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QUANTITAVE MERGING FOR TIME INHOMOGENEOUS MARKOV CHAINS IN NON-DECREASING ENVIRONMENTS VIA FUNCTIONAL INEQUALITIES

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Abstract

We study time-inhomogeneous Markov chains to obtain quantitative results on their asymptotic behavior. We use Poincaré, Nash, and logarithmic-Sobolev inequalities. We assume that our Markov chain admits a finite invariant measure at each time and that the sequence of these invariant measures is non-decreasing. We deduce quantitative bounds on the merging time of the distributions for the chain started at two arbitrary points and we illustrate these new results with examples.

1 Introduction

1.1 Motivation and Background

In this article, we are interested in studying quantitatively time-inhomogeneous Markov chains. The problem of obtaining accurate estimates for the time to reach equilibrium for a Markov chain within a time-homogeneous context is classical and has been extensively studied in the literature see [17]. On the other hand, adapting standard techniques from the time-homogenous to the time-inhomogenous context turns out to be far from straightforward, and indeed time inhomogeneity may produce unexpected behaviors as shown in [15],[14] and [6].

Let us first observe that, unlike a time-homogeneous and aperiodic Markov chain, a time-inhomogenous Markov chain may not converge in law as time tends to infinity. For time-homogenous Markov chains, studying mixing properties or the convergence to equilibrium means investigating the rate of convergence of the laws towards a reference target measure. In the time-inhomogeneous context, there is generally no time-independent reference measure. However, it still makes sense to ask how long one should wait until the law at time t does not depend much on the initial state. Following [15] and [16], this observation leads us to the definition of "merging times".

Let V be a finite or infinite countable discrete set and $(K_t)_{t\geq 1}$ a family of Markov transition

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operators defined on V. Let μ_0 be a probability measure on V. We denote with $(X_t^{\mu_0})_{t\geq 0}$ the V-valued discrete-time Markov chain driven by $(K_t)_{t\geq 1}$ with initial law μ_0 defined by:

$$\mathcal{L}(X_0^{\mu_0}) = \mu_0 \text{ and } \forall t \geq 0, \forall z \in V, \mathbb{P}(X_{t+1}^{\mu_0} = z | X_t^{\mu_0}) = K_{t+1}(X_t^{\mu_0}, z)\,.$$

Furthermore, we denote

$$K_{s,t} := K_{s+1}...K_t \text{ and } K_{0,t} = K_1...K_t\,.$$

Let $\mu_t^{\mu_0} := \mu_0 K_{0,t}$ be the law of $X_t^{\mu_0}$. When μ_0 is the Dirac mass at point x, we use the shorthand notation μ_t^x instead of $\mu_t^{\mu_0}$.

Merging corresponds to the property of forgetting the initial law of the Markov chain. To quantify this property, we control the distance between the distributions after t steps of two chains driven by the same Markov transition operators and started at two distinct initial conditions. Therefore, we aim to find bounds as precise as possible on the following quantities:

$$d_{TV}(\mu_t^x, \mu_t^y) \text{ or } s(\mu_t^x, \mu_t^y)$$

where d_{TV} and s are respectively the classical total variation distance and the separation distance.

Imitating [13], we define the following quantities:

Definition 1. *Let η be in $(0;1)$. Let x, y in V. Define the η-merging time related to x, y by:*

$$T_{mer}(x, y, \eta) = \inf\{t \geq 1 : \quad d_{TV}(\mu_t^x, \mu_t^y) \leq \eta\}$$

and the relative-sup η-merging time related to x, y is:

$$T_{mer}^{\infty}(x, y, \eta) = \inf\{t \geq 1 : \quad s(\mu_t^x, \mu_t^y) \leq \eta\}\ .$$

Furthermore, one may want to bound the merging uniformly over x, y. Hence, we consider the following quantities. First let the η-merging time be:

$$T_{mer}(\eta) = \max_{(x,y)\in V^2} T_{mer}(x, y, \eta)$$

and the relative-sup η-merging time be:

$$T_{mer}^{\infty}(\eta) = \max_{(x,y)\in V^2} T_{mer}^{\infty}(x, y, \eta)\,.$$

Observe that the merging time $T_{\text{mer}}^{\infty}(\eta)$ coincides with the classical mixing time in the time-homogeneous context up to a constant.

In order to quantify merging, we shall rely on several functional inequalities. We start discussing Poincaré inequalities. Let us recall some results.

First, when π is a finite measure, we denote by $\tilde{\pi}$ the probability measure given by π that is to say:

$$\tilde{\pi} := \frac{\pi}{\pi(V)}.$$

Besides, given a Markov transition operator K with stationary probability $\tilde{\pi}$, we define the following operators:

- K^* is the adjoint operator of K from $\ell^2(\tilde{\pi})$ to $\ell^2(\tilde{\pi})$,
- $Q := K^*K$ from $\ell^2(\tilde{\pi})$ to $\ell^2(\tilde{\pi})$ the multiplicative symmetrisation.

We recall the definition of the variance linked to $\tilde{\pi}$ and the Dirichlet form linked to $Q, \tilde{\pi}$:

$$\mathrm{Var}_{\tilde{\pi}}(f) = \sum_{x\in V} f^2(x)\tilde{\pi}(x) - (\sum_{x\in V} f(x)\tilde{\pi}(x))^2 \quad \text{and}$$
$$\mathcal{E}_{Q,\tilde{\pi}}(f,f) = \sum_{x\in V}\sum_{y\in V}(f(x)-f(y))^2\tilde{\pi}(x)Q(x,y), \qquad \text{for all } f : V \to \mathbb{R}.$$

The Poincaré constant of Q is defined as follows:

Definition 2. *Let $\gamma(Q) \geq 0$ be the optimal constant in the inequality:*

$$\gamma\,\mathrm{Var}_{\tilde{\pi}}(f) \leq \mathcal{E}_{Q,\tilde{\pi}}(f,f), \qquad \textit{for all } f : V \to \mathbb{R}.$$

Note that $1-\gamma(Q)$ is an eigenvalue of Q.
It is then immediate to estimate the merging of a time-homogeneous Markov chain using this constant, see Corollary 2.1.5 in [17]:

Theorem 1. *Let K be an aperiodic and irreducible Markov transition operator with invariant probability $\tilde{\pi}$. Let Q be the multiplicative symmetrisation of K from $\ell^2(\tilde{\pi})$ to $\ell^2(\tilde{\pi})$ and γ be $\gamma(Q)$, then for all x, y elements of V :*

$$d_{TV}\left(\mu_t^x, \mu_t^y\right) \leq \frac{1}{2}\left(\frac{1}{\sqrt{\tilde{\pi}(x)}} + \frac{1}{\sqrt{\tilde{\pi}(y)}}\right)(1-\gamma)^{t/2}.$$

Therefore, for all η in $(0;1)$:

$$T_{mer}(x,y,\eta) \leq \frac{2}{\gamma}\left[\log(\frac{1}{\eta}) + \log(\frac{1}{\sqrt{\tilde{\pi}(x)}} + \frac{1}{\sqrt{\tilde{\pi}(y)}})\right].$$

It is important to note that when the Markov transition operators share the same invariant probability, Theorem 1 still holds, see [18].
A classical example of application of Theorem 1 is as follows.
Consider a Markov transition operator K^N on $\{0, \ldots, N\}$ such that:

- $K^N(x,y) \in [1/4, 3/4]$ if $|x-y| \leq 1$,
- K^N has an invariant probability measure $\tilde{\pi}^N$ satisfying:
  $1/4 \leq (N+1)\tilde{\pi}^N(x) \leq 4, \forall x \in V_N$.

Then, a comparison argument with the symmetric nearest neighbor random walk on $\{0, \ldots, N\}$ such as in section 2.1.2 implies that $\gamma((K^N)^*K^N)$ is of order $\frac{1}{N^2}$.
Therefore, for all η in $(0;1)$, an imediate application of Theorem 1 gives the following bound on the η-merging time :

$$T_{mer}^N(\eta) \leq \kappa N^2[\log(\frac{1}{\eta}) + \log(N)]$$

where κ is a constant independent of N.

Let us now turn to the time-inhomogeneous context. Let, for each t, $\tilde{\pi}_t$ denote an invariant probability measure for K_t and γ_t be the Poincaré constant. Assume that $\gamma > 0$ is a common lower bound for all the γ_t's. One might naively expect a similar bound as in Theorem 1. However, R. Huang provides a counter-example in [6], see the following :

Theorem 2.
Let V_N be $\{0, \ldots, N\}$. There exists $Q_N := (K_t^N)_{t\geq 1}$ a sequence of nearest-neighbor Markov transition operators on V_N satisfying:

- *for $t \geq 1$, $K_t^N(x,y) \in [1/4, 3/4]$ if $|x-y| \leq 1$,*
- *for $t \geq 1$, K_t^N has an invariant probability measure $\tilde{\pi}_t^N$ satisfying:*
  $1/4 \leq (N+1)\tilde{\pi}_t^N(x) \leq 4$, for all x in V_N.

Then, for all $t \geq 1$, the Poincaré constant $\gamma((K_t^N)^* K_t^N)$ is of order $\frac{1}{N^2}$ but however,

$$\liminf_{N\to+\infty} \frac{\log(T_{mer}^N(0, N, \frac{1}{2}))}{N} > 0\,.$$

This counterexample illustrates the difficulty in grasping the merging time. Indeed, in this case, we assume bounds on the spectral gaps, kernels, and invariant probabilities that are uniform both with respect to time and space. Finally, to each transition operator, one can associate a set of conductances with values between 1 and 2. However, despite all this and contrary to intuition, the merging time is not polynomial but exponential in N. In conclusion, the naive extension of Theorem 1 to the time-inhomogeneous context does not work. It is therefore necessary to impose hypotheses that exclude this type of case.

One approach is with c-stablity. In a series of papers, L. Saloff-Coste and J. Zúñiga introduce the notion of c-stability for time-inhomogeneous Markov chains. Under this assumption, one controls the fluctuations of the invariant probabilities. Checking c-stability is a challenging task, as noted by the authors. Indeed, the c-stability property involves the laws that are unknown and need to be estimated. Examples of time-inhomogeneous Markov chains for which it was possible to check c-stability are small perturbations of time-homogeneous Markov chains that furthermore satisfy some symmetries.
Under the assumption of c-stability, a minor modification of Theorem 1 holds. Its proof relies on singular values theory.
Effective tools for studying ℓ^2-merging times and complementary to spectral techniques are advanced functional inequalities: Nash inequalities and logarithmic Sobolev inequalities. L. Sallof-Coste and J. Zuniga successfully applied these functional inequalities under the c-stability assumptions to estimate the merging time of a time-inhomogeneous chain.

In this article, we work under a different assumption: the existence of a non-decreasing finite environment.

Definition 3. *Let $(K_t)_{t\geq 1}$ be a sequence of Markov transition operators and let $(\pi_t)_{t\geq 1}$ be a sequence of positive measure.*
We say that $\{(K_t,\pi_t)\}_{t\geq 1}$ is a non-decreasing environment when the sequence $(\pi_t)_{t\geq 1}$ satisfies the following:

- *for all $t\geq 1, \pi_t K_t = \pi_t$,*
- *for all $t\geq 1, \pi_t(V) < +\infty$,*
- *for all $t\geq 1$, for all x in $V, \pi_{t+1}(x)\geq \pi_t(x)$.*

Note that in Definition 3, the π_t's are measures but not necessarily probability measures. Indeed, if they were all probability measures then they would all be the same.
Moreover, when a non-decreasing finite environment exists, an infinite number of non-decreasing finite environments exists. Indeed, let $(a_t)_{t\geq 1}$ be a non-decreasing sequence of positive real numbers, if $\{(K_t,\pi_t)\}_{t\geq 1}$ is a finite non decreasing environment then, $\{(K_t,a_t\pi_t)\}_{t\geq 1}$ is also a finite non decreasing environment.
In the examples, see section 2, we will leverage the availability of multiple choices for invariant measures and play with these different options to, for instance, facilitate the study of the PoincarÃ© constants or to ensure that total masses are not too large.

Theorem 3. *Let $(K_t)_{t\geq 1}$ be a sequence of irreducible aperiodic Markov transition operators and $(\pi_t)_{t\geq 1}$ a sequence of finite measures. Assume $\{(K_t,\pi_t)\}_{t\geq 1}$ is a finite non-decreasing environment. Let $\gamma_t=\gamma(K_t^*K_t)$ be the Poincaré constant associated to $K_t^*K_t$.*
Then, for all x,y elements of V :

$$d_{TV}(\mu_t^x,\mu_t^y)\leq \frac{1}{2}\sqrt{\frac{\pi_t(V)}{\pi_1(V)}}\left(\frac{1}{\sqrt{\tilde{\pi}_1(x)}}+\frac{1}{\sqrt{\tilde{\pi}_1(y)}}\right)\prod_{s=1}^{t}\sqrt{1-\gamma_s}\,. \tag{1}$$

Therefore, for all η in $(0;1)$:

$$T_{mer}(x,y,\eta)\leq \min\{t\geq 1, \frac{1}{2}\sqrt{\frac{\pi_t(V)}{\pi_1(V)}}\left(\frac{1}{\sqrt{\tilde{\pi}_1(x)}}+\frac{1}{\sqrt{\tilde{\pi}_1(y)}}\right)\prod_{s=1}^{t}\sqrt{1-\gamma_s}\leq \eta\}\,. \tag{2}$$

Note that these estimates are very close to the conclusions of Theorem 1.
Besides, when $(\pi_t)_{t\geq 1}$ and $(K_t)_{t\geq 1}$ are constant, we retrieve the exact statement of Theorem 1.
To ensure the assumption of a finite non-decreasing environment, the challenge lies in computing explicit invariant probability measures and our ability to compare them. Roughly speaking, there is a competition between the ratio of the total masses and the product of eigenvalues to achieve a reasonably small merging time. When the ratio of the total masses is too large, the right-hand side of Equation 1 might not tend to 0, see example 2.1.4. In fact, the existence of a non-decreasing environment does not ensure that merging will occur see 2.1.4. Our Estimate 2 is not always sharp, see example 2.1 .

When V is a finite set and $(K_t)_{t\geq 1}$ is a family of irreducible Markov transition operators, then there always exists a family of finite measures $(\pi_t)_{t\geq 1}$ such that $\{(K_t,\pi_t)\}_{t\geq 1}$ is a finite non-decreasing environment. However, the ratio of the total masses can increase very fast and this deteriorates our estimate.

A class of examples where the hypothesis on the existence of a finite non-decreasing environment is easy to check is Markov chains on electrical networks, see [9] for further details. Then, the sum of the conductances at a given vertex provides an explicit invariant measure. Therefore, a sequence of electrical network with non-decreasing conductances forms a non-decreasing environment.
More precisely, a set of conductances is a function c satisfiying:

- $c : V \times V \mapsto \mathbb{R}_+$,
- for all (x, y) in $V, c(x, y) = c(y, x)$.

To this, we add the following assumption:

$$\sum_{x\in V}\sum_{y\in V} c(x,y) < +\infty\,. \tag{3}$$

Let us consider the Markov transition operator P defined with:

$$\forall (x,y) \in V^2,\ \ P(x,y) = \frac{c(x,y)}{\sum_{z\in V} c(x,z)}\,.$$

Let π be the measure defined by:

$$\forall x \in V,\ \ \pi(x) = \sum_{y\in V} c(x,y)\,. \tag{4}$$

By Assumption (3), π is a finite measure. It is straightforward to see that P is reversible with respect to π. It implies that π is an invariant finite measure for P.

Moreover, when the Markov transition operator P is symmetric from ℓ^2 to ℓ^2 with respect to some measure then, there exists a set of conductances c so that:

$$\forall (x,y) \in V^2,\ \ P(x,y) = \frac{c(x,y)}{\sum_{z\in V} c(x,z)}\,.$$

Note that multiple choices of conductances can be associated with P. More precisely, if c is a set of conductances associated with P, then for all λ element of $\mathbb{R}_+^*$, λc is a set of conductances associated with P.

A typical example of a time-inhomogeneous Markov process to which we can apply Theorem 3 is as follows: we fix a reference set of conductances c and a positive constant M. Consider a family of non-decreasing conductance sets $(c_t)_{t\geq 1}$ such that $c_1 \geq c$ and $c_t \leq Mc$ for each t. Let P_t be the associated Markov transition operator with each c_t for $t \geq 1$. At time $t-1$, we apply the operator $K_t := \frac{1}{2}(I + P_t)$. We will see that this type of example has several advantages, see example 2.1. The first advantage is that the sequence $(\pi_t)_{t\geq 1}$ given by (4) is a non-decreasing environment. The second advantage is if the operator associated with c satisfies a Poincaré condition, then K_t does too with the same constants.
One method for generalizing electrical networks is to consider graphs with cycles, see [10] and [7] . Additionally, note that we may construct examples of sequences of electrical networks where the π_t's are non-decreasing but the conductances themselves are not monotonous see example 2.2.

Now, we present the results obtained using Nash and logarithmic Sobolev inequalities. Let us recall the expression of the Dirichlet form subordinated to $(Q_t, \tilde{\pi}_t)$:

$$\forall f : V \to \mathbb{R},\ \mathcal{E}_{Q_t,\tilde{\pi}_t}(f,f) = \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(x)-f(y))^2 Q_t(x,y)\tilde{\pi}_t(x)\,.$$

We recall that a Nash inequality is defined as follows:

Definition 4. *Let $T \geq 1$. $Q, \tilde{\pi}$ satisfies the $\mathcal{N}(C,D,T)$ Nash inequality for $C, D > 0$ if for $1 \leq t \leq T$, we have:*

$$\forall f : V \to \mathbb{R}, \quad \|f\|_{\ell^2(\tilde{\pi}_t)}^{2+1/D} \leq C(\mathcal{E}_{Q_t,\tilde{\pi}_t}(f,f) + \frac{1}{T}\|f\|_{\ell^2(\tilde{\pi}_t)}^2\Big)\,\|f\|_{\ell^1(\tilde{\pi}_t)}^{1/D}.$$

If we assume that, for all t, $K_t^*K_t, \tilde{\pi}_t$ satisfies the same Nash inequality, then one can control the merging time of a time-inhomogenous Markov chain.

Theorem 4. *Let $(K_t)_{t\geq 1}$ be a sequence of Markov transition operators and $(\pi_t)_{t\geq 1}$ a sequence of finite measures.*
Assume $\{(K_t, \pi_t)\}_{t\geq 1}$ is a finite non-decreasing environment.
Assume that it exists $T, C, D > 0$ such that at each time t, $(Q_t, \tilde{\pi}_t)$ satisfies the $\mathcal{N}(C,D,T)$ Nash inequality.
Moreover, assume $\pi_1(V) \geq 1$. Let $\gamma_t = \gamma(K_t^*K_t)$ be the PoincarÃ© constant associated to $K_t^*K_t$ and let $B = B(D,T) = (1+1/T)(1+[4D])$.
Then, for $t \geq r$, $T \geq r$ and for x, y elements of V:

$$d_{TV}(\mu_t^x, \mu_t^y) \leq \sqrt{\frac{\pi_t(V)\pi_r(V)}{\pi_1(V)}}\left(\frac{4CB}{r+1}\right)^D \prod_{l=r+1}^{t}\sqrt{1-\gamma_l}\,. \tag{5}$$

Therefore, for all η in $(0;1)$:

$$T_{mer}(\eta) \leq \min\{t \geq 0 \text{ such that } \exists\, r \leq t\ ,\ r \leq T \text{ and } \sqrt{\frac{\pi_t(V)\pi_r(V)}{\pi_1(V)}}\left(\frac{4CB}{r+1}\right)^D \prod_{l=r+1}^{t}\sqrt{1-\gamma_l} \leq \eta\}\,.$$

Moreover, if V is finite, then for $t = 2r+u$, with $r \leq T$:

$$\max_{x\in V}\{s(\mu_t^x, \mu_t^{\tilde{\pi}_1})\} \leq 4\pi_t(V)\left(\frac{4CB}{r+1}\right)^{2D} \prod_{l=r+1}^{r+u}\sqrt{1-\gamma_l}\,. \tag{6}$$

Therefore, for all η in $(0;\frac{1}{2})$:

$$T_{mer}^{\infty}(\eta) \leq \min\{u+2r \text{ such that } r \in [[0;T]],\ u \geq 0 \text{ and } 16\pi_{2r+u}(V)\left(\frac{4CB}{r+1}\right)^{2D} \prod_{l=r+1}^{r+u}\sqrt{1-\gamma_l} \leq \eta\}\,.$$

Finally, we state a bound on the merging time in finite non-decreasing environments that satisfy a logarithmic Sobolev inequality at each time. We recall the expression of the entropy:

Definition 5. *Let $\tilde{\pi}$ be a probability measure on V. Let f be a function in $\ell^2(\pi)$ that does not vanish. The entropy of f with respect to $\tilde{\pi}$ is defined with:*

$$\mathcal{L}(f|\tilde{\pi}) := \sum_{x\in V} f^2(x)\log(\frac{f^2(x)}{\tilde{\pi}(f^2)})\tilde{\pi}(x)\,.$$

The standard logarithmic Sobolev constant of a transition operator Q is the following:

Definition 6. *Let $\alpha(Q) > 0$ be the optimal constant in the inequality:*

$$\frac{\mathcal{E}_{Q,\tilde{\pi}}(f,f)}{\mathcal{L}(f|\tilde{\pi})} \geq \alpha, \qquad \textit{for all } f : V \to \mathbb{R},\ f - \textit{non constant.}$$

Thanks to hypercontractivity, we can control the merging times for finite non-decreasing environments with the following:

Theorem 5. *Let $(K_t)_{t\geq 1}$ be a sequence of irreducible aperiodic Markov transitions operators and $(\pi_t)_{t\geq 1}$ a sequence of positive measures. Assume $\{(K_t,\pi_t)\}_{t\geq 1}$ is a finite non-decreasing environment. Let $\gamma_t = \gamma(K_t^*K_t)$ be the Poincaré constant associated to $K_t^*K_t$ and let α_t be the Logarithmic Sobolev constant $\alpha(K_t^*K_t)$.*
For an integer s and z in V, we define the following quantities:

$$q_s = 2\prod_{u=1}^{s}(1+\alpha_u) \textit{ and } r_z = \min\{s\geq 1,\ \log(q_s) \geq \log(\log(\frac{1}{\tilde{\pi}_0(z)}))\}\,.$$

Consider x, y elements of V. Let $r = r(x,y) = \max(r_x, r_y)$.
Then,

$$\forall t \geq r,\ d_{TV}(\mu_t^x,\mu_t^y) \leq e\frac{\sqrt{\pi_t(V)}}{\pi_0(V)^{1/q_r}}\prod_{l=r+1}^{t}\sqrt{1-\gamma_l}\,. \tag{7}$$

Therefore, for all η in $(0;1)$:

$$T_{mer}(x,y,\eta) \leq r + \min\{u\geq 1, e\frac{\sqrt{\pi_t(V)}}{\pi_0(V)^{1/q_r}}\prod_{l=r+1}^{r+u}\sqrt{1-\gamma_l}\leq \eta\}\,. \tag{8}$$

The preceeding result is general. In the case of a finite set and when the logarithmic Sobolev constants are uniformly bounded from below, we can control the relative-sup merging time and state a clearer statement:

Theorem 6. *Let $(K_t)_{t\geq 1}$ be a sequence of irreducible aperiodic Markov transition operators and $(\pi_t)_{t\geq 1}$ a sequence of positive measures. Assume $\{(K_t,\pi_t)\}_{t\geq 1}$ is a finite non-decreasing environment. Assume that V is finite and that $\pi_1(V)\geq 1$. Let $\gamma_t = \gamma(K_t^*K_t)$ be the PoincarÃ© constant associated to $K_t^*K_t$ and let α_t be the Logarithmic Sobolev constant $\alpha(K_t^*K_t)$.*
Assume that α_t is bounded from below by $\alpha > 0$.
Moreover, assume that it exists $\rho > 0$ such that :

$$\forall t\geq 1,\quad \underset{x\in V}{min}\{\tilde{\pi}_t(x)\}\geq \rho\,.$$

Let r be defined by $r = [\frac{\log(\frac{\log(1/\rho)}{2})}{\log(1+\alpha)}] + 1$.
Let u be an integer and set $t = 2r + u$. Then,

$$\max_{x \in V} \{s(\mu_t^x, \mu_t^{\tilde{\pi}_1})\} \leq e^2 \sqrt{\pi_t(V)\pi_{r+u}(V) \prod_{l=r+1}^{r+u} 1 - \gamma_l} \,.$$

Therefore, for all η in $(0; \frac{1}{2})$:

$$T_{mer}^{\infty}(\eta) \leq 2r + \min\{u \geq 0, 4e^2 \sqrt{\pi_{2r+u}(V)\pi_{r+u}(V) \prod_{l=r+1}^{r+u} 1 - \gamma_l} \leq \eta\} \,.$$

In practice, we will use Theorem 6 rather than the Theorem 5, see example 2.5.

Remark 1. *The concept of monotonous invariant measures already appeared in the literature to deal with issues of recurrence versus transience, see [1].*
Besides, in [4], the idea of non-decreasing invariant measure is highlighted. A. Dembo, R. Huang, and T. Zheng establish two-sided Gaussian estimates for the transition kernel of a time-inhomogeneous Markov chain on a non-decreasing sequence of electrical networks satisfying a suitable uniform Poincaré inequality and the volume doubling property.
In our paper, in the proofs, we have succeeded in adapting ideas of [4] such as backward induction in the Poincaré case. However, unlike in [4], some extra care is needed because we have to center our test functions with respect to measures that evolve over time.

1.2 Outline of the paper

This article is divided into three parts. Section 1 presents the motivations, background, and main theorems. Each theorem adapts the results of classical functional inequalities to the non-decreasing finite environment hypothesis. The first theorem, Theorem 3, assumes a Poincaré condition at each time, in addition to the non-decreasing finite environment hypothesis, and states control of the merging time in total variation distance. The second theorem, Theorem 4, assumes a Nash inequality at each instance and states control of the merging time in both total variation distance and infinity distance when the set V is finite. Finally, Theorem 5 assumes a Logarithmic Sobolev inequality at each time and similarly states control of the merging time in total variation distance and infinity distance when the set V is finite with Theorem 6. Theorems 3,4 and 6 are new tools for understanding finite Markov chains in the inhomogeneous time context.

Section 2 deals with examples and investigates five instances within the context of electrical networks. The first example serves an educational purpose, while the subsequent two illustrate the application of Theorem 3. The final two examples are within the classical framework for the application of Nash inequalities and Logarithmic Sobolev inequalities.

The third section delves into the proofs and is divided into four sub-sections. The first one revisits the preliminaries and classical definitions from analysis and probability theory. The second part focuses on proving Theorem 3 and deals with the Poincaré constants. It includes a crucial proposition regarding the duality of operators within the inhomogeneous time context. The third segment contains the proof of Theorem 4, reiterating the definition of a Nash inequality. Finally, in the fourth subsection, we adopt an entropic approach and prove Theorem 5 and 6.

1.3 Aknowledgement

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2 Examples

In this section, we discuss pedagogical examples and other interesting cases. Each example consists of a first part where we introduce the model, a second part where we investigate inequalities such as PoincarÃ©'s ones or Logarithmic Sobolev ones, and a third part where we compare these results with the homogeneous case.

2.1 Conductances on a stick

2.1.1 Description

We first start with a basic example. Let $\mathcal{G}_N := (V_N, E_N)$ be the standart Cayley graph of $\mathbb{Z}/N\mathbb{Z}$:

- the set V_N is equal to $\mathbb{Z}/N\mathbb{Z}$. It is a finite circle.
- the edges E_N are $\{\{x,y\}$, such that $(x,y) \in V_N^2$ and $|x-y| \leq 1\}$.

We assume that a family of conductances $(c_t)_{t\geq 1}$ on E_N is given that satisfies:

- for all e in E_N, for all $t \geq 1, c_t(e) \geq 1$.
- for all e in E_N, the function $t \mapsto c_t(e)$ is non decreasing.

For $t \geq 1$, let P_t be the Markov transition operator on V_N associated to c_t and its expression is given by:

$$\forall (x,y) \in V_N^2, \ P_t(x,y) = \frac{c_t(x,y)}{\sum_{z\in V_N} c_t(x,z)} \, .$$

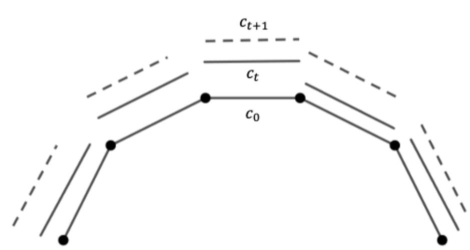

Figure 1: Non-decreasing conductances

We study the Markov transition operators $(K_t)_{t\geq 1}$ given by:

$$\forall t \geq 1, \ K_t = \frac{1}{2}(I + P_t) \, .$$

We now define the family of finite measure $(\pi_t)_{t\geq 1}$ by:

$$\forall t \geq 1, \forall x \in V_N, \ \pi_t(x) = \sum_{z\in V_N} c_t(e) \, .$$

It is easy to check that P_t and K_t are reversible with respect to π_t.

2.1.2 Lower bounds for the spectral gap

In this subsection, we are going to estimate the spectral gap of $Q_t := K_t^2$ under the assumption:

$$\forall t \geq 1, \forall e \in E_N, c_t(e) \leq M < +\infty \,.$$

We define P the following Markov transition operator:

$$\forall (x,y) \in V_N^2 \text{ such that } |x-y| = 1 \,, \text{ then } P(x,y) = \frac{1}{2} \,.$$

Define the set of edges $E_N^+ = \{\{x,y\}, \text{such that} \, |x-y|=1\}$. It is easy to check that P is reversible with respect to $u \equiv 1$ and P is associated to the set of conductances c_0 defined by:

$$\forall e \in E_N^+, c_0(e) = 1 \text{ and } \forall e \notin E_N^+, c_0(e) = 0 \,.$$

Let $\tilde{u}$ be the uniform probability on V_N and denote with γ_u the constant $\gamma(P)$. It is well known see [8] that there exists $\kappa > 0$ such that:

$$\gamma_u = \frac{\kappa}{N} = \operatorname*{Inf}_{f \in \ell^2(u), \ f \text{ non-constant}} \left\{ \frac{\mathcal{E}_{P,\tilde{u}}(f)}{\mathrm{Var}_{\tilde{u}}(f)} \right\} \,.$$

Besides, for each t, denote with γ_t the constant $\gamma(Q_t)$:

$$\gamma_t = \operatorname*{Inf}_{f \in \ell^2(\tilde{\pi}_t), \ f \text{ non-constant}} \left\{ \frac{\mathcal{E}_{Q_t,\tilde{\pi}_t}(f)}{\mathrm{Var}_{\tilde{\pi}_t}(f)} \right\} \,.$$

Furthermore, we define for π a finite measure:

$$\mathrm{Var}_\pi(f) := \pi(V) \mathrm{Var}_{\tilde{\pi}}(f), \quad \forall f : V_N \to \mathbb{R} \,.$$

Then, note that:

$$\gamma_u = \operatorname*{Inf}_{f \in \ell^2(u), \ f \text{ non-constant}} \left\{ \frac{\mathcal{E}_{P,u}(f)}{\mathrm{Var}_u(f)} \right\} \text{ and}$$
$$\forall t \geq 1, \ \gamma_t = \operatorname*{Inf}_{f \in \ell^2(\tilde{\pi}_t), \ f \text{ non-constant}} \left\{ \frac{\mathcal{E}_{Q_t,\pi_t}(f)}{\mathrm{Var}_{\pi_t}(f)} \right\} \,.$$

We now use a comparison method in order to estimate the spectral gap of Q_t. We have for the Dirichlet forms, for all f in $\ell^2(\tilde{\pi}_t)$:

$$\begin{aligned}
\mathcal{E}_{P,u}(f,f) &= \frac{1}{2} \sum_{x,y} (f(x) - f(y))^2 u(x) P(x,y) \\
&= \frac{1}{2} \sum_{x,y,\{x,y\} \in E_N^+} (f(x) - f(y))^2 \\
&\leq \frac{1}{2} \sum_{x \neq y} (f(x) - f(y))^2 c_t(x,y) \text{ since } c_t(e) \geq 1 \\
&\leq \frac{1}{2} \sum_{x \neq y} (f(x) - f(y))^2 \pi_t(x) P_t(x,y) \\
&= \mathcal{E}_{P_t,\pi_t}(f,f) \\
&\leq 2 \mathcal{E}_{Q_t,\pi_t}(f,f) \,.
\end{aligned}$$

The last inequality holds because P_t and K_t are reversible with respect to π_t. More precisely, the following fact is true:

$$\mathcal{E}_{P_t,\pi_t}(f,f) = 2\mathcal{E}_{K_t,\pi_t}(f,f) \leq 2\mathcal{E}_{K_t^2,\pi_t}(f,f)\,.$$

Moreover, remark that $\pi_t \leq Mu$ so comparison of variances gives:

$$\forall f \in \ell^2(\tilde{\pi}_t), \quad \mathrm{Var}_{\pi_t}(f) \leq M\,\mathrm{Var}_u(f)\,.$$

We finally find that:

$$\forall t \geq 1, \quad \gamma_t \geq \frac{1}{2M}\gamma_u\,.$$

2.1.3 Bound on the merging time

Let us summarize:

- $t \mapsto c_t(e)$ is non-decreasing and for each t, $1 \leq c_t(e) \leq M$,
- $t \mapsto \pi_t(x)$ is non-decreasing,
- $\forall t \geq 1, \quad \gamma_t \geq \frac{\kappa}{2MN^2}$.

An application of Theorem 3 yields that it exists a constant A independent of N, M such that:

$$\forall \eta \in (0;1), \quad T_{\mathrm{mer}}(\eta) \leq AMN^2[\log(N) + \log(M) + \log(\frac{1}{\eta})]\,.$$

Let $\hat{K}$ be a Markov transition operator associated with a set of conductances c satisfying $1 \leq c \leq M$. Denote $\hat{\pi}$ the invariant probability of $\hat{K}$ and let $\hat{\mu}_t^x$ be the law of the chain started at x and driven by $\hat{K}^t$. For $\eta > 0$, we recall that the η-mixing time is given by:

$$T_{\mathrm{mix}}(\eta) := \inf\left\{t \geq 1,\ \sup\left\{x \in V_N, d_{TV}(\hat{\mu}_t^x, \hat{\pi}) \leq \eta\right\}\right\}\,.$$

Theorem 1 gives the following bound on the mixing time of the chain driven by $\hat{K}$:

$$\forall \eta \in (0;1), \quad T_{\mathrm{mix}}(\eta) \leq 4\kappa MN^2[\log(N) + \log(\frac{1}{\eta})]\,.$$

Thus, we obtain an estimate of the merging time similar to the mixing time. The only difference is an extra $\log(M)$.
An argument based on a Nash inequality in the time-homogeneous case allows us to refine this bound. We will do so for this example in the time-inhomogenous case, see 2.4.

2.1.4 Unbounded conductances

We keep the same notation as in 2.1.1. We still assume that $c_t(e) \geq 1$ but now we consider the case when:

$$\sup_{t\geq 1}\sup_{x\in V_N}\left\{\pi_t(x)\right\} = +\infty\,. \tag{9}$$

For $t \geq 1$, let $\phi(t)$ be equal to $\sup_{x \in V_N} \{\pi_t(x)\}$.
Imitating the calculations of 2.1.2, if t is a solution of the following inequality:

$$t \geq 4\kappa\phi(t)N^2[\log(\phi(t)) + \log(N) + \log(\frac{1}{\eta}) + \log(2)]\,, \tag{10}$$

then, we have

$$t \geq T_{\text{mer}}(\eta)\,.$$

We distinguish several regimes.
First, if $\phi(t) \geq t$ then equation (10) has no solution and in this scenario, merging may not occur.
Indeed, let us consider the case $N = 2$ and the three edges : $\{\{0; 1\}; \{1; 1\}; \{0; 0\}\}$.
Fix $\epsilon > 0$ and consider the following family of Markov transition operators $(P_t)_{t\geq 1}$:

$$\forall t \geq 1,\ P_t(0,0) = P_t(1,1) = \frac{t^{1+\epsilon}}{t^{1+\epsilon}+1} \text{ and } P_t(1,0) = P_t(0,1) = \frac{1}{t^{1+\epsilon}+1}\,.$$

Here, one can consider the set of conductances $(c_t)_{t\geq 1}$ defined with:

$$\forall t \geq 1,\ c_t(0,0) = c_t(1,1) = t^{1+\epsilon} \text{ and } c_t(1,0) = c_t(0,1) = 1\,.$$

It is imediate that $(c_t)_{t\geq 1}$ is associated to $(P_t)_{t\geq 1}$, then, we define the invariant measure π_t's by:

$$\pi_t(1) = \pi_t(0) = t^{1+\epsilon} + 1\,.$$

With this choice, $\{(P_t, \pi_t)\}_{t\geq 1}$ is a finite non decreasing environment. Note that for all t, $\tilde{\pi}_t$ does not depend of t, indeed :

$$\forall t \geq 1, \tilde{\pi}_t(0) = \tilde{\pi}_t(1) = \frac{1}{2}\,.$$

For x in $\{0; 1\}$, denote $(Z_t^x)_{t\geq 1}$ the Markov process $\{0; 1\}$-valued driven by $(P_t)_{t\geq 1}$ with:

$$Z_0^x = x \text{ and } \forall t \geq 0, \forall z \in \{0; 1\}, \mathbb{P}(Z_{t+1}^x = z | Z_t^x) = K_{t+1}(Z_t, z)\,.$$

Remark that for all x in $\{0; 1\}$, for all $t \geq 1$:

$$\log(\mathbb{P}(\forall s \in [[0; t]], Z_s^x = x | Z_0^x = x)) = \sum_{s=1}^{t} \log(1 - \frac{1}{t^{1+\epsilon}+1})\,.$$

By comparison with a convergent Riemann series, we obtain the following result:

$$\forall x \in \{0; 1\},\ \mathbb{P}(\forall s \geq 0, Z_s^x = x | Z_0^x = x) > 0\,.$$

There is no merging.

Second, consider $\alpha \in (0; 1)$ and $\phi(t) = t^\alpha$. Then, equation (10) becomes:

$$t^{(1-\alpha)} \geq 4\kappa N^2[\alpha \log(t) + \log(N) + \log(\frac{1}{\eta}) + \log(2)]\,.$$

Observe that $t = (4\kappa N^2[(\alpha+1)\log(N) + 2\log(\frac{1}{\eta}) + \log(2)])^{1/(1-\alpha)}$ solves the equation above.

Then, it exists a constant $A := A(\alpha)$ independent of N:

$$T_{\mathrm{mer}}(\eta) \leq A(N^2[(\alpha+1)\log(N) + \log(\frac{1}{\eta})])^{1/(1-\alpha)}\,.$$

We obtain a polynomial bound on the merging time.

Question: Can one build an example on V_N with non-decreasing conductances where $\phi(t) = t^\alpha$ and prove that the bound $T_{\mathrm{mer}}(\eta) = B(\alpha)N^{2/(1-\alpha)}(1+\log(\frac{1}{\eta}))$ is sharp?

2.2 From a simple random walk to another

Consider a finite set V and two sets of edges E_1 and E_2. Let us assume that the graphs $\mathcal{G}_1 := (V, E_1)$ and $\mathcal{G}_2 := (V, E_2)$ are connected. Let $P(1)$ and $P(2)$ be the Markov transition operators associated with the simple random walk on $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.
Our objective is to study the merging time of a time-inhomogeneous Markov chain which interpolates between these two simple random walks.

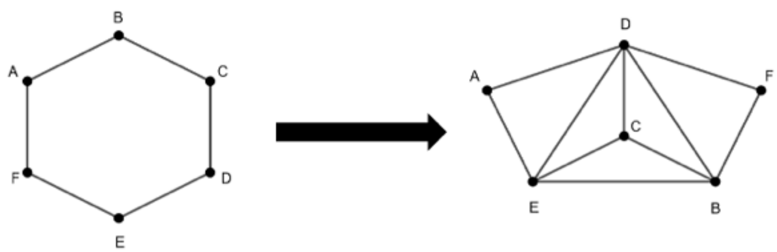

Figure 2: A 6 vertices example

To these Markov transition operators, two sets of conductances $c(1)$ and $c(2)$ can be respectively associated by:

$$\forall e \in E_1 \cap E_2, c(1)(e) = c(2)(e) = 1, \quad \forall e \in E_1 \cap E_2^c, c(1)(e) = 1,\ c(2)(e) = 0,$$
$$\text{and } \forall e \in E_1^c \cap E_2, c(1)(e) = 0,\ c(2)(e) = 0\,.$$

Define $\nu(1)$ and $\nu(2)$ two finite measures on V with:

$$\forall x \in V,\ \nu(1)(x) = \sum_{y \in V} c(1)(x,y) \text{ and } \nu(2)(x) = \sum_{y \in V} c(2)(x,y)\,.$$

It is straightforward that $P(1)$ is reversible with respect to $\nu(1)$ and $P(2)$ is reversible with respect to $\nu(2)$.
Assume that $P(1)$ and $P(2)$ satisfy a PoincarÃ© condition and consider the two following quantities:

$$\gamma(1) := \operatorname*{Inf}_{f \in \ell^2(\nu(1)),\ f \text{ non constant}} \left\{ \frac{\mathcal{E}_{P(1),\nu(1)}(f)}{\mathrm{Var}_{\nu(1)}(f)} \right\} \text{ and}$$
$$\gamma(2) := \operatorname*{Inf}_{f \in \ell^2(\nu(2)),\ f \text{ non constant}} \left\{ \frac{\mathcal{E}_{P(2),\nu(2)}(f)}{\mathrm{Var}_{\nu(2)}(f)} \right\}.$$

In the sequel, we assume the following:

$$\gamma(1) > 0 \text{ and } \gamma(2) > 0\,, \tag{11}$$

and we consider the family of conductances $(c_t)_{t\geq 1}$ defined with for all $t \geq 1$:

$$c_t(e) = \frac{1}{t} c(1)(e) + (1 - \frac{1}{t}) c(2)(e)\,.$$

Let $(K_t)_{t\geq 1}$ be the family of Markov transition operators associated with $(c_t)_{t\geq 1}$ and $\frac{1}{2}$-lazy. We are interested in the time-inhomogeneous Markov chain driven by $(K_t)_{t\geq 1}$.

First, note that if $\mathcal{G}_1$ and $\mathcal{G}_2$ have constant degrees d_1 and d_2 respectively. Then, it is easy to check that $\tilde{\nu}(1) = \tilde{\nu}(2) \equiv \frac{1}{\mathrm{card}(V)}$. In this case, for all $t \geq 1$:

$$K_t = \frac{1}{2}I + \frac{1}{2}[\frac{1}{t}P(1) + (1 - \frac{1}{t})P(2)]\,.$$

Besides for each t, the K_t's share the same invariant probability: $\frac{1}{\mathrm{card}(V)}$. Therefore, the Markov chain driven by $(K_t)_{t\geq 1}$ is much simpler to study. More precisely, mixing and merging for this chain coincides and one can apply Theorem 1. In order to apply Theorem 1, one needs to estimate the Poincaré constant of $K_t^* K_t$. A way to obtain an estimate on the spectral gaps is to consider for each t, the measure π_t defined with $\frac{1}{t}\nu(1) + (1 - \frac{1}{t})\nu(2)$ and

$$\gamma_t := \operatorname*{Inf}_{f \in \ell^2(\pi_t),\ f \text{ non-constant}} \left\{ \frac{\mathcal{E}_{K_t,\pi_t}(f)}{\mathrm{Var}_{\pi_t}(f)} \right\}.$$

A straightforward comparison argument such as in 2.1.2 gives :

$$\gamma_t \geq \gamma_* := \frac{1}{2}\min(\gamma(1); \gamma(2))\,.$$

Then, Theorem 1 gives, for all x in V:

$$\forall t \geq 1, d_{TV}(\mu_t^x, \frac{1}{\mathrm{card}(V)}) \leq \frac{1}{\sqrt{\mathrm{card}(V)}}(1 - \gamma_*)^{t/2}\,.$$

Here, we performed a convex interpolation of the two graphs and the interpolation does not depend on the point x or of the edge e.

From now on, we do not assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ have constant degrees but we still perform a convex interpolation.
Note that

- for all e in $E_2|E_1$, $c_1(e) = 0$ and $t \mapsto c_t(e)$ is non-decreasing.
- for all e in $E_1|E_2$, $c_1(e) = 1$, $\ t \mapsto c_t(e)$ is non-increasing and $\lim_{t\to+\infty} c_t(e) = 0$.

One can write π_t as:

$$\forall t \geq 1, \forall x \in V,\ \ \pi_t(x) = \sum_{z \in V} c_t(x,z) = \nu(2)(x) + \frac{1}{t}(\nu(1)(x) - \nu(2)(x))\,. \tag{12}$$

For all x in V, we assume that :

$$\nu(2)(x) \geq \nu(1)(x)\,. \tag{13}$$

Then, $t \mapsto \pi_t(x)$ is non-decreasing and once again a comparison argument gives:

$$\forall t \geq 1,\ \ \gamma_t \geq \gamma_*\,.$$

An application of Theorem 3 yields the following bound on the merging time:

$$\forall \eta \in (0;1),\quad T_{\mathrm{mer}}(\eta) \leq \frac{2\frac{\nu(2)(V)}{\nu(1)(V)}}{\gamma_*}[\log(N) + \log(\frac{\nu(2)(V)}{\nu(1)(V)}) + 2\log(\frac{1}{\eta})]\,.$$

This example is not a non-decreasing conductance one.

2.3 A birth and death process on $\mathbb{N}$ with extra connections to 0

2.3.1 Description

In this example, we consider the graph $\mathcal{G} = (V, E)$ where:

- the set V is $\mathbb{N}$,
- and the set of edges E is $\{\{x, y\}$, such that $|x - y| \leq 1\} \cup \cup_{n\geq 1}\{0, n\}$.

Let P be a Markov transition operator on V which satisfies:

- for all (x, y) in V if $|x - y| > 1$ then $P(x, y) = 0$,
- the chain driven by P is positive recurrent and irreducible on V,
- u, the unique invariant probability of P, satisfies $u(0) = \max\limits_{x\in V}\{u(x)\}$,
- $\gamma(P) > 0$.

It is well known that one can consider a set of conductances c_0 such that:

$$\forall (x, y) \in V^2, P(x, y) = \frac{c_0(x, y)}{\sum_{z\in V} c_0(x, z)} \text{ and } \forall x \in V, u(x) = \sum_{z\in V} c_0(x, z)\,.$$

In the sequel, we are given:

- a sequence of positive integers $(x_t)_{t\geq 1}$,
- and a sequence of positive reals $(w_t)_{t\geq 1}$.

Roughly speaking, at each time, we add an edge from 0 to x_t with a weight proportional to w_t. We now build, by induction, the family of Markov transition operators $(P_t)_{t\geq 0}$ that formalizes the previously described process. More precisely, we build $(P_t)_{t\geq 0}$ a set of conductances associated to $(P_t)_{t\geq 0}$.

We initialize with $P_0 = P$ and c_0. Given P_t and c_t, we build c_{t+1} with:

$$\begin{aligned} c_{t+1}(x_{t+1}) = c_t(x_t) + u(x_{t+1})w_{t+1}\,,\ c_{t+1}(0) = c_t(0) + u(x_{t+1})w_{t+1} \\ \text{and if } y \neq x_{t+1}, \quad c_{t+1}(y) = c_t(y)\ . \end{aligned}$$

Then, we find:

$$\begin{aligned} P_{t+1}(0, x_{t+1}) &= \frac{c_t(0, x_{t+1}) + c_0(x_{t+1})w_{t+1}}{c_{t+1}(0)} \quad, \quad P_{t+1}(0, y) = \frac{c_t(0, y)}{c_{t+1}(0)}, \\ P_{t+1}(x_{t+1}, 0) &= \frac{c_t(0, x_{t+1}) + c_0(x_{t+1})w_{t+1}}{c_{t+1}(x_{t+1})} \text{ and } P_{t+1}(x_{t+1}, y) = \frac{c_t(x_{t+1}, y)}{c_{t+1}(x_{t+1})}\,. \end{aligned}$$

And, for all y, z elements of V with $z \neq 0$, $y \neq 0$, $z \neq x_{t+1}$ and $y \neq x_{t+1}$:

$$P_{t+1}(y, 0) = \frac{c_t(y, 0)}{c_t(y)} \text{ and } P_{t+1}(z, y) = \frac{c_t(z, y)}{c_t(z)}\,.$$

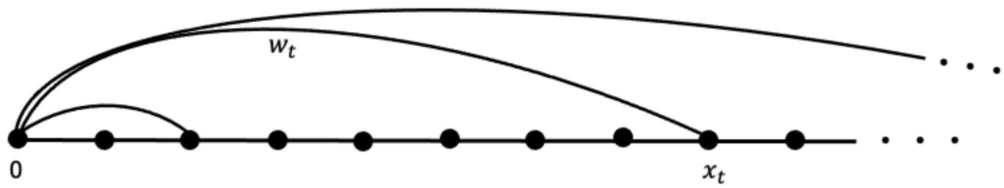

Figure 3: Extra connection with 0

Furthermore, we consider $(\pi_t)_{t\geq 0}$ defined with:

$$\forall t \geq 0, \forall x \in V,\ \ \pi_t(x) = \sum_{z\in V} c_t(x,z)\,.$$

Note that P_t is reversible with respect to π_t and $t \mapsto \pi_t$ is non-decreasing.

2.3.2 Bounded case

In the first instance, we assume that:

$$M := \sum_{t\geq 1} w_t < +\infty\,.$$

Let M' be equal to $\frac{M+1}{u(0)}$, one can remark that by construction:

$$\forall t \geq 0,\ \ u \leq \pi_t \leq M'u\,.$$

Define, for each t:

$$\gamma_t = \underset{f\in\ell^2(\tilde{\pi}_t),\ f \text{ non-constant}}{\mathrm{Inf}} \left\{ \frac{\mathcal{E}_{K_t^2,\tilde{\pi}_t}(f)}{\mathrm{Var}_{\tilde{\pi}_t}(f)} \right\}\,.$$

Then, a comparison argument such as in 2.1.2, gives that K_t^2 satisfies a Poincaré condition with:

$$\forall t \geq 0,\ \ \gamma_t \geq \frac{1}{2M'}\gamma(P)\,.$$

Moreover, it is straightforward that:

- for all $t \geq 0$, $\pi_t(V) \leq M'$,
- and for all $t \geq 0$, for all x in V, $\tilde{\pi}_t(x) \geq \frac{1}{M'}u(x)$.

Let μ_t^z be the law of the chain started at z and driven by $(K_t)_{t\geq 1}$.
An application of Theorem 3 yields for all x, y in V:

$$d_{TV}(\mu_t^x, \mu_t^y) \leq \frac{1}{2}\sqrt{M'}\left(\frac{1}{\sqrt{u(x)}} + \frac{1}{\sqrt{u(y)}}\right)\left(1 - \frac{1}{2M'}\gamma(P)\right)^{t/2}\,.$$

Finally, for all η in $(0;1)$:

$$T_{\mathrm{mer}}(x,y,\eta) \leq \frac{2M'}{\gamma(P)}\left[\log(M') + \log\left(\frac{1}{u(x)}\right) + \log\left(\frac{1}{u(y)}\right) + 2\log\left(\frac{1}{\eta}\right)\right]\,.$$

2.3.3 Unbounded case

In this section, we assume that:

$$\sum_{t\geq 1} w_t = +\infty\,. \tag{14}$$

Let M'_t be equal to $\frac{1+\sum_{r=1}^{t} w_r}{u(0)}$. The same calculations as in 2.3.2 gives:

$$\forall r \in [[1;t]],\ \ \gamma_r \geq \frac{1}{2M'_t}\gamma(P) \text{ and } \pi_r(V) \leq M'_t\,.$$

Fix x and y two elements of V and η in $(0;1)$.
If t is a solution of the following inequality:

$$M'_t \exp(\frac{-t\gamma(P)}{2M'_t}) \leq 4[\tilde{u}(x)+\tilde{u}(y)]\eta^2\,. \tag{15}$$

Then, Theorem 3 gives $T_{\text{mer}}(x,y,\eta) \leq t$.
If

$$\liminf_{t\to+\infty} \frac{M'_t \log(M'_t)}{t} = 0\,, \tag{16}$$

then, equation (15) has a solution for any choice of x, y, and η.

2.3.4 An always merging condition

Let us define the first Dirichlet eigenvalue of P:

$$\hat{\gamma} = \inf\left\{\frac{\mathcal{E}_{P,u}\,(f,f)}{\|f\|^2_{\ell^2(u)}} : f(0)=0\,,\ f \text{ non-constant and } f\in\ell^2(u)\right\}.$$

Assume $\hat{\gamma} > 0$. Below, see Inequation (17). We provide an estimate of T_{mer} that holds for any choice of $(x_t)_{t\geq 1}$ and $(w_t)_{t\geq 1}$. This gives an estimate of the merging time even if Condition (14) is satisfied. This set of examples includes situations when the Markov transition operators $(K_t)_{t\geq 1}$ do not have a limit. However, merging still occurs.

First, use $\|f-f(0)\|^2_{\ell^2(u)} \geq \text{Var}_u(f)$ to find:

$$\begin{aligned}\text{Var}_u(f) &= \text{Var}_u(f-f(0)) \leq \|f-f(0)\|^2_{\ell^2(u)}\\ &\leq \frac{1}{\hat{\gamma}}\mathcal{E}_{P,u}(f-f(0),f-f(0)) = \frac{1}{\hat{\gamma}}\mathcal{E}_{P,u}(f,f)\end{aligned}$$

and

$$\gamma(P) \geq \hat{\gamma}\,.$$

For each t, consider the first Dirichlet eigenvalue of K_t^2:

$$\hat{\gamma}_t = \inf\left\{\frac{\mathcal{E}_{K_t^2,\pi_t}\,(f,f)}{\|f\|^2_{\ell^2(\pi_t)}} : f(0)=0\,,\ f \text{ non-constant and } f\in\ell^2(\pi_t)\right\}.$$

Note that:

$$
\begin{aligned}
\|f\|^2_{\ell^2(\pi_t)} &= \|f\|^2_{\ell^2(u)} + \sum_{x \in A_t} f^2(x) w_t(x,0) \\
&\le \frac{1}{\hat{\gamma}} \mathcal{E}_{P,u}(f,f) + \sum_{x \in A_t} (f(x) - f(0))^2 w_t(x,0) \\
&\le 2(\frac{1}{\hat{\gamma}} + 1) \mathcal{E}_{K_t^2, \pi_t}(f,f) \,.
\end{aligned}
$$

Let $\gamma := \frac{1}{2(\frac{1}{\hat{\gamma}}+1)}$. Then, one finds a uniform lower bound in time on the γ_t's:

$$\gamma_t \ge \gamma \,.$$

Finally, Theorem 3 yields that merging occurs and one can bound it. Indeed, for all x, y in V and for all η in $(0;1)$:

$$T_{\mathrm{mer}}(x,y,\eta) \le \frac{2}{\gamma} [\log(\frac{1}{\eta}) + \log\left(\frac{1}{\sqrt{u(x)}} + \frac{1}{\sqrt{u(y)}}\right)] \,. \tag{17}$$

2.4 Nash Inequalities for a box in $\mathbb{Z}^d$

In this section, we deal with the generalization of example 2.1: a time-inhomogeneous Markov chain in $(\mathbb{Z}/N\mathbb{Z})^d$. Once again, we consider time-dependent electric networks with non-decreasing conductances. More precisely, let $\mathcal{G}_N^d := (V_N^d, E_N^d)$ where:

- the set V_N^d is $(\mathbb{Z}/N\mathbb{Z})^d$.
- the edges E_N^d are $\{\{x, y\}$, such that $(x, y) \in (V_N^d)^2$ and $|x - y| \leq 1\}$.

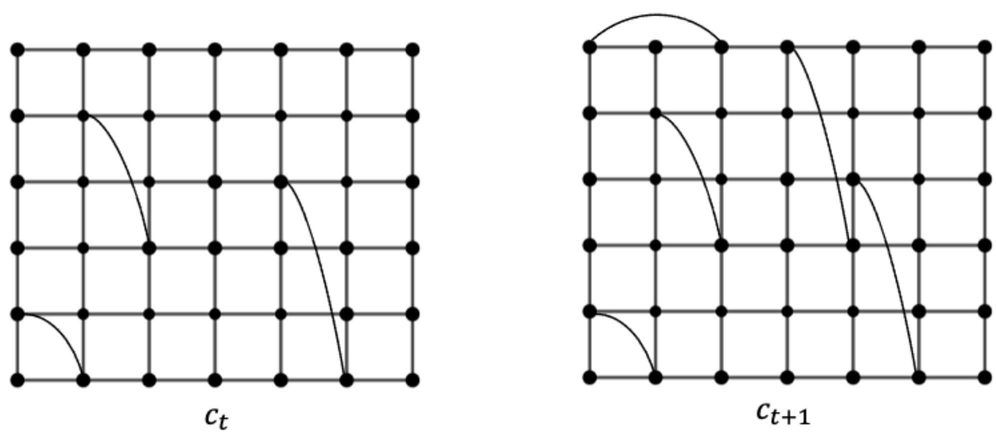

Figure 4: An example in $\mathbb{Z}^2$

We assume that a family of conductances $(c_t)_{t\geq 1}$ on E_N^d is given that satisfies:

- for all e in E_N^d, for all $t \geq 1$, $c_t(e) \geq 1$.
- for all e in $E_N^d, t \mapsto c_t(e)$ is non decreasing.

For $t \geq 1$ a positive integer, we consider P_t the Markov transition operator on V_N^d associated with c_t. The expression of P_t is given by:

$$\forall (x, y) \in (V_N^d)^2,\ P_t(x, y) = \frac{c_t(x, y)}{\sum_{z \in V_N} c_t(x, z)}\,.$$

We consider the family of lazy Markov transition operators $(K_t)_{t\geq 1}$ given by:

$$\forall t \geq 1,\ K_t = \frac{1}{2}(I + P_t)\,.$$

We now define the family of finite measures $(\pi_t)_{t\geq 1}$ by:

$$\forall t \geq 1, \forall x \in V_N^d,\ \pi_t(x) = \sum_{z \in V_N} c_t(x, z)\,.$$

It is easy to check that, for each t, P_t and K_t are reversible with respect to π_t.

2.4.1 Nash inequality at each time

Now, we fix N and d and we are going to prove that, at each time t, K_t satisfies a Nash inequality under the following assumption:

$$1 \le c_t \le M < +\infty\,.$$

Let P be the Markov transition operator on V_N^d associated to the simple random walk on $\mathcal{G}_N$, P is defined by:

$$\forall (x,y) \in (V_N^d)^2, P(x,y) = \frac{1}{2d}\mathbf{1}_{|x-y|=1}\,.$$

It is easy to check that P is reversible with respect to the uniform probability measure $u \equiv \frac{1}{N^d}$.
In [5], P. Diaconis and L. Saloff-Coste show that (P,u) satisfies the $\mathcal{N}(\kappa T, d/4, T)$ Nash inequality :

$$\forall f : V_N^d \to \mathbb{R},\ \|f\|_{\ell^2(u)}^{2+4/d} \le \kappa T \left(\mathcal{E}_{P,u}(f,f) + \frac{1}{T}\|f\|^2_{\ell^2(u)}\right)\|f\|^{4/d}_{\ell^1(u)} \tag{18}$$

where κ is independent of (N,d) and $T = \frac{1}{dN^2}$.

We divide the conductances $(c_t)_{t\ge 1}$ by a factor $(N+1)^d$ and we keep the same notation. One finds that:

$$\forall t \ge 1,\ \ 1 \le \pi_t(V_N^d) \le (2d+1)M$$

Note that we can compare $\tilde{\pi}_t$ and u as follow:

$$\forall t \ge 1, \quad \frac{1}{M'}u \le \tilde{\pi}_t \le M'u\,.$$

where M' is $(2d+1)M$.
Then, one finds for all $f : V_N^d \to \mathbb{R}$ and for all $t \ge 1$:

- $\|f\|^{4/d}_{\ell^1(u)} \le M'^{4/d}\|f\|^{4/d}_{\ell^1(\tilde{\pi}_t)}$,
- $\frac{1}{M'}\|f\|^2_{\ell^2(\tilde{\pi}_t)} \le \|f\|^2_{\ell^2(u)} \le M'\|f\|^2_{\ell^2(\tilde{\pi}_t)}$,
- $\mathcal{E}_{P,u}(f,f) \le 2M'\mathcal{E}_{Q_t,\tilde{\pi}_t}(f,f)$.

We finally use a comparison method to show that $(K_t^2, \tilde{\pi}_t)$ satisfies the $\mathcal{N}(\kappa' T, d/4, T)$. We get the following Nash inequality at time t:

$$\forall f : V_N^d \to \mathbb{R}, \forall t \ge 1,\ \|f\|^{2+4/d}_{\ell^2(\tilde{\pi}_t)} \le \kappa' T\left(\mathcal{E}_{K_t^2,\tilde{\pi}_t}(f,f) + \frac{1}{T}\|f\|^2_{\ell^2(\tilde{\pi}_t)}\right)\|f\|^{4/d}_{\ell^1(\tilde{\pi}_t)}$$

where $\kappa' = \kappa 2M'^{(1+4/d)(1+2/d)}$.

On the other hand, we can obtain a lower bound on the spectral gap of K_t^2. Indeed, let γ_t be the Poincaré constant $\gamma(K_t^* K_t)$. A comparison method gives:

$$\forall t \geq 1,\ \ \gamma_t \geq \frac{1}{M'} \gamma_u$$

where $\gamma_u = \gamma(P^2)$.
It is easy to check that:

$$\gamma_u = \frac{a}{dN^2}\,.$$

Then, we conclude with:

$$\forall t \geq 1,\ \ \gamma_t \geq \frac{a}{M'dN^2}\,.$$

2.4.2 Bound of merging time

In order to use Theorem 4, we summarize:

- for all $t \geq 1,\ \ 1 \leq \pi_t(V) \leq M'$,
- $t \mapsto \pi_t$ is non-decreasing.

Denote by μ_t^z the law of the chain started at z and driven by $(K_t)_{t\geq 1}$.
Therefore, an application of Theorem 4 yields :

$$\forall t \geq 2T,\ \ \max_{(x,y)\in V^2} \{s(\mu_t^x, \mu_t^y)\} \leq bM'(1 - \frac{a}{M'dN^2})^{t-2T}\,.$$

Finally, we find that it exists B, C two constants independent of N, d and M such that for all η in $(0; \frac{1}{2})$:

$$T_{\text{mer}}^{\infty}(\eta) \leq BdN^2 + CdM'N^2(\log(\frac{1}{\eta}) + \log(M'))\,.$$

In this example, the Markov transition operator K_t convergences to a Markov transition operator with conductances.

We may compare the previous result with the time-homogeneous situation. Let $\hat{K}$ be a Markov transition operator associated with a set of conductances c satisfying $1 \leq c \leq M$. Denote by $\hat{\pi}$, the invariant probability of $\hat{K}$, and $\hat{\mu}_t^x$, the law of the chain started at x and driven by $\hat{K}^t$.
A Nash inequality argument gives the following bound on the mixing time of the chain driven by $\hat{K}$:

$$\forall \eta \in (0; \frac{1}{2}),\ \ T_{\text{mix}}^{\infty}(\eta) \leq \kappa M N^2 [C' + \log(\frac{1}{\eta})]\,.$$

We conclude that the two estimates on $T_{\text{mix}}^{\infty}(\eta)$ and $T_{\text{mer}}^{\infty}(\eta)$ we just obtained are similar. The only differences are an additional mass term and in the constants.

2.5 Hypercube with conductances

2.5.1 Description

In this example, we consider the N-dimensional hypercube. More precisely, let $\mathcal{G}^N := (V^N, E^N)$ where

- V^N is equal to $\{0,1\}^N$.
- x, y elements of V^N are neighbors or $x \sim y$ if $\sum_{i=1}^N |x_i - y_i| = 1$, where x_i is the i th coordinate.

We assume that a family of conductances $(c_t)_{t\geq 1}$ on E^N is given and satisfies:

- for all e in E^N, for all $t \geq 1$, $1 \leq c_t(e) \leq M$,
- for all e in E^N, $t \mapsto c_t(e)$ is non-decreasing.

For each $t \geq 1$, we consider P_t the Markov transition operator on V^N associated to c_t. We recall that its expression is given by:

$$\forall (x,y) \in (V^N)^2,\ P_t(x,y) = \frac{c_t(x,y)}{\sum_{z\in V^N} c_t(x,z)} .$$

We consider $(K_t)_{t\geq 1}$ the family of Markov transition operators given by:

$$\forall t \geq 1,\ K_t = \frac{1}{2}(I + P_t) .$$

We now define the family of finite measures $(\pi_t)_{t\geq 1}$ by:

$$\forall t \geq 1, \forall x \in V^N,\ \pi_t(x) = \sum_{z\in V} c_t(x,z) .$$

It is easy to check that, for each $t \geq 1$, P_t and K_t are reversible with respect to π_t. By assumption on $(c_t)_{t\geq 1}$, π_t is non-decreasing.

2.5.2 Control of the logarithmic Sobolev constant

In this subsection, we give a lower bound on the logarithmic Sobolev constant through the simple random walk on $\mathcal{G}^N$.
Recall that the simple random walk on V^N is driven by the Markov transition operator

$$P(x,y) = \begin{cases} \frac{1}{N}, & \text{if } x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Let u be the finite measure on V^N defined by $u = 1$. It is easy to check that u is stationary for P and that $\tilde{u}$ is constant equal to $\frac{1}{N}$. It is straightforward that the set of conductances $c_0 \equiv 1$ is associated to P.
Let $K = \frac{1}{2}(I + P)$ and define α by:

$$\alpha = \operatorname*{Inf}_{f\in \ell^2(u),\ f \text{ non-constant}} \left\{ \frac{\mathcal{E}_{K,\tilde{u}}(f)}{\mathcal{L}(f|\tilde{u})} \right\} .$$

It is well-known that, see [17] :

$$\exists \kappa > 0,\ \ \alpha = \frac{\kappa}{N}\,.$$

Let :

$$\alpha_t := \operatorname*{Inf}_{f\in\ell^2(u),\ f\ \text{non-constant}} \left\{ \frac{\mathcal{E}_{K_t^2,\tilde{\pi}_t}(f,f)}{\mathcal{L}(f|\tilde{\pi}_t)} \right\}\,.$$

Once again, we use a comparison method to bound α_t from below.
Define $\mathcal{S}_t^N$ the set of functions defined on V such that $\mathcal{L}(f|\pi_t)$ exists. Let $\mathcal{L}(f|\pi_t)$ be equal to $\pi_t(V)\mathcal{L}(f|\tilde{\pi}_t)$.
Now, note the following facts, for each $t \geq 1$:

- for all f in $\mathcal{S}_t^N$, $\mathcal{L}(f|\pi_t) \leq M\mathcal{L}(f|u)$,
- for all f in $\mathcal{S}_t^N$, $\mathcal{E}_{K_t,\pi_t}(f,f) \geq \mathcal{E}_{K,u}(f,f)$,
- for all f in $\mathcal{S}_t^N$, $\mathcal{E}_{K_t^2,\tilde{\pi}_t}(f,f) \geq \frac{1}{2}\mathcal{E}_{K_t,\tilde{\pi}_t}(f,f)$.

The standart logarithmic Sobolev inequality for K implies:

$$\forall t \geq 1, \forall f \in \mathcal{S}_t^N,\ \ \frac{\mathcal{E}_{K_t^2,\pi_t}(f,f)}{\mathcal{L}(f|\pi_t)} = \frac{\mathcal{E}_{K_t^2,\tilde{\pi}_t}(f,f)}{\mathcal{L}(f|\tilde{\pi}_t)} \geq \frac{1}{2M}\alpha\,.$$

We conclude with the following:

$$\forall t \geq 1,\ \ \alpha_t \geq \frac{1}{2M}\alpha\,.$$

2.5.3 Bound on the merging time

In order to apply Theorem 5, we divide π_t by $u(V^N)$ and still note it π_t. Let us summarize:

- for each $t \geq 1$, $1 \leq \pi_t(V) \leq M$,
- for each $t \geq 1$, $\min\limits_{x\in V^N} \tilde{\pi}_t(x) \geq \frac{1}{M2^N}$,
- and $t \mapsto \pi_t$ is non-decreasing.

Let s be an integer, we define :

$$q_s = 2(1+\frac{1}{2M}\alpha)^s$$

Note that:

$$q_s \leq 2\prod_{u=1}^{s}(1+\alpha_j)\,.$$

Then, by choosing $s = 1 + [\frac{2MN}{\kappa}\log(\log(M2^N))]$, we obtain:

$$\log(q_s) \geq \log(\log(M2^N))\,.$$

Denote γ_t the Poincaré constants $\gamma(K_t^*K_t)$ and recall that $\gamma_t \geq 2\alpha_t$, then, one finds :

$$\forall t \geq 1,\ \ \gamma_t \geq \frac{\kappa}{MN}\,.$$

Denote with μ_t^z the law of the chain started at z and driven by $(K_t)_{t\geq 1}$.
Therefore, an application of Theorem 6 yields :

$$\forall t \geq r, \quad \max_{(x,y)\in V^2} \{s(\mu_t^x, \mu_t^y)\} \leq 4e^2 M(1 - \frac{\kappa}{MN})^{(t-2r)} \,.$$

Finally, there exists a constant A independent of N such that for all η in $(0; \frac{1}{2})$:

$$T_{\text{mer}}^{\infty}(\eta) \leq AMN[\log(\log(M)) + \log(\log(N)) + \log(\frac{1}{\eta}) + \log(M)] \,.$$

In this example too, the Markov transition operators K_t have a limit and we find a merging time similar to the mixing time in the time-homogenous case.

3 Proof

3.1 Preliminaries

In all the sequels, we consider a set V at the most countable and discrete set. We will sometimes need to assume that V is finite. In this case, we will mention it. Let $\mathcal{M}_{<+\infty}(V)$ be the set of finite measures on V and let $\mathcal{P}(V)$ be the subset of $\mathcal{M}_{<+\infty}(V)$ containing the probability measures on V.
We will be working with several classical distances on $\mathcal{P}(V)$, -the total variation distance is defined as:

$$\forall(\mu,\nu)\,,\, d_{TV}(\mu,\nu)=\frac{1}{2}\sum_{x\in V}|\mu(x)-\nu(x)|\,,$$

When V is finite, it is interesting to consider the separation distance:

$$s(\mu,\nu)=\max_{x\in V}\left\{1-\frac{\mu(x)}{\nu(x)}\right\}\,.$$

Recall that K is said to be a Markov transition operator on V when K is a map $V\times V\to[0;1]$ satisfying :

- $\forall y\in V, K(.,y):V\to[0;1]$ is measurable.
- $\forall x\in V, K(x,.):\mathcal{P}(V)\to[0;1]$ is a probability measure.

As usual, one can see K as a right linear operator on $\mathcal{F}(V,\mathbb{R}):=\{f:V\to\mathbb{R}\}$ with:

$$\forall f\in\mathcal{F}(V,R),(Kf)(x)=\sum_{y\in V}K(x,y)f(y)\,. \tag{19}$$

and as an left linear operator which acts on $\mathcal{M}_{<+\infty}(V)$ as follows:

$$\forall\mu\in\mathcal{M}_{<+\infty}(V),\forall y\in V,\ (\mu K)(y)=\sum_{x\in V}\mu(x)K(x,y)\,. \tag{20}$$

For π an element of $\mathcal{M}_{<+\infty}(V)$, let $\tilde{\pi}:=\frac{\pi}{\pi(V)}$ denote the element of $\mathcal{P}(V)$ given by π.

Moreover, given p in $[1;+\infty[$, we denote conventionally $\ell^p(\pi)\subset\mathcal{F}(V,\mathbb{R})$ the set of functions f such that :

$$\sum_{x\in V}\pi(x)|f(x)|^p<+\infty$$

and we consider the following p-norm:

$$\|f\|_{\ell^p(\pi)}=(\sum_{x\in V}\pi(x)|f(x)|^p)^{\frac{1}{p}}\,.$$

In the $\ell^2(\pi)$ case, we write:

$$\forall(f,g)\in\ell^2(\pi),\langle f,g\rangle_\pi=\sum_{x\in V}f(x)g(x)\pi(x)$$
$$\text{and }\ \langle f,g\rangle_{\tilde{\pi}}=\sum_{x\in V}f(x)g(x)\tilde{\pi}(x)\,.$$

Classicaly, we define $\ell^\infty(\pi) \subset \mathcal{F}(V, \mathbb{R})$ the set of bounded functions on V and:

$$\|f\|_{\ell^\infty(\pi)} = \sup_{x \in V} \{|f(x)|\} \ .$$

Using the fact that we consider only proper measure is proper, the previous quantity is independent the measure. It is also independent of the fact that the measure is a probability or not. However, we will retain the dependency on it in the proofs to help the reader.
An important notation is the following:

$$\forall f \in \ell^1(\pi),\ \ \pi(f) = \sum_{x \in V} \pi(x) f(x) \ \text{ and } \ \ \tilde{\pi}(f) = \sum_{x \in V} \tilde{\pi}(x) f(x) \,.$$

Furthermore, we use the notation Var_π when π is not a probability measure. We define the variance of a function with respect to a finite measure that is not a probability as follows:

$$\forall f \in \ell^2(\pi),\ \ \mathrm{Var}_\pi(f) = \sum_{x \in V} (f(x) - \tilde{\pi}(f))^2 \pi(x) \,.$$

Note that:

$$\begin{aligned}\forall f \in \ell^2(\pi), \mathrm{Var}_\pi(f) &= \frac{1}{2\pi(V)} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))^2 \pi(x)\pi(y) \\ &= \sum_{x \in V} f^2(x)\pi(x) - \pi(V)\tilde{\pi}(f)^2 \,.\end{aligned}$$

Moreover, consider the Markov transition operator Δ defined with:

$$\forall \mu \in \mathcal{P}(V), \ \ \ \mu\Delta = \tilde{\pi} \,.$$

Then, we find another expression of the variance:

$$\forall f \in \ell^2(\pi), \mathrm{Var}_\pi(f) = \langle (I - \Delta) f, f \rangle_\pi \ .$$

One of the characteristics of the time-inhomogeneous context is that the invariant measures of the operators evolve with time. So, we have to consider operator between ℓ^p spaces with sometimes different measures in the domain and target spaces and other times the same measures.
More precisely, we will juggle between:

$$K_t : \ell^p(\pi_t) \to \ell^q(\pi_t)$$

and

$$K_t : \ell^p(\pi_t) \to \ell^q(\pi_{t-1}) \,.$$

One of the advantages of the finite non-decreasing context is the following remark:

Remark 2. *if $t \mapsto \pi_t$ is non-decreasing then*

$$\forall t \geq 1, \forall p \in [1; +\infty], \ell^p(\pi_{t+1}) \subset \ell^p(\pi_t) \,. \tag{21}$$

However, $\ell^p(\pi_{t+1}) \subset \ell^p(\pi_t)$ does not imply π_t dominates π_{t-1}.

Finally, we are going to distinguish the duals of K_t. The classical dual of $K_t : \ell^2(\pi_t) \to \ell^2(\pi_t)$ is denoted $K_t^* : \ell^2(\pi_t) \to \ell^2(\pi_t)$ and defined with:

$$\forall (f,g) \in \ell^2(\pi_t)^2, \langle K_t f, g \rangle_{\pi_t} = \langle f, K_t^* g \rangle_{\pi_t} .$$

We recall that:

$$\forall (x,y) \in V^2, K_t^*(x,y) = \frac{\pi_t(y)}{\pi_t(x)} K_t(y,x) = \frac{\tilde{\pi}_t(y)}{\tilde{\pi}_t(x)} K_t(y,x) .$$

Then, remark that K_t^* is also the dual of $K_t : \ell^2(\tilde{\pi}_t) \to \ell^2(\tilde{\pi}_t)$.

Besides, we denote $K_t^{\Rightarrow} = K_{t-1,t}^{\Rightarrow} : \ell^2(\pi_{t-1}) \to \ell^2(\pi_t)$ the dual of $K_t : \ell^2(\pi_t) \to \ell^2(\pi_{t-1})$. It is given by:

$$\forall f \in \ell^2(\pi_t), \forall g \in \ell^2(\pi_{t-1}), \langle K_t f, g \rangle_{\pi_{t-1}} = \langle f, K_t^{\Rightarrow} g \rangle_{\pi_t} .$$

We find that:

$$\forall (x,y) \in V^2, K_t^{\Rightarrow}(x,y) = \frac{\pi_{t-1}(y)}{\pi_t(x)} K_t(y,x) .$$

It is important to remark that in general, we can not substitute $\tilde{\pi}_t$ to π_t that is to say if $\pi_t(V) \neq \pi_{t-1}(V)$ then:

$$\begin{aligned} \forall (x,y) \in V^2, K_t^{\Rightarrow}(x,y) &\neq \frac{\tilde{\pi}_{t-1}(y)}{\tilde{\pi}_t(x)} K_t(y,x) \\ \text{but } K_t^{\Rightarrow}(x,y) &= \frac{\pi_t(V)}{\pi_{t-1}(V)} \frac{\tilde{\pi}_{t-1}(y)}{\tilde{\pi}_t(x)} K_t(y,x) \\ \text{and } K_t^{\Rightarrow}(x,y) &= \frac{\pi_t(V)}{\pi_{t-1}(V)} K_t^{\rightarrow}(x,y) . \end{aligned}$$

where $K_t^{\rightarrow} : \ell^2(\tilde{\pi}_{t-1}) \to \ell^2(\tilde{\pi}_t)$ is the dual of $K_t : \ell^2(\tilde{\pi}_t) \to \ell^2(\tilde{\pi}_{t-1})$ in these precise spaces.

3.2 Poincaré case

We recall that we work with the next Dirichlet form subordinated to the measure π_t and the kernel $Q_t = K_t^* K_t$ with:

$$\begin{aligned} \forall (f,g) \in \ell^2(\pi_t)^2, \mathcal{E}_{Q_t,\pi_t}(f,g) &= \langle (I - Q_t) f, g \rangle_{\pi_t} \\ &= \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))(g(x) - g(y)) Q_t(x,y) \pi_t(x) . \end{aligned}$$

We recall a fundamental concept in spectral analysis:

Definition 7. $1 - \lambda_t$ *is called the spectral gap and if* $1 - \lambda_t > 0$, Q_t *is said to satisfy a* ***Poincaré condition***.

In sections 1.1 and 2, we use the notation $\gamma(Q_t) = 1 - \lambda_t$. *Then, one can write:*

$$(1 - \lambda_t) \, Var_{\pi_t}(f) \leq \mathcal{E}_{Q_t,\pi_t}(f,f) . \tag{22}$$

Now, we recall the variational form of $\lambda(Q_t, \pi_t)$ *:*

$$1 - \lambda(Q_t, \pi_t) = \underset{f \in \ell^2(\pi_t),\ f \text{ non-constant}}{Inf} \left\{ \frac{\mathcal{E}_{Q_t,\pi_t}(f,f)}{Var_{\pi_t}(f)} \right\} . \tag{23}$$

Remark 3. *It is straightforward that:*

$$1 - \lambda(Q_t, \pi_t) = \underset{f \in \ell^2(\tilde{\pi}_t),\ f\ non\text{-}constant}{Inf} \left\{ \frac{\mathcal{E}_{Q_t, \tilde{\pi}_t}(f)}{Var_{\tilde{\pi}_t}(f)} \right\}$$

The main result is the following, it is the key to prove Theorem 3 and the proof paves the way to the logarithmic Sobolev case:

Theorem 7. *Let $(K_t)_{t \geq 1}$ be a sequence of Markov transitions operators and $(\pi_t)_{t \geq 1}$ a sequence of elements of $\mathcal{M}_{<+\infty}(V)$. Assume $\{(K_t, \pi_t)\}_{t \geq 1}$ is a finite non-decreasing environment. Let λ_t be the second highest eigenvalue of Q_t.*
Then,

$$\forall t \geq 0, \forall f \in \ell^2(\pi_t),\ Var_{\tilde{\pi}_1}(K_{0,t} f) \leq \frac{\pi_t(V)}{\pi_1(V)}\, Var_{\tilde{\pi}_t}(f) \prod_{s=1}^{t} \lambda_s\,. \tag{24}$$

And,

$$\|K_{0,t} - \tilde{\pi}_1(K_{0,t})\|_{\ell^2(\tilde{\pi}_t) \to \ell^2(\tilde{\pi}_1)} \leq \sqrt{\frac{\pi_t(V)}{\pi_1(V)} \prod_{s=1}^{t} \lambda_s}\,. \tag{25}$$

Remark 4. *The proof of Theorem 7 is not difficult at all. It uses a backward induction and the well-known following fact:*

$$Var_{\pi_t}(f) = \underset{c \in \mathbb{R}}{Inf} \left\{ \|f - c\|^2_{\ell^2(\pi_t)} \right\}\,. \tag{26}$$

Proof of Theorem 7. First, we impose $\pi_0 := \pi_1$ and remark (2) implies that:

$$\forall t \geq 1, \forall s \in [[1;t]], \ell^2(\pi_t) \subset \ell^2(\pi_s)\,.$$

Fix $t \geq 1$ and f element of $\ell^2(\pi_t)$. Denote:

$$\forall s \in [[0;t]],\ \ a_t(s) = \|K_{t-s,t} f - \tilde{\pi}_{t-s}(K_{t-s,t} f)\|^2_{\ell^2(\pi_{t-s})}\,.$$

Given s element of $[[0;t-1]]$, one can remark that:

$$a_t(s+1) \leq \lambda_{t-s} a_t(s)\,. \tag{27}$$

Indeed,

$$\begin{aligned} a_t(s+1) &= \|K_{t-s} K_{t-s,t} f - \tilde{\pi}_{t-s-1}(K_{t-s} K_{t-s,t} f)\|^2_{\ell^2(\pi_{t-s-1})} \text{ by variance's} \\ &\leq \|K_{t-s} K_{t-s,t} f - \tilde{\pi}_{t-s}(K_{t-s} K_{t-s,t} f)\|^2_{\ell^2(\pi_{t-s-1})} \quad \text{minimality} \\ &\leq \|K_{t-s} K_{t-s,t} f - \tilde{\pi}_{t-s}(K_{t-s} K_{t-s,t} f)\|^2_{\ell^2(\pi_{t-s})} \text{ using } \pi_{t-s} \geq \pi_{t-s-1} \\ &\leq \lambda_{t-s} a_t(s) \text{ by Poincaré Inequality at time } t-s\,. \end{aligned}$$

By iterating equation (27), a reverse induction argument yields:

$$a_t(t) = \mathrm{Var}_{\pi_0}(K_{0,t} f) \leq \mathrm{Var}_{\pi_t}(f) \prod_{s=1}^{t} \lambda_s = a_t(0) \prod_{s=0}^{t-1} \lambda_{t-s}\,. \tag{28}$$

It yields inequality (24).
In order to find inequality (25), remark that:

$$\|f - \tilde{\pi}_t(f)\|_{\ell^2(\tilde{\pi}_t)} \leq \|f\|_{\ell^2(\tilde{\pi}_t)} .$$

□

To apply Theorem 7 to the study of merging-times, we need the operators defined in Proposition 1:

Proposition 1. *In the study of merging times, a family of operators naturally arises, denoted as $O_{s,t}$:*

$$\forall s \geq 0, \forall t \geq s, O_{s,t} := K_{s,t}. - \tilde{\pi}_s(K_{s,t}.) : \ell^2(\tilde{\pi}_t) \to \ell^2(\tilde{\pi}_s) .$$

We denote their dual in precise spaces:

$$\forall s \geq 0, \forall t \geq s, O_{s,t}^{\rightarrow} : \ell^2(\tilde{\pi}_s) \to \ell^2(\tilde{\pi}_t) .$$

First, $(O_{s,t})_{s\leq t}$ is a semi-group.
Second,

$$\forall s \geq 0, \forall t \geq s, O_{s,t}^{\rightarrow} = K_{s,t}^{\rightarrow} - K_{s,t}^{\rightarrow} 1 \tilde{\pi}_s()$$

where 1 is the function constant equal to 1.

Let μ_t^z be the law of the chain started at z and driven by $(K_t)_{t\geq 1}$ and denote $h_t^z = \frac{\mu_t^z}{\tilde{\pi}_t}$ and impose $\pi_0 = \pi_1$.
Third,

$$h_t^z = K_{0,t}^{\rightarrow} h_0^z \text{ and } K_{0,t}^{\rightarrow} 1 \tilde{\pi}_0(h_0^z) = K_{0,t}^{\rightarrow} 1 .$$

Remark that $K_{0,t}^{\rightarrow} 1 \tilde{\pi}_0(h_0^z)$ does not depend of z.
Then,

$$O_{s,t}^{\rightarrow}(h_s^z) = h_t^z - K_{0,t}^{\rightarrow} 1 .$$

Proof. We first prove that $(O_{s,t})_{s\leq t}$ is a semi-group.
Indeed,

$$\begin{aligned}\forall (s,r,t) \subset \mathbb{N}, s \leq r \leq t,\ O_{s,r}O_{r,t} = {} & K_{s,r}K_{r,t}. - \tilde{\pi}_s(K_{s,r}.)K_{r,t}. \\ & - K_{s,r}\tilde{\pi}_r(K_{r,t}.) + \tilde{\pi}_s(K_{s,r}.)\tilde{\pi}_r(K_{r,t}.)\end{aligned}$$

Use the fact that $K_{s,r}K_{r,t} = K_{s,t}$ to find:

$$\tilde{\pi}_s(K_{s,r}.)K_{r,t} = \tilde{\pi}_s(K_{s,t}.) .$$

Moreover, it is clear that:

$$-K_{s,r}\tilde{\pi}_r(K_{r,t}.) + \tilde{\pi}_s(K_{s,r}.)\tilde{\pi}_r(K_{r,t}.) = -\tilde{\pi}_r(K_{r,t}.) + \tilde{\pi}_r(K_{r,t}.) = 0 .$$

Finally, we find:

$$O_{s,r}O_{r,t} = O_{s,t} .$$

Second, it is straightforward that:

$$O^{\rightarrow}_{s,t} = K^{\rightarrow}_{s,t} - [\tilde{\pi}_s(K_{s,t}.)]^{\rightarrow}\,.$$

Now, we make this operator $[\tilde{\pi}_s(K_{s,t}.)]^{\rightarrow} : \ell^2(\tilde{\pi}_s) \to \ell^2(\tilde{\pi}_t)$ explicit:

$$\begin{aligned}\forall f \in \ell^2(\tilde{\pi}_t), \forall g \in \ell^2(\tilde{\pi}_s), \langle \tilde{\pi}_s(K_{s,t}f)|g\rangle_{\tilde{\pi}_s} &= \langle f|[\tilde{\pi}_s(K_{s,t})]^{\rightarrow} g\rangle_{\tilde{\pi}_t}\\ &= \tilde{\pi}_s(K_{s,t}f)\tilde{\pi}_s(g)\,.\end{aligned}$$

And

$$\begin{aligned}\forall f \in \ell^2(\tilde{\pi}_t), \forall g \in \ell^2(\tilde{\pi}_s), \tilde{\pi}_s(K_{s,t}f)\tilde{\pi}_s(g) &= \langle K_{s,t}f|1\rangle_{\tilde{\pi}_s}\tilde{\pi}_s(g)\\ &= \left\langle f|\tilde{\pi}_s(g)K^{\rightarrow}_{s,t}1\right\rangle_{\tilde{\pi}_t}\,.\end{aligned}$$

Finally,

$$\forall f \in \ell^2(\tilde{\pi}_t), O^{\rightarrow}_{s,t}(f) = K^{\rightarrow}_{s,t}f - K^{\rightarrow}_{s,t}\mathbf{1}\tilde{\pi}_s(f)\,.$$

Third, it is straightforward that:

$$\forall y \in V, h^z_t(y) = \frac{K_{0,t}(z,y)}{\tilde{\pi}_t(y)} = \frac{K^{\rightarrow}_{0,t}(y,z)}{\tilde{\pi}_0(z)} = K^{\rightarrow}_{0,t}h^z_0(y)\,.$$

Note that:

$$\forall z \in V, \quad K^{\rightarrow}_{0,t}1\tilde{\pi}_0(h^z_0) = K^{\rightarrow}_{0,t}1\,.$$

Finally, we find:

$$O^{\rightarrow}_{0,t}(h^z_0) = K^{\rightarrow}_{0,t}h^z_0 - K^{\rightarrow}_{0,t}1\,.$$

□

In the following, we provide a proof of Theorem 3:

Proof of Theorem 3. Let μ^z_t be the law of the chain started at z and driven by $(K_t)_{t\geq 1}$ and let m_t be $K^{\rightarrow}_{0,t}1$ and denote $\frac{\mu^z_t}{\tilde{\pi}_t}$ with h^z_t.
The triangle inequality gives:

$$\begin{aligned}\forall (x,y) \in V^2, d_{TV}(\mu^x_t, \mu^y_t) &\leq \frac{1}{2}\left\|h^x_t - h^y_t\right\|_{\ell^2(\tilde{\pi}_t)}\\ &\leq \frac{1}{2}\left\|h^x_t - m_t\right\|_{\ell^2(\tilde{\pi}_t)} + \frac{1}{2}\left\|h^x_t - m_t\right\|_{\ell^2(\tilde{\pi}_t)}\,.\end{aligned}$$

Then, Proposition 1 gives:

$$\|h^z_t - m_t\|^2_{\ell^2(\tilde{\pi}_t)} \leq \left\|O^{\rightarrow}_{0,t}\right\|^2_{\ell^2(\tilde{\pi}_0)\to\ell^2(\tilde{\pi}_t)}\|h^z_0\|^2_{\ell^2(\tilde{\pi}_0)}\,.$$

Then, using duality and Theorem 7 one finds:

$$\begin{aligned}\left\|O^{\rightarrow}_{0,t}\right\|^2_{\ell^2(\tilde{\pi}_0)\to\ell^2(\tilde{\pi}_t)} &= \|K_{0,t} - \tilde{\pi}_0(K_{0,t})\|^2_{\ell^2(\tilde{\pi}_t)\to\ell^2(\tilde{\pi}_0)}\\ &\leq \frac{\pi_t(V)}{\pi_0(V)}\prod_{s=1}^{t}\lambda_s\,.\end{aligned}$$

Finally, remark that $\|h^z_0\|_{\ell^2(\tilde{\pi}_0)} = \frac{1}{\sqrt{\tilde{\pi}_0(z)}}$ to conclude that:

$$d_{TV}(\mu^x_t, \mu^y_t) \leq \frac{1}{2}\left(\frac{1}{\sqrt{\tilde{\pi}_0(x)}} + \frac{1}{\sqrt{\tilde{\pi}_0(y)}}\right)\sqrt{\frac{\pi_t(V)}{\pi_0(V)}\prod_{s=1}^{t}\lambda_s}\,.$$

□

3.3 Nash Inequalities and Poincaré

Some of the best bounds on L^2 mixing times were shown by the use of Nash inequalities. Diaconis and Saloff-Coste introduced them in [5] to study mixing. In the time-homogenous case, a Nash inequality is a tool used to show that when the variance of the density is extremely high then the walk converges even faster than predicted by the Poincaré constant. In this part, we show how one can use this tool in the time-inhomogenous case.

We there give a key lemma:

Proposition 2. *The finite non-decreasing context allows the following control of the Dirichlet form:*

$$\begin{aligned}&\forall r \geq 1, \forall g \in \ell^2(\pi_r),\\ &\qquad \mathcal{E}_{Q_r,\pi_r}(g,g) \leq \|g - \tilde{\pi}_r(g)\|^2_{\ell^2(\pi_r)} - \|K_r g - \tilde{\pi}_{r-1}(K_r g)\|^2_{\ell^2(\pi_{r-1})} .\end{aligned}$$

Proof. First, write:

$$\mathcal{E}_{Q_r,\pi_r}(g,g) = \|g - \tilde{\pi}_r(g)\|^2_{\ell^2(\pi_r)} - \|K_r g - \tilde{\pi}_r(K_r g)\|^2_{\ell^2(\pi_r)} .$$

Then, write:

$$\begin{aligned}\|K_r g - \tilde{\pi}_r(K_r g)\|^2_{\ell^2(\pi_r)} &\geq \|K_r g - \tilde{\pi}_r(K_r g)\|^2_{\ell^2(\pi_{r-1})}\\ &\geq \|K_r g - \tilde{\pi}_{r-1}(K_r g)\|^2_{\ell^2(\pi_{r-1})} .\end{aligned}$$

Merge the two equations above to conclude. □

Under Nash inequalities, we control the norm of operators $O_{r,t}$ with the following:

Theorem 8. *Let $T \geq 1$, assume $\pi_1(V) \geq 1$ and that there are constants $C, D > 0$ such that for $1 \leq t \leq T$, we have $(Q_t, \tilde{\pi}_t)$ satisfies the $\mathcal{N}(C, D, T)$ Nash inequality.:*

$$\begin{aligned}\forall f : V \to \mathbb{R}, \quad \|f\|^{2+1/D}_{\ell^2(\tilde{\pi}_t)} \leq C(&\mathcal{E}_{Q_t,\tilde{\pi}_t}(f,f)\\ &+\frac{1}{T}\|f\|^2_{\ell^2(\tilde{\pi}_t)}\Big) \|f\|^{1/D}_{\ell^1(\tilde{\pi}_t)}.\end{aligned}$$

Then, for $0 \leq r \leq t \leq T$,

$$\|K_{r,t} - \tilde{\pi}_r(K_{r,t}\cdot)\|_{\ell^1(\tilde{\pi}_t)\to\ell^2(\tilde{\pi}_r)} \leq \left(\frac{4C(1+1/T)(1+\lceil 4D\rceil)}{t-r+1}\right)^D \frac{\pi_t(V)}{\sqrt{\pi_r(V)}},$$

Moreover, if V is finite:

$$\|K_{r,t} - \tilde{\pi}_r(K_{r,t}\cdot)\|_{\ell^2(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_r)} \leq 2\left(\frac{4C(1+1/T)(1+\lceil 4D\rceil)}{t-r+1}\right)^D \sqrt{\pi_t(V)}\,.$$

Proof. First of all, remark that for $1 \leq s \leq T$ the Nash inequality with respect to $\tilde{\pi}_s$ gives a Nash inequality with respect to π_s with the same constants.
Indeed, $\pi_s(V) \geq 1$ implies $\frac{\pi_s(V)^{1+1/2D}}{\pi_s(V)^{1+1/D}} \leq 1$, it gives:

$$\begin{aligned}\forall f : V \to \mathbb{R}, \quad \|f\|^{2+1/D}_{\ell^2(\pi_t)} \leq C(&\mathcal{E}_{Q_t,\pi_t}(f,f)\\ &+\frac{1}{T}\|f\|^2_{\ell^2(\pi_t)}\Big) \|f\|^{1/D}_{\ell^1(\pi_t)}.\end{aligned}$$

For s in $[[1,t]]$, we define the next quantities:

$$a_t(s) = \|K_{t-s,t}f - \tilde{\pi}_{t-s}(K_{t-s,t}f)\|^2_{\ell^2(\pi_{t-s})}.$$

In Proof 2.5.3, it is proven $(a_t(s))_{s=0}^t$ is non-increasing.

Then, fix f element of $\ell^1(\pi_T)$ and apply the Nash inequality to the function $K_{t-s,t}f - \tilde{\pi}_{t-s}(K_{s,t}f)$ to get for any $0 \le s-1 \le t \le T$:

$$\begin{aligned} a_t(s+1)^{1+1/(2D)} &\le \|K_{t-s}K_{t-s,t}f - \tilde{\pi}_{t-s}(K_{t-s}K_{t-s,t}f)\|^{2(1+1/(2D))}_{\ell^2(\pi_{t-s})} \\ &\le C\left(\mathcal{E}_{Q_{t-s},\pi_{t-s}}(K_{t-s,t}f, K_{t-s,t}f) + a_t(s)/T\right)\|K_{t-s,t}f - \tilde{\pi}_{t-s}(K_{s,t}f)\|^{1/D}_{\ell^1(\pi_{t-s})} \\ &\le C\left(a_t(s) - a_t(s+1) + a_t(s)/T\right)\|K_{t-s,t}f - \tilde{\pi}_{t-s}(K_{s,t}f)\|^{1/D}_{\ell^1(\pi_{t-s})}. \end{aligned}$$

Proposition 2 proves the last inequality with $g = K_{t-s,t}f$ and $r = t-s$.

We prove by induction that the quantities $||K_{t-s,t}f||_{\ell^1(\pi_{t-s})} := n_t(s)$ are non-increasing.
Indeed, use successively the dominance of π_{t-s+1} on π_{t-s} and the fact that $K_{t-s} : \ell^1(\pi_{t-s}) \to \ell^1(\pi_{t-s})$ is a contraction to find:

$$n_t(s) = \|K_{t-s+1}K_{t-s+1,t}f\|_{\ell^1(\pi_{t-s})} \le \|K_{t-s+1}K_{t-s+1,t}f\|_{\ell^1(\pi_{t-s+1})} \le n_t(s-1)$$

Then, for s in $[[0;t]]$:

$$n_t(s) \le n_t(0) = \|f\|_{\ell^1(\pi_t)} \tag{29}$$

Finally, let $C_f := C\|f\|^{1/D}_{\ell^1(\pi_t)}$ and merge Equations (29) and (2.5.3) and find:

$$a_t(s+1)^{1+1/(2D)} \le C_f\left(a_t(s) - a_t(s+1) + a_t(s)/T\right),$$

Corollary 3.1 of [5] then yields that

$$a_t(r) \le \left(\frac{C_f B}{t-r+1}\right)^{2D}, \quad 0 \le r \le t \le T,$$

where $B = B(D,T) = (1+1/T)(1+\lceil 4D \rceil)$.
In particular, if $0 \le r \le t \le T$:

$$\|K_{r,t}f - \tilde{\pi}_s(K_{s,t}f)\|_{\ell^2(\pi_r)} \le ((CB)/(t-r+1))^D\|f\|_{\ell^1(\pi_t)}.$$

And

$$\|K_{r,t} - \tilde{\pi}_r(K_{r,t}.)\|_{\ell^1(\tilde{\pi}_t)\to\ell^2(\tilde{\pi}_r)} \le ((CB)/(t-r+1))^D\frac{\pi_t(V)}{\sqrt{\pi_r(V)}}.$$

Now, we assume that V is finite. We recall that $O_{s,t} = K_{s,t} - \tilde{\pi}_s(K_{s,t}.)$.
We want to bound $\left\|O^{\Rightarrow}_{s,t}\right\|_{\ell^1(\pi_s)\to\ell^\infty(\pi_t)}$ for $0 \le s \le t \le T$.
First, by duality for $0 \le s \le t \le T$,

$$\left\|O^{\rightarrow}_{s,t}\right\|_{\ell^2(\tilde{\pi}_s)\to\ell^\infty(\tilde{\pi}_t)} \le ((CB)/(t-s+1))^D\frac{\pi_t(V)}{\sqrt{\pi_s(V)}}.$$

We consider the quantity $M(T)$ defined by

$$M(T) = \max_{0 \le s \le t \le T} \left\{ (t-s+1)^{2D} \left\| O^{\Rightarrow}_{s,t} \right\|_{\ell^1(\pi_s) \to \ell^\infty(\pi_t)} \right\}.$$

Let $l = \lfloor \frac{t-s}{2} \rfloor + s$, so that $0 \le s \le l \le t \le T$. We find:

$$\begin{aligned} \left\| O^{\Rightarrow}_{s,t} \right\|_{\ell^1(\pi_s) \to \ell^\infty(\pi_t)} &\le \left\| O^{\Rightarrow}_{s,l} \right\|_{\ell^1(\pi_s) \to \ell^2(\pi_l)} \left\| O^{\Rightarrow}_{l,t} \right\|_{\ell^2(\pi_l) \to \ell^\infty(\pi_t)} \\ &\le \left\| O^{\Rightarrow}_{s,l} \right\|_{\ell^1(\pi_s) \to \ell^2(\pi_l)} ((CB)/(t-l+1))^D. \end{aligned}$$

Note that for all $0 \le s \le l \le T$:

$$\left\| O^{\Rightarrow}_{s,l} \right\|_{\ell^1(\pi_s) \to \ell^2(\pi_l)} \le \left\| O^{\Rightarrow}_{s,l} \right\|^{1/2}_{\ell^1(\pi_s) \to \ell^\infty(\pi_l)} \left\| O^{\Rightarrow}_{s,l} \right\|^{1/2}_{\ell^1(\pi_s) \to \ell^1(\pi_l)}.$$

This follows from the fact that for any function g:

$$\|g\|_{\ell^2(\pi_l)} \le \|g\|^{1/2}_{\ell^\infty(\pi_l)} \|g\|^{1/2}_{\ell^1(\pi_l)}.$$

Moreover, note that:

$$\left\| O^{\Rightarrow}_{s,l} \right\|_{\ell^1(\pi_s) \to \ell^1(\pi_l)} = \|O_{s,l}\|_{\ell^\infty(\pi_l) \to \ell^\infty(\pi_s)} \le 2 \tag{30}$$

So,

$$\begin{aligned} \left\| O^{\Rightarrow}_{s,t} \right\|_{\ell^1(\pi_s) \to \ell^\infty(\pi_t)} &\le \sqrt{2} \left(\frac{CB}{t-l+1} \right)^D \left\| O^{\Rightarrow}_{s,t} \right\|^{1/2}_{\ell^1(\pi_s) \to \ell^\infty(\pi_l)} \\ &\le \sqrt{2} \left(\frac{CB}{(t-l+1)(l-s+1)} \right)^D M(T)^{1/2} \\ &\le \sqrt{2} \left(\frac{4CB}{(t-s+1)^2} \right)^D M(T)^{1/2}. \end{aligned}$$

The last inequality follows from the fact that

$$t-l+1 \ge \frac{t-s+1}{2} \text{ and } l-s+1 \ge \frac{t-s+1}{2}.$$

Then $M(T) = +\infty$ or $M(T) \le 2(4CB)^{2D}$.
V is finite therefore $M(T) \le 2(4CB)^{2D}$ and it follows that for all $0 \le s \le t \le T$

$$\left\| O^{\Rightarrow}_{s,t} \right\|_{\ell^1(\pi_s) \to \ell^\infty(\pi_t)} \le 2 \left(\frac{4CB}{t-s+1} \right)^{2D}.$$

By duality, we find

$$\|O_{s,t}\|_{\ell^1(\pi_t) \to \ell^\infty(\pi_s)} \le 2 \left(\frac{4CB}{t-s+1} \right)^{2D}.$$

Let $\theta = \frac{1}{2}$, $p_1 = 1$, $q_1 = q_2 = p_2 = \infty$ so $\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $\frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, then an application of the Riesz-Thorin interpolation Theorem gives

$$\|O_{s,t}\|_{\ell^2(\pi_t) \to \ell^\infty(\pi_s)} \le 2 \left(\frac{4CB}{t-s+1} \right)^{D}.$$

Finally, one finds:

$$\|O_{s,t}\|_{\ell^2(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_s)} \leq 2\left(\frac{4CB}{t-s+1}\right)^D \sqrt{\pi_t(V)}.$$

□

In the following, we provide a proof of Theorem 4. This proof follows the same outline as the proof of Theorem 3. We will reuse some of its arguments. But first, we need the following fact:

Lemma 1. *Let η in $(0;\frac{1}{2})$. Fix $t \geq 1$, we recall that $\mu_t^x = \delta^x K_{0,t}$ if*

$$\max_{x,z\in V}\left\{\left|\frac{\mu_t^x(z)}{\tilde{\pi}_0 K_{0,t}(z)} - 1\right|\right\} \leq \eta \leq 1/2\,,$$

then,

$$\max_{x,y,z\in V}\left\{\left|\frac{\mu_t^x(z)}{\mu_t^y(z)} - 1\right|\right\} \leq 4\eta\,.$$

Proof. If $1-\eta \leq a/b, c/b \leq 1+\eta$ with $\eta \in (0,1/2)$ then,

$$1-4\eta \leq \frac{1-\eta}{1+\eta} \leq \frac{a}{c} \leq \frac{1+\eta}{1-\eta} \leq 1+4\eta\,.$$

□

Proof of Theorem 4. For time $t=0$, we impose $\pi_0 := \pi_1$ and we let μ_t^z be the law of the chain started at z and driven by $(K_t)_{t\geq 1}$. The triangular inequality gives:

$$\forall (x,y)\in V^2, d_{TV}(\mu_t^x,\mu_t^y) \leq \frac{1}{2}\|h_t^x - m_t\|_{\ell^2(\tilde{\pi}_t)} + \frac{1}{2}\|h_t^x - m_t\|_{\ell^2(\tilde{\pi}_t)} \tag{31}$$

where m_t is equal to $K_{0,t}^{\rightarrow}1$ and h_t^z denotes $\frac{\mu_t^z}{\tilde{\pi}_t}$.
Then, Theorem 8 gives:

$$\begin{aligned}
&\forall z\in V, \forall t\geq s\geq 0 \text{ and } s\leq T,\\
\|h_t^z - m_t\|_{\ell^2(\tilde{\pi}_t)} &\leq \left\|O_{s,t}^{\rightarrow}\right\|_{\ell^2(\tilde{\pi}_s)\to\ell^2(\tilde{\pi}_t)} \left\|O_{0,s}^{\rightarrow}\right\|_{\ell^1(\tilde{\pi}_0)\to\ell^2(\tilde{\pi}_s)} \|h_0^z\|_{\ell^1(\tilde{\pi}_0)}\\
&\leq \sqrt{\frac{\pi_t(V)\pi_s(V)}{\pi_0(V)}}\left(\frac{4CB}{s+1}\right)^D \prod_{u=s+1}^{t}\sqrt{\lambda_u}
\end{aligned}$$

where $B = B(D,T) = (1+1/T)(1+\lceil 4D\rceil)$.
Finally,

$$d_{TV}(\mu_t^x,\mu_t^y) \leq \sqrt{\frac{\pi_t(V)\pi_s(V)}{\pi_0(V)}}\left(\frac{4CB}{s+1}\right)^D \prod_{u=s+1}^{t}\sqrt{\lambda_u}\,.$$

In a second step, we assume V is finite. Note that

$$\max_{x,y\in V}\left\{\left|\frac{\mu_t^x(y)}{\tilde{\pi}_0 K_{0,t}(y)} - 1\right|\right\} = \max_{x,y\in V}\left\{\left|\frac{\delta_t^x K_{0,t}(y)}{\tilde{\pi}_0 K_{0,t}(y)} - 1\right|\right\} = \|O_{0,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_0)}\,.$$

Then, write $t = 2r + u$ with $T \geq r$, an application of Theorem 7 and Theorem 8 give:

$$\begin{aligned}
\|O_{0,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_0)} &\leq \left\|O^{\rightarrow}_{0,r}\right\|_{\ell^1(\tilde{\pi}_0)\to\ell^2(\tilde{\pi}_r)} \times \left\|O^{\rightarrow}_{r,r+u}\right\|_{\ell^2(\tilde{\pi}_r)\to\ell^2(\tilde{\pi}_{r+u})} \\
&\quad \times \left\|O^{\rightarrow}_{r+u,2r+u}\right\|_{\ell^2(\tilde{\pi}_{r+u})\to\ell^\infty(\tilde{\pi}_{2r+u})} \\
&\leq 4\pi_t(V)\left(\frac{4CB}{r+1}\right)^{2D} \prod_{s=r+1}^{r+u} \sqrt{\lambda_s}\,.
\end{aligned}$$

Merge the inequalities above to conclude that:

$$\max_{x,y\in V}\left\{\left|\frac{\mu_t^x(y)}{\tilde{\pi}_0 K_{0,t}(y)} - 1\right|\right\} \leq 4\pi_t(V)\left(\frac{4CB}{r+1}\right)^{2D} \prod_{s=r+1}^{r+u} \sqrt{\lambda_s}\,.$$

Let η in $(0; \frac{1}{2})$. Finally, use Lemma 1 to find if

$$4\pi_t(V)\left(\frac{4CB}{r+1}\right)^{2D} \prod_{s=r+1}^{r+u} \sqrt{\lambda_s} \leq \eta\,.$$

then,

$$\max_{x,y,z\in V}\left\{\left|\frac{\mu_t^x(z)}{\mu_t^y(z)} - 1\right|\right\} \leq 4\eta\,,$$

and one finds a bound on $T^\infty_{\text{mer}}(\eta)$. $\square$

3.4 Logarithmic Sobolev Case

In line with the results obtained with functional inequalities for mixing estimates, hypercontractivity is one of the most powerful. These results adapt in a non-decreasing environment to estimate merging for time-inhomogeneous Markov chains.

3.4.1 Hypercontractivity

First, we recall that we work with the following entropy of f with respect to $\tilde{\pi}$ element of $\mathcal{P}(V)$:

$$\mathcal{L}(f|\tilde{\pi}) = \sum_{x\in V} f^2(x)\log(\frac{f^2(x)}{\tilde{\pi}(f^2)})\tilde{\pi}(x)\,. \tag{32}$$

Besides, we denote $\mathcal{S}_t$ the subset of $\ell^2(\pi_t)$ which contains the functions such that $\mathcal{L}(f|\tilde{\pi}_t)$ is defined. The analog of Remark 2 in the logarithmic Sobolev case is the following:

Remark 5. *if $t \mapsto \pi_t$ is non-decreasing then*

$$\forall t \geq 1, \mathcal{S}_t \subset \mathcal{S}_{t-1}\,. \tag{33}$$

Let us recall classical facts:

Lemma 2. *For any Markov transition operator Q the logarithmic Sobolev constant $\alpha(Q)$ and the Poincaré constant $\gamma(Q)$ satisfy*

$$2\alpha(Q) \leq \gamma(Q)\,.$$

Proposition 3. *Let K be an irreducible aperiodic Markov transition operator on V and $\tilde{\pi}$ its invariant probability. Set $\alpha := \alpha(K^*K)$ the logarithmic Sobolev constant of K^*K where K^* is the standard dual of K in $\ell^2(\tilde{\pi})$.*
For all $q \geq 2$, set $q^* = (1+\alpha)q$. Then,

$$\forall p \leq q^*, \forall f \in \ell^q(\tilde{\pi}),\ \|Kf\|_{\ell^p(\tilde{\pi})} \leq \|f\|_{\ell^q(\tilde{\pi})}\,. \tag{34}$$

Inequality (34) is called a hypercontractivity inequality. For more details, consult [11]. Hypercontractivity adapts to a non-decreasing environment. It takes on a new form. The equivalent of Proposition 3 is the following:

Theorem 9. *Let $(K_t)_{t\geq 1}$ be a sequence of aperiodic and irreducible Markov transitions operators and $(\pi_t)_{t\geq 1}$ a sequence of elements of $\mathcal{M}_{<+\infty}(V)$. Assume $\{(K_t,\pi_t)\}_{t\geq 1}$ is a finite non-decreasing environment.*
Let α_t be the logarithmic Sobolev constants $\alpha(Q_t, \tilde{\pi}_t)$ and assume $\pi_1(V) \geq 1$.
For all $q \geq 2$ and $t \geq 1$, set $q_t = q\prod_{s=1}^{t}(1+\alpha_s)$.
Then,

$$\forall p \leq q_t, \forall t \geq 1, \forall f \in \ell^q(\pi_t),\ \|K_{0,t}f\|_{\ell^p(\tilde{\pi}_1)} \leq \|f\|_{\ell^q(\tilde{\pi}_t)}\frac{\pi_t(V)^{1/q}}{\pi_1(V)^{1/q_t}}\,. \tag{35}$$

Proof of Theorem 9. Set $\pi_0 := \pi_1$ and fix $t \geq 1$. Define for s in $[[0;t]]$:

$$n_t(s) = \|K_{t-s,t}f\|_{\ell^{r_s}(\pi_{t-s})}$$

where $r_s = q\prod_{u=t-s+1}^{t}(1+\alpha_u)$.
Then, an application of Proposition 3 and the dominance of π_{t-s} on π_{t-s-1} give:

$$
\begin{aligned}
n_t(s+1) &= \|K_{t-s}K_{t-s,t}f\|_{\ell^{r_{s+1}}(\pi_{t-s-1})} \\
&\leq \|K_{t-s}K_{t-s,t}f\|_{\ell^{(1+\alpha_{t-s})r_s}(\pi_{t-s})} \\
&\leq \|K_{t-s,t}f\|_{\ell^{r_s}(\pi_{t-s})} \\
&= n_t(s)\,.
\end{aligned}
$$

It is then straightforward that:

$$
n_t(t) = \|K_{0,t}f\|_{\ell^{q_t}(\pi_0)} \leq \|f\|_{\ell^{q}(\pi_t)} = n_t(0)
$$

Renormalize to conclude:

$$
\|K_{0,t}f\|_{\ell^{q_t}(\tilde{\pi}_0)} \leq \|f\|_{\ell^{q}(\tilde{\pi}_t)}\,\frac{\pi_t(V)^{1/q}}{\pi_0(V)^{1/q_t}}\,.
$$

Finally, the announced result is a consequence of the following fact:

$$
\forall p \leq q_t, \forall g : V \to \mathbb{R},\ \|g\|_{\ell^{p}(\tilde{\pi}_0)} \leq \|g\|_{\ell^{q_t}(\tilde{\pi}_0)}\ .
$$

□

In the following, we provide a proof of Theorem 5 and 6. This proof follows the same outline as the proof of Theorem 3 and we will reuse some of its arguments.

Proof of Theorem 5 . For time $t = 0$, we impose $\pi_0 := \pi_1$ and we let μ_t^z be the law of the chain started at z and driven by $(K_t)_{t\geq 1}$.
For $q \geq 2$, we denote q' as the Hölder conjugate of q. Let $m_{s,t}$ be equal to $K_{s,t}^{\rightarrow}1$ and let h_t^z denote $\frac{\mu_t^z}{\tilde{\pi}_t}$. An application of proposition 1 gives :

$$
\begin{aligned}
\forall t \geq s \geq 0, \forall z \in V,\ \|h_t^z - m_{s,t}\|_{\ell^2(\tilde{\pi}_t)} &= \left\|O_{s,t}^{\rightarrow}h_s^z\right\|_{\ell^2(\tilde{\pi}_t)} \\
&\leq \left\|O_{s,t}^{\rightarrow}\right\|_{\ell^2(\tilde{\pi}_s)\to\ell^2(\tilde{\pi}_t)}\left\|K_{0,s}^{\rightarrow}\right\|_{\ell^{q'}(\tilde{\pi}_0)\to\ell^2(\tilde{\pi}_s)}\|h_0^z\|_{\ell^{q'}(\tilde{\pi}_0)}\ .
\end{aligned}
$$

Remark that $m_{0,t}$ is equal to m_t.
Moreover, for z element of V, we denote t_z the following quantity:

$$
t_z = \min\{t \geq 1,\ \sum_{u=1}^{t}\log(1+\alpha_u) \geq \log(\log(\tilde{\pi}_0(z)^{-1}))\}\,. \tag{36}
$$

Using the fact that $\|h_0^z\|_{\ell^{q'}(\tilde{\pi}_0)} = \tilde{\pi}_0(z)^{-1/q}$, one finds that if $s \geq t_z$ then $\|h_0^z\|_{\ell^{q'_s}(\tilde{\pi}_0)} \leq e$.
Finally, for x, y elements of V, let s be equal to $\max\{t_x, t_y\}$. One finds using the previous inequality and Theorems 7 and 9:

$$
\begin{aligned}
\forall t \geq s,\ d_{TV}(\mu_t^x, \mu_t^y) &\leq \frac{1}{2}\|h_t^x - m_{s,t}\|_{\ell^2(\tilde{\pi}_t)} + \frac{1}{2}\|h_t^y - m_{s,t}\|_{\ell^2(\tilde{\pi}_t)} \\
&\leq e\frac{\sqrt{\pi_t(V)}}{\pi_0(V)^{1/q_s}}\prod_{u=s+1}^{t}\sqrt{\lambda_u}\,.
\end{aligned}
$$

□

In the second step, we prove Theorem 6 but first, we need the following result:

Lemma 3. *One can control the following norm of operators:*

$$\begin{aligned}&\forall u \geq 1, \forall a \in [1;\infty], \forall b \in [1;\infty],\\&\|K_u^{\Rightarrow}\|_{\ell^a(\pi_{u-1})\to\ell^b(\pi_u)} \leq \|K_u^*\|_{\ell^a(\pi_u)\to\ell^b(\pi_u)} .\end{aligned}$$

Proof. Denote a' and b' the Hoölder conjugate exponents of a and b and use the dominance of π_u on π_{u-1}:

$$\begin{aligned}\|K_u^{\Rightarrow}\|_{\ell^a(\pi_{u-1})\to\ell^b(\pi_u)} &= \|K_u\|_{\ell^{b'}(\pi_u)\to\ell^{a'}(\pi_{u-1})}\\&\leq \|K_u\|_{\ell^{b'}(\pi_u)\to\ell^{a'}(\pi_u)} = \|K_u^*\|_{\ell^a(\pi_u)\to\ell^b(\pi_u)} .\end{aligned}$$

□

Proof of Theorem 6 . Assume that V is finite and:

$$\forall t \geq 1, \tilde{\pi}_t^\sharp \geq \rho \text{ and } \forall t \geq 1, \alpha_t \geq \alpha .$$

Choose t in the form $2r+u$ (and $s = r+u$).
Write, for $t \geq 1$:

$$\max_{x,y\in V}\left\{\left|\frac{\mu_t^x(y)}{\mu_t^{\tilde{\pi}_1}(y)} - 1\right|\right\} = \|K_{0,t} - \tilde{\pi}_1 K_{0,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_1)} .$$

And,

$$\begin{aligned}\|K_{0,t} - \tilde{\pi}_1 K_{0,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_1)} &= \left\|O_{0,t}^{\to}\right\|_{\ell^1(\tilde{\pi}_1)\to\ell^\infty(\tilde{\pi}_t)}\\&\leq \left\|O_{s,t}^{\to}\right\|_{\ell^2(\tilde{\pi}_s)\to\ell^\infty(\tilde{\pi}_t)} \times \left\|O_{r,s}^{\to}\right\|_{\ell^2(\tilde{\pi}_r)\to\ell^2(\tilde{\pi}_s)}\\&\times \left\|K_{0,r}^{\to}\right\|_{\ell^1(\tilde{\pi}_1)\to\ell^2(\tilde{\pi}_r)} .\end{aligned}$$

Recall that Theorem 7 gives

$$\left\|O_{r,s}^{\to}\right\|_{\ell^2(\tilde{\pi}_r)\to\ell^2(\tilde{\pi}_s)} \leq \sqrt{\frac{\pi_s(V)}{\pi_r(V)}\prod_{u=r+1}^{s}\lambda_u} .$$

It remains to bound $\left\|O_{s,t}^{\to}\right\|_{\ell^2(\tilde{\pi}_s)\to\ell^\infty(\tilde{\pi}_t)}$ and $\left\|K_{0,r}^{\to}\right\|_{\ell^1(\tilde{\pi}_1)\to\ell^2(\tilde{\pi}_r)}$.
First, using Theorem 9, one finds:

$$\begin{aligned}\forall q \geq 2, \left\|K_{0,r}^{\to}\right\|_{\ell^1(\tilde{\pi}_1)\to\ell^2(\tilde{\pi}_r)} &\leq \|K_{0,r}\|_{\ell^2(\tilde{\pi}_r)\to\ell^q(\tilde{\pi}_1)} \|I\|_{\ell^q(\tilde{\pi}_1)\to\ell^\infty(\tilde{\pi}_1)}\\&\leq \frac{\sqrt{\pi_r(V)}}{\pi_1(V)^{\frac{1}{q}}}\frac{1}{\tilde{\pi}_1^{\sharp\frac{1}{q}}} .\end{aligned}$$

The last inequality holds by the following fact:

$$\forall f \in \ell^\infty(\tilde{\pi}_1),\ \|f\|_{\ell^\infty(\tilde{\pi}_1)} \leq \frac{1}{\tilde{\pi}_1^{\sharp\frac{1}{q}}}\|f\|_{\ell^q(\tilde{\pi}_1)} .$$

On the other hand, use the minimality of the variance and Theorem 9 to find:

$$\begin{aligned}\forall \hat{q} \geq 2, \left\| O_{s,t}^{\rightarrow} \right\|_{\ell^2(\tilde{\pi}_s)\to\ell^\infty(\tilde{\pi}_t)} &= \|O_{s,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^2(\tilde{\pi}_s)} = \sup_{\|f\|_{\ell^1(\tilde{\pi}_t)}=1} \{\|O_{s,t} f\|_{\ell^2(\tilde{\pi}_s)}\} \\ &\leq \sup_{\|f\|_{\ell^1(\tilde{\pi}_t)}=1} \{\|K_{s,t} f\|_{\ell^2(\tilde{\pi}_s)}\} = \|K_{s,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^2(\tilde{\pi}_s)} \\ &\leq \left\| K_{s,t}^{\rightarrow} \right\|_{\ell^2(\tilde{\pi}_s)\to\ell^{\hat{q}}(\tilde{\pi}_t)} \|I\|_{\ell^{\hat{q}}(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_t)} \\ &\leq \frac{1}{\tilde{\pi}_t^{\sharp \frac{1}{\hat{q}}}} \left\| K_{s,t}^{\rightarrow} \right\|_{\ell^2(\tilde{\pi}_s)\to\ell^{\hat{q}}(\tilde{\pi}_t)} .\end{aligned}$$

To bound $\left\| K_{s,t}^{\rightarrow} \right\|_{\ell^2(\tilde{\pi}_s)\to\ell^{\hat{q}}(\tilde{\pi}_t)}$, apply Lemma 3 with $a_{s+1} = 2$ and $a_u \leq a_{u+1}$ for u in $[[s+1; t+1]]$, one finds:

$$\begin{aligned}\left\| K_{s,t}^{\Rightarrow} \right\|_{\ell^2(\pi_s)\to\ell^{\hat{q}}(\pi_t)} &\leq \prod_{u=s+1}^{t} \|K_u^{\Rightarrow}\|_{\ell^{a_u}(\pi_u)\to\ell^{a_{u+1}}(\pi_{u-1})} \\ &\leq \prod_{u=s+1}^{t} \|K_u^*\|_{\ell^{a_u}(\pi_u)\to\ell^{a_{u+1}}(\pi_u)} .\end{aligned}$$

Remark that by hypothesis, we find that K_u^* is hyper contractive:

$$\forall u \geq 1, \forall q \geq 2, \forall p \leq q(1+\alpha_u),\ \|K_u^*\|_{\ell^q(\pi_u)\to\ell^p(\pi_u)} \leq 1 .$$

So, we choose a_u by induction with $a_{s+2} = a_{s+1}(1+\alpha_{s+1})$ and so on.
One can choose $\hat{q}' = 2\prod_{u=s+1}^{t} 1+\alpha_u$. However, using that α_t is bounded below by α, we choose $\hat{q} = 2(1+\alpha)^{t-s}$ and we get:

$$\left\| K_{s,t}^{\rightarrow} \right\|_{\ell^2(\tilde{\pi}_s)\to\ell^{\hat{q}}(\tilde{\pi}_t)} \leq \frac{\sqrt{\pi_s(V)}}{\pi_s(V)^{\frac{1}{\hat{q}}}} .$$

Finally, recall that $q_r = 2\prod_{j=1}^{r}(1+\alpha_j)$ and choose r such that:

$$q_r \geq \log(\frac{1}{\rho}) .$$

Then, using the hypothesis on the lower bound of $\tilde{\pi}_t^\sharp$ and $\pi_1(V) \geq 1$, we find:

$$\forall t \geq 1, \frac{1}{\tilde{\pi}_t^{\sharp \frac{1}{q_r}}} \leq e .$$

Merge the inequalities above to control $\|K_{0,t} - \tilde{\pi}_1 K_{0,t}\|_{\ell^1(\tilde{\pi}_t)\to\ell^\infty(\tilde{\pi}_1)}$ and find:

$$s(\mu_t^x, \mu_t^{\tilde{\pi}_1}) \leq e^2 \frac{1}{\pi_1(V)^{\frac{1}{q_r}} \pi_{r+u}(V)^{\frac{1}{q_r}}} \sqrt{\pi_t(V)\pi_{r+u}(V) \prod_{l=r+1}^{r+u} \lambda_r} .$$

By taking the maximum over x and y and using $\pi_1(V) \geq 1$, we obtain the following result:

$$\max_{x\in V}\{s(\mu_t^x, \mu_t^{\tilde{\pi}_1})\} \leq e^2 \sqrt{\pi_t(V)\pi_{r+u}(V) \prod_{l=r+1}^{r+u} \lambda_l} .$$

Let η in $(0;\frac{1}{2})$. Using Lemma 1, one finds if

$$e^2\sqrt{\pi_t(V)\pi_{r+u}(V)\prod_{l=r+1}^{r+u}\lambda_l} \leq \eta\,,$$

then,

$$\max_{x,y\in V}\{s(\mu_t^x,\mu_t^y)\} \leq 4\eta\,.$$

And, one finds a bound on $T_{\mathrm{mer}}^{\infty}(\eta)$. $\square$

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