arXiv:2404.11381v1 [math.CO] 17 Apr 2024

CLUSTER SCATTERING COEFFICIENTS IN RANK 2

THOMAS ELGIN, NATHAN READING, AND SALVATORE STELLA

The purpose of this note is to record and share some conjectures on cluster scattering diagrams of rank 2. We continue the notation of [3, Example 1.15]. Thus, in the language of cluster algebras, we take the exchange matrix to be $\left[\begin{smallmatrix} 0 & c \\ -b & 0 \end{smallmatrix}\right]$ for $b, c > 0$. For $i, j \geq 0$, define $\tau(i,j)$ to be the coefficient of $A_1^{-ib} A_2^{cj}$ on the wall of the cluster scattering diagram that is orthogonal to $ie_1 + je_2$. To specify or emphasize the exchange matrix, we may write $\tau^{b,c}(i,j)$, but generally, we think of $\tau(i,j)$ as a function of indeterminates b and c. We define $g = \frac{\gcd(ib,jc)}{\gcd(i,j)}$.

If i and j are relatively prime positive integers, then the condition that the cluster scattering diagram has no wall orthogonal to $ie_1 + je_2$ is equivalent to the condition that $\tau(ki, kj) = 0$ for all positive integers k.

Example. Several of the $\tau(i,j)$ are shown below, with i changing in the horizontal direction and j changing in the vertical direction and $(0,0)$ at the bottom-left.

$\vdots$				
0	$\frac{g(c-1)(c-2)}{6}$	$\frac{g(c-1)(3bc-3b-2c+1)}{6}$	$\frac{g(3bc-3b-3c+2+g)(3bc-3b-3c+1+g)}{6}$	
0	$\frac{g(c-1)}{2}$	$\frac{g(2bc-2b-2c+1+g)}{6}$	$\frac{g(b-1)(3bc-2b-3c+1)}{6}$	
1	g	$\frac{g(b-1)}{2}$	$\frac{g(b-1)(b-2)}{6}$	
1	1	0	0	$\cdots$

Example. Take $b = 3$ and $c = 2$ so that the exchange matrix is $\left[\begin{smallmatrix} 0 & 2 \\ -3 & 0 \end{smallmatrix}\right]$. Some of the integers $\tau^{3,2}(i,j)$ are shown below, again with i changing in the horizontal direction and j changing in the vertical direction and $(0,0)$ at the bottom-left.

$\vdots$								
0	0	0	1	33	87	286	429	
0	0	0	5	327	143	132	143	
0	0	1	6	33	42	33	6	
0	0	2	6	14	6	2	0	
0	1	14	5	14	1	0	0	
0	1	2	1	0	0	0	0	
1	1	1	0	0	0	0	0	
1	1	0	0	0	0	0	0	$\cdots$

We point out the obvious symmetry $\tau^{b,c}(i,j) = \tau^{c,b}(j,i)$. Another symmetry is less obvious but not hard to prove using mutation of scattering diagrams: Interpreting (i,j) as a vector $ie_1 + je_2$ in the root lattice of a root system with Cartan matrix $\left[\begin{smallmatrix} 2 & -c \\ -b & 2 \end{smallmatrix}\right]$ and simple roots e_1 and e_2, the integers $\tau^{b,c}(i,j)$ are invariant under the action of the Weyl group on (i,j). This amounts to the symmetries $\tau^{b,c}(i,j) = \tau^{b,c}(i, -j+ci)$ and $\tau^{b,c}(i,j) = \tau^{b,c}(-i+bj, j)$.

Conjectures.

1. For $i,j > 0$, the coefficient $\tau(i,j)$ is a polynomial in b, c, and g.
2. The polynomial has g as a factor and its degree in g is $\gcd(i,j)$.
3. Its degree in b is $j-1$ and its degree in c is $i-1$.
4. The polynomial $(\max(i,j))!\ \tau(i,j)$ has integer coefficients.
5. $\tau(1,j) = \dfrac{g}{b}\dbinom{b}{j}$
6. $\tau(i,1) = \dfrac{g}{c}\dbinom{c}{i}$
7. $\tau(i,i) = \dfrac{g}{(b-1)(c-1)i+g}\dbinom{(b-1)(c-1)i+g}{i}$
8. $\tau(i,i) = \displaystyle\sum_{\ell=0}^{\infty} \frac{g}{\ell+1}\binom{i-1}{\ell}\binom{i(bc-b-c)+g-1}{\ell}$
9. $\tau(i,i-1) = \dfrac{g}{i(ib-i+1)}\dbinom{(ib-i+1)(c-1)}{i-1}$
10. $\tau(j-1,j) = \dfrac{g}{j(jc-j+1)}\dbinom{(jc-j+1)(b-1)}{j-1}$
11. $\tau^{b,b}(i,j)$ is a polynomial in b of degree $i+j-1$ that expands positively in the basis $\{\binom{b}{0}, \binom{b}{1}, \binom{b}{2}, \ldots\}$.
12. $\tau^{b,b}(i,j)$ has unimodal log-concave coefficients.
13. $\tau^{1,5}(2j,j) = \dfrac{1}{j}\displaystyle\sum_{\ell=0}^{\infty}\binom{\ell}{j-\ell+1}\binom{j+\ell-1}{\ell}$.

Comments.

- Specializing Conjecture 8 to the case $b = c$ recovers a result of Reineke [4]. (See [3, Example 1.15].)
- Conjecture 8 is equivalent to Conjecture 7.
- Since $c = b$ in Conjectures 11 and 12, also $g = b$.

Assuming Conjecture 1, write $\tau(i,j\,;k)$ for the coefficient of g^k in $\tau(i,j)$ and similarly $\tau^{b,c}(i,j\,;k)$. Because g is also invariant under the action of the Weyl group on (i,j), we have $\tau^{b,c}(i,j;k) = \tau^{b,c}(i,-j+bi;k)$ and $\tau^{b,c}(i,j;k) = \tau^{b,c}(-i+aj,j;k)$.

Conjectures.

14. $\tau(i,j\,;k)$ is a polynomial of degree $j-k$ in b and degree $i-k$ in c and has a term that is a nonzero constant times $b^{j-k}c^{i-k}$.
15. If $\gcd(i,j) = 1$, then $\tau(ik, jk\,;k) = \dfrac{\tau(i,j\,;1)^k}{k!}$.
16. $\tau(k, jk\,;k-1) = \dfrac{\tau(1,j\,;1)^{k-1}\cdot p_j}{(k-2)!}$, where p_j is a polynomial in b and c that depends only on j, not k.

17. $\tau(ik,k\,;k-1) = \dfrac{\tau(i,1\,;1)^{k-1}\cdot p_i}{(k-2)!}$, where p_i is a polynomial in b and c that depends only on i, not k.

18. If $\gcd(i,j)=1$, then $\tau(ki,kj\,;k-1)$ has a factor p_{ij} that depends only on i and j, not on k, and the other factors of $\tau(ki,kj\,;k-1)$ also appear as factors of $\tau(i,j\,;1)$.

Comments.

- Assuming Conjecture 2, for fixed i and j, Conjecture 15 is a formula for $\tau(i,j;k)$ for the largest k such that $\tau(i,j;k)\neq 0$ in terms of some $\tau(\,\cdot\,,\,\cdot\,;1)$.
- In Conjecture 18, the factors of $\tau(i,j\,;1)$ appear to different powers in $\tau(ki,kj\,;k-1)$ for various k. For example,

$$\begin{aligned}
\tau(2,3\,;\,1) &= \frac{(b-1)\,(3cb-2b-3c+1)}{6}\\
\tau(4,6\,;\,1) &= \frac{(b-1)\,p_{23}}{180}\\
\tau(6,9\,;\,2) &= \frac{(b-1)^2\,(3cb-2b-3c+1)\,p_{23}}{1080}\\
\tau(8,12\,;\,3) &= \frac{(b-1)^3\,(3cb-2b-3c+1)^2\,p_{23}}{12960}
\end{aligned}$$

for
$$\begin{aligned}
p_{23} = {} & 330b^4c^3 - 720b^4c^2 - 1530b^3c^3 + 525b^4c + 2880b^3c^2\\
& + 2610b^2c^3 - 128b^4 - 1770b^3c - 4140b^2c^2\\
& - 1950b\,c^3 + 352b^3 + 2085b^2c + 2520b\,c^2\\
& + 540c^3 - 328b^2 - 1005cb - 540c^2 + 122b + 165c - 15
\end{aligned}$$

Computation. These conjectures are backed up by significant computational evidence. The functions $\tau(i,j)$ of b and c can be computed symbolically up to large values of i and j. Thus, for example, the polynomials shown in the first Example above are known to be correct for all b and c, rather than only for some specific values of b and c. Similarly, the conjectures on $\tau(i,j)$ have been checked for many values of i and j, and each case that has been checked is true for all b and c.

Computing the functions $\tau(i,j)$ proceeds by induction on $i+j$, by solving, at each step, the equations that describe consistency of the cluster scattering diagram in degree $i+j$ using known values of $\tau(i',j')$ for $i'+j'<i+j$. Thus, it would be significant progress even to find a recursive description of $\tau(i,j)$ whose recursive step is simpler than solving a system of equations.

Related work. Ryota Akagi [1] has made progress on the goal of explicitly understanding scattering terms in the cluster scattering diagram, but with different approach and conventions. It is likely that some of his results imply some of our conjectures, once the proper translation is made. His work is independent of ours, was posted before we publicized any of our conjectures, and contains results that we did not conjecture.

Tom Bridgeland [2, Theorem 1.5] identifies the cluster scattering diagram with the stability scattering diagram, thus realizing $\tau^{b,c}(i,j)$ as the Euler characteristic of a certain moduli scheme of representations of the associated Jacobi algebra.

Acknowledgments. Thomas Elgin and Nathan Reading were partially supported by the National Science Foundation under award number DMS-2054489. Salvatore Stella was partially supported by INdAM.

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(Thomas Elgin and Nathan Reading) Department of Mathematics, North Carolina State University, Raleigh, NC, USA

(Salvatore Stella) Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, IT