Solution to the iterative differential equation $-\gamma g' = g^{-1}$

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Abstract

Using a Picard-like operator T, we prove that the iterative differential equation $-\gamma g' = g^{-1}$ with parameter $\gamma > 0$ has a solution $g = h\colon [0,1] \to [0,1]$ for only one value $\gamma = \kappa \approx 0.278877$, and that this solution h is unique. As an even stronger result, we exhibit h as the global limit of the operator T.

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1 Introduction

The study of Levine's sequence [6, 10, 9] naturally leads to some differentiable function $g\colon [0,1] \to [0,1]$ satisfying the following property: When we rotate (the graph of) g clockwise by 90° about the origin and subsequently stretch it vertically by a suitable positive factor, then we obtain its derivative g'. Formally speaking, g satisfies the iterative differential equation (IDE)

$$(1) \qquad g\colon [0,1] \to [0,1], \quad -\gamma g' = g^{-1} \quad \text{for some} \quad \gamma > 0$$

where g^{-1} denotes the compositional inverse of g. We call such a function a **unit stribola** (from Greek στρίβω = turn, twist). Every unit stribola, that is, any solution g to (1) will obviously be continuously differentiable and strictly decreasing and satisfy the identities

$$g(0) = 1, \quad g(1) = 0, \quad g'(0) = -\tfrac{1}{\gamma}, \quad \int_0^1 g = \gamma.$$

IDEs similar to (1) have been studied by Eder [3], Fečkan [5], Buică [2], Egri and Rus [4] and Berinde [1], but the techniques employed there appear to be inapt for solving (1).
In [7], we have constructed a unit stribola h by an iterative process. At each step of this process, we perform the following operation, denoted T: Given any decreasing function $f\colon [0,1] \to [0,1]$ with $f(0) = 1$ and non-zero area $\alpha := \int_0^1 f$, we rotate it by 90° about the origin, then stretch it vertically by $\frac{1}{\alpha}$, then integrate, to obtain $Tf\colon [0,1] \to [0,1]$. Starting from the line segment $h_1 = 1 - \mathrm{id}_{[0,1]}$, the sequence of iterates h_1, $h_2 := Th_1$, $h_3 := Th_2, \ldots$ is shown to converge to a unit stribola h. Figure 1 on the next page illustrates the functions h and h' and some of their geometric properties.

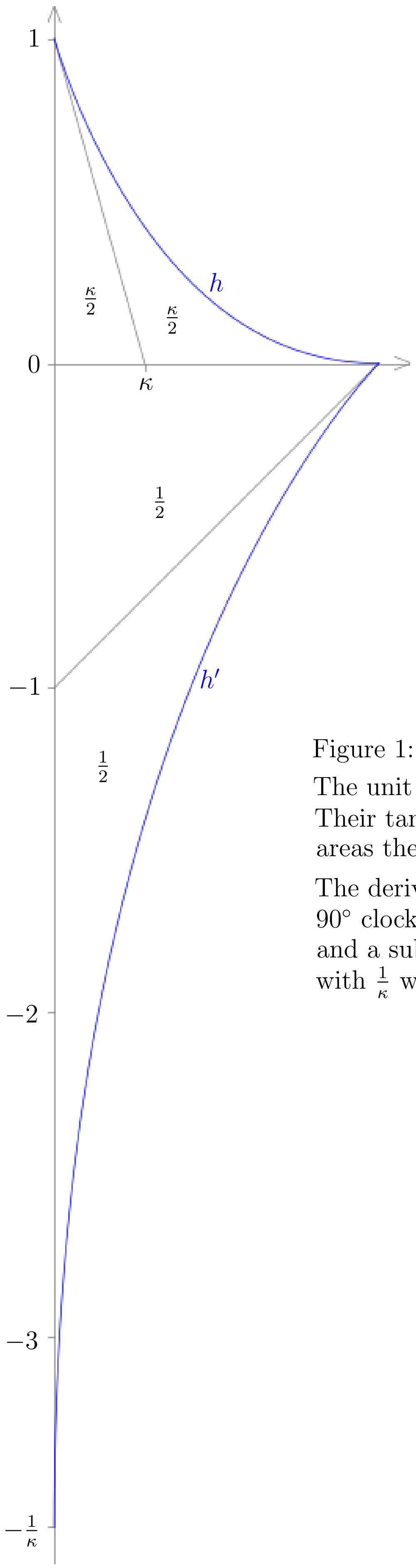

Figure 1:
The unit stribola h and its derivative h'. Their tangents at 0 resp. 1 bisect the areas they enclose with the axes.

The derivative h' arises from h by a 90° clockwise rotation about the origin and a subsequent vertical stretching with $\frac{1}{\kappa}$ where $\kappa = \int_0^1 h \approx 0.278877$.

For the sake of better clarity, we include the existence proof from [7] here in a simplified form. A key ingredient to proving the convergence $h_n \to h$ is the observation that the h_n, when stretched horizontally and vertically by arbitrary positive factors, always "cross" each other at most twice. This feature (addressed by the concept of "domination") automatically extends to their limit. To be explicit, the unit stribola h is "dominated" by each iterate h_n. From this, we can easily prove h to be the only unit stribola. Finally, we will establish global convergence, that is, $\lim_{n\to\infty} T^n f = h$ for any decreasing function $f\colon [0,1] \to [0,1]$ with $f(0) = 1$ and $\int_0^1 f > 0$. We attain this strong result by again and more heavily exploiting the domination structure between the iterates h_n and their limit h.

2 The operator T

For $0 \leq a \leq b \leq 1$ and any (Lebesgue) measurable function $f\colon [0,1] \to [0,\infty)$, we abbreviate $\int_a^b f := \int_a^b f(x)dx$ and $\int f := \int_0^1 f$. We will also conveniently write $\mathrm{id} := \mathrm{id}_{[0,1]}$ for the identity function on $[0,1]$. Our investigations will involve the spaces

$$
\begin{aligned}
\mathcal{M} &:= \{f\colon [0,1] \to [0,\infty) \,:\, f \text{ measurable}, \textstyle\int f > 0\}, \\
\mathcal{E} &:= \{f \in \mathcal{M} \,:\, f \text{ decreasing}, f(0) = 1\}, \\
\mathcal{C} &:= \{f \in \mathcal{E} \,:\, f \text{ continuous}, f(1) = 0\}, & \breve{\mathcal{C}} &:= \{f \in \mathcal{C} \,:\, f \text{ convex}\}, \\
\mathcal{D} &:= \{f \in \mathcal{C} \,:\, f \text{ strictly decreasing}\}, & \breve{\mathcal{D}} &:= \mathcal{D} \cap \breve{\mathcal{C}}, \\
\mathcal{D}' &:= \{f \in \mathcal{D} \,:\, f \text{ continuously differentiable on } (0,1]\}, & \breve{\mathcal{D}}' &:= \mathcal{D}' \cap \breve{\mathcal{C}}, \\
\mathcal{D}^{`} &:= \{f \in \mathcal{D}' \,:\, f'(1) = 0, \lim_{x\to 0} f'(x) \in (-\infty, 0] \text{ exists}\}, & \breve{\mathcal{D}}^{`} &:= \mathcal{D}^{`} \cap \breve{\mathcal{C}}
\end{aligned}
$$

of functions on $[0,1]$. On $\mathcal{E}$ we consider the sup-**metric** d_∞ and the 1-**pseudometric** d_1 defined by

$$
d_\infty(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)| \quad \text{and} \quad d_1(f,g) := \int |f-g| \quad \text{for } f,g \in \mathcal{E}.
$$

Note that d_1 is not a metric on $\mathcal{E}$, but on $\mathcal{C}$ because $d_1(f,g) = 0 \iff f = g$ almost everywhere. With any given $g \in \breve{\mathcal{C}}$ we associate its **stride**

$$
(2) \quad \lrcorner g := \sup\{\alpha \geq 0 \,:\, \alpha - \mathrm{id} \leq \alpha g\} \in [0,1]
$$

and note that the two slopes

$$
(3) \quad g'(0) := -\tfrac{1}{\lrcorner g} \in [-\infty, -1] \quad \text{and} \quad g'(1) = \inf\{\alpha \leq 0 \,:\, \alpha(\mathrm{id} - 1) \leq g\} \in [-1, 0]
$$

are well-defined, where we intentionally allow $g'(0)$ to assume the value $-\infty$.
Given $g \in \mathcal{E}$, we will use its **pseudo-inverse** $g^* \in \mathcal{E}$ defined by

$$
g^*(y) := \sup g^{-1}[y,1] = \sup\{x \in [0,1] \,:\, g(x) \geq y\} \quad \text{for } y \in [0,1].
$$

According to 2.1(c) below, g^* equals the compositional inverse g^{-1} if $g \in \mathcal{D}$. Thus we may, and shall, consistently write g^* in all cases from now on. In [7], the following properties of the pseudo-inverse are established.

2.1 Remark. *For $f, g \in \mathcal{E}$, the following statements hold.*

(a) $f \le g \implies f^* \le g^*$.

(b) *If $g \in \mathcal{C}$, then g^* is strictly decreasing.*

(c) *If $g \in \mathcal{D}$, then $g^* = g^{-1} \in \mathcal{D}$ is the inverse function of g.*

(d) $g \in \breve{\mathcal{D}} \iff g^* \in \breve{\mathcal{D}}$.

(e) $g \in \breve{\mathcal{D}}^{`} \implies g^* \in \breve{\mathcal{D}}'$.

(f) $\int g = \int g^*$.

(g) $d_1(f,g) = \int |f-g| = \int |f^* - g^*| = d_1(f^*, g^*)$.

(h) $\int_{g(t)}^1 g^* = \int_0^t g - tg(t)$ *for all $t \in [0,1]$.*

For given $g \in \mathcal{M}$ and $f \in \breve{\mathcal{D}}^{`}$, we define the continuous functions Ig and Df by setting

$$(Ig)(x) := \frac{\int_x^1 g}{\int g} \quad \text{and} \quad (Df)(x) := \frac{f'(x)}{f'(0)} \quad \text{for } x \in [0,1]$$

and formally introduce the operator $T\colon \mathcal{E} \to \mathcal{C}$ described in the introduction by setting

$$Tf := If^* \text{ for } f \in \mathcal{E}, \quad \text{that is,} \quad (Tf)(x) = \frac{\int_x^1 f^*}{\int f} \text{ for } x \in [0,1]$$

by 2.1(f), as well as its iterations $T^0 = \mathrm{id}_{\mathcal{E}}$ and $T^n := T \circ T^{n-1}$ for $n \in \mathbb{N}$.

2.2 Proposition. *For $g \in \mathcal{C}$, the following statements hold.*

(a) $Tg \in \breve{\mathcal{D}} \setminus \{1 - \mathrm{id}\}$.

(b) $1 - \frac{\mathrm{id}}{\int g} \le Tg \le 1 - \mathrm{id}$.

(c) $(Tg)(g(t)) \cdot \int g = \int_0^t g - tg(t)$ *for all $t \in [0,1]$.*

(d) *If $g|_{[0,1)} > 0$, then $Ig \in \breve{\mathcal{D}}^{`}$ with $-\frac{1}{(Ig)'(0)} = \backslash Ig = \int g$ and $DIg = g$.*

(e) *If $g \in \breve{\mathcal{D}}^{`}$, then $Dg \in \mathcal{C}$ with $(Dg)|_{[0,1)} > 0$, $\int Dg = \backslash g$ and $IDg = g$.*

(f) $\mathcal{C}^{>} := \{f \in \mathcal{C} \,:\, f|_{[0,1)} > 0\} \subseteq \mathcal{D}$, *and $I|_{\mathcal{C}^{>}} \colon \mathcal{C}^{>} \to \breve{\mathcal{D}}^{`}$ is bijective with inverse D.*

Proof. **(a)** Let $0 \le a < b \le 1$. Then $g^*(a) > g^*(b)$ by 2.1(b), hence

$$\int g \cdot \big[(Tg)(a) - (Tg)(b)\big] = \int_a^b g^* > (b-a)g^*(b) \ge 0,$$

showing that Tg is strictly decreasing. Similarly, for $a < x < b$, we obtain

$$\int g \cdot \frac{(Tg)(a) - (Tg)(x)}{x-a} = \frac{\int_a^x g^*}{x-a} > g^*(x) > \frac{\int_x^b g^*}{b-x} = \int g \cdot \frac{(Tg)(x) - (Tg)(b)}{b-x}, \tag{4}$$

hence Tg is convex. Evaluating (4) for $0 = a < x < b = 1$ yields $(Tg)(x) < 1 - x$, which completes the proof of the assertion.

(b) We have $\gamma := \int g = \int g^*$ according to 2.1(f) and $g^* \leq 1$, hence $\gamma(Tg)(x) = \int_x^1 g^* = \gamma - \int_0^x g^* \geq \gamma - x$ for all $x \in [0,1]$, implying the left inequality, while the right one follows from (a).

(c) follows from 2.1(f) and (h).

(d) By the assumptions, $Ig \in \mathcal{D}$, $(Ig)' = -\frac{g}{\int g}$ is continuous and increasing, $(Ig)'(1) = 0$, and we have $-\frac{1}{\backslash Ig} = (Ig)'(0) = -\frac{1}{\int g}$ using (3), so that $Ig \in \breve{\mathcal{D}}^{`}$ and $DIg = \frac{(Ig)'}{(Ig)'(0)} = g$.

(e) From $g \in \breve{\mathcal{D}}^{`}$ we conclude that $g'(0) < -1$ and $Dg = \frac{g'}{g'(0)} : [0,1] \to [0,1]$ is continuous and decreasing with $(Dg)|_{[0,1)} > 0$, $(Dg)(0) = 1$ and $(Dg)(1) = 0$, hence $Dg \in \mathcal{C}$. Moreover, $\int Dg = \frac{g(1)-g(0)}{g'(0)} = \backslash g$ by (3) and $(\int Dg)(IDg)(x) = \frac{\int_x^1 g'}{g'(0)} = \backslash g \cdot g(x)$ for all $x \in [0,1]$, hence $IDg = g$.

(f) follows from (c) and (d). $\square$

We now explicitly state the connection between the operator T and the IDE (1).

2.3 Proposition. *A function $g \in \mathcal{C}$ is a fixed point of the operator T if and only if g solves the IDE (1) for some $\gamma > 0$, and then g also satisfies the following properties:*

(a) $g \in \breve{\mathcal{D}}^{`}$.

(b) $\int g = \gamma$.

(c) $\backslash g = \gamma$.

(d) *g^* and g' are continuously differentiable on the interval $(0,1]$.*

(e) *$g''(1) = 1$ and $(g^*)'(1) = -\gamma$.*

Proof. First we assume that $g = Tg \in \mathcal{C}$ and set $\alpha := \int g$. Using Proposition 2.2(a), we conclude that $g \in \breve{\mathcal{D}}$ and then $g \in \breve{\mathcal{D}}^{`}$ by 2.1(c) and Proposition 2.2(d), settling assertion (a). Differentiating the equation $g(x) = (Tg)(x) = \frac{1}{\alpha}\int_x^1 g^*$, we arrive at $-\alpha g' = g^*$, that is, g solves (1) with $\gamma = \alpha$.
Conversely assume that $g \in \mathcal{C}$ (is differentiable and) solves (1) for some $\gamma > 0$. Integrating (1) while considering 2.1(c) leads to $g(x) = \frac{1}{\gamma}\int_x^1 g^*$. Plugging 0 into this, yields $1 = g(0) = \frac{1}{\gamma}\int g$ by 2.1(f), thereby showing (b) and $g = Tg$.

(a) Plugging 0 into (1) and using 2.1(c) gives $g'(0) = \frac{-g^*(0)}{\gamma} = \frac{-1}{\gamma}$, hence $\backslash g = \gamma$ by (3).

(b) By (a) and 2.1(e), we have $g^* \in \mathcal{D}'$, and the assertion follows from (1).

(c) Plugging 1 into the derivative of (1), yields $-\gamma g''(1) = (g^*)'(g(0)) = \frac{1}{g'(0)} = -\gamma$ by the chain rule and (c), thus $g''(1) = 1$ and $(g^*)'(1) = -\gamma$. $\square$

Now we want to construct a complete T-invariant subset $\mathcal{K}$ of $\breve{\mathcal{C}}$. To this end, we need to bound area and stride of Tg from below.

2.4 Lemma. *Let $g \in \breve{\mathcal{C}}$, $0 < \alpha \leq \backslash g$, $\beta := \inf g^{-1}\{0\}$ and $\gamma := \int g$. Then*

(a) *$\alpha \leq 2\gamma \leq \beta \leq 1$, and $\alpha = 2\gamma \implies 2\gamma = \beta \implies \int Tg = \frac{1}{3}$,*

(b) *$(Tg)' : [0,1] \to (-\infty, 0]$ exists, is continuous, strictly increasing and concave,*

(c) *$Tg \in \breve{\mathcal{D}}^{`}$ with $\backslash Tg = \frac{\gamma}{\beta}$,*

(d) $\frac{\beta}{\gamma}(\mathrm{id}-1) \le (Tg)' \le \frac{\alpha}{\gamma}(\mathrm{id}-1)$,

(e) $\int Tg \le \frac{1}{3}$,

(f) $\alpha\beta - 4\alpha\gamma + 4\gamma^2 \le 6(\beta-\alpha)\gamma \int Tg$.

Proof. **(a)** From $g \in \check{\mathcal{C}}$ and the definition of β, we infer that $g(x) \le 1 - \frac{x}{\beta}$ for all $x \in [0,\beta]$, hence $\frac{\alpha}{2} = \int_0^\alpha (1 - \frac{\mathrm{id}}{\alpha}) \le \int g = \gamma \le \int_0^\beta (1 - \frac{\mathrm{id}}{\beta}) = \frac{\beta}{2}$, settling the asserted inequality chain. From this, we also see that $\frac{\alpha}{2} = \gamma \iff g|_{[\alpha,1]} = 0 \implies \beta = \alpha$ and that $\gamma = \frac{\beta}{2} \implies g^*(y) = \beta(1-y)$ for $y \in (0,1] \implies Tg = (1-\mathrm{id})^2 \implies \int Tg = \frac{1}{3}$.

(b) By its convexity, g is strictly decreasing on $[0,\beta]$. Thus $f(x) := g(\beta x)$ for $x \in [0,1]$ defines a function $f \in \check{\mathcal{D}}$, which satisfies $1 - \frac{\beta}{\alpha}\mathrm{id} \le f \le 1 - \mathrm{id}$. Using 2.1(d), (a) and (f), we infer that $f^* \in \check{\mathcal{D}}$,

$$(5) \qquad \tfrac{\alpha}{\beta}(1-\mathrm{id}) \le f^* \le 1-\mathrm{id} \quad \text{and} \quad \textstyle\int f^* = \int f = \frac{1}{\beta}\int g = \frac{\gamma}{\beta}.$$

Because $\beta f^*(x) = g^*(x)$ for all $x \in (0,1]$, we conclude that $Tg = Tf$ is differentiable with continuous derivative

$$(6) \qquad (Tg)' = (Tf)' = -\tfrac{\beta}{\gamma} f^*,$$

and the assertions follow.

(c) From (3) and (6), we infer that $\backslash Tg = -\frac{1}{(Tg)'(0)} = \frac{\gamma}{\beta}$ and $(Tg)'(1) = 0$, hence $Tg \in \mathcal{D}^{`}$, while $Tg \in \check{\mathcal{D}}$ holds by Proposition 2.2(a).

(d) follows from (5) and (6).

(e) By (a)–(c) and becaus $\int (Tg)' = (Tg)(1) - (Tg)(0) = -1 = \int (2\mathrm{id} - 2)$,

$$s := \sup\{0 < x < 1 \,:\, (Tg)'(x) \le 2x - 2\} \in (0,1]$$

is well-defined, $(Tg)'|_{[0,s]} \le 2\mathrm{id}_{[0,s]} - 2$ and $(Tg)'|_{[s,1]} \ge 2\mathrm{id}_{[s,1]} - 2$. We conclude that $(Tg)(x) \le 1 + \int_0^x (2\mathrm{id} - 2) = (1-x)^2$ for $x \in [0,s]$ and also $(Tg)(x) = -\int_x^1 (Tg)' \le -\int_x^1 (2\mathrm{id}-2) = (1-x)^2$ for $x \in [s,1]$. Hence, $\int Tg \le \int (1-\mathrm{id})^2 = \frac{1}{3}$.

(f) Using (a), the asserted inequality is verified directly if $\alpha \le 2\gamma = \beta$, and we may assume $\alpha < 2\gamma < \beta$. We conclude that $\xi := \frac{2\gamma-\alpha}{\beta-\alpha} \in (0,1)$ and define $b\colon [0,1] \to \mathbb{R}$ by setting

$$b(x) := \begin{cases} b_0(x) := 1 - \frac{\beta}{\gamma}x + \frac{\beta^2 - 2\alpha(\beta-\gamma)}{2(2\gamma-\alpha)\gamma}x^2 & \text{for } x \in [0,\xi], \\ b_1(x) := \frac{\alpha}{2\gamma}(1-x)^2 & \text{for } x \in [\xi,1]. \end{cases}$$

It is straightforward to verify that $b \in \check{\mathcal{D}}^{`}$ with derivative $b'\colon [0,1] \to \mathbb{R}$ given by

$$b'(x) = \begin{cases} b_0'(x) = -\frac{\beta}{\gamma} + \frac{\beta^2 - 2\alpha(\beta-\gamma)}{(2\gamma-\alpha)\gamma}x & \text{for } x \in [0,\xi], \\ b_1'(x) = -\frac{\alpha}{\gamma}(1-x) & \text{for } x \in [\xi,1], \end{cases}$$

which is concave and consists of two lines meeting in the point $(\xi, \frac{\alpha}{\gamma}(\xi - 1))$. Using (b) and (d), we infer that

$$s := \inf\{x \in (0,1] \,:\, (Tg)'(x) \le b'(x)\} \in [0,\xi],$$

$(Tg)'|_{[0,s]} \geq b'|_{[0,s]}$ and $(Tg)'|_{[s,1]} \leq b'|_{[s,1]}$. Thus $(Tg)(x) = 1 + \int_0^x (Tg)' \geq 1 + \int_0^x b' = b(x)$ for $x \in [0,s]$ and also $(Tg)(x) = -\int_x^1 (Tg)' \geq -\int_x^1 b' = b(x)$ for $x \in [s,1]$, hence

$$\textstyle\int Tg \geq \int_0^1 b = \int_0^\xi b_0 + \int_\xi^1 b_1 = \frac{\alpha\beta - 4\alpha\gamma + 4\gamma^2}{6(\beta-\alpha)\gamma}$$

after a tedious but straightforward calculation. $\square$

We are now ready to establish the set

$$\mathcal{K} := \{g \in \breve{\mathcal{C}} \,:\, \measuredangle g, \textstyle\int g \geq \frac{1}{5}\}.$$

and its properties concerning the operator T, needed to prove our main theorems.

2.5 Theorem. *The set $\mathcal{K}$ has the following properties.*

(a) *$T(\mathcal{K}) \subseteq \mathcal{K}$.*

(b) *For each $f \in \mathcal{E}$ there exists $n \in \mathbb{N}$ such that $T^n f \in \mathcal{K}$.*

(c) *The two metrics d_∞ and d_1 are equivalent on $\mathcal{K}$ in the sense that $d_1(f,g) \leq d_\infty(f,g) \leq 5\sqrt{d_1(f,g)}$ for all $f, g \in \mathcal{K}$.*

(d) *The metric space $(\mathcal{K}, d_\infty)$ resp. $(\mathcal{K}, d_1)$ is complete.*

(e) *The restriction $T|_\mathcal{K} \colon \mathcal{K} \to \mathcal{K}$ is continuous when equipping domain and codomain independently with d_∞ or d_1.*

(f) *Every sequence in $(\mathcal{K}, d_\infty)$ resp. $(\mathcal{K}, d_1)$ has a convergent subsequence.*

Proof. **(a)** Let $g \in \mathcal{K}$. Then $\gamma := \int g$, $\measuredangle g \geq \frac{1}{5}$ and $\beta := \inf g^{-1}\{0\} \in [\frac{2}{5}, 1]$ by 2.4(a). With 2.4(f) we infer that $\int Tg \geq u(\gamma)$, where the function $u \colon (0,\infty) \to \mathbb{R}$ satisfies

$$u(x) = \frac{\beta - 4x + 20x^2}{6(5\beta - 1)x} \quad \text{and} \quad u'(x) = \frac{20x^2 - \beta}{6(5\beta-1)x^2} \quad \text{for all } x > 0.$$

We conclude that $\int Tg \geq u\big(\frac{\beta}{2\sqrt{5}}\big) = \frac{2}{3} \cdot \frac{\sqrt{5}-1}{5\beta-1} \geq \frac{\sqrt{5}-1}{6} > \frac{1}{5}$. Moreover, $Tg \in \breve{\mathcal{D}}^{\backprime} \subseteq \breve{\mathcal{C}}$ and $\measuredangle Tg = \frac{\gamma}{\beta} \geq \gamma \geq \frac{1}{5}$ by 2.4(c). In total we have shown $Tg \in \mathcal{K}$.

(b) Let $f \in \mathcal{E}$, and set $f_n := T^{2+n} f$, $\gamma_n := \int f_n$ and $\alpha_n := \measuredangle f_n$ for $n \in \mathbb{N}_0$. For $0 < \alpha < 1$, define $\vartheta(\alpha) := \frac{3}{2}(\alpha + \sqrt{\alpha})$ and the function $u_\alpha \colon (0,\infty) \to (0,\infty)$ given by

$$u_\alpha(\gamma) := \frac{\alpha - 4\alpha\gamma + 4\gamma^2}{6(1-\alpha)\gamma}, \quad \text{thus} \quad u_\alpha'(\gamma) = \frac{4\gamma^2 - \alpha}{6(1-\alpha)\gamma^2} \quad \text{for all } \gamma > 0,$$

which therefore satisfies $u_\alpha(\gamma) \geq u_\alpha\big(\frac{\sqrt{\alpha}}{2}\big) = \frac{\alpha}{\vartheta(\alpha)}$ for all $\gamma > 0$. With Proposition 2.2(a) and (b) and Lemma 2.4(c) and (f), we obtain

$$f_n \in \breve{\mathcal{D}}, \quad \alpha_{n+1} = \gamma_n \quad \text{and} \quad \gamma_{n+1} \geq u_{\alpha_n}(\gamma_n) \geq \tfrac{\alpha_n}{\vartheta(\alpha_n)} \quad \text{for all } n \in \mathbb{N}_0.$$

Because

$$0 < \alpha \leq \tfrac{1}{5} \implies \vartheta(\alpha) \leq \vartheta\big(\tfrac{1}{5}\big) < 1 \quad \text{and} \quad \tfrac{1}{5} \leq \alpha < 1 \implies \tfrac{\alpha}{\vartheta(\alpha)} \geq \tfrac{\sqrt{5}-1}{6} > \tfrac{1}{5},$$

there consequently exists $n \in \mathbb{N}_0$ with $\alpha_n, \gamma_n \geq \frac{1}{5}$, hence $f_n \in \mathcal{K}$.

(c) Let $f, g \in \mathcal{K}$. The estimate $\int|f-g| \leq \sup_{x\in[0,1]} |f(x)-g(x)|$ settles the left inequality. As for the right one, we may assume that $\delta := d_\infty(f,g) = f(x_0)-g(x_0)$ for some $x_0 \in [0,1]$. From $\lrtriangle f, \lrtriangle g \geq \frac{1}{5}$ and $f, g \in \breve{\mathcal{C}}$ we infer that

$$f(x) - g(x) \geq \delta - 5|x - x_0| \quad \text{for all } x \in [0,1],$$

hence $0 \leq a := x_0 - \frac{\delta}{5} \leq b := x_0 + \frac{\delta}{5} \leq 1$ and $d_1(f,g) \geq \int_a^b (f-g) \geq \frac{\delta}{2}(b-a) = \frac{\delta^2}{5}$.

(d) Recall that $\mathcal{C}^0[0,1]$, the $\mathbb{R}$-vector space of continuous functions on the interval $[0,1]$, is complete with respect to the sup-norm. Therefore, each Cauchy sequence $(g_n)_{n\in\mathbb{N}}$ in $(\mathcal{K}, d_\infty)$ converges to some function $g \in \mathcal{C}^0[0,1]$ satisfying $g(0) = 1$ and $g(1) = 0$. Clearly, g is again decreasing and convex, and both inequalities $g \geq 1 - 5\mathrm{id}$ and $\int g \geq \frac{1}{5}$ hold. Hence $g \in \mathcal{K}$, showing that $(\mathcal{K}, d_\infty)$ is complete. The completeness of $(\mathcal{K}, d_1)$ follows with (c).

(e) Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ converging to $g \in \mathcal{K}$ with respect to d_∞ or d_1. Setting $\hat{g} := \int g \cdot Tg$ and $\hat{g}_n := \int g_n \cdot Tg_n$, Remark 2.1(g) yields

$$|\hat{g}(x) - \hat{g}_n(x)| = \left|\int_x^1 (g^* - g_n^*)\right| \leq \int |g^* - g_n^*| = d_1(g, g_n) \leq d_\infty(g, g_n)$$

for all $x \in [0,1]$, implying $\lim_{n\to\infty} d_\infty(\hat{g}, \hat{g}_n) = 0$ and $\lim_{n\to\infty} \int g_n = \int g$ because $(Tg)(0) = 1 = (Tg_n)(0)$. We conclude that $d_1(Tg, Tg_n) \leq d_\infty(Tg, Tg_n) \to 0$ as $n \to \infty$.

(f) Let $g_n \in \mathcal{K}$ for $n \in \mathbb{N}$. Because g_n is convex and $\lrtriangle g_n \geq \frac{1}{5}$, we conclude that $|g_n(x_1) - g_n(x_2)| \leq 5|x_2 - x_1|$ for all $x_1, x_2 \in [0,1]$, $n \in \mathbb{N}$. Therefore the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous, and as it is also uniformly bounded, the Arzelà-Ascoli theorem guarantees that it has a convergent subsequence in $(\mathcal{K}, d_\infty)$. By (d) its limit lies in $\mathcal{K}$, and by (c) the same subsequence converges in $(\mathcal{K}, d_1)$ to the same limit. $\square$

Henceforth, when speaking about convergence in $\mathcal{K}$, we mean convergence in $(\mathcal{K}, d_\infty)$, i.e. uniform convergence, *and*, by 2.5(c) equivalently, convergence in $(\mathcal{K}, d_1)$.

3 Crossing number

The crucial proofs of 2.4(e) and (f) rest on the fact that $(Tg)'$ intersects another derivative at most once. More generally, if $f, g \in \mathcal{D}$, $g \leq f \neq g$ and $(Tg)' - (Tf)' = \frac{f^*}{\int f} - \frac{g^*}{\int g}$ changes its sign only once (from $-$ to $+$ in this case), then we will have $Tg \leq Tf$. To propagate this reasoning to the next iteration step, we would require the difference of $(Tf)^*$ and $(Tg)^*$, after somehow stretching them vertically, to also change sign at most once. But a vertical stretching of, say $(Tg)^*$, corresponds to a horizontal stretching of Tg and thus of $(Tg)' = -\frac{g^*}{\int g}$, which again corresponds to a vertical *and* horizontal stretching of g. Because it is hard to tell the stretching factors in advance, we will consider the difference of f and g after arbitrary horizontal *and* vertical stretching.

As a first step, we want to count how often a given continuous function $\Delta\colon [a,b] \to \mathbb{R}$ defined on a bounded, closed interval $[a,b]$ changes sign. To this end, we call a closed subinterval $[c,d] \subseteq [a,b]$ with $a < c \leq d < b$ and image $\Delta([c,d]) = \{0\}$ a **sign switch** of Δ if there exists $\delta \in (0, \min\{c-a, b-d\}]$ such that $\Delta(c-x) \cdot \Delta(d+x) < 0$ for all $x \in (0, \delta]$. By $\mathcal{X}\Delta$ we denote the set of all sign switches of Δ and by $\chi\Delta := \#\mathcal{X}\Delta \in \mathbb{N}_0 \cup \{\infty\}$ their number.

3.1 Remark. *Let $k \in \mathbb{N}_0$, $a, b, a', b' \in \mathbb{R}$ with $a < b$ and $a' < b'$. Let $u\colon [a', b'] \to [a, b]$ and $\Delta\colon [a, b] \to \mathbb{R}$ be continuous functions, u bijective. The following statements hold.*

(a) $\mathcal{X}(c\Delta) = \mathcal{X}\Delta$ *for every* $c \in \mathbb{R} \setminus \{0\}$.

(b) $\chi(\Delta \circ u) = \chi\Delta$.

(c) $\chi\Delta \geq k$ *if and only if there exist* $a \leq x_0 < \cdots < x_k \leq b$ *such that* $\Delta(x_{i-1}) \cdot \Delta(x_i) < 0$ *for* $i \in \{1, \ldots, k\}$.

(d) *If* $\Delta(a) \cdot \Delta(b) > 0$, *then* $\chi\Delta$ *is even or* ∞.

(e) *If* $\Delta(a) \cdot \Delta(b) < 0$, *then* $\chi\Delta$ *is odd or* ∞.

(f) *Suppose that* Δ *is continuously differentiable. Then* $\chi\Delta' \geq \chi\Delta - 1$. *If* $\Delta(a) \cdot \Delta'(a) > 0$ *in addition, then* $\chi\Delta' \geq \chi\Delta$.

(g) *Let* $\Delta_n\colon [a, b] \to \mathbb{R}$ *for* $n \in \mathbb{N}$ *be continuous functions such that* $(\Delta_n)_{n\in\mathbb{N}}$ *converges uniformly to* Δ. *Then* $\chi\Delta \leq \sup_{n\in\mathbb{N}} \chi\Delta_n$.

Proof. **(a)**–**(c)** are immediate from the definition of sign switches.

(a) and **(e)** follow from (c).

(b) Suppose that $k := \chi\Delta \in \mathbb{N}_0$. Then there are $a \leq x_0 < \cdots < x_k \leq b$ as in (c). By the mean value theorem, we can find $y_i \in [x_{i-1}, x_i]$ with $\Delta'(y_i) \cdot \Delta(x_i) > 0$ for $i \in \{1, \ldots, k\}$. This shows $\chi\Delta' \geq k - 1$ according to (c).

If $\Delta(a) \cdot \Delta'(a) > 0$, then we can find $y_0 \in [a, x_0]$ with $\Delta'(y_0) \cdot \Delta(x_0) > 0$; hence $\chi\Delta' \geq k$ by (c) again.

(c) Let $\chi\Delta \geq k$ with $x_0, \ldots, x_k$ as in (c). By assumption, we can choose $n \in \mathbb{N}$ such that $|\Delta_n(x_i) - \Delta(x_i)| < \min\{|\Delta(x_0)|, \ldots, |\Delta(x_k)|\}$, hence $\Delta_n(x_i) \cdot \Delta(x_i) > 0$ for $i \in \{0, \ldots, k\}$. This implies $\Delta_n(x_{i-1}) \cdot \Delta_n(x_i) < 0$ for $i \in \{1, \ldots, k\}$ which, by (c), is equivalent to $\chi\Delta_n \geq k$. $\square$

Given two functions $f, g \in \mathcal{D}$ and $a, b > 0$, we introduce the function

$$f \odot a := f \circ (a\mathrm{id})\colon [0, \tfrac{1}{a}] \to [0, 1], \quad x \mapsto f(ax)$$

obtained by stretching f horizontally with the factor $\frac{1}{a}$, and consider the continuous function $f \odot a - bg\colon [0, \min\{1, \frac{1}{a}\}] \to \mathbb{R}$. The next lemma tells us how its number of sign switches behaves under swapping f with g and under the operators * and I.

3.2 Lemma. *Let $a, b > 0$ and $f, g \in \mathcal{D}$. The following statements hold.*

(a) $\chi(f \odot a - bg) = \chi(g \odot \frac{1}{a} - \frac{1}{b} f)$.

(b) $\chi(f \odot a - bg) = \chi(g^* \odot \frac{1}{b} - \frac{1}{a} \cdot f^*)$.

(c) *Let* $\hat{\Delta} := If \odot a - bIg$, $b' := \frac{b \int f}{a \int g}$ *and* $\Delta := f \odot a - b'g$. *Then* $\chi\hat{\Delta} \leq 1 + \chi\Delta$. *If* $b < 1 < b'$ *or* $b' < 1 < b$, *then* $\chi\hat{\Delta} \leq \chi\Delta$.

(d) *If either* $a, b < 1$ *or* $a, b > 1$, *then* $\chi(f \odot a - bg)$ *is even or* ∞.

(e) *If* $a < 1 < b$ *or* $b < 1 < a$, *then* $\chi(f \odot a - bg)$ *is odd or* ∞.

(f) *If* $a, b \leq 1$ *and* $g \leq f$, *then* $\chi(f \odot a - bg) = 0$.

Proof. **(a)** Let $a' := \min\{1, \frac{1}{a}\}$, $\Delta := f\circledcirc a - bg$ and $\tilde{\Delta} := g\circledcirc\frac{1}{a} - \frac{1}{b}f$. Then

$$b\tilde{\Delta}(ax) = b \cdot \big(g(\tfrac{ax}{a}) - \tfrac{1}{b}f(ax)\big) = bg(x) - f(ax) = -\Delta(x)$$

for all $x \in [0, a']$. Hence, $\chi\Delta = \chi\tilde{\Delta}$ by 3.1(a) and (b).

(b) Let $a' := \min\{1, \frac{1}{a}\}$, $b' := \min\{1, b\}$, $\Delta := f\circledcirc a - bg\colon [0, a'] \to \mathbb{R}$ and

$$\tilde{\Delta} := g^*\circledcirc\tfrac{1}{b} - \tfrac{1}{a}f^*\colon [0, b'] \to \mathbb{R}.$$

Because the function $u\colon [0, a'] \to [0, b']$, $x \mapsto \min\{f(ax), bg(x)\}$ is bijective by 2.1(c) and $\mathcal{X}(\tilde{\Delta} \circ u) = \mathcal{X}\Delta$, the assertion follows from 3.1(b).

(c) According to its definition, $\hat{\Delta}$ is differentiable with continuous derivative

$$\hat{\Delta}' = a(If)'\circledcirc a - b(Ig)' = -\tfrac{a}{\int f}\Delta,$$

hence $\chi\Delta = \chi\hat{\Delta}' \geq \chi\hat{\Delta} - 1$ by 3.1(a) and (f). If $b < 1 < b'$ or $b' < 1 < b$, then $\hat{\Delta}(0) \cdot \hat{\Delta}'(0) = (1 - b) \cdot \frac{a}{\int f} \cdot (b' - 1) > 0$, hence $\chi\Delta = \chi\hat{\Delta}' \geq \chi\hat{\Delta}$, again by 3.1(a) and (f).

(d) Set $\Delta := f\circledcirc a - bg$. If $a, b < 1$, then $\Delta(0) = f(0) - bg(0) = 1 - b > 0$ and $\Delta(1) = f(a) > 0$. If $a, b > 1$, then $\Delta(0) = f(0) - bg(0) = 1 - b < 0$ and $\Delta(\frac{1}{a}) = -g(\frac{1}{a}) < 0$. In both cases, the assertion follows from 3.1(d).

(e) Set $\Delta := f\circledcirc a - bg$. If $a < 1 < b$, then $\Delta(0) = f(0) - bg(0) = 1 - b < 0$ and $\Delta(1) = f(a) > 0$. If $b < 1 < a$, then $\Delta(0) = f(0) - bg(0) = 1 - b > 0$ and $\Delta(\frac{1}{a}) = -g(\frac{1}{a}) < 0$. In both cases, the assertion follows from 3.1(e).

(f) From $f, g \in \mathcal{D}$, $a, b \leq 1$ and $g \leq f$, we conclude $f(ax) \geq f(x)$ and $bg(x) \leq g(x)$, hence $(f\circledcirc a - bg)(x) = f(ax) - bg(x) \geq f(x) - g(x) \geq 0$ for all $x \in [0, 1]$, so that $\chi(f\circledcirc a - bg) = 0$. ☐

Given $f, g \in \mathcal{D}$, we define the **crossing number**

$$\chi(f, g) := \sup\{\chi(f\circledcirc a - bg) \,:\, a, b > 0\} \in \mathbb{N}_0 \cup \{\infty\}$$

of f with g. We write $g \lhd f$ or $f \rhd g$ and say that f **dominates** g if $\chi(f, g) = 2$ and $g \leq f$.

3.3 Lemma. *For $f, g \in \mathcal{D}$, the following statements hold.*

(a) $\chi(f, g) = \chi(g, f) \geq 1$.

(b) $g \lhd f \iff g^* \lhd f^*$.

(c) *$\chi(f, f)$ is odd or ∞.*

(d) $\chi(f, g) \leq 2 \implies \chi(f - g) = 0 \iff f \leq g$ *or* $g \leq f \implies f = g$ *or* $\chi(f, g) \geq 2$.

(e) $\chi(f, g) = 2 \iff f \neq g$ *and either* $f \lhd g$ *or* $g \lhd f$.

(f) *If $g \lhd f$ and $0 < \min\{a, b\} \leq 1$, then $\chi(f\circledcirc a - bg) \leq 1$.*

Proof. **(a)** follows from 3.2(a) and (e).

(b) follows from 3.2(b) and 2.1(a).

(c) If $a, b \leq 1$ or $a, b \geq 1$, then $\chi(f \circledcirc a - bf) = 0$ by 3.2(a) and (f). Therefore, the assertion follows with 3.2(e).

(d) We prove the first implication by contraposition. Suppose that $\chi(f - g) > 0$, that is, neither $f \leq g$ nor $g \leq f$ holds. Then we may assume w.l.o.g. that there are $0 < x_1 < x_2 < 1$ with $f(x_1) > g(x_1)$ and $f(x_2) < g(x_2)$, and $\Delta := f \circledcirc (1 - \varepsilon) - (1 + \varepsilon) g$ by continuity still satisfies $\Delta(x_1) > 0$ and $\Delta(x_2) < 0$ for sufficiently small $\varepsilon > 0$. But then $\Delta(0) = -\varepsilon < 0$ and $\Delta(1) = f(1 - \varepsilon) > 0$, so that $\chi\Delta \geq 3$ by 3.1(c). The last implication is obvious.

(e) follows from (c), (d) and the definition of $\lhd$.

(f) Set $\Delta = f \circledcirc a - bg$, then $\chi\Delta \leq 2$. The case $a, b \in (0, 1]$ is covered by 3.2(f). If $0 < a \leq 1 < b$, then $\Delta(0) = b - 1 > 0$, and $\chi(\Delta) = 2$ would imply $\Delta(x) < 0$ for some $x \in (0, 1)$, where we may assume $a < 1$ by continuity, in violation of 3.2(e). If $0 < b \leq 1 < a$, then $\Delta(\frac{1}{a}) = -bg(\frac{1}{a}) < 0$, and $\chi(\Delta) = 2$ would imply $\Delta(x) > 0$ for some $x \in (0, \frac{1}{a})$, where we may assume $b < 1$ by continuity, violating 3.2(e) again. $\square$

Now we can show that domination in $\mathcal{D}$ is preserved by the operators I and T.

3.4 Theorem. *Let $f, g \in \mathcal{D}$ such that $g \lhd f$. Then $Ig \lhd If$ and $Tg \lhd Tf$.*

Proof. By continuity and 2.2(d), we have

(7) $$\llcorner Ig = \int g < \int f = \llcorner If.$$

Let $a, b > 0$ and $b' := \frac{b \int f}{a \int g}$. Then $k' := \chi(f \circledcirc a - b'g) \leq 2$ and $k := \chi(If \circledcirc a - bIg) \leq 1 + k'$ according to 3.2(c). Because of 3.2 (d), we need only consider the two cases $a \leq 1 \leq b$ and $b < 1 \leq a$, in order to show that $k \leq 2$.

- If $a \leq 1 \leq b$, then $b' > 1$ by (7), hence $k \leq 1 + k' \leq 2$ by 3.3(f).
- Now suppose that $b < 1 \leq a$. If $b' \leq 1$, then $k \leq 1 + k' \leq 2$ by 3.3(f), and if $b' > 1$, then $k \leq k' \leq 2$ by 3.2(c).

In view of 3.3(a) and 3.2(d) we have altogether proved $\chi(If, Ig) = 2$, hence $Ig \lhd If$ by (7). Because $f^*, g^* \in \mathcal{D}$ by 2.1 and $g^* \lhd f^*$ by 3.3(b), we also obtain $Tg = Ig^* \lhd If^* = Tf$. $\square$

We conclude this section by observing that domination is also preserved under taking limits.

3.5 Proposition. *Let $f_n, g_n \in \mathcal{K}$ for all $n \in \mathbb{N}$ such that the limits $f := \lim_{n\to\infty} f_n$ and $g := \lim_{n\to\infty} g_n$ exist. The following statements hold.*

(a) *$\chi(f, g) \leq \sup_{n\in\mathbb{N}} \chi(f_n, g_n)$.*

(b) *If $g_n \lhd f_n$ for all $n \in \mathbb{N}$, then either $f = g$ or $g \lhd f$.*

Proof. **(a)** Let $a, b > 0$, set $\Delta := f \circledcirc a - bg$ and $\Delta_n := f_n \circledcirc a - bg_n$ for all $n \in \mathbb{N}$. Then $(\Delta_n)_{n\in\mathbb{N}}$ converges uniformly to Δ by Theorem 2.5. Hence, $\chi\Delta \leq \sup_{n\in\mathbb{N}} \chi\Delta_n$ due to 3.1(g).

(b) From the assumption and (b), we infer that $g \leq f$ and $\chi(f, g) \leq 2$. With 3.3(a), (d) and (e), we conclude that either $f = g$ or $g \lhd f$. $\square$

4 Existence, uniqueness and global convergence

We will now construct a unit stribola, that is, a solution to the IDE (1). To this end, we define the **canonical stribolic iterates** and their **areas**

(8) $\quad h_n := T^n 1_{[0,1]} \in \mathcal{E}, \quad \kappa_n := \int h_n \quad \text{for } n \in \mathbb{N}_0.$

In particular, we have $h_0(x) = 1 = h_0^*(x)$, $h_1(x) = (Ih_0^*)(x) = 1 - x = h_1^*(x)$, $h_2(x) = (Ih_1^*)(x) = (1-x)^2$, $h_2^*(x) = 1 - \sqrt{x}$, $h_3(x) = (Ih_2^*)(x) = 1 - 3x + 2x^{\frac{3}{2}}$ for $x \in [0,1]$, and $(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \dots) = (1, \frac{1}{2}, \frac{1}{3}, \frac{3}{10}, \dots)$.

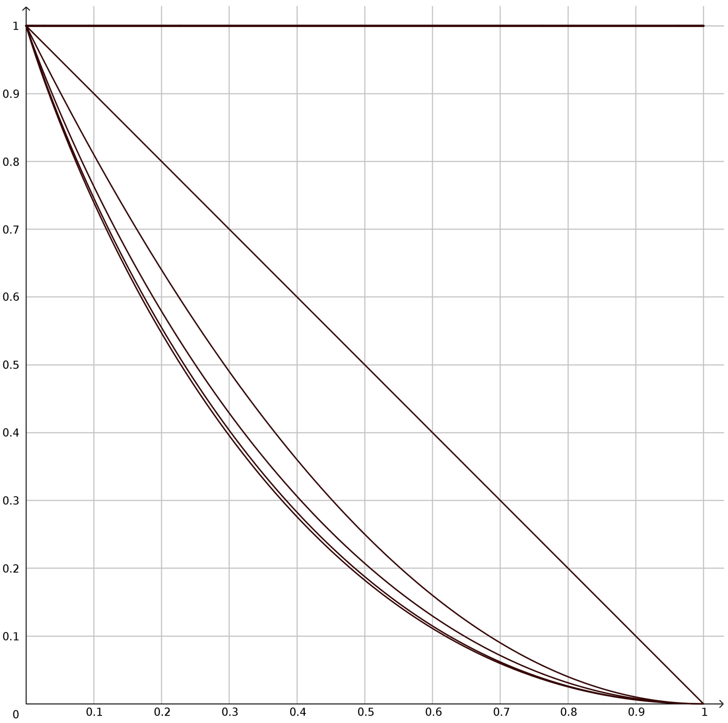

Figure 2: Graphs of $h_0, \dots, h_5$

Repeated application of Theorem 3.4 will show that these iterates descend to a unit stribola.

4.1 Theorem. *The sequence $(h_n)_{n\in\mathbb{N}_0}$ from (8) satisfies $h_{n+1} \lhd h_n \in \mathcal{K}$ for all $n \in \mathbb{N}$ and converges to a function $h \in \mathcal{K}$. Moreover, $h = Th$ solves (1) with $\gamma = \int h$.*

Proof. Because $h_1 \in \mathcal{K}$ is linear, it dominates the non-linear, convex function $h_2 \in \mathcal{K}$. With Theorems 3.4 and 2.5(a), we inductively conclude that

(9) $\quad h_{n+1} \lhd h_n \in \mathcal{K} \quad \text{for all } n \in \mathbb{N}.$

By 2.5(f), there are positive integers $n_1 < n_2 < \cdots$ and $h \in \mathcal{K}$ such that $\lim_{k\to\infty} h_{n_k} = h$, which implies $\lim_{n\to\infty} h_n = h$ because of (9). Finally, using 2.5(e), (8) and Proposition 2.3, we see that

$$Th = T\Big(\lim_{n\to\infty} h_n\Big) = \lim_{n\to\infty} Th_n = \lim_{n\to\infty} h_{n+1} = h$$

solves (1) with $\gamma = \int h$. $\square$

Until the end of this paper, we shall focus on the unit stribola $h := \lim_{n\to\infty} h_n \in \breve{\mathcal{D}}^{\grave{}}$ and its area

$$(10)\quad \kappa := \textstyle\int h = -\dfrac{1}{h'(0)} = \lrcorner h = \lim\limits_{n\to\infty} \kappa_n = \inf\limits_{n\in\mathbb{N}_0} \kappa_n,$$

as established in Theorem 4.1.

4.2 Corollary. *For $m, n \in \mathbb{N}$ with $m < n$, we have*

(a) $h_n \lhd h_m$,

(b) $h \lhd h_m$,

(c) $\chi(h_m, h_m) = 1$,

(d) $\chi(h, h) = 1$.

Proof. **(a)** Clearly, $h_n \lhd h_1$ because h_1 is linear, $h_n \neq h_1$ and h_n is convex. The assertion follows with Theorem 3.4 by induction on m.

(b) holds according to (a) and Proposition 3.5(b).

(c) We proceed by induction on m. Obviously, $\chi(h_1, h_1) = 1$. Assume that $\chi(h_m, h_m) = 1$. Because $h_{m+1} = Ih_m^*$, we conclude that $\chi(h_{m+1}, h_{m+1}) \leq 2$ using 3.2(b) and (c), hence $\chi(h_{m+1}, h_{m+1}) = 1$ by 3.3(c).

(d) follows from (c) with 3.5(a) and 3.3(a). □

As an aside, we observe that the function

$$(11)\quad \tilde{h} := \tfrac{1}{\kappa} h \circ \kappa : [0, \tfrac{1}{\kappa}] \to [0, \tfrac{1}{\kappa}] \quad \text{satisfies} \quad -\tilde{h}' = \tilde{h}^*,$$

that is, $\tilde{h}$ becomes its own derivative when rotated clockwise about the origin by 90°. A function defined on $[0, a]$ for some $a > 0$ and satisfying the above IDE shall be called a **standard stribola**.

Our next goals are to prove that h is the only unit stribola and that $(T^n f)_{n\in\mathbb{N}}$ converges to h for every $f \in \mathcal{E}$ (global convergence). We begin with a result frequently used in the sequel, that involves the stride, see (2) and (3).

4.3 Stride Lemma. *Let $f, g \in \breve{\mathcal{D}}$ satisfy $g \lhd f$ and $\lrcorner g > 0$. Then*

$$(12)\quad f \circ \lrcorner f \leq g \circ \lrcorner g$$

and $\lrcorner g \cdot \int f < \lrcorner f \cdot \int g$.

Proof. We have $0 < \lrcorner g \leq \lrcorner f$ by assumption. Suppose (12) were wrong. Then $f(\frac{\lrcorner f}{\lrcorner g} x_2) > g(x_2)$ for some $x_2 \in (0, \frac{\lrcorner g}{\lrcorner f})$. By continuity, there is $a > \frac{\lrcorner f}{\lrcorner g}$ such that $f(ax_2) > g(x_2)$ still holds. But because $\lrcorner(f \circ a) = \frac{\lrcorner f}{a} < \lrcorner g$, we can find $x_1 \in (0, x_2)$ with $f(ax_1) < g(x_1)$. Again by continuity, there is $b \in (0, 1)$ such that $\Delta := f \circ a - bg$ still satisfies $\Delta(x_1) < 0$ and $\Delta(x_2) > 0$. Since also $\Delta(0) = 1 - b > 0$ and $\Delta(\frac{1}{a}) = -bg(\frac{1}{a}) < 0$, we would have $\chi(f, g) \geq 3$ by 3.1(c). Therefore (12) holds. Because $f \circ \lrcorner f = g \circ \lrcorner g$ would imply $f = g$, we conclude that $\lrcorner g \cdot \int f < \lrcorner f \cdot \int g$. □

4.4 Corollary. *If the unit stribola h dominates $g \in \breve{\mathcal{D}}$, then $\lrcorner g < \int g = \lrcorner Tg < \kappa$.*

Proof. We may assume that $\lrcorner g > 0$ and obtain $\kappa \cdot \lrcorner g = \lrcorner g \cdot \int h < \lrcorner h \cdot \int g = \kappa \int g$ from 4.3, hence $\lrcorner g < \int g = \lrcorner Tg$ by 2.2(d) and 2.1(f). Moreover, $g \lhd h$ implies $\int g < \int h = \kappa$. $\square$

4.5 Corollary. *The sequence $\left(\frac{\kappa_n}{\kappa_{n-1}}\right)_{n\in\mathbb{N}}$ is strictly increasing and converges to* 1.

Proof. Let $n \in \mathbb{N}$. Applying the Stride Lemma 4.3 to $h_{n+1} \lhd h_n$ yields

$$\kappa_n^2 = \lrcorner h_{n+1} \cdot \textstyle\int h_n < \lrcorner h_n \cdot \int h_{n+1} = \kappa_{n-1}\kappa_{n+1},$$

hence $\frac{\kappa_n}{\kappa_{n-1}} < \frac{\kappa_{n+1}}{\kappa_n}$. The convergence to 1 holds because of (10). $\square$

We shall use the following lemma as a tool to estimate areas from below.

4.6 Hammock Lemma. *Let $n \in \mathbb{N}$ and $g \in \breve{\mathcal{D}}$ satisfy $g \lhd h, h_n$ and $\int g \leq \frac{\kappa_n}{\kappa} \lrcorner g$. Then $\kappa_{n+1} - 1 + \frac{\kappa}{\kappa_n} < \int Tg$.*

Proof. From $c := \frac{\kappa_n}{\int g} \geq \frac{\kappa}{\lrcorner g}$ and the Stride Lemma 4.3, we obtain

$$h \leq g \circ \tfrac{1}{c} =: \tilde{g} \quad \text{and} \quad c \textstyle\int g^* = c \int g = \int_0^c \tilde{g} = \kappa_n = \int h_n = \int h_n^*.$$

Because $\chi(g, h_n) \leq 2$, there is $x \in (0,1)$ such that $h_n|_{[0,x]} \geq \tilde{g}|_{[0,x]}$ and $h_n|_{[x,1]} \leq \tilde{g}|_{[x,1]}$. Therefore,

$$0 \leq \kappa_n(h_{n+1} - Tg) \leq \delta := \textstyle\int_{\tilde{g}(x)}^1 (h_n^* - cg^*) = \int_0^x (h_n - \tilde{g}) < \kappa_n - \kappa,$$

which proves $\kappa_{n+1} - 1 + \frac{\kappa}{\kappa_n} < \int Tg$ upon integration. $\square$

Due to Theorem 2.5, for any $f \in \mathcal{E}$, the limit set

$$\mathcal{L}(f) := \{\lim_{k\to\infty} T^{m_k} f \,:\, 0 < m_1 < m_2 < \cdots \text{ and } (T^{m_k} f)_{k\in\mathbb{N}} \text{ is Cauchy}\}$$

of the sequence $(T^n f)_{n\in\mathbb{N}_0}$ is well-defined and satisfies $\emptyset \neq \mathcal{L}(f) \subseteq \mathcal{K}$.

4.7 Lemma. *Let $f \in \mathcal{E}$ and $g \in \mathcal{L}(f)$. Then*

(a) *$T^m g \in \mathcal{L}(f)$ for all $m \in \mathbb{N}$,*

(b) *$g \lhd h_n$ for all $n \in \mathbb{N}$,*

(c) *$g = h$ or $g \lhd h$,*

(d) *$h \in \mathcal{L}(f)$ implies $\lim_{n\to\infty} T^n f = h$.*

Proof. By Theorem 2.5(b), we may assume that $h_1 \neq f \in \mathcal{K}$ and conclude that

$$f_n := T^{n-1} f \lhd h_1, \ldots, h_n \tag{13}$$

for all $n \in \mathbb{N}$ using Theorem 3.4 with induction. By assumption, there are positive integers $m_1 < m_2 < \cdots$ such that $g = \lim_{k\to\infty} f_{m_k}$, implying (b) and (c) by (13) and Proposition 3.5(b). Due to 2.5(e), T^m is continuous on $\mathcal{K}$, hence $T^m g = \lim_{k\to\infty} T^m f_{m_k} =$

$\lim_{k\to\infty} f_{m_k+m} \in \mathcal{L}(f)$ for all $m \in \mathbb{N}$, proving (a). As for (d), let $\varepsilon > 0$. Since $h \in \mathcal{L}(f)$ and $(\kappa_n)_{n\in\mathbb{N}_0}$ is decreasing with limit κ, we can choose $m \in \mathbb{N}$ such that

$$\textstyle\int f_m > \kappa - \varepsilon \quad \text{and} \quad \kappa_m < \dfrac{\kappa - \varepsilon}{\kappa - 2\varepsilon}\kappa.$$

Using (13) and applying the Stride Lemma 4.3 to $f_{m+1} \lhd h_{m+1}, \ldots, f_n \lhd h_n$ yields

$$\textstyle\int f_n > \dfrac{\kappa_n}{\kappa_m}\int f_m > \dfrac{\kappa}{\kappa_m}\int f_m > \dfrac{\kappa - 2\varepsilon}{\kappa - \varepsilon}\int f_m > \kappa - 2\epsilon$$

for all $n > m$. We conclude that $\lim_{n\to\infty} f_n = h$ using (13) again. ☐

Although the strong uniqueness established in the next theorem would follow from the global convergence, we feel like proving it directly because it drops out easily from a small subset of our previous results.

4.8 Theorem. *Suppose that $r \in \mathbb{N}$ and $g \in \mathcal{E}$ satisfy $T^r g = g$. Then $g = h$.*

Proof. The assumptions and 2.5(b) imply $\mathcal{L}(g) = \{g, Tg, \ldots, T^{r-1}g\} \subseteq \mathcal{K}$. By Lemma 4.7(a) and (c), we either have $g = h$ or $g, Tg, \ldots, T^{r-1}g \lhd h$. But the latter option entails the contradictory inequality chain

$$\lrcorner g < \textstyle\int g = \lrcorner Tg < \cdots < \int T^{r-1}g = \lrcorner T^r g = \lrcorner g$$

according to Corollary 4.4. ☐

4.9 Corollary. *The IDE (1) has a solution only for $\gamma = \kappa$, and h is the only unit stribola. Furthermore, the function $\tilde{h}$ from (11) is the only standard stribola.*

Proof. The assertions concerning (1) and h follow immediately from Theorem 4.8. As for the standard stribola, suppose that $a > 0$ and $\tilde{g}\colon [0,a] \to [0,a]$ satisfies $-\tilde{g}' = \tilde{g}^*$. Then $g := \frac{1}{a}\tilde{g}\circ a$ satisfies (1) with $\gamma = \frac{1}{a}$. Hence $g = h$, $a = \frac{1}{\kappa}$ and $\tilde{g} = \tilde{h}$. ☐

We are now also ready to prove the global convergence.

4.10 Theorem. *We have $\lim_{n\to\infty} T^n f = h$ for every $f \in \mathcal{E}$.*

Proof. Let $f \in \mathcal{E}$. In view of Lemma 4.7(d), it suffices to show that $h \in \mathcal{L}(f)$. To this end, let us assume that $h \neq g_1 \in \mathcal{L}(f)$. Then, by Lemma 4.7(a)–(c) and Corollary 4.4,

(14) $\quad g_m := T^{m-1}g_1 \in \mathcal{L}(f)$,

(15) $\quad g_m \lhd h, h_n \quad \text{and}$

(16) $\quad \gamma_{m-1} := \lrcorner g_m < \int g_m = \gamma_m = \lrcorner g_{m+1} < \kappa$

for all $m, n \in \mathbb{N}$. In particular, $\lim_{n\to\infty} \frac{\gamma_m}{\gamma_{m-1}} = 1$, hence, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\frac{\gamma_m}{\gamma_{m-1}} \le \frac{\kappa_n}{\kappa}$ and thus $\kappa_{n+1} - 1 + \frac{\kappa}{\kappa_n} < \gamma_{m+1}$ according to 4.5, (15) and the Hammock Lemma 4.6. With (15) and (16), we conclude that $(\gamma_m)_{m\in\mathbb{N}}$ converges to κ and that $(g_m)_{m\in\mathbb{N}}$ converges to h, implying that $h \in \mathcal{L}(f)$ by (14). ☐

The following variation of the Hammock Lemma 4.6 allows us to estimate κ from below.

4.11 Lemma. *Let $f, g \in \breve{\mathcal{D}}$ satisfy $g \lhd f$, $g \lhd Tf \le f$ and $\searrow g \ge \int g$. Then $\int Tf - 1 + \frac{\int Tf}{\int f} < \int Tg < \int Tf$.*

Proof. From $g \lhd f$, we conclude that $c := \frac{\int f}{\int g} > 1$, and also $Tg \lhd Tf$ by 3.4, hence $\int Tg < \int Tf$, settling the last inequality. Since $\searrow Tf = \int f > 0$, we can apply the Stride Lemma 4.3 to g and Tf and obtain

$$Tf \le g \odot \tfrac{\searrow g}{\searrow Tf} \le g \odot \tfrac{1}{c} =: \tilde{g} \quad \text{and} \quad c \int g^* = c \int g = \int_0^c \tilde{g} = \int f = \int f^*.$$

Because $\chi(f, g) \le 2$, there is $0 < x < 1$ such that $f|_{[0,x]} \ge \tilde{g}|_{[0,x]}$ and $f|_{[x,1]} \le \tilde{g}|_{[x,1]}$. Therefore,

$$0 \le \int f \cdot (Tf - Tg) \le \delta := \int_{f(x)}^1 (f^* - cg^*) = \int_0^x (f - \tilde{g}) < \int f - \int Tf,$$

which proves $\int Tf - 1 + \frac{\int Tf}{\int f} < \int Tg$ upon integration. □

4.12 Corollary. *For all $n \in \mathbb{N}$, we have $\kappa_n - 1 + \frac{\kappa_n}{\kappa_{n-1}} < \kappa$. In particular,*

$$0.2788770612338 < \kappa_{23} - 1 + \frac{\kappa_{23}}{\kappa_{22}} < \kappa < \kappa_{23} < 0.2788770613941.$$

Proof. The assertion holds trivially for $n = 1$. Let $n \in \mathbb{N}$, then $h \lhd h_n, Th_n$ by 4.2(b), so that $\kappa_{n+1} - 1 + \frac{\kappa_{n+1}}{\kappa_n} = \int Th_n - 1 + \frac{\int Th_n}{\int h_n} < \int Th = \kappa$ follows from 4.11. The values for $\kappa_0, \ldots, \kappa_{23}$ are determined in [8]. □

References

[1] V. Berinde. *Existence and approximation of solutions of some first order iterative differential equations.* Miskolc Mathematical Notes **11**/1 (2010) 13–26.

[2] A. Buică. *Existence and continuous dependence of solutions of some functional-differential equations.* Seminar on Fixed Point Theory Cluj-Napoca **3** (1995) 1–14.

[3] E. Eder. *The functional-differential equation $x'(t) = x(x(t))$.* J. Differential Equations **54**/3 (1984) 390–400.

[4] E. Egri, I. A. Rus. *First order iterative functional-differential equation with parameter.* Stud. Univ. Babeş-Bolyai Math. **52**/4 (2007) 67–80.

[5] M. Fečkan. *On a certain type of functional-differential equations.* Math. Slovaca **43**/1 (1993) 39–43.

[6] L. Levine. *Levine's sequence.* `https://oeis.org/A011784`

[7] R. Miyamoto, J. W. Sander. *Solving the Iterative Differential Equation $-\gamma g' = g^{-1}$.* In: H. Maier, J. & R. Steuding (Hrsg.), Number Theory in Memory of Eduard Wirsing, Springer, 2023, 223–236.

[8] R. Miyamoto. *Polynomial parametrisation of the canonical iterates to the solution of $-\gamma g' = g^{-1}$.* arXiv:2402.06618 [math.CO], 2024.

[9] N. J. A. Sloane. *My Favorite Integer Sequences.* In: C. Ding, T. Helleseth, H. Niederreiter (eds): Sequences and their Applications. Discrete Mathematics and Theoretical Computer Science. Springer, London, 1999, p. 118.

[10] C. L. Mallows, B. Poonen, N. J. A. Sloane. *Discussion of A011784.* [scans of pages 150–155 and 164 of Sloane's notebook "Lattices 77" from June–July 1997] https://oeis.org/A011784/a011784.pdf