# SELF-AFFINITY OF DISCS UNDER GLASS-CUT DISSECTIONS 

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#### Abstract

A topological disc is called $n$-self-affine if it has a dissection into $n$ affine images of itself. It is called $n$-gc-self-affine if the dissection is obtained by successive glass-cuts, which are cuts along segments splitting one disc into two. For every $n \geq 2$, we characterize all $n$-gc-self-affine discs. All such discs turn out to be either triangles or convex quadrangles. All triangles and trapezoids are $n$-gc-self-affine for every $n$. Non-trapezoidal quadrangles are not $n$-gc-selfaffine for even $n$. They are $n$-gc-self-affine for every odd $n \geq 7$, and they are $n$-gc-self-affine for $n=5$ if they aren't affine kites. Only four one-parameter families of quadrangles turn out to be 3-gc-self-affine.

In addition, we show that every convex quadrangle is $n$-self-affine for all $n \geq 5$.


## 1. Background and main results

A topological disc $D$ in the Euclidean plane is an image of a closed circular disc under an affine transformation of the plane. The disc $D$ is called $n$-self-affine if it has a dissection into discs $D_{1}, \ldots, D_{n}$, called pieces, that are all affine images of $D$. Here we speak of a dissection if $D$ is the union of all pieces and any two pieces have disjoint interiors. We call $D$ self-affine if it is $n$-self-affine for some integer $n \geq 2$. The concept of self-affinity generalizes self-similarity and is prominent in fractal geometry, where the affine transformations are supposed to be contractions [4. Section 9.4]. But it is also fruitful in the elementary geometry of polygons, see e.g. [2, 6, 7, 11]. Figure 1 presents some examples. The non-convex ones are adopted from [5]. Self-affine convex polygons have at most five vertices as follows from [1, Theorem 5], see also [6, Satz 1]. Self-affinity of triangles is trivial. All convex quadrangles are 5 -self-affine. This goes back to Attila Pór, see [7, Proposition 1]. There exist self-affine convex pentagons [7] Proposition 4], but the regular pentagon and, more generally, pentagons whose inner angles are close to $\frac{3 \pi}{5}=108^{\circ}$ are not self-affine [7, Proposition 3], [2, Theorem 1].


Figure 1. Realizations of self-affinities and gc-self-affinities

In the present paper our main focus is on so-called glass-cut dissections. A glass-cut (also guillotine cut) dissects a disc $D$ along a line segment into two discs. That process can be repeated finitely many times with some of the resulting pieces. The outcome is called a glass-cut dissection (or gc-dissection for short) of $D$. Accordingly, we obtain the concepts of $n$-gc-self-affine and gc-self-affine discs. Glass-cut dissections are less flexible than general ones, but are more accessible to algorithmic approaches, see e.g. [3, 8, 9, 10, 13 for applications. Only the last two dissections in Figure 1 are based on glass-cuts.

Our main result is the following characterization of all $n$-gc-self-affine discs for every $n=2,3, \ldots$
Theorem 1. (i) Every gc-self-affine topological disc is a triangle or a convex quadrangle.
(ii) Every triangle is $n$-gc-self-affine for all $n=2,3, \ldots$
(iii) Let $n \in \mathbb{Z}, n \geq 2$, be even. Then a convex quadrangle is $n$-gc-self-affine if and only if it is a trapezoid.
(iv) Let $n \in \mathbb{Z}, n \geq 7$, be odd. Then every convex quadrangle is $n$-gc-self-affine.
(v) A convex quadrangle is not 5-gc-self-affine if and only if it is an affine image of a kite, but no parallelogram.
(vi) The 3-gc-self-affine convex quadrangles are given by four one-parameter families, see Theorem 18 for details.

Theorem 1 shows in particular that gc-self-affinity of every non-trapezoidal quadrangle $Q$ is non-trivial in so far as there is no number $n_{0}$ such that $Q$ is $n$-gc-selfaffine for all $n \geq n_{0}$. Let us point out that the situation is different for self-affinity based on general dissections.

Theorem 2. Every convex quadrangle is n-self-affine for every $n \in \mathbb{Z}, n \geq 5$.
Since every dissection of a convex quadrangle into two quadrangles is done by a glass-cut, the only 2 -self-affine convex quadrangles are trapezoids by Theorem 1 (see also [6, Satz 3]). In view of Theorem[2] the following question arises.

Problem 3. What convex quadrangles are $n$-self-affine for $n=3,4$ ?
First systematic considerations of that problem can be found in [6, 14].
The remainder of the present paper is organized as follows. In Section 2 we show that qc-self-affine topological discs are necessarily convex. The short Section 3 proves Theorem 1(i). Theorem 1(ii) is trivial. In Section 4 we start the discussion of quadrangles by introducing a appropriate parametrization of affine types of quadrangles. In Section 5we analyse the parameters of a quadrangle that is composed by glueing together two given quadrangles along a common side. Section 6 is devoted to the proof of Theorem 1 (iii), Section 7 concerns Theorem 1 (iv) and (v), and Section 8 gives Theorem 1(vi). Finally, Section 9 proves Theorem 2,

We use the following notations. Open and closed intervals in $\mathbb{R}$ are denoted by $(\xi, \eta)$ and $[\xi, \eta]$, respectively. The line segment in $\mathbb{R}^{2}$ with endpoints $x$ and $y$ is denoted by $x y$, its length by $|x y|$. The straight line through $x$ and $y$ is $l(x y)$. We write $B\left(x_{0}, r\right)$ for the closed circular disc (or ball) $\left\{x \in \mathbb{R}^{2}:\left|x x_{0}\right| \leq r\right\}$ of radius $r>0$ centred at $x_{0} \in \mathbb{R}^{2}$. Interior, boundary, convex hull and area of a plane set $X$ are $\operatorname{int}(X), \operatorname{bd}(X), \operatorname{conv}(X)$ and area $(X)$, respectively.

## 2. All gC-SElf-AFfine discs are convex

Let the disc $D$ be $n$-gc-self-affine based on a dissection into affine images $\varphi_{1}(D)$, $\ldots, \varphi_{n}(D)$. For every integer $k \geq 1$, the affine images $\varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D), i_{1}, \ldots, i_{k} \in$ $\{1, \ldots, n\}$, form an (iterated) $n^{k}$-gc-self-affinity of $D$. Every sequence $\left(i_{k}\right)_{k=1}^{\infty} \subseteq$ $\{1, \ldots, n\}$ gives rise to a decreasing sequence $\left(\varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D)\right)_{k=1}^{n}$ of compact sets, whose limit set in the Hausdorff metric is $S=\bigcap_{k=1}^{\infty} \varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D) \subseteq D$ (see [12, Lemma 1.8.2]). Since the determinants of (the linear maps associated to) $\varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}, k=1,2, \ldots$, tend to zero, $S$ is either a singleton or a non-degenerate line segment. In the latter case we speak of a limit segment.

Lemma 4. The following are satisfied for every qc-self-affine disc D.
(i) Every $x \in D$ belongs to some limit set.
(ii) Two limit segments $S_{1}$ and $S_{2}$ do not cross in the sense that $S_{1} \cap S_{2}$ is a singleton in the relative interiors of both $S_{1}$ and $S_{2}$.
(iii) If some limit set $S$ satisfies $S \subseteq \operatorname{int}(D)$, then $D$ is convex.

Proof. For (i), one easily checks that there is a sequence $\left(i_{k}\right)_{k=1}^{\infty}$ such that $x \in$ $\varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D)$ for $k=1,2, \ldots$
For (ii), assume that the Hausdorff limits $S_{1}=\bigcap_{k=1}^{\infty} \varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D)$ and $S_{2}=$ $\bigcap_{k=1}^{\infty} \varphi_{j_{1}} \circ \ldots \circ \varphi_{j_{k}}(D)$ cross. Hence, for all $k$, the pieces $\varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D)$ and $\varphi_{j_{1}} \circ \ldots \circ \varphi_{j_{k}}(D)$ of the $k$-th iterated dissection share some inner points. Then $\left(i_{k}\right)_{k=1}^{\infty}=\left(j_{k}\right)_{k=1}^{\infty}$ and in turn $S_{1}=S_{2}$, a contradiction.
For (iii), note that, if $S=\bigcap_{k=1}^{\infty} \varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k}}(D) \subseteq \operatorname{int}(D)$, then $D_{0}=\varphi_{i_{1}} \circ \ldots \circ$ $\varphi_{i_{k_{0}}}(D) \subseteq \operatorname{int}(D)$ for some sufficiently large $k_{0}$. Since the piece $D_{0}$ of the $k_{0}$-th iterated gc-dissection is in $\operatorname{int}(D)$, its boundary is formed by finitely many line segments and its inner angles are smaller than $\pi$. Hence $D_{0}=\varphi_{i_{1}} \circ \ldots \circ \varphi_{i_{k_{0}}}(D)$ is a convex polygon, and so is $D$.

Unfortunately, there are gc-self-affinities of some discs $D$ that do not give rise to limit sets completely contained in the interior of $D$, see e.g. the right-most example in Figure 1. Therefore Lemma 4(iii) is not enough to show convexity of every gc-self-affine disc. A deeper analysis is needed.

Lemma 5. Let $D$ be a non-convex gc-self-affine topological disc. Then there exists a limit segment $S=a b$ such that $S \cap \operatorname{bd}(D) \subseteq\{a, b\}$.

Proof. We divide the proof into two parts.
Claim 1. There exist $r>0$ and uncountably many limit segments, all placed on mutually different straight lines and being of length at least $r$, and each meeting $\operatorname{bd}(D)$.

We pick some $x \in \operatorname{int}(D)$ and $r>0$ such that $B(x, 2 r) \subseteq \operatorname{int}(D)$, see the left-hand part of Figure 2, By Lemma 4(i) and (iii), every point from $B(x, r)$ belongs to some limit segment that meets $\operatorname{bd}(D)$. By the triangle inequality, their lengths are larger than $r$. Since countably many straight lines cannot cover $B(x, r)$, there are uncountably many such limit segments placed on different lines. Figure 2 illustrates three examples and their intersections with $\operatorname{bd}(D)$.


Figure 2. Proof of Lemma 5

Claim 2. At most countably many limit segments ab found in Claim 1 satisfy $a b \cap \mathrm{bd}(D) \nsubseteq\{a, b\}$.

We shall prove Claim 2 by showing that, for every $\varepsilon>0$, at most finitely many of the segments $S=a b$ described in Claim 1 contain some $x \in S \cap \operatorname{bd}(D)$ such that $\min \{|a x|,|b x|\}>\varepsilon$.
Suppose that, contrary to the last assertion, there are infinitely many limit segments $S_{i}=a_{i} b_{i}$ and points $x_{i} \in\left(S_{i} \cap \operatorname{bd}(D)\right) \backslash\left(B\left(a_{i}, \varepsilon\right) \cup B\left(b_{i}, \varepsilon\right)\right), i=1,2, \ldots$ Since $\operatorname{bd}(D)$ is compact, we can assume that $\left(x_{i}\right)_{i=1}^{\infty}$ converges to some $x_{0} \in \operatorname{bd}(D)$ and that $\left(x_{i}\right)_{i=1}^{\infty} \subseteq B\left(x_{0}, \frac{\varepsilon}{2}\right)$, see the right-hand part of Figure 2, By the triangle inequality, the end-points of each $S_{i}$ are outside $B\left(x_{0}, \frac{\varepsilon}{2}\right)$. By Lemma4(ii), distinct segments $S_{i}$ do not meet in $B\left(x_{0}, \frac{\varepsilon}{2}\right)$.
Since $x_{0} \in \operatorname{bd}(D), \operatorname{bd}(D) \cap B\left(x_{0}, \frac{\varepsilon}{2}\right)$ contains a connected component $\Gamma$ of $\operatorname{bd}(D)$ that contains $x_{0}$ in its relative interior. Since $\left(x_{i}\right)_{i=1}^{\infty} \subseteq \operatorname{bd}(D)$ and $\lim _{i \rightarrow \infty} x_{i}=x_{0}$, we get $\left(x_{i}\right)_{i=i_{0}}^{\infty} \subseteq \Gamma$ for some $i_{0}$. But then the arc $\Gamma \subseteq B\left(x_{0}, \frac{\varepsilon}{2}\right)$ has to connect all the points $x_{i}, i=i_{0}, i_{0}+1, \ldots$, without crossing any of the segments $S_{i}$, because $S_{i} \subseteq D$ and $\Gamma \subseteq \operatorname{bd}(D)$. This is impossible. The proof of Claim 2 and of Lemma 5 is complete.

Lemma 6. Let $D$ be a gc-self-affine topological disc having a limit segment $S=a b$ such that $S \cap \operatorname{bd}(D) \subseteq\{a, b\}$. Then, for every $\varepsilon \in\left(0, \frac{1}{2}\right]$, there is an equiaffine map $\alpha_{\varepsilon}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ such that
(i) $[-1,1] \times\{0\} \subseteq \alpha_{\varepsilon}(D) \subseteq[-1-\varepsilon, 1+\varepsilon] \times \mathbb{R}$,
(ii) $\alpha_{\varepsilon}(D) \cap([-1+\varepsilon, 1-\varepsilon] \times \mathbb{R})$ is convex,
(iii) $\alpha_{\varepsilon}(D) \cap([-1+\varepsilon, 1-\varepsilon] \times \mathbb{R}) \subseteq[-1+\varepsilon, 1-\varepsilon] \times[-2 \operatorname{area}(D), 2 \operatorname{area}(D)]$.

Proof. W.l.o.g., i.e., after some equiaffine transformation, we have $S=[-1,1] \times\{0\}$, i.e., $a=\binom{-1}{0}$ and $b=\binom{1}{0}$. Since $S$ is the Hausdorff limit of a decreasing sequence of pieces in certain gc-self-affinities of $D$ and since $S \backslash\left\{\binom{-1}{0},\binom{1}{0}\right\} \subseteq \operatorname{int}(D)$, there exist some $\delta=\delta(\varepsilon) \in(0, \varepsilon]$ and an affine transformation $\beta_{\varepsilon}$ such that

$$
\begin{gather*}
{[-1,1] \times\{0\} \subseteq \beta_{\varepsilon}(D) \subseteq[-1-\varepsilon, 1+\varepsilon] \times \mathbb{R}}  \tag{1}\\
\beta_{\varepsilon}(D) \cap([-1+\varepsilon, 1-\varepsilon] \times \mathbb{R}) \subseteq[-1+\varepsilon, 1-\varepsilon] \cap[-\delta, \delta] \subseteq \operatorname{int}(D) \tag{2}
\end{gather*}
$$

Since the piece $\beta_{\varepsilon}(D)$ belongs to a gc-dissection of $D, \operatorname{bd}\left(\beta_{\varepsilon}(D)\right) \cap \operatorname{int}(D)$ is a finite union of polygonal arcs with inner angles smaller than $\pi$. Hence (2) implies

$$
\begin{equation*}
\beta_{\varepsilon}(D) \cap([-1+\varepsilon, 1-\varepsilon] \times \mathbb{R}) \text { is convex. } \tag{3}
\end{equation*}
$$

We define the equiaffine map $\alpha_{\varepsilon}$ as $\alpha_{\varepsilon}=\gamma \circ \beta_{\varepsilon}$ with $\gamma\binom{\xi_{1}}{\xi_{2}}=\left(\begin{array}{c}\frac{\xi_{1}}{\operatorname{det}\left(\beta_{\varepsilon}\right)} \xi_{2}\end{array}\right)$. Then (11) implies (i) and (3) yields (ii). Finally, since $\varepsilon \leq \frac{1}{2}$, (i) gives

$$
\left[-\frac{1}{2}, \frac{1}{2}\right] \times\{0\} \subseteq \alpha_{\varepsilon}(D) \cap([-1+\varepsilon, 1-\varepsilon] \times \mathbb{R})
$$

By (ii) and area $\left(\alpha_{\varepsilon}(D)\right)=\operatorname{area}(D)$, this shows (iii).
Every equiaffine map $\alpha: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ has a unique representation $\alpha\binom{\xi_{1}}{\xi_{2}}=\binom{\alpha_{11} \xi_{1}+\alpha_{12} \xi_{2}+\alpha_{13}}{\alpha_{21} \xi_{1}+\alpha_{22} \xi_{2}+\alpha_{23}}$ with $\left(\alpha_{11}, \ldots, \alpha_{23}\right) \in \mathbb{R}^{6}$. This way the set $A f f{ }_{2}^{1}$ of all equiaffine maps corresponds to a closed subset of $\mathbb{R}^{6}$, and the natural topology of $\mathbb{R}^{6}$ gives rise to a respective topology on $\mathrm{Aff}_{2}^{1}$. The following lemma is routine, based on elementary topological properties of $\mathbb{R}^{n}$ and the localization of the Hausdorff convergence in the sense of [12, Theorem 1.8.8].
Lemma 7. (i) For every $x_{0} \in \mathbb{R}^{2}$, every $r>0$ and every compact set $C \subseteq \mathbb{R}^{2}$, $\left\{\alpha \in \mathrm{Aff}_{2}^{1}: \alpha\left(B\left(x_{0}, r\right)\right) \subseteq C\right\}$ is compact.
(ii) Suppose that $\alpha_{0}, \alpha_{1}, \ldots \in$ Aff $2_{2}^{1}$ satisfy $\lim _{i \rightarrow \infty} \alpha_{i}=\alpha_{0}$. Then, for every non-empty compact set $C \subseteq \mathbb{R}^{2}, \lim _{i \rightarrow \infty} \alpha_{i}(C)=\alpha_{0}(C)$ in the Hausdorff metric.

Lemma 8. Let $D$ be a topological disc. Then there exist a disc $B\left(x_{0}, r\right) \subseteq D$, $x_{0} \in D, r>0$, and $\mu \in(0,1)$ such that, for every affine functional $\beta: \mathbb{R}^{2} \rightarrow \mathbb{R}$,

$$
\begin{aligned}
& {\left[\min \beta\left(B\left(x_{0}, r\right)\right), \max \beta\left(B\left(x_{0}, r\right)\right)\right]} \\
& \quad \subseteq[(1-\mu) \min \beta(D)+\mu \max \beta(D),(1-\mu) \max \beta(D)+\mu \min \beta(D)]
\end{aligned}
$$

Proof. We fix $x_{0} \in \operatorname{int}(D)$ and $R>r>0$ such that

$$
B\left(x_{0}, r\right) \subseteq B\left(x_{0}, 2 r\right) \subseteq D \subseteq B\left(x_{0}, R\right)
$$

Since $\beta$ is affine, we have

$$
\begin{gather*}
\frac{3}{2} \min \beta\left(B\left(x_{0}, r\right)\right)-\frac{1}{2} \max \beta\left(B\left(x_{0}, r\right)\right)=\min \beta\left(B\left(x_{0}, 2 r\right)\right)  \tag{4}\\
\frac{r-R}{2 r} \min \beta\left(B\left(x_{0}, r\right)\right)+\frac{r+R}{2 r} \max \beta\left(B\left(x_{0}, r\right)\right)=\max \beta\left(B\left(x_{0}, R\right)\right) \tag{5}
\end{gather*}
$$

By (4) and $\min \beta\left(B\left(x_{0}, 2 r\right)\right) \geq \min \beta(D)$,

$$
3(r+R) \min \beta\left(B\left(x_{0}, r\right)\right)-(r+R) \max \beta\left(B\left(x_{0}, r\right)\right) \geq 2(r+R) \min \beta(D)
$$

Similarly, by (5) and $\max \beta\left(B\left(x_{0}, R\right)\right) \geq \max \beta(D)$,

$$
(r-R) \min \beta\left(B\left(x_{0}, r\right)\right)+(r+R) \max \beta\left(B\left(x_{0}, r\right)\right) \geq 2 r \max \beta(D)
$$

Adding the last two inequalities and dividing by $4 r+2 R$, we get

$$
\begin{equation*}
\min \beta\left(B\left(x_{0}, r\right)\right) \geq \frac{r+R}{2 r+R} \min \beta(D)+\frac{r}{2 r+R} \max \beta(D) \tag{6}
\end{equation*}
$$

In the same way we arrive at

$$
\begin{equation*}
\max \beta\left(B\left(x_{0}, r\right)\right) \leq \frac{r+R}{2 r+R} \max \beta(D)+\frac{r}{2 r+R} \min \beta(D) \tag{7}
\end{equation*}
$$

Inequalities (6) and (7) yield the claim of Lemma 8 with $\mu=\frac{r}{2 r+R}$.
Lemma 9. If a topological disc $D$ is gc-self-affine such that there is a limit segment $S=a b$ with $S \cap \operatorname{bd}(D) \subseteq\{a, b\}$, then $D$ is convex.

Proof. We use the equiaffine maps $\alpha_{\varepsilon}, 0<\varepsilon \leq \frac{1}{2}$, obtained from Lemma 6 and the circular disc $B\left(x_{0}, r\right) \subseteq D$ and the real $\mu$ given by Lemma 8 .
Let $\varepsilon_{0}=\min \left\{\frac{2 \mu}{2-\mu}, \frac{1}{2}\right\}$. First we show that, for all $\varepsilon \in\left(0, \varepsilon_{0}\right]$,

$$
\begin{equation*}
\alpha_{\varepsilon}\left(B\left(x_{0}, r\right)\right) \subseteq[-1,1] \times[-2 \operatorname{area}(D), 2 \operatorname{area}(D)] \tag{8}
\end{equation*}
$$

For that, let $\varepsilon \in\left(0, \varepsilon_{0}\right]$ be fixed and let $\pi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ denote the projection onto the first coordinate. Then Lemma 8 and Lemma 6(i) give

$$
\begin{aligned}
\min \pi \circ \alpha_{\varepsilon}\left(B\left(x_{0}, r\right)\right) & \geq(1-\mu) \cdot \min \pi \circ \alpha_{\varepsilon}(D)+\mu \cdot \max \pi \circ \alpha_{\varepsilon}(D) \\
& \geq(1-\mu)(-1-\varepsilon)+\mu \cdot 1 \\
& =-1+(2 \mu-(1-\mu) \varepsilon) \\
& \geq-1+\varepsilon
\end{aligned}
$$

where the last estimate comes from $\varepsilon \leq \varepsilon_{0} \leq \frac{2 \mu}{2-\mu}$. In the same way we get

$$
\max \pi \circ \alpha_{\varepsilon}\left(B\left(x_{0}, r\right)\right) \leq 1-\varepsilon
$$

Consequently, the first coordinates of all points from $\alpha_{\varepsilon}\left(B\left(x_{0}, r\right)\right)$ are in $[-1+\varepsilon, 1-$ $\varepsilon]$. Hence

$$
\alpha_{\varepsilon}\left(B\left(x_{0}, r\right)\right) \subseteq([-1+\varepsilon, 1-\varepsilon] \times \mathbb{R}) \cap \alpha_{\varepsilon}(D)
$$

and Lemma (iii) yields (8).
For $\varepsilon_{n}=\frac{1}{n}$, (8) gives

$$
\alpha_{\frac{1}{n}}\left(B\left(x_{0}, r\right)\right) \subseteq[-1,1] \times[-2 \operatorname{area}(D), 2 \operatorname{area}(D)]
$$

provided $n \geq n_{0}=\left\lceil\frac{1}{\varepsilon_{0}}\right\rceil$. Lemma $7(\mathrm{i})$ shows that the sequence $\left(\alpha_{\frac{1}{n}}\right)_{n=n_{0}}^{\infty}$ is in a compact set of equiaffine transformations, in turn having a convergent subsequence. W.l.o.g., $\lim _{n \rightarrow \infty} \alpha_{\frac{1}{n}}=\alpha_{0} \in$ Aff $_{2}^{1}$. By Lemma 7 (ii),

$$
\lim _{n \rightarrow \infty} \alpha_{\frac{1}{n}}(D)=\alpha_{0}(D)
$$

in the Hausdorff metric. In particular,
$\alpha_{0}(D) \cap((-1,1) \times \mathbb{R})=((-1,1) \times \mathbb{R}) \cap \lim _{n \rightarrow \infty} \alpha_{\frac{1}{n}}(D) \cap\left(\left[-1+\frac{1}{n}, 1-\frac{1}{n}\right] \times \mathbb{R}\right)$.
Now Lemma (ii) shows that $\alpha_{0}(D) \cap((-1,1) \times \mathbb{R})$ is convex. Moreover, Lemma 6 (i) yields $\alpha_{0}(D) \subseteq[-1,1] \times \mathbb{R}$. Therefore $\alpha_{0}(D)$ is the closure of the convex set $\alpha_{0}(D) \cap((-1,1) \times \mathbb{R})$, because $\alpha_{0}(D)$ is a topological disc. Thus $\alpha_{0}(D)$ is convex, as well as $D=\alpha_{0}^{-1}\left(\alpha_{0}(D)\right)$.

Lemmas 5 and 9 together show that every gc-self-affine topological disc is convex.

## 3. Reduction to triangles and quadrangles

Proof of Theorem 1 (i). Let the disc $D$ be gc-self-affine. Then it is convex as we have seen above. Moreover, it is a convex polygon by [11].

A dissection of a convex $k_{0}$-gon $D_{0}$ along some line segment splits it into a convex $k_{1}$ gon $D_{0}^{*}$ and a convex $k_{2}$-gon $D_{0}^{* *}$ with $k_{1}+k_{2} \in\left\{k_{0}+2, k_{0}+3, k_{0}+4\right\}$. In particular, $\min \left\{k_{1}, k_{2}\right\} \leq \frac{k_{0}+4}{2}$, whence $\min \left\{k_{1}, k_{2}\right\} \leq k_{0}$ if $k_{0} \in\{3,4\}$ and $\min \left\{k_{1}, k_{2}\right\}<k_{0}$ if $k_{0} \geq 5$. Iteration of this argument shows that each gc-dissection of a convex $k_{0}$-gon $D_{0}$ with $k_{0} \geq 5$ has a tile with less than $k_{0}$ vertices. Consequently, the gc-self-affine disc $D$ must be a triangle or a convex quadrangle.


Figure 3. Parametrization of non-trapezoids.

## 4. Parametrization of Quadrangles

We call a side of a convex quadrangle opening (or closing or constant, respectively) if the inner angles adjacent to that side sum up to a value less than $\pi$ (or larger than $\pi$ or equal to $\pi$, respectively). Suppose that the sides $b c$ and $c d$ of the quadrangle $a b c d$ are closing, and let $s$ be the intersection point of the straight lines $l(a b)$ and $l(c d)$. Then $0<\frac{|b s|}{|a s|}<\frac{|c s|}{|d s|}<1$. We use $\alpha:=\frac{|b s|}{|a s|}$ and $\beta:=\frac{|c s|}{|d s|}$ for parametrizing the quadrangle and denote its affine type by $Q(\alpha, \beta)$ (over the closing side $b c$ ). The same can be done over the closing side $c d$. If $t$ is the intersection of $l(b c)$ and $l(a d)$, we obtain $0<\bar{\alpha}:=\frac{|d t|}{|a t|}<\bar{\beta}:=\frac{|c t|}{|b t|}<1$. We get the second parametrization $Q(\bar{\alpha}, \bar{\beta})$ with

$$
\begin{equation*}
(\bar{\alpha}, \bar{\beta})=\frac{1-\beta}{(1-\alpha) \beta}(\alpha, \beta) \tag{9}
\end{equation*}
$$

see Figure 3 for the situation $a=(0,0), s=(1,0), t=(0,1)$. Both $Q(\alpha, \beta)$ and $Q(\bar{\alpha}, \bar{\beta})$ describe the same affine type of quadrangles. For further operations we keep both parametrizations and call them flips of each other; $Q(\alpha, \beta)^{F}=Q(\bar{\alpha}, \bar{\beta})$ and $Q(\bar{\alpha}, \bar{\beta})^{F}=Q(\alpha, \beta)$. Note that $Q(\alpha, \beta)$ and $Q(\bar{\alpha}, \bar{\beta})$ give rise to the same value $0<\frac{\alpha}{\beta}=\frac{\bar{\alpha}}{\beta}<1$, that we call the affine quotient of that class of quadrangles.
Clearly, for every choice of $0<\alpha<\beta<1, Q(\alpha, \beta)$ describes a unique affine type of convex quadrangles, and all non-trapezoidal convex quadrangles are covered this way.
Now let the quadrangle $a b c d$ be a trapezoid with (unique) closing side $b c$. Again let $s$ be the common point of $l(a b)$ and $l(c d)$. Then $0<\frac{|b s|}{|a s|}=\frac{|c s|}{|d s|}<1$, we use
$\alpha:=\frac{|b s|}{|a s|}$ for parametrizing the affine type of our trapezoid and write $T(\alpha)$ for the respective class. We call 1 the affine quotient of $T(\alpha)$ (motivated by $\frac{\alpha}{\alpha}=1$ ).
Clearly, all choices of $0<\alpha<1$ give rise to mutually different classes $T(\alpha)$, and this way all trapezoids are covered except for parallelograms. Note that the ratio of the lengths of the parallel sides of trapezoids of type $T(\alpha)$ is $\alpha$, too.
Finally, we write $P$ for the class of all parallelograms and call 1 their affine quotient.
In the sequel we will sometimes write $Q(\alpha, \beta), T(\gamma)$ and $P$ for particular representatives of these classes of quadrangles.

## 5. Parameters of compositions of two quadrangles

When studying a gc-dissection of a disc $D$, we associate a (not necessarily unique) binary dissection tree to its (not necessarily unique) process of dissection. The original disc $D$ is its root. In a first step $D$ is dissected into two discs, the children of $D$. Similarly, if a disc is cut further into two discs, these appear as children of that disc. Accordingly, the leafs of the tree are the tiles of the final dissection of $D$.

Lemma 10. All vertices of a dissection tree associated to the gc-dissection representing a gc-self-affinity of a quadrangle are quadrangles. In particular, in every step of dissection the cut is made between relative inner points of opposite sides of the respective quadrangle.

Proof. Suppose in one step a quadrangle is not cut into two quadrangles. Then one of its children is a triangle. However, every further cut of a triangle produces at least one triangle. So, finally, one leaf of the tree is a triangle, a contradiction.

As we know that a qc-self-affinity of a quadrangle is obtained by successively cutting parent quadrangles into two descending quadrangles, we study now how two quadrangles can be glued together along a common side to obtain a qc-dissection of a parent quadrangle.

Lemma 11. All possible affine types of parent quadrangles admitting a gc-dissection into two descending quadrangles of prescribed affine types are the following.
(i) Combinations of $Q\left(\alpha_{1}, \beta_{1}\right)$ with $Q\left(\alpha_{2}, \beta_{2}\right), 0<\alpha_{i}<\beta_{i}<1$ for $i=1,2$ (with the notation $\left(\bar{\alpha}_{i}, \bar{\beta}_{i}\right)=\frac{1-\beta_{i}}{\left(1-\alpha_{i}\right) \beta_{i}}\left(\alpha_{i}, \beta_{i}\right)$ as in (9) ):

| notation | (one) parametrization | affine <br> quotient |
| :---: | :---: | :---: |
| $Q\left(\alpha_{1}, \beta_{1}\right) \cdot Q\left(\alpha_{2}, \beta_{2}\right)$ | $Q\left(\alpha_{1} \alpha_{2}, \beta_{1} \beta_{2}\right)$ | $\frac{\alpha_{1}}{\beta_{1}} \cdot \frac{\alpha_{2}}{\beta_{2}}$ |
| $Q\left(\alpha_{1}, \beta_{1}\right): Q\left(\alpha_{2}, \beta_{2}\right)$ | - if $\frac{\alpha_{1}}{\beta_{1}}<\frac{\alpha_{2}}{\beta_{2}}: Q\left(\alpha_{1} \beta_{2}, \beta_{1} \alpha_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}=\frac{\alpha_{2}}{\beta_{2}}: T\left(\alpha_{1} \beta_{2}\right)=T\left(\beta_{1} \alpha_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}>\frac{\alpha_{2}}{\beta_{2}}: Q\left(\beta_{1} \alpha_{2}, \alpha_{1} \beta_{2}\right)$ | $\begin{aligned} & \frac{\alpha_{1}}{\beta_{1}}: \frac{\alpha_{2}}{\beta_{2}} \\ & 1 \\ & \frac{\alpha_{2}}{\beta_{2}}: \frac{\alpha_{1}}{\beta_{1}} \end{aligned}$ |
| $\begin{aligned} & Q\left(\alpha_{1}, \beta_{1}\right) \cdot Q\left(\alpha_{2}, \beta_{2}\right)^{F} \\ & =Q\left(\alpha_{1}, \beta_{1}\right) \cdot Q\left(\bar{\alpha}_{2}, \bar{\beta}_{2}\right) \end{aligned}$ | $Q\left(\alpha_{1} \bar{\alpha}_{2}, \beta_{1} \bar{\beta}_{2}\right)$ | $\frac{\alpha_{1}}{\beta_{1}} \cdot \frac{\alpha_{2}}{\beta_{2}}$ |
| $\begin{aligned} & Q\left(\alpha_{1}, \beta_{1}\right): Q\left(\alpha_{2}, \beta_{2}\right)^{F} \\ & =Q\left(\alpha_{1}, \beta_{1}\right): Q\left(\bar{\alpha}_{2}, \bar{\beta}_{2}\right) \end{aligned}$ | - if $\frac{\alpha_{1}}{\beta_{1}}<\frac{\alpha_{2}}{\beta_{2}}: Q\left(\alpha_{1} \bar{\beta}_{2}, \beta_{1} \bar{\alpha}_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}=\frac{\alpha_{2}}{\beta_{2}}: T\left(\alpha_{1} \bar{\beta}_{2}\right)=T\left(\beta_{1} \bar{\alpha}_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}>\frac{\alpha_{2}}{\beta_{2}}: Q\left(\beta_{1} \bar{\alpha}_{2}, \alpha_{1} \bar{\beta}_{2}\right)$ | $\begin{aligned} & \frac{\alpha_{1}}{\beta_{1}}: \frac{\alpha_{2}}{\beta_{2}} \\ & 1 \\ & \frac{\alpha_{2}}{\beta_{2}}: \frac{\alpha_{1}}{\beta_{1}} \end{aligned}$ |
| $\begin{aligned} & Q\left(\alpha_{1}, \beta_{1}\right)^{F} \cdot Q\left(\alpha_{2}, \beta_{2}\right) \\ & =Q\left(\bar{\alpha}_{1}, \bar{\beta}_{1}\right) \cdot Q\left(\alpha_{2}, \beta_{2}\right) \end{aligned}$ | $Q\left(\bar{\alpha}_{1} \alpha_{2}, \bar{\beta}_{1} \beta_{2}\right)$ | $\frac{\alpha_{1}}{\beta_{1}} \cdot \frac{\alpha_{2}}{\beta_{2}}$ |
| $\begin{aligned} & Q\left(\alpha_{1}, \beta_{1}\right)^{F}: Q\left(\alpha_{2}, \beta_{2}\right) \\ & =Q\left(\bar{\alpha}_{1}, \bar{\beta}_{1}\right): Q\left(\alpha_{2}, \beta_{2}\right) \end{aligned}$ | - if $\frac{\alpha_{1}}{\beta_{1}}<\frac{\alpha_{2}}{\beta_{2}}: Q\left(\bar{\alpha}_{1} \beta_{2}, \bar{\beta}_{1} \alpha_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}=\frac{\alpha_{2}}{\beta_{2}}: T\left(\bar{\alpha}_{1} \beta_{2}\right)=T\left(\bar{\beta}_{1} \alpha_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}>\frac{\alpha_{2}}{\beta_{2}}: Q\left(\bar{\beta}_{1} \alpha_{2}, \bar{\alpha}_{1} \beta_{2}\right)$ | $\begin{aligned} & \frac{\alpha_{1}}{\beta_{1}}: \frac{\alpha_{2}}{\beta_{2}} \\ & 1 \\ & \frac{\alpha_{2}}{\beta_{2}}: \frac{\alpha_{1}}{\beta_{1}} \end{aligned}$ |
| $\begin{aligned} & Q\left(\alpha_{1}, \beta_{1}\right)^{F} \cdot Q\left(\alpha_{2}, \beta_{2}\right)^{F} \\ & =Q\left(\bar{\alpha}_{1}, \bar{\beta}_{1}\right) \cdot Q\left(\bar{\alpha}_{2}, \bar{\beta}_{2}\right) \end{aligned}$ | $Q\left(\bar{\alpha}_{1} \bar{\alpha}_{2}, \bar{\beta}_{1} \bar{\beta}_{2}\right)$ | $\frac{\alpha_{1}}{\beta_{1}} \cdot \frac{\alpha_{2}}{\beta_{2}}$ |
| $\begin{aligned} & Q\left(\alpha_{1}, \beta_{1}\right)^{F}: Q\left(\alpha_{2}, \beta_{2}\right)^{F} \\ & =Q\left(\bar{\alpha}_{1}, \bar{\beta}_{1}\right): Q\left(\bar{\alpha}_{2}, \bar{\beta}_{2}\right) \end{aligned}$ | - if $\frac{\alpha_{1}}{\beta_{1}}<\frac{\alpha_{2}}{\beta_{2}}: Q\left(\bar{\alpha}_{1} \bar{\beta}_{2}, \bar{\beta}_{1} \bar{\alpha}_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}=\frac{\alpha_{2}}{\beta_{2}}: T\left(\bar{\alpha}_{1} \bar{\beta}_{2}\right)=T\left(\bar{\beta}_{1} \bar{\alpha}_{2}\right)$ <br> - if $\frac{\alpha_{1}}{\beta_{1}}>\frac{\alpha_{2}}{\beta_{2}}: Q\left(\bar{\beta}_{1} \bar{\alpha}_{2}, \bar{\alpha}_{1} \bar{\beta}_{2}\right)$ | $\begin{aligned} & \frac{\alpha_{1}}{\beta_{1}}: \frac{\alpha_{2}}{\beta_{2}} \\ & 1 \\ & \frac{\alpha_{2}}{\beta_{2}}: \frac{\alpha_{1}}{\beta_{1}} \end{aligned}$ |

(ii) Combinations of $Q(\alpha, \beta), 0<\alpha<\beta<1$, with $T(\gamma), 0<\gamma<1$ (with the notation $(\bar{\alpha}, \bar{\beta})=\frac{1-\beta}{(1-\alpha) \beta}(\alpha, \beta)$ as in (9) ):

| notation | (one) parametrization | affine <br> quotient |
| :--- | :--- | :--- |
| $Q(\alpha, \beta) \cdot T(\gamma)$ | $Q(\alpha \gamma, \beta \gamma)$ | $\frac{\alpha}{\beta}$ |
| $Q(\alpha, \beta)^{F} \cdot T(\gamma)=Q(\bar{\alpha}, \bar{\beta}) \cdot T(\gamma)$ | $Q(\bar{\alpha} \gamma, \bar{\beta} \gamma)$ | $\frac{\alpha}{\beta}$ |

(iii) There are no possible combinations of $Q(\alpha, \beta), 0<\alpha<\beta<1$, with $P$.
(iv) Combinations of $T\left(\gamma_{1}\right)$ with $T\left(\gamma_{2}\right), 0<\gamma_{i}<1$ for $i=1,2$ :

| notation | possible parametrizations | affine <br> quotient |
| :--- | :--- | :--- |
| $T\left(\gamma_{1}\right) \cdot T\left(\gamma_{2}\right)$ | $T\left(\gamma_{1} \gamma_{2}\right)$ | 1 |
| $T\left(\gamma_{1}\right)^{F} \cdot T\left(\gamma_{2}\right)^{F}$ | $\bullet$ if $\gamma_{1} \neq \gamma_{2}:\left\{T(\gamma): \min \left\{\gamma_{1}, \gamma_{2}\right\}<\gamma<1\right\} \cup\{P\}$ | 1 |
|  | $\bullet$ if $\gamma_{1}=\gamma_{2}:\left\{T(\gamma): \min \left\{\gamma_{1}, \gamma_{2}\right\} \leq \gamma<1\right\} \cup\{P\}$ | 1 |

(v) Combinations of $T\left(\gamma_{0}\right), 0<\gamma_{0}<1$, with $P$ :

| notation | possible parametrizations | affine quotient |
| :--- | :--- | :--- |
| $T\left(\gamma_{0}\right)^{F} \cdot P$ | $\left\{T(\gamma): \gamma_{0}<\gamma<1\right\}$ | 1 |



Figure 4. $Q\left(\alpha_{1}, \beta_{1}\right) \cdot Q\left(\alpha_{2}, \beta_{2}\right)$ and $Q\left(\alpha_{1}, \beta_{1}\right): Q\left(\alpha_{2}, \beta_{2}\right)$.
(vi) The only possible combination of $P$ with $P$ is $P$ and has the affine quotient 1.

Proof. (i). Assume that, as in Figure 3, $Q\left(\alpha_{i}, \beta_{i}\right)$ is realized as the quadrangle $a_{i} b_{i} c_{i} d_{i}$ with closing sides $b_{i} c_{i}$ and $c_{i} d_{i}$ and with the intersection $s_{i}$ of $l\left(a_{i} b_{i}\right)$ with $l\left(c_{i} d_{i}\right)$, such that the parameters (over $b_{i} c_{i}$ ) are

$$
\alpha_{i}=\frac{\left|b_{i} s_{i}\right|}{\left|a_{i} s_{i}\right|} \quad \text { and } \quad \beta_{i}=\frac{\left|c_{i} s_{i}\right|}{\left|d_{i} s_{i}\right|} .
$$

When both $Q\left(\alpha_{1}, \beta_{1}\right)$ and $Q\left(\alpha_{2}, \beta_{2}\right)$ together form a gc-dissection of a third quadrangle, a closing side of one quadrangle must be glued together with an opening side of the other one.
First, suppose that $b_{1} c_{1}$ is glued with $a_{2} d_{2}$. Then necessarily $s_{1}=s_{2}=: s$. There are still two possibilities, namely either $b_{1}=a_{2}$ and $c_{1}=d_{2}$ or $b_{1}=d_{2}$ and $c_{1}=a_{2}$, see Figure 4. In the former situation we obtain a quadrangle $Q\left(\alpha_{1}, \beta_{1}\right) \cdot Q\left(\alpha_{2}, \beta_{2}\right)$ with vertices $a_{1} b_{2} c_{2} d_{1}$ and parameters

$$
0<\alpha=\frac{\left|b_{2} s\right|}{\left|a_{1} s\right|}=\frac{\left|b_{1} s_{1}\right|}{\left|a_{1} s_{1}\right|} \cdot \frac{\left|b_{2} s_{2}\right|}{\left|a_{2} s_{2}\right|}=\alpha_{1} \alpha_{2}<\beta=\frac{\left|c_{2} s\right|}{\left|d_{1} s\right|}=\frac{\left|c_{1} s_{1}\right|}{\left|d_{1} s_{1}\right|} \cdot \frac{\left|c_{2} s_{2}\right|}{\left|d_{2} s_{2}\right|}=\beta_{1} \beta_{2}<1
$$

which yields the first line in the table of (i). In the latter situation we obtain a quadrangle $Q\left(\alpha_{1}, \beta_{1}\right): Q\left(\alpha_{2}, \beta_{2}\right)$ with vertices $a_{1} c_{2} b_{2} d_{1}$. The parameters are computed by
$0<\frac{\left|c_{2} s\right|}{\left|a_{1} s\right|}=\frac{\left|b_{1} s_{1}\right|}{\left|a_{1} s_{1}\right|} \cdot \frac{\left|c_{2} s_{2}\right|}{\left|d_{2} s_{2}\right|}=\alpha_{1} \beta_{2}<1$ and $0<\frac{\left|b_{2} s\right|}{\left|d_{1} s\right|}=\frac{\left|c_{1} s_{1}\right|}{\left|d_{1} s_{1}\right|} \cdot \frac{\left|b_{2} s_{2}\right|}{\left|a_{2} s_{2}\right|}=\beta_{1} \alpha_{2}<1$.
Depending on whether $\alpha_{1} \beta_{2} \lesseqgtr \alpha_{2} \beta_{1}$ we obtain the parametrization and the affine quotient as in the second line in the table of (i).
Note that the resulting affine types are exactly the same if $a_{1} d_{1}$ is glued together with $b_{2} c_{2}$. So we obtain

$$
\begin{align*}
& Q\left(\alpha_{1}, \beta_{1}\right) \cdot Q\left(\alpha_{2}, \beta_{2}\right)=Q\left(\alpha_{2}, \beta_{2}\right) \cdot Q\left(\alpha_{1}, \beta_{1}\right), \\
& Q\left(\alpha_{1}, \beta_{1}\right): Q\left(\alpha_{2}, \beta_{2}\right)=Q\left(\alpha_{2}, \beta_{2}\right): Q\left(\alpha_{1}, \beta_{1}\right) . \tag{10}
\end{align*}
$$

The remainder of (i) is obtained by glueing $b_{1} c_{1}$ with $a_{2} b_{2}$ (or, equivalently, $a_{1} b_{1}$ with $b_{2} c_{2}$ ), $c_{1} d_{1}$ with $a_{2} d_{2}$ (or $a_{1} d_{1}$ with $c_{2} d_{2}$ ), and $c_{1} d_{1}$ with $a_{2} b_{2}$ (or $a_{1} b_{1}$ with


Figure 5. Combining $T\left(\gamma_{1}\right)$ with $T\left(\gamma_{2}\right)$ along constant sides.
$c_{2} d_{2}$ ). The distinction of cases and the resulting affine quotient can be formulated in terms of $\frac{\alpha_{i}}{\beta_{i}}$, since $\frac{\bar{\beta}_{i}}{\bar{\alpha}_{i}}=\frac{\alpha_{i}}{\beta_{i}}$.
(ii). Here the arguments are as in (i). The situation simplifies, because $T(\gamma)$ has only one parameter and only one opening and one closing side.
(iii). $Q(\alpha, \beta)$ and $P$ cannot be combined, since opposite sides of $P$ are parallel, whereas opposite sides of $Q(\alpha, \beta)$ are not.
(iv). The first line of the table of (iv), i.e. the quadrangle $T\left(\gamma_{1}\right) \cdot T\left(\gamma_{2}\right)$, is obtained by glueing the closing side of $T\left(\gamma_{1}\right)$ together with the opening side of $T\left(\gamma_{2}\right)$ or vice versa.
Alternatively, $T\left(\gamma_{1}\right)$ and $T\left(\gamma_{2}\right)$ can be glued along constant sides, see Figure5. This type of glueing is called $T\left(\gamma_{1}\right)^{F} \cdot T\left(\gamma_{2}\right)^{F}$. We can assume that, w.l.o.g., $\gamma_{1} \leq \gamma_{2}$ and the parallel sides of $T\left(\gamma_{1}\right)$ have lengths 1 and $\gamma_{1}$. Then the lengths of the parallel sides of the combined trapezoids are either $\gamma_{1}+\lambda \gamma_{2}$ and $1+\lambda$ or $\gamma_{1}+\lambda$ and $1+\lambda \gamma_{2}$ with arbitrary $\lambda>0$. Their respective ratios are

$$
\frac{\gamma_{1}+\lambda \gamma_{2}}{1+\lambda}=\frac{1}{1+\lambda} \gamma_{1}+\frac{\lambda}{1+\lambda} \gamma_{2}
$$

which is $\gamma_{1}$ if $\gamma_{1}=\gamma_{2}$ or ranges in the interval $\left(\gamma_{1}, \gamma_{2}\right)$ if $\gamma_{1}<\gamma_{2}$, and

$$
\frac{\gamma_{1}+\lambda}{1+\lambda \gamma_{2}}=\frac{1}{1+\lambda \gamma_{2}} \gamma_{1}+\frac{\lambda \gamma_{2}}{1+\lambda \gamma_{2}} \frac{1}{\gamma_{2}},
$$

which ranges in $\left(\gamma_{1}, \frac{1}{\gamma_{2}}\right) \supseteq\left(\gamma_{1}, 1\right]$. Consequently, the affine types obtained by $T\left(\gamma_{1}\right)^{F} \cdot T\left(\gamma_{2}\right)^{F}$ are $T(\gamma), \gamma_{1} \leq \gamma<1$, and $P$ if $\gamma_{1}=\gamma_{2}$, and $T(\gamma), \gamma_{1}<\gamma<1$, and $P$ if $\gamma_{1}<\gamma_{2}$.
(v). This is obtained as (iv). Only glueing along constant sides is possible.
(vi). This is trivial.

Remark 12. (i) The multiplicative operational notation comes with a multiplication of the parameters as well as of the affine quotients. The divisional notation corresponds to a division of the affine quotients and to an exchange of the parameters of one quadrangle.
(ii) We do not use a flip in our operational notation if the respective glueing is made along the closing side over which the parametrization is made (or along its opposite opening side). We use a flip if the glueing is made along the neighbouring closing (or opening) side in case of a non-trapezoid or along a constant side in case of a trapezoid.
(iii) The operational notations $T\left(\gamma_{1}\right)^{F} \cdot T\left(\gamma_{2}\right)^{F}$ and $T\left(\gamma_{0}\right)^{F} \cdot P$ stand for infinitely many possible results, whereas the other notations produce a unique outcome. The notations are commutative (see (10) in the proof of (i)).
(iv) Lemma 11 allows to classify all 2 -gc-self-affine quadrangles (and all 2 -selfaffine convex quadrangles, since this is equivalent). Indeed, every trapezoid is 2-gc-self-affine, as follows from Lemma 11(iv) and (vi) (but in fact is trivial, see Figure 11). On the other hand, non-trapezoids are not 2 -gc-self-affine, because Lemma 11(i) shows that a combination of two copies of $Q(\alpha, \beta), 0<\alpha<\beta<1$, has an affine quotient $\left(\frac{\alpha}{\beta}\right)^{2}$ or 1 , which is different from the quotient $\frac{\alpha}{\beta}$ of $Q(\alpha, \beta)$.

## 6. $n$-GC-SELF-AFFINE QUADRANGLES WITH EVEN $n$ ARE TRAPEZOIDS

Lemma 13. Suppose that a quadrangle $Q$ has a gc-dissection into $n$ copies of $Q(\alpha, \beta), 0<\alpha<\beta<1$. Then the affine quotient of $Q$ is of the form $\left(\frac{\alpha}{\beta}\right)^{k}$ with $k \in\{0,1, \ldots, n\}$, where $k$ is even if $n$ is even and $k$ is odd if $n$ is odd.

Proof. This is obtained by mathematical induction over the number $n$ of tiles (or of leafs in the dissection tree). The claim is trivial for the trivial dissection $(n=1)$. In case of a finer dissection $Q$ is split into two quadrangles $Q^{*}$ and $Q^{* *}$ who themselves are dissected into $n_{1}$ and $n_{2}$ tiles with $n_{1}+n_{2}=n$ and $0<n_{1}, n_{2}<n$. By the induction hypothesis, their affine quotients are $\left(\frac{\alpha}{\beta}\right)^{k_{1}}$ and $\left(\frac{\alpha}{\beta}\right)^{k_{2}}$ with $k_{i} \in$ $\left\{0,1, \ldots, n_{i}\right\}$ and $k_{i} \equiv n_{i} \bmod 2, i=1,2$. Now Lemma 11 shows that $Q=$ $Q^{*} \cup Q^{* *}$ has an affine quotient $\left(\frac{\alpha}{\beta}\right)^{k}$ with $k \in\left\{k_{1}+k_{2},\left|k_{1}-k_{2}\right|\right\}$. Accordingly, $k \in\left\{0,1, \ldots, k_{1}+k_{2}\right\} \subseteq\left\{0,1, \ldots, n_{1}+n_{2}\right\}$ and $k \equiv k_{1}+k_{2} \equiv n_{1}+n_{2}=n$ $\bmod 2$.

Proof of Theorem 1 (iii). It is clear that every trapezoid is $n$-gc-self-affine by repeated application of Lemma 11(iv) and (vi), see also Figure 1 On the other hand, a non-trapezoid $Q(\alpha, \beta), 0<\alpha<\beta<1$, cannot be gc-dissected into an even number $n$ of copies of $Q(\alpha, \beta)$, because then Lemma 13 implied that $Q(\alpha, \beta)$ had an affine quotient $\left(\frac{\alpha}{\beta}\right)^{k}$ with even $k$, which is different from the actual quotient $\left(\frac{\alpha}{\beta}\right)^{1}$.

## 7. $n$-GC-SELF-AFFINE QUADRANGLES FOR ODD $n \geq 5$

Lemma 14. Let $0<\alpha<\beta<1$.
(i) $T(\alpha \beta)$ has a gc-dissection into two copies of $Q(\alpha, \beta)$.
(ii) For every real number $\gamma$ with $\alpha \beta \min \left\{\frac{1-\beta}{(1-\alpha) \beta}, 1\right\} \leq \gamma<1$ and every even integer $k \geq 4, T(\gamma)$ has a gc-dissection into $k$ copies of $Q(\alpha, \beta)$.

Proof. (i). This comes from $Q(\alpha, \beta): Q(\alpha, \beta)$ in Lemman1(i).
(ii). First suppose that $\frac{1-\beta}{(1-\alpha) \beta} \geq 1$. By (i), we can produce $\frac{k}{2}$ copies of $T(\alpha \beta)$, each composed of two copies of $Q(\alpha, \beta)$. Repeated use of the first part of Lemma 11(iv) shows that $T\left((\alpha \beta)^{\frac{k}{2}-1}\right)$ can be obtained from $\frac{k}{2}-1$ copies of $T(\alpha \beta)$, that is,
from $k-2$ copies of $Q(\alpha, \beta)$. Finally, we use the second part of Lemma 11(iv) for writing $T(\gamma)$ with $\alpha \beta \leq \gamma<1$ as $T\left((\alpha \beta)^{\frac{k}{2}-1}\right)^{F} \cdot T(\alpha \beta)^{F}$, this way obtaining a gc-dissection into $(k-2)+2=k$ copies of $Q(\alpha, \beta)$.
If $\frac{1-\beta}{(1-\alpha) \beta}<1$, we recall that $Q(\alpha, \beta)$ represents the same quadrangles as

$$
Q(\alpha, \beta)^{F}=Q\left(\frac{1-\beta}{(1-\alpha) \beta}(\alpha, \beta)\right)=Q(\bar{\alpha}, \bar{\beta})
$$

where $0<\bar{\alpha}<\bar{\beta}<1$ and $\frac{1-\bar{\beta}}{(1-\bar{\alpha}) \beta}=\left(\frac{1-\beta}{(1-\alpha) \beta}\right)^{-1}<1$. As above, we see that $T(\gamma)$ has a gc-dissection into $k$ copies of $Q(\bar{\alpha}, \bar{\beta})$ whenever $\gamma$ is in the interval

$$
[\bar{\alpha} \bar{\beta}, 1)=\left[\left(\frac{1-\beta}{(1-\alpha) \beta}\right)^{2} \alpha \beta, 1\right) \supseteq\left[\frac{1-\beta}{(1-\alpha) \beta} \alpha \beta, 1\right)
$$

In the sequel affine images of (convex) kites turn out to be crucial. We call them affine kites for short. Of course, a trapezoid is an affine kite if and only if it is a parallelogram.
Lemma 15. Let $0<\alpha<\beta<1$. The following are equivalent.
(i) $Q(\alpha, \beta)$ is an affine kite.
(ii) $\frac{1-\beta}{(1-\alpha) \beta}=1$ (or, equivalently, $\alpha=2-\frac{1}{\beta}$ or $\beta=\frac{1}{2-\alpha}$ ).
(iii) The parametrization $Q(\alpha, \beta)$ is invariant under flipping: $Q(\alpha, \beta)^{F}=Q(\alpha, \beta)$.

In particular, every affine kite has a unique parametrization.
Proof. We refer to the representation of $Q(\alpha, \beta)$ given in Figure 3. Then $Q(\alpha, \beta)$ is an affine kite if and only if the diagonal $a c$ meets the midpoint of $b d$. Equivalently, the linear map given by $a \mapsto a, s \mapsto t$ and $t \mapsto s$ maps $c=(1-\beta, 1-\bar{\beta})$ onto itself. Since that map is nothing but the reflection exchanging the coordinates, $Q(\alpha, \beta)$ is an affine kite if and only if $\bar{\beta}=\beta$. Taking into account that $(\bar{\alpha}, \bar{\beta})=\frac{1-\beta}{(1-\alpha) \beta}(\alpha, \beta)$, we see the equivalence of (i), (ii) and (iii).

Lemma 16. (i) If $Q(\alpha, \beta), 0<\alpha<\beta<1$, is no affine kite, then $Q(\alpha, \beta)$ is $n$-gc-self-affine for every odd $n \geq 5$.
(ii) Every affine kite is $n$-gc-self-affine for every odd $n \geq 7$.

Proof. (i). By Lemma [15, $\frac{1-\beta}{(1-\alpha) \beta} \neq 1$. As in the proof of Lemma 14, we can assume that $\frac{1-\beta}{(1-\alpha) \beta}<1$, since otherwise we consider $Q(\alpha, \beta)^{F}$ instead of $Q(\alpha, \beta)$. By Lemma 14(ii), $T\left(\frac{1-\beta}{(1-\alpha) \beta}\right)$ has a gc-dissection into $n-1$ copies of $Q(\alpha, \beta)$. Now Lemma 11(ii) shows that

$$
Q(\alpha, \beta)^{F}=Q\left(\frac{1-\beta}{(1-\alpha) \beta}(\alpha, \beta)\right)=Q(\alpha, \beta) \cdot T\left(\frac{1-\beta}{(1-\alpha) \beta}\right)
$$

has a gc-dissection into $n$ copies of $Q(\alpha, \beta)$, which yields (i).
(ii). The claim is trivial for trapezoidal affine kites (i.e., for parallelograms). It remains to consider a kite $Q(\alpha, \beta), 0<\alpha<\beta<1$. By Lemma 15, $\beta=\frac{1}{2-\alpha}$. An

$L L$ Necessities

Case I.c



Case I.a


Case I.b


Case I.d


Case II

Figure 6. Particular extended dissection trees for the proof of Lemma 17.
easy calculation shows that the number $\gamma=\frac{\left(1-\alpha^{2} \beta\right) \beta}{1-\alpha \beta^{2}}$ satisfies $\alpha \beta \leq \gamma<1$. Then Lemma 14 shows that $T(\alpha \beta)$ has a gc-dissection into two copies of $Q(\alpha, \beta)$ and $T(\gamma)$ has a gc-dissection into $n-3$ copies of $Q(\alpha, \beta)$. Another simple calculation gives

$$
Q(\alpha, \beta)=(Q(\alpha, \beta) \cdot T(\alpha \beta))^{F} \cdot T(\gamma)
$$

which in turn yields that $Q(\alpha, \beta)$ has a gc-dissection into $(1+2)+(n-3)=n$ copies of $Q(\alpha, \beta)$.

Lemma 17. If $Q(\alpha, \beta), 0<\alpha<\beta<1$, is an affine kite, then $Q(\alpha, \beta)$ is not 5-gc-self-affine.

Proof. Suppose that some affine kite $Q\left(\alpha, \frac{1}{2-\alpha}\right), 0<\alpha<1$ (cf. Lemma 15), is 5 -gc-self-affine.
Step 1. Reduction to five extended dissection trees. The dissection tree has five leafs, called $L$, and a root $R$. $L$ and $R$ have the unique parametrization $Q\left(\alpha, \frac{1}{2-\alpha}\right)$, see Lemma 15(iii). We extend the tree as follows. Given a vertex $Q$ that is not a leaf, a parametrization of $Q$ is obtained from the parametrizations of its two children $C_{1}$ and $C_{2}$ by an operation • or :, possibly preceded by a flip on $C_{1}$ and/or $C_{2}$, see Lemma 11. We mark the operation • or : at the vertex $Q$ and flips, in case they exist, at the corresponding edges between $Q$ and its children, see Figure 6
We collect necessary properties of the extended dissection tree.
(A) W.l.o.g., no edge emanating from a leaf has a flip.
(B) On the path between any leaf $L$ and the root $R$ there is at least one flip.

For (A), note that leafs represent kites, that are invariant under flips.
For (B), suppose that there is no flip on the path from some $L$ to $R$, both with unique parametrization $Q\left(\alpha, \frac{1}{2-\alpha}\right)$. Then, by Lemma 11, in each operation $\cdot$ or : on the path from $L$ to $R$, the smaller parameter of the respective quadrangle decreases strictly, since it is multiplied by some positive number less than one. Consequently, the smaller parameter of $R$ is smaller than that of $L$, contradicting that both have the same unique parametrization $Q\left(\alpha, \frac{1}{2-\alpha}\right)$.
It follows from (A) and (B) that no child of $R$ is a leaf. Hence there are two subtrees below $R$, one with 3 and one with 2 leafs. Using the commutativity of and $:\left(\right.$ cf. (10)), the root $R$, leafs $L$, additional vertices $Q_{1}, Q_{2}, Q_{3}$ and the edges of the tree are as in the upper left illustration in Figure 6. Moreover, by (A) and (B) we have no flips on the edges emanating from leafs and we have flips on both edges emanating from $R$.
(C) The extended dissection tree has the structure displayed under Necessities in Figure 6. In particular, we have the operations : at $R$ and $\cdot$ at $Q_{3}$, and only for the edge between $Q_{1}$ and $Q_{2}$ it is not yet determined if there is a flip.

The situation for flips is already shown.
For the operations at $R$ and $Q_{3}$, suppose first that we have $\cdot$ at $R$. By Lemma 13 the affine quotient of $Q_{2}$ and $Q_{3}$ are $[\alpha(2-\alpha)]^{k_{2}}$ with $k_{2} \in\{1,3\}$ and $[\alpha(2-\alpha)]^{k_{3}}$ with $k_{3} \in\{0,2\}$, respectively. Since $R=Q_{2}^{F} \cdot Q_{3}^{F}$, we have the quotient $\alpha(2-\alpha)=$ $[\alpha(2-\alpha)]^{k_{2}} \cdot[\alpha(2-\alpha)]^{k_{3}}$ by Lemma [11, which yields $k_{2}+k_{3}=1$ and in turn $k_{2}=1$ and $k_{3}=0$. By $k_{3}=0$, we have : at $Q_{3}$ and $Q_{3}=T$ is a trapezoid. But then the equation $R=Q_{2}^{F} \cdot T^{F}$ implies that $Q_{2}^{F}$ must be a flipped trapezoid or a parallelogram, since these are the only partners for multiplication with flipped trapezoids by Lemma 11. However $Q_{2}^{F}$ cannot be of that type, because its affine quotient is $[\alpha(2-\alpha)]^{1} \neq 1$. This contradiction shows that the operation at $R$ is :.

Finally, to see that we have • at $Q_{3}$, assume to the contrary that there is : at $Q_{3}$. But then $Q_{3}$ is a trapezoid $T$, and an operation $R=Q_{2}^{F}: T^{F}$ does not exist. So (C) is verified.

Now we discuss cases depending on the operations at $Q_{1}$ and $Q_{2}$ and on the existence of a flip between $Q_{1}$ and $Q_{2}$.
Case I, there is . at $Q_{1}$. Then we get Case I.a, ..., Case I.d from Figure 6 .
Case II, there is : at $Q_{1}$. Then $Q_{1}$ is a trapezoid $T$, which does not allow : at $Q_{2}$. So there is • at $Q_{2}$. Moreover, there is no flip between $Q_{1}=T$ and $Q_{2}$, because this would give the impossibility $Q_{2}=T^{F} \cdot L$, since $L$ is neither a parallelogram nor a flipped trapezoid. We obtain the last situation illustrated in Figure 6

Step 2. Discussion of the five remaining cases. For every case we compute the parametrization of $R$ along the extended dissection tree, based on $L=Q\left(\alpha, \frac{1}{2-\alpha}\right)$, and we analyse the condition $R=Q\left(\alpha, \frac{1}{2-\alpha}\right)$. The computations can be made by hand or by some computer algebra system.

Case I.a. Here $R=((L \cdot L) \cdot L)^{F}:(L \cdot L)^{F}$ gives

$$
R=Q\left(\frac{\left(\alpha^{2}-5 \alpha+7\right)(3-\alpha) \alpha^{2}}{\left(\alpha^{2}+\alpha+1\right)(\alpha+1)(2-\alpha)^{2}}\left(\alpha, \frac{1}{2-\alpha}\right)\right)
$$

The condition $R=Q\left(\alpha, \frac{1}{2-\alpha}\right)$ amounts to $1=\frac{\left(\alpha^{2}-5 \alpha+7\right)(3-\alpha) \alpha^{2}}{\left(\alpha^{2}+\alpha+1\right)(\alpha+1)(2-\alpha)^{2}}$. Subtraction of 1 , multiplication by the denominator and division by the non-zero term $2(1-\alpha)$ yields the contradiction

$$
0=\alpha^{4}-4 \alpha^{3}+6 \alpha^{2}-4 \alpha-2=(1-\alpha)^{4}-3<1-3=-2
$$

because $0<\alpha<1$. Therefore Case I.a does not give a 5 -gc-self-affinity of an affine kite.
Case I.b. Here $R=\left((L \cdot L)^{F} \cdot L\right)^{F}:(L \cdot L)^{F}$ gives

$$
R=Q\left(\frac{\left(-\alpha^{3}+4 \alpha^{2}-2 \alpha-5\right)(3-\alpha) \alpha^{2}}{\left(\alpha^{3}-2 \alpha^{2}-2 \alpha-1\right)(\alpha+1)(2-\alpha)^{2}}\left(\alpha, \frac{1}{2-\alpha}\right)\right)
$$

The condition $R=Q\left(\alpha, \frac{1}{2-\alpha}\right)$ reads as $1=\frac{\left(-\alpha^{3}+4 \alpha^{2}-2 \alpha-5\right)(3-\alpha) \alpha^{2}}{\left(\alpha^{3}-2 \alpha^{2}-2 \alpha-1\right)(\alpha+1)(2-\alpha)^{2}}$. Subtracting 1 , multiplying by the denominator and dividing by $2(1-\alpha)$ gives the contradiction

$$
0=\alpha^{4}-4 \alpha^{3}+\alpha^{2}+6 \alpha+2=\alpha^{2}(2-\alpha)^{2}-3(1-\alpha)^{2}+5>0-3+5=2
$$

for $0<\alpha<1$.
Case I.c. Here $R=((L \cdot L): L)^{F}:(L \cdot L)^{F}$ gives

$$
R=Q\left(\frac{(4-\alpha)(3-\alpha) \alpha}{(\alpha+2)(\alpha+1)(2-\alpha)}\left(\alpha, \frac{1}{2-\alpha}\right)\right)
$$

The condition $R=Q\left(\alpha, \frac{1}{2-\alpha}\right)$ is now $1=\frac{(4-\alpha)(3-\alpha) \alpha}{(\alpha+2)(\alpha+1)(2-\alpha)}$. Subtracting 1 , multiplying by the denominator and dividing by $2(1-\alpha)$ gives the contradiction

$$
0=-\alpha^{2}+2 \alpha-2=-(1-\alpha)^{2}-1<-1
$$

for $0<\alpha<1$.
Case I.d. Here $R=\left((L \cdot L)^{F}: L\right)^{F}:(L \cdot L)^{F}$ gives

$$
R=Q\left(\frac{\left(\alpha^{2}-\alpha-4\right)(3-\alpha) \alpha}{\left(\alpha^{2}-3 \alpha-2\right)(\alpha+1)(2-\alpha)}\left(\alpha, \frac{1}{2-\alpha}\right)\right)
$$

Now $R=Q\left(\alpha, \frac{1}{2-\alpha}\right)$ amounts to $1=\frac{\left(\alpha^{2}-\alpha-4\right)(3-\alpha) \alpha}{\left(\alpha^{2}-3 \alpha-2\right)(\alpha+1)(2-\alpha)}$. Subtracting 1, multiplying by the denominator and dividing by $2(1-\alpha)$ gives the contradiction $0=2$. Case II. Here $R=((L: L) \cdot L)^{F}:(L \cdot L)^{F}$ gives

$$
R=Q\left(\frac{(4-\alpha)(3-\alpha) \alpha}{(\alpha+2)(\alpha+1)(2-\alpha)}\left(\alpha, \frac{1}{2-\alpha}\right)\right)
$$

as in Case I.c.
Proof of Theorem [1(iv) and (v). Clearly, every trapezoid is $n$-gc-self-affine for all $n \geq 2$, see the rightmost illustration in Figure 1. The positive claims of Theorem 1 (iv) and (v) for non-trapezoids are given by Lemma 16 . The remaining negative result for kites in Theorem 1(v) is the claim of Lemma 17.

 Case I.a Case I.b Case I.c Case II.a


Case II.b Case II.c Case II.d Case II.e Case II.f

Figure 7. Particular extended dissection trees for the proof of Theorem 18.

## 8. 3-GC-SELF-AFFINE QUADRANGLES

Theorem 18. A convex quadrangle is 3-gc-self-affine if and only if it belongs to one of the following families.
(I) $\{T(\alpha): 0<\alpha<1\} \cup\{P\}$ (the family of all trapezoids including parallelograms).
(II) $\left\{Q\left(\alpha, \frac{-1+\sqrt{1+4 \alpha-4 \alpha^{2}}}{2 \alpha(1-\alpha)}\right): 0<\alpha<1\right\}$.

Here $\alpha<\frac{-1+\sqrt{1+4 \alpha-4 \alpha^{2}}}{2 \alpha(1-\alpha)}<1$ whenever $0<\alpha<1$.
(III) $\left\{Q\left(\alpha, \frac{1-3 \alpha+\alpha^{2}+\sqrt{1-2 \alpha+7 \alpha^{2}-6 \alpha^{3}+\alpha^{4}}}{2(1-\alpha)}\right): 0<\alpha<1\right\}$.

Here $\alpha<\frac{1-3 \alpha+\alpha^{2}+\sqrt{1-2 \alpha+7 \alpha^{2}-6 \alpha^{3}+\alpha^{4}}}{2(1-\alpha)}<1$ whenever $0<\alpha<1$.
(IV) $\{Q(\alpha, \beta): 0<\alpha<\beta<1$,

$$
\left.\left(\alpha-\alpha^{2}\right) \beta^{3}+\left(1-2 \alpha+2 \alpha^{2}\right) \beta^{2}+\left(-1+2 \alpha-4 \alpha^{2}+\alpha^{3}\right) \beta+\alpha^{2}=0\right\}
$$

Here $\{(\alpha, \beta): 0<\alpha<\beta<1$,

$$
\left.\left(\alpha-\alpha^{2}\right) \beta^{3}+\left(1-2 \alpha+2 \alpha^{2}\right) \beta^{2}+\left(-1+2 \alpha-4 \alpha^{2}+\alpha^{3}\right) \beta+\alpha^{2}=0\right\}
$$

is the graph of a function $\beta:(0,1) \rightarrow(0,1)$ with $\alpha<\beta(\alpha)<1$ whenever $0<\alpha<1$.

Proof. We restrict our consideration to the classes $Q(\alpha, \beta), 0<\alpha<\beta<1$, because the situation is trivial for trapezoids.

Step 1. Reduction to nine extended dissection trees. If $R$ is the root of a dissection tree with leafs $L=Q(\alpha, \beta)$ of a 3-gc-self-affinity, then one child of $R$ is a leaf and the other child is a quadrangle $Q$ whose both children are leafs. By the commutativity of the operations • and :, w.l.o.g., $Q$ is the left child of $R$. Moreover, the following are satisfied for the extended dissection tree, see the left-hand part of Figure 7
(A) W.l.o.g., there is no flip on the edge directly connecting $R$ with a leaf $L$.
(B) If there is: at $Q$ then there is no flip between $Q$ and $R$.
(C) If there is exactly one flip below $Q$ then, w.l.o.g., it is on the left edge below $Q$.
(D) There is • at $R$ if and only if there is : at $Q$.

For (A), note that if there is a dissection tree with leafs $L=Q(\alpha, \beta)$, then the same dissection is described by leafs $L=Q(\alpha, \beta)^{F}$ when flips on edges emanating from leafs are replaced by non-flips and vice versa.

For (B), if there is : at $Q$ and a flip between $Q$ and $R$, then $Q=T(\gamma)$ is a trapezoid and the operation at $R$ combines $T(\gamma)^{F}$ with a non-trapezoid. This is impossible.
Claim (C) follows from the commutativity of $\cdot$ and :.
For (D), assume first that we have $\cdot$ at $R$ and $Q$ simultaneously. Then $R$ has the affine quotient $\left(\frac{\alpha}{\beta}\right)^{3}$ and cannot be of the same affine type as $L$, whose quotient is $\frac{\alpha}{\beta}$, a contradiction. If there is : at both $R$ and $Q$, then $Q=T(\gamma)$ is a trapezoid. By (A) and (B), there are no flips directly below $R$. Thus $R=T(\gamma): Q(\alpha, \beta)=$ $Q(\alpha, \beta): T(\gamma)$, which does not make sense.
The restrictions (A), (B), (C) and (D) reduce the dissection trees to Cases I.a-c if there is • at $R$ (whence there is : at $Q$ ) and to Cases II.a-f in the opposite situation, see Figure 7
Step 2. Discussion of the nine trees.
(E) Any of the remaining trees with leafs $L=Q(\alpha, \beta)$ describes a 3-gc-selfaffinity of a quadrangle if and only if the parametrization of its root $R$ obtained by computation along the tree from $L=Q(\alpha, \beta)$ coincides with $Q(\alpha, \beta)^{F}=Q\left(\frac{1-\beta}{(1-\alpha) \beta}(\alpha, \beta)\right)$.

We have a 3 -gc-self-affinity if and only if $R$ is of the same affine type as $L=Q(\alpha, \beta)$; that is, if the parametrization of $R$ is one of $Q(\alpha, \beta)$ or $Q(\alpha, \beta)^{F}$. In all cases the parametrization of $R$ is obtained by either • or : with one operand being $L=Q(\alpha, \beta)$. This way we obtain a parameter by multiplication of $\alpha$ with some positive number less than one. But then the parametrization of $R$ is not $Q(\alpha, \beta)$, because its smaller parameter is smaller than $\alpha$. Consequently, $R$ and $L$ have the same affine type if and only if the parametrization of $R$ is $Q(\alpha, \beta)^{F}$.
In the following we compute the parametrization of $R$ from $L=Q(\alpha, \beta)$ for all trees determined in Step 1 and analyse criterion (E). Support by a computer algebra system is useful, but not necessary.
Case I.a. Here $R=Q(\alpha \beta(\alpha, \beta))$ and (E) amounts to $\alpha \beta=\frac{1-\beta}{(1-\alpha) \beta}$. This is equivalent to $p_{\alpha}(\beta):=\beta^{2}+\frac{1}{\alpha(1-\alpha)} \beta-\frac{1}{\alpha(1-\alpha)}=0$. For every $\alpha \in(0,1)$, the quadratic function $p_{\alpha}(\beta)$ has a unique root $\beta \in(\alpha, 1)$, because $p_{\alpha}(\alpha)=\frac{\alpha^{3}-1}{\alpha}<$ $0<1=p_{\alpha}(1)$. That root is $\beta(\alpha)=\frac{-1+\sqrt{1+4 \alpha-4 \alpha^{2}}}{2 \alpha(1-\alpha)}$, and we obtain family (II) from Theorem 18
Case I.b. Now $R=Q\left(\frac{(1-\beta) \alpha}{1-\alpha}(\alpha, \beta)\right)$ and (E) reads as $\frac{(1-\beta) \alpha}{1-\alpha}=\frac{1-\beta}{(1-\alpha) \beta}$. This is equivalent to $\alpha \beta=1$, contradicting $0<\alpha<\beta<1$.

Case I.c. We get $R=Q\left(\frac{(1-\beta)^{2} \alpha}{(1-\alpha)^{2} \beta}(\alpha, \beta)\right)$ and (E) gives $\frac{(1-\beta)^{2} \alpha}{(1-\alpha)^{2} \beta}=\frac{1-\beta}{(1-\alpha) \beta}$. Multiplication with $\frac{(1-\alpha)^{2} \beta}{1-\beta}$ and subtraction of the right-hand side yields $(1-\beta) \alpha-(1-\alpha)=$ $-\left((1-\alpha)^{2}+\alpha(\beta-\alpha)\right)=0$, again contradicting $0<\alpha<\beta<1$.
Case II.a. We have $R=Q(\alpha \beta(\alpha, \beta))$ as in Case I.a.
Case II.b. Here $R=Q\left(\frac{\left(1-\beta^{2}\right) \alpha}{\left(1-\alpha^{2}\right) \beta}(\alpha, \beta)\right)$ and (E) is $\frac{\left(1-\beta^{2}\right) \alpha}{\left(1-\alpha^{2}\right) \beta}=\frac{1-\beta}{(1-\alpha) \beta}$. Multiplying with $\frac{\left(1-\alpha^{2}\right) \beta}{1-\beta}$ and subtracting $\alpha$ we obtain $\alpha \beta=1$, contradicting $0<\alpha<\beta<1$.

Case II.c. We get $R=Q\left(\frac{(1-\beta) \alpha}{1-\alpha}(\alpha, \beta)\right)$ as in Case I.b.
Case II.d. Here $R=Q\left(\frac{((1-\alpha)-(1-\beta) \beta) \alpha}{(1-\alpha) \beta-(1-\beta) \alpha^{2}}(\alpha, \beta)\right)$ and $(\mathrm{E})$ is $\frac{((1-\alpha)-(1-\beta) \beta) \alpha}{(1-\alpha) \beta-(1-\beta) \alpha^{2}}=$ $\frac{1-\beta}{(1-\alpha) \beta}$. Note that the left-hand denominator is positive, because $1-\alpha>1-\beta>0$ and $\beta>\alpha^{2}>0$. By multiplying with the denominators and subtracting the right-hand side, ( E ) is equivalent to

$$
\begin{equation*}
\left(\alpha-\alpha^{2}\right) \beta^{3}+\left(1-2 \alpha+2 \alpha^{2}\right) \beta^{2}+\left(-1+2 \alpha-4 \alpha^{2}+\alpha^{3}\right) \beta+\alpha^{2}=0 \tag{11}
\end{equation*}
$$

This describes family (IV) from Theorem 18 To see that, for every $\alpha \in(0,1)$, equation (11) has a unique solution $\beta=\beta(\alpha) \in(\alpha, 1)$, we denote the left-hand side of (11) by $h_{\alpha}(\beta)$. There is a solution, since $h_{\alpha}(\alpha)=-\alpha(1-\alpha)^{4}<0<$ $\alpha(1-\alpha)^{2}=h_{\alpha}(1)$. The solution is unique, because $h_{\alpha}$ is convex on the interval $(\alpha, 1)$ by $\left.h_{\alpha}^{\prime \prime}(\beta)=6 \alpha(1-\alpha) \beta+2\left((1-\alpha)^{2}+\alpha^{2}\right)\right)>0$.
Case II.e. We have $R=Q\left(\frac{(1-\beta)^{2} \alpha}{(1-\alpha)^{2} \beta}(\alpha, \beta)\right)$ as in Case I.c.
Case II.f. Here $R=Q\left(\frac{(2-\alpha-\beta) \alpha \beta}{(1-\alpha) \beta+(1-\beta) \alpha}(\alpha, \beta)\right)$ and (E) reads as $\frac{(2-\alpha-\beta) \alpha \beta}{(1-\alpha) \beta+(1-\beta) \alpha}=$ $\frac{1-\beta}{(1-\alpha) \beta}$. By subtracting the right-hand side and multiplying with the denominators and the positive term $\frac{1}{(1-\alpha)(1-\alpha \beta)}$, we arrive at the equivalent formulation $q_{\alpha}(\beta):=$ $\beta^{2}-\frac{1-3 \alpha+\alpha^{2}}{1-\alpha} \beta-\frac{\alpha}{1-\alpha}=0$. For every $\alpha \in(0,1)$, the quadratic function $q_{\alpha}(\beta)$ has a unique root $\beta \in(\alpha, 1)$, because $q_{\alpha}(\alpha)=-2 \alpha(1-\alpha)<0<\alpha=q_{\alpha}(1)$. Noting that the smaller root $\beta_{1}=\frac{1-3 \alpha+\alpha^{2}-\sqrt{\left(1-3 \alpha+\alpha^{2}\right)^{2}+4 \alpha(1-\alpha)}}{2(1-\alpha)}$ is negative, the above mentioned relevant root is $\beta_{2}=\frac{1-3 \alpha+\alpha^{2}+\sqrt{\left(1-3 \alpha+\alpha^{2}\right)^{2}+4 \alpha(1-\alpha)}}{2(1-\alpha)}$. This way we get family (III) from Theorem 18 .

Simple numerical computations show that all families in Theorem 18 are relevant: not all quadrangles represented by one of the families (II), (III) and (IV) are contained in the other families.

## 9. An observation on self-affine convex quadrangles

Proof of Theorem 2, By Theorem (1), it remains to consider the case $n=5$ and the cases with even $n \geq 6$. Moreover, the situation for trapezoids is trivial, so that we restrict our consideration to quadrangles $Q(\alpha, \beta), 0<\alpha<\beta<1$.
The case $n=5$. This dissection can be found in [7, Proposition 1] and goes back to Attila Pór: Let $Q(\alpha, \beta)$ be represented by a quadrangle $Q=a b c d$ with $a=(0,0)$ as in Figure 3, and let $\varrho$ be a contraction with centre $a$ and ratio $\alpha \beta$. Then $Q$


Figure 8. $n$-self-affinity for even $n \geq 6$.
splits into the contracted quadrangle $\varrho(Q)$ and two trapezoids $T_{1}=\varrho(b) b c \varrho(c)$ and $T_{2}=\varrho(c) c d \varrho(d) . T_{1}$ and $T_{2}$ are of affine type $T(\alpha \beta)$ and can be gc-dissected into two copies of $Q(\alpha, \beta)$ by Lemma 14(i). This way $Q(\alpha, \beta)$ is dissected into five affine copies of itself.
The case of even $n \geq 6$. Let the quadrangle $Q^{*}=a b c d$ represent $Q(\alpha, \beta)$ as in Figure [3. Now let $\varphi$ be the affine map defined by $\varphi(a)=c, \varphi(b)=b$ and $\varphi(d)=d$. Note that $\varphi(c)$ is between $a$ and $c$ and, in particular, $\varphi\left(Q^{*}\right) \subseteq Q^{*}$; see the left-hand part of Figure 8, Next let $\sigma$ be a homothety with centre $a$ such that $\sigma(\varphi(c))=c$; see the right-hand part of Figure 8 Then $\sigma\left(Q^{*}\right)$ is dissected into the quadrangles $Q^{*}$ and $Q^{* *}=\sigma\left(\varphi\left(Q^{*}\right)\right)$ and two triangles $T_{1}=b \sigma(b) c$ and $T_{2}=c \sigma(d) d$. Finally, let $\tau_{\nu}$ denote a homothety with centre $a$ and ratio $\nu$ with $\frac{|a c|}{|a \sigma(c)|}<\nu<1$. Then the quadrangle $\tau_{\nu}\left(\sigma\left(Q^{*}\right)\right)$ (dashed in Figure (8) is of affine type $Q(\alpha, \beta)$ and dissected into the quadrangles $Q^{*}$ and $Q^{* *} \cap \tau_{\nu}\left(\sigma\left(Q^{*}\right)\right)$ of type $Q(\alpha, \beta)$ and into two trapezoids $T_{1} \cap \tau_{\nu}\left(\sigma\left(Q^{*}\right)\right)$ and $T_{2} \cap \tau_{\nu}\left(\sigma\left(Q^{*}\right)\right)$ of the same type $T(\mu(\nu))$. Here the parameter $\mu(\nu)$ depends continuously on $\nu$ with $\lim _{\nu \downarrow \frac{|a c|}{|a \sigma(c)|}} \mu(\nu)=1$ and $\lim _{\nu \uparrow 1} \mu(\nu)=0$. So we find $\nu_{0}$ such that $\mu\left(\nu_{0}\right)=\alpha \beta$. Consequently, the quadrangle $Q=\tau_{\nu_{0}}\left(\sigma\left(Q^{*}\right)\right)$ of type $Q(\alpha, \beta)$ splits into two quadrangles of type $Q(\alpha, \beta)$ and two trapezoids of type $T(\alpha \beta)$. By Lemma 14, one of the trapezoids can be dissected into two quadrangles of type $Q(\alpha, \beta)$ and the other one can be dissected into $n-4$ quadrangles of type $Q(\alpha, \beta)$. This proves the $n$-self-affinity of $Q$.

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