 0$ belongs to the resolvent set of S_ν as a consequence of (45).

Statements and declarations

Declaration of competing interest: The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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