## Solving the Yang-Baxter, tetrahedron and higher simplex equations using Clifford algebras

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#### Abstract

Bethe Ansatz was discoverd in 1932. Half a century later its algebraic structure was unearthed: Yang-Baxter equation was discovered, as well as its multidimensional generalizations [tetrahedron equation and d-simplex equations]. Here we describe a universal method to solve these equations using Clifford algebras. The Yang-Baxter equation (d = 2), Zamalodchikov's tetrahedron equation (d = 3) and the Bazhanov-Stroganov equation (d = 4) are special cases. Our solutions form a linear space. This helps us to include spectral parameters. Potential applications are discussed.

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# 1 Introduction

Quantum integrable models in 1+1 dimensions are described and solved using the Yang-Baxter equation [51, 4] and the associated quantum inverse scattering [46, 48] or algebraic Bethe ansatz method [28]. The Yang-Baxter equation appears in several contexts : in the theory of quantum groups [37], knot theory and braid groups [50, 22], in exactly solvable models in statistical mechanics [5] and in factorizable scattering theory [54]. An instinctive question is the extension of these ideas to higher dimensions. The first steps in this direction were taken by Zamalodchikov in introducing the three dimensional version known as the tetrahedron equation [52,53]. This equation describes the scattering of straight strings in 2+1spacetime dimensions. The initial years afters its introduction saw a flurry of papers studying various aspects of the model starting from its solutions and corresponding three dimensional integrable models. Examples of the latter are found in [11, 6, 7, 47, 25, 43, 38, 49, 23, 36, 21]. The first non-trivial solutions to the tetrahedron equation appeared in [52] which was further explained in [3]. Subsequently several other solutions have been constructed [26, 9, 15, 27, 8]. Solutions can be obtained from cluster algebras [14,18], and quantum groups [33]. Many of the solutions in these works are obtained from the so called Tetrahedron Zamalodchikov algebra  $(TZA)^1$  which is the relation

$$R_{12}^{a}R_{13}^{b}R_{23}^{c} = \sum_{d,e,f=0}^{1} (S_{123})_{def}^{abc} R_{23}^{f}R_{13}^{e}R_{12}^{d}.$$
 (1.1)

The S operator in this equation satisfies the tetrahedron equation in (2.3). More solutions obtained from R-matrices include [29,30]. Along these lines reflection equations for the tetrahedron case can also be solved [32,34,19]. As a result solving the tetrahedron equation helps us obtain an infinite number of Yang-Baxter solutions [8]. The higher  $(d \ge 4)$  simplex equations first appeared in [10,35]. However the amount of work surrounding these equations is far lesser than the tetrahedron equation. Some solutions can be found in [16, 12, 13, 2]. The general philosophy in these works is also to use the d - 1-simplex operators to generate the solutions of the d-simplex equations.

In this paper we will use an entirely different approach that is completely algebraic. This method is universal in the sense that it provides the algorithm to construct the d-simplex operators for all the d. The starting point is a pair of anticommuting operators A and B that

<sup>&</sup>lt;sup>1</sup>See [24] for a short description and [44,45] for its role in the integrability of the 1D Hubbard model.

act on the local Hilbert space. First we write down the simplest solutions of the constant [independent of spectral parameter] d-simplex equations. Next we show that these solutions form a linear space. The coefficients in their linear combinations are interpreted as spectral parameters. Thus we also solve the spectral parameter dependent d-simplex equations. For easier navigation we summarise the main results.

### Summary of main results

The following summary is intended for the reader who wishes to go directly to the solutions of the various d-simplex operators. A key feature of our solutions is that they form a linear space. All solutions are representation independent.

- 1. d = 2 or Yang-Baxter operators : Solutions in (3.2) and (3.10).
- 2. d = 3 or Tetrahedron operators : Solutions in (3.33), (3.38) and (3.40).
- 3. 4-simplex operators : Solutions in (3.46), (3.52) and (3.47).
- 4. 5-simplex operators : Solutions in (3.58) and (3.59).
- 5. The qubit versions can be found in Sec. 4.

### Organisation of the paper

There is no unique form for the higher simplex equations. The various forms and index structures are discussed in Sec. 2. We solve the vertex forms of the higher simplex equations. Starting from d = 2 we present the method to solve all the *d*-simplex equations using Clifford algebras in Sec. 3. The solutions in this section are algebraic, meaning they are independent of the choice of representation of the local Hilbert space. For each choice we get a different set of solutions. The linear structure of the space of solutions is discussed for each *d*. The operators obtained by choosing the qubit representation is shown in Sec. 4. We conclude with a summary of all the solutions and future directions in Sec. 5.

We include several important appendices that present the versatility of the method developed in the main text.

1. Appendix A is a technical section which shows that the A and B operators need to either commute or anticommute to solve the d-simplex equations for  $d \ge 3$ .

- 2. In Appendix B we define and solve the anti-*d*-simplex equation. This is the equation where the right hand side is multiplied by an overall negative sign. This naturally occurs in this construction.
- 3. The tetrahedron equation takes different forms. We show that the Clifford algebra method provides a solution for each of these forms in Appendix C.
- 4. Examples of solutions to the higher reflection equations using Clifford algebras are shown in Appendix D.
- 5. The methodology to generate solutions extends to non-Clifford solutions as well. This is the subject of the last Appendix E.
- 6. The Mathematica codes to verify the *d*-simplex solutions in the qubit representation are shown in Appendix F.

# 2 Labeling schemes

Our first task is to write down the higher simplex equations. The *d*-simplex equations describe the scattering of extended (d-2 dimensional) objects in (d-1)+1 dimensions. Corresponding to the simplices generated in these processes there are different ways of labelling them. This is reflected in the index structure of these equations as we shall now see. We will closely follow [35, 15, 17] to establish the conventions.

We begin with the first non-trivial simplex equation, the tetrahedron equation. This equation describes the scattering of straight strings or particles at the intersection of strings. It was first formulated and solved by Zamalodchikov in [52,53], with subsequent simplifications and solutions [3,11]. There are multiple forms of the tetrahedron equation. The source of this ambiguity lies in the ordering of particles in two dimensions. This is visibly absent for the Yang-Baxter equation (star-triangle relations) in one dimension where an analogous ordering takes place without ambiguity. The basic object is the scattering matrix of three particles. We will denote it by

# $R_{j_1 j_2 j_3}^{k_1 k_2 k_3}$

with the j's (k's) indexing the incoming (outgoing) particles respectively. The tetrahedron equation describes the scattering of four strings with six intersection points. The two ways of scattering six particles should yield the same result leading to the consistency condition

described by the tetrahedron equation  $^2$ :

$$R_{j_1j_2j_3}^{k_1k_2k_3} R_{k_1j_4j_5}^{l_1k_4k_5} R_{k_2k_4j_6}^{l_2l_4k_6} R_{k_3k_5k_6}^{l_3l_5l_6} = R_{j_3j_5j_6}^{k_3k_5k_6} R_{j_2j_4k_6}^{k_2k_4l_6} R_{j_1k_4k_5}^{k_2l_4l_5} R_{k_1k_2k_3}^{l_1l_2l_3}$$

$$(2.1)$$

The k indices are summed over using the Einstein summation convention. In operator form the above equation is written as :

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}.$$
(2.2)

This is also known as the vertex-form of the tetrahedron equation. We expect six indices in this equation as they correspond to the six particles formed at the intersection of the four strings. Thus this equation acts on  $\bigotimes_{j=1}^{6} V_j$ , with V being the local Hilbert space. It is natural to label the scattering matrix with the string segments of the four scattering strings. This leads to the next form of the tetrahedron equation :

$$R_{123}R_{124}R_{134}R_{234} = R_{234}R_{134}R_{124}R_{123}.$$
(2.3)

It acts on  $\bigotimes_{j=1}^{4} V_j$  corresponding to the four scattering strings. Finally we have the vacuum or cell labelling where the equation is given in terms of Boltzmann weights. We will omit writing down this expression<sup>3</sup> and refer the reader to [15] for the explicit form.

While the above three forms are inspired by physical scattering processes, the last form we will write down is derived from algebraic considerations. Let us assume the Yang-Baxter equation to be weakly true, that is it is true up to a conjugation. The 'conjugator' is a three indexed object satisfying a consistency condition which gives another form of the tetrahedron equation :

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}.$$
(2.4)

This is also known as the quantized Yang-Baxter equation in [31]. This consistency equation can be written down in eight different ways arising due to the ambiguity in the reversal process [17]. This is extensively studied in the higher category and higher braid group literature [1,20]. In this work we will study the vertex form of the tetrahedron equation (2.2) in the main text

 $<sup>^{2}</sup>$ The scattering process can be depicted pictorially. As the considerations in this paper are algebraic we direct the interested reader to [15] for these pictures.

 $<sup>^{3}</sup>$ The resulting equation describes the classical three dimensional integrable model. Since that is not the subject of this work we will skip this form.

and reserve the forms in (2.3) and (2.4) for Appendix C.

The next higher simplex equation is the 4-simplex equation. This is also known as the Bazhanov-Stroganov equation [10]. The vertex form of this equation reads :

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}.$$
(2.5)

The index structure of the 4-simplex equation suggests the pattern for the vertex form of the higher simplex equations [35]. For example the vertex form of the 5-simplex equation is

$$R_{12345}R_{16789}R_{26,10,11,12}R_{37,10,13,14}R_{48,11,13,15}R_{59,12,14,15}$$
  
=  $R_{59,12,14,15}R_{48,11,13,15}R_{37,10,13,14}R_{26,10,11,12}R_{16789}R_{12345}.$  (2.6)

For the sake of completion we write the algorithm to construct the vertex form of the *d*-simplex equation for arbitrary *d*. The *d*-simplex operator acts non-trivially on *d* sites. On each side of the equation there are a product of d + 1, *d*-simplex operators. They act on

$$\bigotimes_{j=1}^{\frac{d(d+1)}{2}} V_j,$$

with V being the local Hilbert space. The number  $\frac{d(d+1)}{2}$  corresponds to the number of particles formed at the vertices of the scattering process of d+1 extended objects. Now we construct the left hand side of the *d*-simplex equation. Begin with the operator  $R_{1,2,\dots,d}$ . The second operator starts with the index 1 and acquires d-1 new indices to become  $R_{1,d+1,\dots,2d-1}$ . The third operator begins with the index 2 and acquires d-2 new indices to become  $R_{2,d+1,2d,\dots,3d-3}$ . This pattern can be extended to find that the *k*th operator begins with the index k-1 and acquires d-k+1 new indices to become  $R_{k-1,d+k-2,\dots,kd-\frac{k(k-1)}{2}}$ . The values of *k* run up to d+1. The right hand side of the equation is just the left hand side in reverse order.

The *d*-simplex equations written here do not depend on spectral parameters, or are the constant versions. Like in the Yang-Baxter equation it is physically important for the *d*-simplex equations to depend on spectral parameters that appear in the scattering processes. This dependence will be shown in the next section when we construct their solutions.

# 3 Solutions

We will now present an algebraic method to generate solutions of the higher simplex equations. For each d we solve the vertex form of the higher simplex equation as described in [35]. We call the solutions as d-simplex operators. The technique is a generalisation of the one used in [39]. As a result we will see that the latter reduce to special cases of the solutions presented here.

The ansatzes for the *d*-simplex operators are constructed with two operators A and B. These operators act on a space V, which is the local Hilbert space labeled by the indices appearing in the higher simplex equations. We assume they satisfy

$$AB = \alpha BA,\tag{3.1}$$

for some parameter  $\alpha \in \mathbb{C}$ . Additionally the operators A and B can depend on a set of parameters. For simplicity we will ignore them in the proofs to follow. These ansatzes can also be generalised to include more number of local operators as we shall see.

The technique is well illustrated for the 2-simplex equation or the **Yang-Baxter equation**. This will serve as a warm-up for the higher simplex cases. We begin with two simple solutions of the Yang Baxter equation<sup>4</sup> :

$$A_i A_j$$
;  $B_i B_j$ .

Next we consider the R-matrix,

$$R_{ij} = A_i A_j + B_i B_j. aga{3.2}$$

This ansatz is plugged into the constant [independent of spectral parameter]<sup>5</sup> Yang-Baxter equation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. (3.3)$$

This is the vertex form of the Yang-Baxter equation. Now we check the conditions on  $\alpha$  for

<sup>&</sup>lt;sup>4</sup>Such solutions are discussed in [39].

<sup>&</sup>lt;sup>5</sup>This is also known as the braid relation.

(3.2) to satisfy (3.3).

$$R_{12}R_{13}R_{23} = A_2A_3\left(A_1A_2 + \frac{1}{\alpha}B_1B_2\right)\left(A_1A_3 + \frac{1}{\alpha}B_1B_3\right) + B_2B_3\left(\alpha A_1A_2 + B_1B_2\right)\left(\alpha A_1A_3 + B_1B_3\right) = \left(A_2A_3 + \alpha^2B_2B_3\right)\left(A_1A_2 + \frac{1}{\alpha}B_1B_2\right)\left(A_1A_3 + \frac{1}{\alpha}B_1B_3\right) = \left(A_2A_3 + \alpha^2B_2B_3\right)\left[A_1A_3\left(A_1A_2 + \frac{1}{\alpha^2}B_1B_2\right) + B_1B_3\left(A_1A_2 + \frac{1}{\alpha^2}B_1B_2\right)\right] = \left(A_2A_3 + \alpha^2B_2B_3\right)R_{13}\left(A_1A_2 + \frac{1}{\alpha^2}B_1B_2\right) = R_{23}R_{13}R_{12} ; \text{ iff } \alpha^2 = 1.$$
(3.4)

Thus we find that when

$$\alpha = \pm 1 \tag{3.5}$$

the *R*-matrix in (3.2) satisfies the constant Yang-Baxter equation (3.3). We note that the  $\alpha = 1$  case, corresponding to commuting *A* and *B*, are the solutions considered in [39]. When  $\alpha = -1$ , the operators anticommute and hence can be considered to be part of Clifford algebras of arbitrary orders. Note that the set of solutions found above form a **linear space**. Thus linear combinations of the two operators in (3.2)

$$R_{ij}(\mu_i, \mu_j) = \mu_i A_i A_j + \mu_j B_i B_j,$$
(3.6)

satisfy the Yang-Baxter equation.

$$R_{12}(\mu_1,\mu_2)R_{13}(\mu_1,\mu_3)R_{23}(\mu_2,\mu_3) = R_{23}(\mu_2,\mu_3)R_{13}(\mu_1,\mu_3)R_{12}(\mu_1,\mu_2).$$
(3.7)

This is the Yang-Baxter equation in the non-additive form. Here  $\mu$ 's are some complex numbers, we can identify them with the spectral parameters.

We can further generalise these solutions by including more local operators. Consider two sets of mutually anticommuting operators :

$$\{A^{(m)}|m \in (1, \cdots, r)\}; \{B^{(n)}|n \in (1, \cdots, s)\}.$$
(3.8)

These operators satisfy,

$$\left[A^{(m_1)}, A^{(m_2)}\right] = \left[B^{(n_1)}, B^{(n_2)}\right] = \left\{A^{(m_1)}, B^{(n_1)}\right\} = 0 \; ; \; \forall \; m_1, m_2, n_1, n_2. \tag{3.9}$$

Using the linear structure of the solutions we have the following R-matrix

$$R_{ij} = \sum_{m_1,m_2=1}^{r} \mu_{m_1m_2} \left( A^{(m_1)} \right)_i \left( A^{(m_2)} \right)_j + \sum_{n_1,n_2=1}^{s} \nu_{n_1n_2} \left( B^{(n_1)} \right)_i \left( B^{(n_2)} \right)_j, \quad (3.10)$$

as the most general Clifford solution to the 2-simplex equation in the non-additive form. This solutions contains  $r^2 + s^2$  parameters.

Henceforth we will assume that the local operators A and B anticommute while considering higher simplex operators. Next we show that such anticommuting operators can naturally be constructed out of basis elements of a Clifford algebra. The method to generate the d-simplex operators only assumes that A and B anticommutes and thus the following subsection can be skipped by the reader who wishes to go directly to the solutions.

### 3.1 Clifford algebras

The operators A and B appearing in (3.1) are required to either commute or anticommute for the R-matrix in (3.2) to satisfy the 2-simplex or the constant Yang-Baxter equation. The commuting case is the subject of [39]. The commuting case also satisfies the higher simplex equations. The anticommuting case will be the focus of this paper. The condition of anticommutativity places constraints on the operators A and B. There are three possibilities.

- 1. Both A and B are invertible.
- 2. One of them is invertible and the other non-invertible.
- 3. Both A and B are non-invertible.

**Case 1 :** We assume A and B are invertible and they generate a closed algebra. The algebra is taken to be finite dimensional. Then we have

$$AB = -BA \implies A^{-1}B = -BA^{-1}. \tag{3.11}$$

As A and B form a closed algebra, the operator  $A^{-1}$  has to be a linear combination of A

and  $B^6$ ,

$$A^{-1} = \alpha \ A + \beta \ B. \tag{3.12}$$

Substituting this into (3.11) we find that

$$\beta B^2 = 0 \implies \beta = 0. \tag{3.13}$$

Thus  $A^{-1} \propto A$ . We take the proportionality constant to be  $\alpha = \pm 1$  without loss of generality. Any other positive or negative constant can be absorbed into A, thus scaling  $\alpha$  to  $\pm 1$ . The above arguments imply

$$A^2 = \pm 1. (3.14)$$

The roles of A and B can be reversed in (3.11) to show that

$$B^2 = \pm \mathbb{1}.\tag{3.15}$$

Thus we find that when two invertible operators A and B, forming a closed algebra, anticommute with each other, their squares are either  $\pm 1$ . Operators satisfying these relations form a Clifford algebra.

**Definition 1.** Consider a vector space V with a degenerate quadratic form denoted by a multiplication. The quadratic form has a signature p + q with p and q being positive integers. A Clifford algebra is an associative algebra generated by p + q elements denoted

$$\{\Gamma_1,\cdots,\Gamma_{p+q}\}.$$

They satisfy the relations

$$\Gamma_i^2 = 1 \text{ for } 1 \le i \le p, \tag{3.16}$$

$$\Gamma_i^2 = -\mathbb{1} \text{ for } p+1 \le i \le p+q, \tag{3.17}$$

$$\Gamma_i \Gamma_j = -\Gamma_j \Gamma_i \text{ for } i \neq j. \tag{3.18}$$

We denote the Clifford algebra by  $\mathbf{CL}(p,q)$  with order p+q.

The identity operator 1 is also included in the algebra. It is obtained from the generators. The vector space can be real or complex. This matters for the representations of  $\mathbf{CL}(p,q)$ .

<sup>&</sup>lt;sup>6</sup>A more general assumption here can be  $A^{-1} = \alpha A + \beta B + \gamma AB$ . Using (3.11) and  $AA^{-1} = A^{-1}A = \mathbb{1}$ , we can show that both  $\beta = \gamma = 0$ .

The Clifford relations (3.16)-(3.18) help determine the dimension of  $\mathbf{CL}(p,q)$ . An arbitrary element of the basis element is an unordered word in the generators. This can be written as  $\Gamma_1^{j_1} \cdots \Gamma_{p+q}^{j_{p+q}}$  with  $(j_1, \cdots, j_{p+q}) \in \{0, 1\}$ . Thus each generator is included or not in each word making the total number of words  $2^{p+q} = \dim \mathbf{CL}(p,q)$ . Each word (element) of the Clifford algebra has a definite grade.

**Definition 2.** The grade of an element or word of CL(p,q) is the number of unique generators needed to generate the word.

For example the scalars are grade 0, the set of generators ( $\Gamma_i$ ) are of grade 1. Bilinears<sup>7</sup> in the generators ( $\Gamma_i\Gamma_j$ ) are of grade 2 and so on. We can now write down the basis elements of  $\mathbf{CL}(p,q)$  by distinguishing words according to their grade. The different spaces along with their dimensions are

$$G_{0} = k \ \mathbb{1} \ k \in \mathbb{C} \text{ or } \mathbb{R}, \ \dim = \binom{p+q}{0}$$

$$G_{1} = \{\Gamma_{i} | i \in \{1, \cdots, p+q\}\}, \ \dim = \binom{p+q}{1}$$

$$G_{2} = \{\Gamma_{i}\Gamma_{j} | i \neq j \in \{1, \cdots, p+q\}\}, \ \dim = \binom{p+q}{2}$$

$$\vdots$$

$$G_{p+q} = \Gamma_{1} \cdots \Gamma_{p+q}, \ \dim = \binom{p+q}{p+q}.$$
(3.19)

The sum of the dimensions of each of these graded spaces

$$\sum_{j=0}^{p+q} \binom{p+q}{j} = 2^{p+q}$$
(3.20)

as expected. With this the Clifford algebra becomes

$$\mathbf{CL}(p,q) = G_0 \oplus G_1 \oplus G_2 \oplus \dots \oplus G_{p+q}.$$
(3.21)

The different elements of  $\mathbf{CL}(p,q)$  are either even or odd. In particular we find that the even elements of the Clifford algebra form a subalgebra. The grading structure is useful to identify the mutually anticommuting operators in (3.8) with elements of  $\mathbf{CL}(p,q)$ . Let us consider two

 $<sup>^7\</sup>mathrm{For}$  real Clifford algebras these are also called bivectors and have a natural interpretation in geometric algebra.

examples.

*Example 1*: Consider a Clifford algebra of order 2. This can be one of three possibilities -  $\mathbf{CL}(2,0)$ ,  $\mathbf{CL}(1,1)$ ,  $\mathbf{CL}(0,2)$ . Barring the identity operator there are three non-trivial elements  $\{\Gamma_1, \Gamma_2, \Gamma_1\Gamma_2\}$ . Any of the three Clifford algebras will generate *R*-matrices in (3.2). This is done by choosing the *A* and *B* operators as shown in Table 1. When using Clifford algebras

A	В
$\Gamma_1$	$\Gamma_2$
$\Gamma_1$	$\Gamma_1\Gamma_2$
$\Gamma_2$	$\Gamma_1\Gamma_2$

Table 1: Possible choices of the A and B operators from a Clifford algebra of order 2.

of order 2 there are not enough operators to construct the R-matrices in (3.10). For this we need to use Clifford algebras of order higher than 2.

*Example 2*: For this case we choose a Clifford algebra of order 3. There are four such examples -  $\mathbf{CL}(3,0)$ ,  $\mathbf{CL}(2,1)$ ,  $\mathbf{CL}(1,2)$  and  $\mathbf{CL}(0,3)$ . The basis elements are given by

$$\{\mathbb{1} ; \Gamma_1, \Gamma_2, \Gamma_3 ; \Gamma_1\Gamma_2, \Gamma_1\Gamma_3, \Gamma_2\Gamma_3 ; \Gamma_1\Gamma_2\Gamma_3\}.$$

The choices of A and B resulting in the R-matrix in (3.2) are shown in Table 2. Next

A	В
$\Gamma_1$	$\Gamma_2/\Gamma_3/\Gamma_1\Gamma_2/\Gamma_1\Gamma_3$
$\Gamma_2$	$\Gamma_3/\Gamma_1\Gamma_2/\Gamma_2\Gamma_3$
$\Gamma_3$	$\Gamma_1\Gamma_3/\Gamma_2\Gamma_3$

Table 2: Possible choices of the A and B operators for the R-matrix in (3.2) from a Clifford algebra of order 3. Interchanging A and B gives another set of R-matrices as discussed in the text.

we realise the anticommuting sets of operators in (3.8) using the basis elements of an order 3 Clifford algebra. This helps us construct the *R*-matrices in (3.10). Note that we cannot use the grade 3 element  $\Gamma_1\Gamma_2\Gamma_3$  in any of the sets as this element is central<sup>8</sup>. The choice of the *A*'s and *B*'s for r = s = 2 is shown in Table 3.

 $<sup>^{8}\</sup>mathrm{An}$  analogous statement holds for all Clifford algebras of odd order.

A	В
$(\Gamma_1,\Gamma_2\Gamma_3)$	$(\Gamma_3,\Gamma_1\Gamma_2)$
$(\Gamma_1,\Gamma_2\Gamma_3)$	$(\Gamma_2,\Gamma_1\Gamma_3)$
$(\Gamma_2,\Gamma_1\Gamma_3)$	$(\Gamma_3,\Gamma_1\Gamma_2)$

Table 3: Possible choices of the A's and B's for r = s = 2 to construct the R-matrix in (3.10) from a Clifford algebra of order 3. Interchanging A and B gives another set of R-matrices as discussed in the text.

**Case 2**: Let A be non-invertible and B invertible<sup>9</sup>. From the arguments in case 1 we can show that

$$B^2 = \pm \mathbb{1}$$

As A and B anticommute we have

$$A^2 B = B A^2. aga{3.22}$$

Since A is non-invertible, its determinant is 0. We consider two possibilities<sup>10</sup> :

- 1. A is nilpotent :  $A^2 = 0$ .
- 2. A is a projector :  $A^2 = A$ .

For the first case we consider Clifford algebras generalised to include nilpotent operators. The new definition involves a set of y generators that are nilpotent. The resulting Clifford algebra will be denoted  $\mathbf{CL}(p,q,y)^{11}$ . Thus they can be used to realise the A and B operators.

We could also realise the first case with Clifford algebras of the type  $\mathbf{CL}(p,q)$ . As an example consider an order 2 Clifford algebra,  $\mathbf{CL}(2,0)$ . Then

$$A = \Gamma_1 + \Gamma_1 \Gamma_2 \; ; \; B = \Gamma_2, \tag{3.23}$$

satisfies the conditions required of A and B.

In the second case (3.22) reduces to

$$AB = BA = 0. \tag{3.24}$$

But since B is invertible the only way this equation can be satisfied is when A = 0. This is a

<sup>&</sup>lt;sup>9</sup>Interchanging the roles of A and B does not produce anything new.

 $<sup>^{10}</sup>$ We are being conservative here as there can be more possibilities for a non-invertible matrix. However for 2 by 2 matrices, which we will mostly interested in, these are the only possibilities.

<sup>&</sup>lt;sup>11</sup>The set of nilpotent anticommuting elements is also called the Grassmann algebra.

trivial solution.

Finally we consider another situation. A non-invertible operator A should satisfy a characteristic equation of the form

$$A\left(\sum_{n} k_{n}A^{n}\right) = 0.$$

From this substitute for  $A^2$  into (3.22). We find that the even coefficients of this sum are 0. We are then left with two cases.

- 1. A is nilpotent :  $A^2 = 0$ .
- 2. A satisfies  $\sum_{n \in \text{odd}} k_n A^{n-1} = 0.$

The first case has already been considered. The characteristic equation in the second case can be chosen in a particular representation. In this case the A operators can still be expanded in terms of Clifford basis elements.

**Case 3**: Next we turn to the case when A and B are not invertible. This implies that (3.11) no longer holds. However (3.22) is still true. From the arguments of Case 2, we have that

$$A^2 B = 0 \; ; \; A B^2 = 0. \tag{3.25}$$

Subtracting these two we obtain

$$(A+B)AB = 0. (3.26)$$

We want A and B to be independent of each other. So they must satisfy AB = 0. The operators A and B can either be projectors or they can be nilpotent. When the operators are projectors they cannot be thought of as generators of a Clifford algebra. Nevertheless we can realise such operators using the basis elements of a Clifford algebra of an appropriate order. As an example we realise the two operators from the order 2 Clifford algebra,  $\mathbf{CL}(2,0)$  to check these situations.

*Example 1*: Consider A as nilpotent and B as a projector. We require AB = BA = 0. This is not possible as A and B will have to be orthogonal to each other, but we have taken one of them to be nilpotent. So this is a contradiction and hence this case is ruled out. *Example 2*: Both A and B are projectors. Their product is zero when they are orthogonal to each other,

$$A = \frac{\mathbb{1} + \Gamma_1}{2} \; ; \; B = \frac{\mathbb{1} - \Gamma_1}{2}. \tag{3.27}$$

*Example 3*: The third situation where A and B are realised using CL(2,0) and are both nilpotent is not possible. However if we use CL(3,0) we can find such A's and B's. One example is

$$A = \Gamma_1 - \mathrm{i}\Gamma_3 \; ; \; B = \Gamma_1\Gamma_2 + \mathrm{i}\Gamma_2\Gamma_3, \tag{3.28}$$

with  $i = \sqrt{-1}$ .

#### 3.1.1 Summary of the Clifford algebra

We considered the situations when two operators A and B anticommute. There are three possibilities.

- 1. Case 1 : When A and B are both invertible,  $A^2 = \pm 1$  and  $B^2 = \pm 1$ . Then both A and B satisfy the relations of a Clifford algebra  $\mathbf{CL}(p,q)$ .
- 2. Case 2: When A is non-invertible and B is invertible,  $A^2 = 0$  and  $B^2 = \pm 1$ . The operators can be realised using a more general Clifford algebra that also includes nilpotent generators,  $\mathbf{CL}(p,q,r)$ . The nilpotent operator A can also be obtained using the basis elements of  $\mathbf{CL}(p,q)$ .
- 3. Case 3: When both A and B are non-invertible we have AB = 0. This is non-trivially satisfied when they are either nilpotent or they are projectors. In all the cases the Clifford algebra  $\mathbf{CL}(p,q)$  can be used to realise them.

### **3.2** 3-Simplex or Zamalodchikov's tetrahedron equations

The 3-simplex equation or the tetrahedron equation is the first higher simplex equation that will admit solutions constructed out of Clifford algebras<sup>12</sup>. As in the 2-simplex case we will demonstrate this with some simple ansatzes. Subsequently we will generalise this by showing that these solutions form a linear space.

<sup>&</sup>lt;sup>12</sup>As with the d = 2 or Yang-Baxter equation we could consider an ansatz where the  $\alpha$  in (3.1) is not  $\pm 1$ . However it turns out that the 3-simplex equation is satisfied only when  $\alpha = \pm 1$ . This is shown in Appendix A.

Consider the 3-simplex operator,

$$R_{ijk} = A_i A_j B_k. \tag{3.29}$$

This solves the constant 3-simplex equation in the vertex form,

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}, (3.30)$$

as both sides of this equation simplify to

$$A_1^2 A_2^2 (B_3 A_3) A_4^2 (B_5 A_5) A_6^2$$

In a similar vein the choices

$$A_i B_j A_k$$
,  $B_i A_j A_k$ ,

also satisfy the tetrahedron equation in (3.30). Notice that the three operators  $A_i A_j B_k$ ,  $A_i B_j A_k$  and  $B_i A_j A_k$  commute with each other. Naively we expect a linear combination of these three operators to also be a solution to the tetrahedron equation (3.30). We check this systematically starting with the sum

$$R_{ijk} = A_i A_j B_k + A_i B_j A_k. aga{3.31}$$

The proof that this solves the tetrahedron equation (3.30) goes as follows :

$$R_{123}R_{145}R_{246}R_{356}$$

$$= A_3A_5B_6 (-A_1A_2B_3 + A_1B_2A_3) (-A_1A_4B_5 + A_1B_4A_5) (A_2A_4B_6 - A_2B_4A_6)$$

$$+ A_3B_5A_6 (-A_1A_2B_3 + A_1B_2A_3) (A_1A_4B_5 - A_1B_4A_5) (-A_2A_4B_6 + A_2B_4A_6)$$

$$= R_{356} (-A_1A_2B_3 + A_1B_2A_3) (-A_1A_4B_5 + A_1B_4A_5) (A_2A_4B_6 - A_2B_4A_6)$$

$$= R_{3456} [A_2A_4B_6 (-A_1A_2B_3 - A_1B_2A_3) (-A_1A_4B_5 - A_1B_4A_5)]$$

$$- A_2B_4A_6 (-A_1A_2B_3 - A_1B_2A_3) (A_1A_4B_5 + A_1B_4A_5)]$$

$$= R_{356}R_{246} (A_1A_2B_3 + A_1B_2A_3) (A_1A_4B_5 + A_1B_4A_5)$$

$$= R_{356}R_{246} (A_1A_2B_3 + A_1B_2A_3) (A_1A_4B_5 + A_1B_4A_5)$$

$$(3.32)$$

This confirms the linear structure. The nature of the proof shows that the linear structure will hold for the entire set of three product 3-simplex operators. We apply this method for

the next generalisation,

$$R_{ijk} = A_i A_j B_k + A_i B_j A_k + B_i A_j A_k. ag{3.33}$$

The proof that this solves the tetrahedron equation is straightforward but the notation makes it cumbersome. To simplify things we introduce a new notation.

#### Notation to simplify proofs

The proofs in the rest of the paper are virtually impossible to write down without some form of shorthand notation. We will introduce this for the 3-simplex operators. The generalisation to the higher simplex cases is straightforward.

$$R_{ijk}(\kappa_1,\kappa_2,\kappa_3) = \kappa_1 A_i A_j B_k + \kappa_2 A_i B_j A_k + \kappa_3 B_i A_j A_k \equiv (\kappa_1,\kappa_2,\kappa_3)_{ijk} ; \kappa_1,\kappa_2,\kappa_3 \in \{+,-\}.$$
(3.34)

The  $\kappa_j$ 's are the coefficients of the terms appearing in the 3-simplex operator. For the ansatz in (3.33) the coefficients are just one and so in this case the  $\kappa_j$ 's specify their signs. Thus the shorthand for this 3-simplex operator becomes  $(+, +, +)_{ijk}$ . Note that the  $\kappa_j$ 's are not to be confused with spectral parameters. The sole purpose of their introduction is to simplify the upcoming proofs. We are now ready to check that (3.33) satisfies the 3-simplex equation (3.30) :

$$R_{123}R_{145}R_{246}R_{356} = A_3A_5B_6(-, +, +)_{123}(-, +, +)_{145}(+, -, -)_{246} + A_3B_5A_6(-, +, +)_{123}(+, -, -)_{145}(-, +, +)_{246} + B_3A_5A_6(+, -, -)_{123}(-, +, +)_{145}(-, +, +)_{246} = R_{356}(+, -, -)_{123}(+, -, -)_{145}(+, -, -)_{246} = R_{356}[A_2A_4B_6(+, +, -)_{123}(+, +, -)_{145} - A_2B_4A_6(+, +, -)_{123}(-, -, +)_{145} - B_2A_4A_6(-, -, +)_{123}(+, +, -)_{145}] = R_{356}R_{246}(+, +, -)_{123}(+, +, -)_{145} = R_{356}R_{246}[A_1A_4B_5(+, +, +)_{123} + A_1B_4A_5(+, +, +)_{123} - B_1A_4A_5(-, -, -)_{123}] = R_{356}R_{246}R_{145}R_{123}.$$
(3.35)

This establishes the linearity of these solutions. Thus we can generalise the 3-simplex operator (3.33) by taking linear combinations of the three operators in the sum in (3.33) to get

$$R_{ijk}(\mu_i, \mu_j, \mu_k) = \mu_i \ A_i A_j B_k + \mu_j \ A_i B_j A_k + \mu_k \ B_i A_j A_k, \tag{3.36}$$

where the  $\mu$ 's are identified as spectral parameters. This satisfies a spectral parameter dependent 3-simplex equation

$$R_{123}(\mu_1, \mu_2, \mu_3) R_{145}(\mu_1, \mu_4, \mu_5) R_{246}(\mu_2, \mu_4, \mu_6) R_{356}(\mu_3, \mu_5, \mu_6)$$
  
=  $R_{356}(\mu_3, \mu_5, \mu_6) R_{246}(\mu_2, \mu_4, \mu_6) R_{145}(\mu_1, \mu_4, \mu_5) R_{123}(\mu_1, \mu_2, \mu_3).$  (3.37)

Next we take an arbitrary linear combination of the two sets of anticommuting operators in (3.8) to obtain the most general 3-simplex operator constructed out of Clifford algebras

$$R_{ijk}(\boldsymbol{\mu}_{i},\boldsymbol{\nu}_{j},\boldsymbol{\omega}_{k}) = \sum_{m_{1},m_{2}=1}^{r} \sum_{n=1}^{s} \left[ \left( \mu_{m_{1}m_{2}n} \right)_{i} \left( A^{(m_{1})} \right)_{i} \left( A^{(m_{2})} \right)_{j} \left( B^{(n)} \right)_{k} + \left( \nu_{m_{1}m_{2}n} \right)_{j} \left( A^{(m_{1})} \right)_{i} \left( B^{(n)} \right)_{j} \left( A^{(m_{2})} \right)_{k} + \left( \omega_{m_{1}m_{2}n} \right)_{k} \left( B^{(n)} \right)_{i} \left( A^{(m_{1})} \right)_{j} \left( A^{(m_{2})} \right)_{k} \right].$$

$$(3.38)$$

Here the spectral parameter  $\boldsymbol{\mu}$  is the tuple  $\mu_{m_1m_2n}$  with  $r^2s$  elements. Thus the solution in (3.38) contains a total of  $3r^2s$  parameters. It satisfies a more general form of the spectral parameter dependent 3-simplex equation

$$R_{123}(\boldsymbol{\mu}_{1}, \boldsymbol{\nu}_{2}, \boldsymbol{\omega}_{3}) R_{145}(\boldsymbol{\mu}_{1}, \boldsymbol{\nu}_{4}, \boldsymbol{\omega}_{5}) R_{246}(\boldsymbol{\mu}_{2}, \boldsymbol{\nu}_{4}, \boldsymbol{\omega}_{6}) R_{356}(\boldsymbol{\mu}_{3}, \boldsymbol{\nu}_{5}, \boldsymbol{\omega}_{6})$$
  
=  $R_{356}(\boldsymbol{\mu}_{3}, \boldsymbol{\nu}_{5}, \boldsymbol{\omega}_{6}) R_{246}(\boldsymbol{\mu}_{2}, \boldsymbol{\nu}_{4}, \boldsymbol{\omega}_{6}) R_{145}(\boldsymbol{\mu}_{1}, \boldsymbol{\nu}_{4}, \boldsymbol{\omega}_{5}) R_{123}(\boldsymbol{\mu}_{1}, \boldsymbol{\nu}_{2}, \boldsymbol{\omega}_{3}).$  (3.39)

We close this subsection with two remarks.

**Remark 1**: Interchanging the A and B operators in (3.33) leads to another solution,

$$\tilde{R}_{ijk}(\boldsymbol{\mu}_{i},\boldsymbol{\nu}_{j},\boldsymbol{\omega}_{k}) = \sum_{n_{1},n_{2}=1}^{s} \sum_{m=1}^{r} \left[ \left( \mu_{n_{1}n_{2}m} \right)_{i} \left( B^{(n_{1})} \right)_{i} \left( B^{(n_{2})} \right)_{j} \left( A^{(m)} \right)_{k} + \left( \nu_{n_{1}n_{2}m} \right)_{j} \left( B^{(n_{1})} \right)_{i} \left( A^{(m)} \right)_{j} \left( B^{(n_{2})} \right)_{k} + \left( \omega_{n_{1}n_{2}m} \right)_{k} \left( A^{(m)} \right)_{i} \left( B^{(n_{1})} \right)_{j} \left( B^{(n_{2})} \right)_{k} \right].$$

$$(3.40)$$

This satisfies the spectral parameter dependent 3-simplex equation in (3.39). When r = s = 1, this solution can be obtained by interchanging A and B in (3.36) by an invertible operator. When this is the case the two 3-simplex operators are equivalent to each other by a local invertible operator<sup>13</sup>:

$$\tilde{R}_{ijk}(\mu_i, \mu_j, \mu_k) = Q_i Q_j Q_k R_{ijk}(\mu_i, \mu_j, \mu_k) Q_k^{-1} Q_j^{-1} Q_i^{-1}.$$
(3.41)

It is easy to see that  $\tilde{R}$  also satisfies the 3-simplex equation (3.37). Due to this equivalence we do not consider  $\tilde{R}$  and R to be different solutions of the 3-simplex equation. When no such Q exists the two 3-simplex operators are taken to be inequivalent solutions. We will see explicit examples of both these situations in Sec. 4. The above arguments can be extended for the case when  $r = s \neq 1$ . For example consider the case r = s = 2. Possible choices for A and B are listed in Table 3. The operator  $\frac{\Gamma_1 + \Gamma_3}{\sqrt{2}}$  interchanges the two columns in the first row of Table 3. This is the analog of the Hadamard gate for an order 3 Clifford algebra.

**Remark 2** : An alternate choice to the ansatz in (3.29) is

$$R_{ijk} = A_i A_j A_k. \tag{3.42}$$

This is easily seen to solve the tetrahedron equation in (3.30). In a similar manner the operator  $B_i B_j B_k$  also solves the tetrahedron equation. Such a choice exists for all the *d*-simplex operators. For the d = 2 case see [39]. These choices anticommute with each other. Hence we do not expect a linear combination of the type

$$A_i A_j A_k + B_i B_j B_k$$

to solve the tetrahedron equation of (3.30). Nevertheless it solves the anti-tetrahedron equation as discussed in Appendix B.

### **3.3** 4 - Simplex or the Bazhanov-Stroganov's equation

Constructing the solutions for the 4-simplex equation will guide us in generalising the Clifford solutions to arbitrary even d. As in the previous cases we start with the simplest examples and systematically generalise them to more complicated ones by exploiting their linear structure.

 $<sup>^{13}\</sup>mathrm{These}$  two  $R\mathrm{-matrices}$  are gauge equivalent in old terminology.

Consider the following ansatzes for  $R_{ijkl}$ 

$$A_i A_j A_k A_l, \ B_i B_j B_k B_l. \tag{3.43}$$

They satisfy the constant 4-simplex equation

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}, aga{3.44}$$

as both sides simplify to

$$\prod_{j=1}^{10} A_j^2, \ \prod_{j=1}^{10} B_j^2,$$

respectively. Another set of non-trivial ansatzes for  $R_{ijkl}$  is given by

$$A_i A_j B_k B_l, \ A_i B_j A_k B_l, \ B_i A_j A_k B_l, \ A_i B_j B_k A_l, \ B_i A_j B_k A_l, \ B_i B_j A_k A_l.$$
(3.45)

These satisfy the 4-simplex equation (3.44) by reducing both sides of the equation to terms like

$$A_1^2 A_2^2 (B_3 A_3) (B_4 A_4) A_5^2 (B_6 A_6) (B_7 A_7) B_8^2 B_9^2 B_{10}^2$$

We expect any linear combination of these two sets, (3.43) and (3.45), to be solutions of the 4simplex equation in (3.44). However to organise the solutions and simplify the upcoming proofs we will make a split among these operators. We consider two types of 4-simplex operators to show the linear structure :

$$R_{ijkl} = A_i A_j A_k A_l + B_i B_j B_k B_l$$

$$R_{ijkl} = A_i A_j B_k B_l + A_i B_j A_k B_l + B_i A_j A_k B_l$$

$$+ A_i B_j B_k A_l + B_i A_j B_k A_l + B_i B_j A_k A_l.$$
(3.47)

The proofs that they solve (3.44), though straightforward, are once again cumbersome due to growing number of indices and terms. To simplify the proofs we will use notation introduced

in (3.34) adapted to the 4-simplex operators. Thus the shorthand notations become

$$R_{ijkl}(\kappa_1, \kappa_2) = \kappa_1 A_i A_j A_k A_l + \kappa_2 B_i B_j B_k B_l \equiv (\kappa_1, \kappa_2)_{ijkl}$$
(3.48)  

$$R_{ijkl}(\kappa_1, \cdots \kappa_6) = \kappa_1 A_i A_j B_k B_l + \kappa_2 A_i B_j A_k B_l + \kappa_3 B_i A_j A_k B_l$$
  

$$+ \kappa_4 A_i B_j B_k A_l + \kappa_5 B_i A_j B_k A_l + \kappa_6 B_i B_j A_k A_l$$
  

$$\equiv (\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6)_{ijkl}.$$
(3.49)

Here the  $\kappa_i$ 's take values in (+, -). As before, these parameters are not to be confused with spectral parameters. Using this notation the proofs that the 4-simplex operators (3.46) and (3.47) satisfy the 4-simplex equation (3.44) are

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = R_{479,10} (+, -)_{1234} (+, -)_{1567} (+, -)_{2589} (+, -)_{368,10}$$
  
=  $R_{479,10}R_{368,10} (+, +)_{1234} (+, +)_{1567} (+, +)_{2589}$   
=  $R_{479,10}R_{368,10}R_{2589} (+, -)_{1234} (+, -)_{1567}$   
=  $R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}$ , (3.50)

and

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = R_{479,10}(+,+,+,-,-,-,-)_{1234}(+,+,+,-,-,-,-)_{1567}$$

$$\times (+,+,+,-,-,-)_{2589}(+,+,+,-,-,-)_{368,10}$$

$$= R_{479,10}R_{368,10}(+,-,-,-,-,-,+)_{1234}$$

$$\times (+,-,-,-,-,+)_{1567}(+,-,-,-,-,+)_{2589}$$

$$= R_{479,10}R_{368,10}R_{2589}$$

$$\times (+,+,-,+,-,-)_{1234}(+,+,-,+,-,-)_{1567}$$

$$= R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}$$
(3.51)

respectively.

As in the previous cases these solutions can be generalised to include arbitrary coefficients and more number of operators from the anticommuting sets of (3.8). The most general form

of the 4-simplex operator in (3.46) is

$$R_{ijkl}(\boldsymbol{\mu}_{ijkl},\boldsymbol{\nu}_{ijkl}) = \sum_{m_1,m_2,m_3,m_4=1}^{r} (\mu_{m_1m_2m_3m_4})_{ijkl} (A^{(m_1)})_i (A^{(m_2)})_j (A^{(m_3)})_k (A^{(m_4)})_l + \sum_{n_1,n_2,n_3,n_4=1}^{s} (\nu_{n_1n_2n_3n_4})_{ijkl} (B^{(n_1)})_i (B^{(n_2)})_j (B^{(n_3)})_k (B^{(n_4)})_l, \quad (3.52)$$

where the  $\mu$ 's and  $\nu$ 's are spectral parameters. This satisfies the spectral parameter dependent 4-simplex equation

$$R_{1234} (\boldsymbol{\mu}_{1234}, \boldsymbol{\nu}_{1234}) R_{1567} (\boldsymbol{\mu}_{1567}, \boldsymbol{\nu}_{1567}) R_{2589} (\boldsymbol{\mu}_{2589}, \boldsymbol{\nu}_{2589})$$

$$\times R_{368,10} (\boldsymbol{\mu}_{368,10}, \boldsymbol{\nu}_{368,10}) R_{479,10} (\boldsymbol{\mu}_{479,10}, \boldsymbol{\nu}_{479,10})$$

$$= R_{479,10} (\boldsymbol{\mu}_{479,10}, \boldsymbol{\nu}_{479,10}) R_{368,10} (\boldsymbol{\mu}_{368,10}, \boldsymbol{\nu}_{368,10}) R_{2589} (\boldsymbol{\mu}_{2589}, \boldsymbol{\nu}_{2589})$$

$$\times R_{1567} (\boldsymbol{\mu}_{1567}, \boldsymbol{\nu}_{1567}) R_{1234} (\boldsymbol{\mu}_{1234}, \boldsymbol{\nu}_{1234}). \qquad (3.53)$$

The generalised 4-simplex operators corresponding to (3.47) are quite tedious to write down due to the large number of indices on the coefficients. Nevertheless the expressions are rather straightforward when we follow the logic used in constructing the solutions in the previous cases. We hope this is clear to the reader by now and so we avoid writing down such long expressions in the rest of this paper. Two remarks are in order before we proceed to the higher simplex cases.

**Remark 1 :** As the solutions are symmetric in the number of A's and B's in each term, their interchange leaves them invariant up to the coefficients multiplying each term. Nevertheless there could be local invertible operators<sup>14</sup> that relate the two R-matrices. For example for the R-matrices in (3.52) we have

$$\tilde{R}_{ijkl}(\boldsymbol{\mu}_{ijkl}, \boldsymbol{\nu}_{ijkl}) = Q_i Q_j Q_k Q_l \ R_{ijkl}(\boldsymbol{\mu}_{ijkl}, \boldsymbol{\nu}_{ijkl}) \ Q_l^{-1} Q_k^{-1} Q_j^{-1} Q_i^{-1}.$$
(3.54)

As both R and  $\tilde{R}$  solve the 4-simplex equation they are not taken to be different solutions if the above holds.

**Remark 2**: The 4-site operators that solve the 4-simplex equation are made up of an even number of A's and B's. We will see that this is the general requirement for the ansatzes to solve the appropriate higher simplex equations. For the d = 4 case this leaves out operators

<sup>&</sup>lt;sup>14</sup>See also **Remark 1** of Sec. 3.2.

of the form

$$A_i A_j A_k B_l, \ A_i A_j B_k A_l, \ A_i B_j A_k A_l, \ B_i A_j A_k A_l,$$

and those obtained by interchanging A and B in the above set. We will show that linear combinations of these operators satisfy the anti-4-simplex equation in Appendix B.

### **3.4** 5 - Simplex equation

The logic to construct the *d*-simplex operators for odd *d* will become clear when we write down the Clifford solutions for the 5-simplex equation. As always we begin with the simplest solutions, increasing the complexity in a linear manner. Following the pattern observed in the d = 2, 3 and 4 cases we make ansatzes with 5-site operators that have an even number of either the *A* or *B* operators. There are two such sets given by :

$$A_{i_1}A_{i_2}A_{i_3}A_{i_4}B_{i_5}, \ A_{i_1}A_{i_2}A_{i_3}B_{i_4}A_{i_5}, \ A_{i_1}A_{i_2}B_{i_3}A_{i_4}A_{i_5}, \ A_{i_1}B_{i_2}A_{i_3}A_{i_4}A_{i_5}, \ B_{i_1}A_{i_2}A_{i_3}A_{i_4}A_{i_5},$$

$$(3.55)$$

and

$$B_{i_1}B_{i_2}A_{i_3}A_{i_4}A_{i_5}, \ B_{i_1}A_{i_2}B_{i_3}A_{i_4}A_{i_5}, \ B_{i_1}A_{i_2}A_{i_3}B_{i_4}A_{i_5}, B_{i_1}A_{i_2}A_{i_3}A_{i_4}B_{i_5}, \ A_{i_1}B_{i_2}B_{i_3}A_{i_4}A_{i_5}, \ A_{i_1}B_{i_2}A_{i_3}B_{i_4}A_{i_5}, A_{i_1}B_{i_2}A_{i_3}A_{i_4}B_{i_5}, \ A_{i_1}A_{i_2}B_{i_3}B_{i_4}A_{i_5}, \ A_{i_1}A_{i_2}B_{i_3}A_{i_4}B_{i_5}, A_{i_1}A_{i_2}A_{i_3}B_{i_4}B_{i_5}.$$

$$(3.56)$$

It is easy to check that each of these operators satisfy the constant 5-simplex equation

$$R_{12345}R_{16789}R_{26,10,11,12}R_{37,10,13,14}R_{48,11,13,15}R_{59,12,14,15}$$
  
=  $R_{59,12,14,15}R_{48,11,13,15}R_{37,10,13,14}R_{26,10,11,12}R_{16789}R_{12345}.$   
(3.57)

Thus any linear combination of the operators in the two sets, (3.55) and (3.56), is expected to satisfy the 5-simplex equation. We will split this into two ansatzes which are easier to prove than the most general ansatz. They also help organise the solutions in the higher simplex

$$R_{i_{1}i_{2}i_{3}i_{4}i_{5}} = A_{i_{1}}A_{i_{2}}A_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}A_{i_{3}}B_{i_{4}}A_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}A_{i_{5}} + A_{i_{1}}B_{i_{2}}A_{i_{3}}A_{i_{4}}A_{i_{5}} + B_{i_{1}}A_{i_{2}}A_{i_{3}}A_{i_{4}}A_{i_{5}} = B_{i_{1}}B_{i_{2}}A_{i_{3}}A_{i_{4}}A_{i_{5}} + B_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}A_{i_{5}} + B_{i_{1}}A_{i_{2}}A_{i_{3}}B_{i_{4}}A_{i_{5}} + B_{i_{1}}A_{i_{2}}A_{i_{3}}A_{i_{4}}A_{i_{5}} + A_{i_{1}}B_{i_{2}}A_{i_{3}}B_{i_{4}}A_{i_{5}} + A_{i_{1}}B_{i_{2}}A_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}B_{i_{2}}A_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}B_{i_{2}}A_{i_{3}}B_{i_{4}}A_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}B_{i_{4}}A_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}B_{i_{4}}A_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}B_{i_{4}}A_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}B_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}B_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}} + A_{i_{1}}A_{i_{2}}B_{i_{3}}A_{i_{4}}B_{i_{5}$$

Next we present the proof that the 5-simplex operator in (3.58) satisfies (3.57). We again use a shorthand notation to keep the proofs tidy. The shorthand notations in (3.34) and (3.48)are appropriately modified. As it is straightforward we omit the details. Using this notation the proof goes as

$$\begin{aligned} R_{12345}R_{16789}R_{26,10,11,12}R_{37,10,13,14}R_{48,11,13,15}R_{59,12,14,15} \\ = R_{59,12,14,15}(+,-,-,-,-,-)_{12345}(+,-,-,-,-,-)_{16789} \\ \times (+,-,-,-,-,-)_{26,10,11,12}(+,-,-,-,-)_{37,10,13,14} \\ \times (+,-,-,-,-)_{48,11,13,15} \\ = R_{59,12,14,15}R_{48,11,13,15}(+,+,-,-,-)_{12345} \\ \times (+,+,-,-,-,-)_{16789}(+,+,-,-,-,-)_{26,10,11,12} \\ \times (+,+,-,-,-)_{37,10,13,14} \\ = R_{59,12,14,15}R_{48,11,13,15}R_{37,10,13,14} \\ \times (+,+,+,-,-)_{12345}(+,+,+,-,-)_{16789} \\ \times (+,+,+,+,-,-)_{26,10,11,12} \\ = R_{59,12,14,15}R_{48,11,13,15}R_{37,10,13,14}R_{26,10,11,12} \\ \times (+,+,+,+,-,-)_{26,10,11,12} \\ = R_{59,12,14,15}R_{48,11,13,15}R_{37,10,13,14}R_{26,10,11,12} \\ \times (+,+,+,+,-)_{12345}(+,+,+,+,-)_{16789} \\ = R_{59,12,14,15}R_{48,11,13,15}R_{37,10,13,14}R_{26,10,11,12}R_{16789}R_{12345}. \end{aligned}$$
(3.60)

A similar proof can be written down for the second 5-simplex operator in (3.59). We omit this as the entire philosophy of these solutions is rather clear now.

The generalisations of both the 5-simplex operators, (3.58) and (3.59) by including arbitrary linear combinations and more operators follow in the same manner as in the earlier cases.

However writing them down is rather cumbersome. Hence we will skip this part with the understanding that they exist. These generalised solutions will satisfy a 5-simplex equation with spectral parameters.

The solutions obtained by interchanging the A and B operators in (3.58) and (3.59) may or may not be equivalent to each other by local invertible operators<sup>15</sup>. The operators  $A_{i_1}A_{i_2}A_{i_3}A_{i_4}A_{i_5}$  and  $B_{i_1}B_{i_2}B_{i_3}B_{i_4}B_{i_5}$  also solve the 5-simplex equation trivially. However their linear combinations solve the anti-5-simplex equation as shown in Appendix B.

### **3.5** General *d* - simplex equation

The solutions for the *d*-simplex equation for an arbitrary *d* vary for odd and even *d*. This pattern can be understood from the solutions constructed so far. In each of the examples we see that the *d*-simplex operator is a sum of terms with each term acting on *d* local spaces. For the operator to be a solution we require an even number of either the *A* or *B* operators in each of these terms. Thus the solutions get classified according to these numbers. Let the number of *A* (*B*) operators in each term be *a* (*b*). Then we have

$$a+b=d$$

This implies that the number of solutions is just the binary partitions of d with the condition that either of a or b has to be even. Henceforth we denote the solutions by the pair (a, b). This makes the 2-simplex operator in (3.2) (2,0), the 3-simplex operator (3.33) (2,1), the 4-simplex operators (3.46) and (3.47) (4,0) and (2,2) respectively.

So when d is odd we find that except (d, 0) and  $(0, d)^{16}$ , all other pairs result in d-simplex operators. There are d - 1 such solutions. When d is even all the pairs with both a and b odd are excluded. The remaining  $\frac{d}{2}$  pairs yield distinct solutions. The set of solutions in both cases form a linear space. So linear combinations of any two allowed pairs (a, b), for a given d, also forms a solution. A few examples are shown in Table 4.

Each of the *d*-simplex operators corresponding to an allowed pair (a, b) can be generalised with the inclusion of arbitrary linear combinations and more number of operators from the anticommuting sets of operators in (3.8). They satisfy the spectral parameter dependent *d*-simplex equations.

<sup>&</sup>lt;sup>15</sup>See **Remark 1** in Secs. 3.2 and 3.3.

<sup>&</sup>lt;sup>16</sup>Note that when d is even the pairs (d, 0) and (0, d) result in equivalent d-simplex operators.

d	(a,b)	Maximum number of terms in $R$
2	(2,0)	2
3	(2, 1)	$\binom{3}{2}$
4	(4, 0)	2
4	(2, 2)	$\binom{4}{2}$
5	(4, 1)	$\binom{5}{4}$
5	(3, 2)	$\binom{5}{3}$
6	(6, 0)	2
6	(4, 2)	$\binom{6}{4}$
7	(6, 1)	$\binom{7}{6}$
7	(5, 2)	$\binom{7}{5}$
7	(4, 3)	$\binom{7}{4}$

Table 4: This table shows the possible (a, b) pairs for a given d. The choice (b, a) corresponds to the interchange of the A and B operators. These may or may not result in equivalent solutions. They are not shown here. The maximum number of terms in the d-simplex operator in each case is given by  $\binom{d}{a}$ , except when b = 0. The extension to higher d is explained in the main text. For a given d a linear combination of the compatible (a, b) pairs result in another solution. This is the linear structure in the space of solutions.

This discussion concludes our construction of the solutions of the *d*-simplex equations. Before we move to particular representations of the *d*-simplex operators we remark on the anti-*d*-simplex equation. Consider the case when both of *a* and *b* in the pair (a, b) are odd<sup>17</sup>. This occurs when *d* is even. The operators corresponding to these pairs solve the anti-*d*-simplex equations. The latter are defined and discussed in Appendix B.

## 4 Examples : Qubit solutions

The *d*-simplex operators presented so far are representation independent. In this section we will choose  $V = \mathbb{C}^2$  for the local Hilbert space and write down the qubit representations of the *d*-simplex operators. In what follows we will use a pair of anticommuting operators, *A* and *B* to construct the solutions. In other words we take r = s = 1 in (3.8). The solutions below can be generalised to the case where there are two sets of mutually anticommuting operators as in (3.8).

<sup>&</sup>lt;sup>17</sup>We also include the case  $(d, 0) \equiv (0, d)$  when d is odd.

**Case 1 :** The operators A and B satisfy the conditions stated in Case 1 of Sec. 3.1. That is they anticommute and both of them square to 1. A simple qubit representation for A and B are

$$A = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
;  $B = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,

the first and third Pauli matrices respectively. In this case the operators A and B can be interchanged using the Hadamard gate :

$$H = \frac{X+Z}{\sqrt{2}}$$
;  $H^2 = \mathbb{1}$ .

Applying this operator on every site gives us another d-simplex operator that is equivalent to the original d-simplex operator. As discussed in Secs. 3.2, 3.3, these solutions fall in the same equivalence class defined by such local invertible operators.

 A 2-simplex or Yang-Baxter operator : The qubit representation of the operator in (3.6) becomes <sup>18</sup>

$$R(\mu_1, \mu_2) = \mu_1 \ X \otimes X + \mu_2 \ Z \otimes Z = \begin{pmatrix} \mu_2 & 0 & 0 & \mu_1 \\ 0 & -\mu_2 & \mu_1 & 0 \\ 0 & \mu_1 & -\mu_2 & 0 \\ \mu_1 & 0 & 0 & \mu_2 \end{pmatrix}.$$
 (4.1)

These are known as X-type solutions. Their entangling properties are studied in [41].

2. A 3-simplex or tetrahedron operator : The qubit representation of the tetrahedron operator in (3.36) is

$$R(\mu_{1},\mu_{2},\mu_{3}) = \mu_{1} X \otimes X \otimes Z + \mu_{2} X \otimes Z \otimes X + \mu_{3} Z \otimes X \otimes X \qquad (4.2)$$

$$= \begin{pmatrix} 0 & 0 & 0 & \mu_{3} & 0 & \mu_{2} & \mu_{1} & 0 \\ 0 & 0 & \mu_{3} & 0 & \mu_{2} & 0 & 0 & -\mu_{1} \\ 0 & \mu_{3} & 0 & 0 & \mu_{1} & 0 & 0 & -\mu_{2} \\ \mu_{3} & 0 & 0 & 0 & -\mu_{1} - \mu_{2} & 0 \\ 0 & \mu_{2} & \mu_{1} & 0 & 0 & 0 & 0 & -\mu_{3} \\ \mu_{2} & 0 & 0 & -\mu_{1} & 0 & 0 & -\mu_{3} & 0 \\ \mu_{1} & 0 & 0 & -\mu_{2} & 0 & -\mu_{3} & 0 & 0 \\ 0 & -\mu_{1} - \mu_{2} & 0 & -\mu_{3} & 0 & 0 & 0 \end{pmatrix}.$$

<sup>18</sup>This form was first mentioned to us by Kun Zhang [55].

3. **4-simplex operators** : The qubit representations of the (4,0) type 4-simplex operator in (3.46) and the (2,2) type 4-simplex operator in (3.47) are

$R(\mu_1, \cdots \mu_6) = \mu_1 \ X \otimes X \otimes Z \otimes Z + \mu_2 \ X \otimes Z \otimes X \otimes Z + \mu_3 \ Z \otimes X \otimes X \otimes Z$																	
$+ \mu$	$\iota_4 X$	$\otimes Z$	$\otimes Z$	$\otimes X$ -	$+ \mu_5$	$Z \otimes I$	$X \otimes Z$	$Z\otimes X$	$X + \mu$	$_6 Z $	$\otimes Z \otimes$	$X \otimes$	X				(4.4)
	0	0	0	$\mu_6$	0	$\mu_5$	$\mu_3$	0	0	$\mu_4$	$\mu_2$	0	$\mu_1$	0	0	0	
	0	0	$\mu_6$	0	$\mu_5$	0	0	$-\mu_3$	$\mu_4$	0	0	$-\mu_2$	0	$-\mu_1$	0	0	
	0	$\mu_6$	0	0	$\mu_3$	0	0	$-\mu_5$	$\mu_2$	0	0	$-\mu_4$	0	0	$-\mu_1$	0	
	$\mu_6$	0	0	0	0	$-\mu_3$	$-\mu_5$	0	0	$-\mu_2$	$-\mu_4$	0	0	0	0	$\mu_1$	
	0	$\mu_5$	$\mu_3$	0	0	0	0	$-\mu_6$	$\mu_1$	0	0	0	0	$-\mu_4$	$-\mu_2$	0	
	$\mu_5$	0	0	$-\mu_3$	0	0	$-\mu_6$	0	0	$-\mu_1$	0	0	$-\mu_4$	0	0	$\mu_2$	
	$\mu_3$	0	0	$-\mu_5$	0	$-\mu_6$	0	0	0	0	$-\mu_1$	0	$-\mu_2$	0	0	$\mu_4$	
=	0	$-\mu_3$	$-\mu_5$	0	$-\mu_6$	0	0	0	0	0	0	$\mu_1$	0	$\mu_2$	$\mu_4$	0	
	0	$\mu_4$	$\mu_2$	0	$\mu_1$	0	0	0	0	0	0	$-\mu_6$	0	$-\mu_5$	$-\mu_3$	0	
	$\mu_4$	0	0	$-\mu_2$	0	$-\mu_1$	0	0	0	0	$-\mu_6$	0	$-\mu_5$	0	0	$\mu_3$	
	$\mu_2$	0	0	$-\mu_4$	0	0	$-\mu_1$	0	0	$-\mu_6$	0	0	$-\mu_3$	0	0	$\mu_5$	
	0	$-\mu_2$	$-\mu_4$	0	0	0	0	$\mu_1$	$-\mu_6$	0	0	0	0	$\mu_3$	$\mu_5$	0	
	$\mu_1$	0	0	0	0	$-\mu_4$	$-\mu_2$	0	0	$-\mu_5$	$-\mu_3$	0	0	0	0	$\mu_6$	
	0	$-\mu_1$	0	0	$-\mu_4$	0	0	$\mu_2$	$-\mu_5$	0	0	$\mu_3$	0	0	$\mu_6$	0	
	0	0	$-\mu_1$	0	$-\mu_2$	0	0	$\mu_4$	$-\mu_3$	0	0	$\mu_5$	0	$\mu_6$	0	0	
	0	0	0	$\mu_1$	0	$\mu_2$	$\mu_4$	0	0	$\mu_3$	$\mu_5$	0	$\mu_6$	0	0	0 /	

Similar solutions occur as generalised Yang-Baxter operators [42,40].

**Case 2**: The *A* and *B* operators now satisfy the conditions of the second case in Sec. 3.1. That is they anticommute and  $A^2 = 0$  and  $B^2 = 1$ . A possible qubit representation is given by :

$$A = X\left(\frac{\mathbb{1}+Z}{2}\right) = \begin{pmatrix} 0 & 0\\ 1 & 0 \end{pmatrix} ; B = Z = \begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix}.$$

These operators have unequal eigenvalues and hence cannot be related by a similarity transform. Thus for these choices of A and B we can obtain inequivalent d-simplex operators by interchanging them. We will explicitly see this in each of the examples below. 1. A 2-simplex or Yang-Baxter operator : The qubit representation of the operator in (3.6) becomes

$$R(\mu_1, \mu_2) = \mu_1 \ A \otimes A + \mu_2 \ Z \otimes Z = \begin{pmatrix} \mu_2 & 0 & 0 & 0 \\ 0 & -\mu_2 & 0 & 0 \\ 0 & 0 & -\mu_2 & 0 \\ \mu_1 & 0 & 0 & \mu_2 \end{pmatrix}$$
(4.5)

Interchanging A and Z in the expression (4.5) amounts to interchanging the parameters  $\mu_1$  and  $\mu_2$ . The eigenvalues of the two operators are

$$\{-\mu_1, -\mu_1, \mu_1, \mu_1\}; \{-\mu_2, -\mu_2, \mu_2, \mu_2\}$$

respectively.

2. A 3-simplex or tetrahedron operator : The qubit representation of the tetrahedron operators in (3.36) and with A and Z interchanged are

$$R(\mu_{1},\mu_{2},\mu_{3}) = \mu_{1} \ Z \otimes Z \otimes A + \mu_{2} \ Z \otimes A \otimes Z + \mu_{3} \ A \otimes Z \otimes Z \qquad (4.7)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{2} & -\mu_{1} & 0 & 0 & 0 & 0 & 0 \\ \mu_{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{3} & 0 & -\mu_{1} & 0 & 0 & 0 \\ 0 & 0 & -\mu_{3} & 0 & -\mu_{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{3} & 0 & \mu_{2} & \mu_{1} & 0 \end{pmatrix}.$$

These are non-invertible.

3. **4-simplex operators** : The qubit representation of the (4,0) type 4-simplex operator in (3.46) is

1	$R(\mu_1)$	$, \mu_2)$	$= \mu_1$	A (	$\otimes A \otimes$	$\triangleright A$	$\otimes$ $\square$	$A + \mu$	$\iota_2 Z$ (	$\otimes Z$	$\otimes$	$Z\otimes Z$	Ζ			
	$(\mu_2)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0)
	0	$-\mu_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	$-\mu_2$	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	$\mu_2$	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	$-\mu_2$	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	$\mu_2$	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	$\mu_2$	0	0	0	0	0	0	0	0	0
=	0	0	0	0	0	0	0	$-\mu_2$	0	0	0	0	0	0	0	0
_	0	0	0	0	0	0	0	0	$-\mu_2$	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	$\mu_2$	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	$\mu_2$	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	$-\mu_2$	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	$\mu_2$	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\mu_2$	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\mu_2$	0
	$\left( \mu_1 \right)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mu_2$

Interchanging A and Z in (4.8) results in an inequivalent operator with  $\mu_1$  and  $\mu_2$  interchanged. The qubit representation of the (2, 2) type 4-simplex operator in (3.47) is

Interchanging A and Z in (4.9) results in an inequivalent operator with the  $\mu$ 's shuffled. The (2, 2) type 4-simplex operator is not invertible while the (4, 0) type is invertible. We find that a linear combination of the two operators is invertible<sup>19</sup>:

$$R(\mu_{1}, \cdots \mu_{6}; \nu_{1}, \nu_{2})$$

$$= \mu_{1} A \otimes A \otimes Z \otimes Z + \mu_{2} A \otimes Z \otimes A \otimes Z + \mu_{3} Z \otimes A \otimes A \otimes Z$$

$$+ \mu_{4} A \otimes Z \otimes Z \otimes A + \mu_{5} Z \otimes A \otimes Z \otimes A + \mu_{6} Z \otimes Z \otimes A \otimes A$$

$$+ \nu_{1} A \otimes A \otimes A \otimes A + \nu_{2} Z \otimes Z \otimes Z \otimes Z. \qquad (4.10)$$

**Case 3 :** Each of operators A and B is either nilpotent or a projector, satisfying the conditions of Case 3 in Sec. 3.1. It turns out that the resulting d-simplex operators for d = 2, 3, 4cases are non-invertible in the qubit representation. So we do not write them down.

<sup>&</sup>lt;sup>19</sup>Note that the linear combination is still a solution of the 4-simplex equation.

# 5 Conclusion

In this work we have introduced and developed a technique to generate solutions to the dsimplex equations using Clifford algebras. This provides solutions to the Yang-Baxter equation (d = 2) and its higher dimensional counterpart, the tetrahedron equation (d = 3). Their generalisations are far less studied. As far as we know there are few solutions for the dsimplex equations for  $d \ge 4$  [16,2]. The Clifford algebra method is unique as it solves all of these equations, including many of their variations. The set of solutions also form a linear space that changes with representation. We summarise this here.

Our method is based on two anticommuting operators A and B (generalised to a pair of anticommuting sets (3.8)). These operators are shown to be realised by Clifford algebras of the form  $\mathbf{CL}(p,q)$  (more generally  $\mathbf{CL}(p,q,y)$  that includes nilpotent operators as well). The solutions are algebraic. They do not depend on the particular representation chosen for the Clifford algebras. We find that for every pair of integers (a, b), with either one of a or b being even, we can write down a solution of the d = a + b-simplex equation (See Sec. 3.5). The space of solutions is linear. The solutions for the d = 2, 3, 4, 5 cases are summarised below.

#### Summary of the new solutions

- 1. The simplest solution to the 2-simplex or the **Yang-Baxter equation** is provided by the formula (3.2), while the most general solution can be found in the formula (3.10).
- 2. For the 3-simplex or the **tetrahedron equation**, the simplest product solution is given by the formula (3.29). It can be slightly generalised to a non-product solution in (3.33). The most general solution is given by formulae (3.38) and (3.40).
- 3. The 4-simplex equation has two sets of product solutions given by (3.43) and (3.45). A slightly more general solution, in (3.46), paves way for the most general solution in (3.52). An analogous general solution to (3.47) can also be written down.
- 4. The product solutions for the **5-simplex equation** are given by (3.55) and (3.56). The constituents of the latter are used to construct the more non-trivial solutions in (3.58) and (3.59) respectively. The most general solutions can be written down following the logic used in the previous cases.
- 5. As examples, the **qubit representations** for the d = 2, 3, 4 cases are shown in Sec. 4.

The scope of the Clifford method is manifested in a series of Appendices. Their contents are summarised below.

#### Summary of content in the Appendices

- 1. For every even d there are a number of d-simplex operators that correspond to the pair (a, b), with both a and b odd. These operators solve a modified version of the d-simplex equations, where the right hand side of the equations is multiplied by an overall negative sign. We call these the **anti-d-simplex equations**. Their solutions are the subject of Appendix B.
- 2. The different forms of the **tetrahedron equation** are solved using the Clifford algebra method in Appendix C.
- 3. Next we consider **reflection equations** for d = 2, 3. These help model integrable systems on open manifolds. We show that these can be solved with the Clifford method in Appendix D. We believe that the solutions to the higher simplex versions of the reflection equations can also be obtained in a similar manner.
- 4. We show that there exists **non-Clifford solutions** for the higher simplex equations in Appendix E. With some restrictions the product Clifford solutions in the main text are special cases of these solutions.
- 5. The Mathematica codes to verify the d-simplex solutions are shown in Appendix F.

We close with a few remarks on the potential applications of the higher simplex operators. There are three applications which are worth further study :

- 1. An important application of the Yang-Baxter *R*-matrices is in finding novel integrable spin chains. The higher simplex operators give the framework for constructing the higher dimensional analogs. The solutions constructed in this paper thus provides these examples. These have implications for quantum phases in higher dimensions.
- 2. In the recent decade there is a surge in interest for using the *R*-matrices as quantum gates in information processing. These are believed to result in less noisy quantum circuits due to the many integrals of motion. It would be interesting to use the higher simplex operators as quantum gates in quantum circuits. These can be achieved once we identify the unitary solutions among the higher simplex operators. This problem is far from obvious. Nevertheless we observe that there are already some unitary operators

among the solutions studied in this work. The simplest solutions for each d, namely the operators that are of the factorised type, are all unitary as the generators of CL(p,q) are Hermitian and they square to 1. These include operators of the type (3.29), (3.43), (3.45), (3.55), (3.56) and their higher simplex analogs. A more non-trivial solution for the d = 4 case is given by

$$R_{ijkl} = \frac{1}{2} \left[ A_i B_j A_k B_l + B_i A_j A_k B_l - A_i B_j B_k A_l + B_i A_j B_k A_l \right].$$
(5.1)

3. Two applications in mathematics : The higher simplex operators can help find the invariant polynomials corresponding to higher dimensional versions of knots. As the higher simplex equations appear as constraint equations in higher categories, the solutions shown in this work can help construct examples of such structures.

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# A Proof that $\alpha = \pm 1$ to satisfy the *d*-simplex equation

Assume that (3.1) holds. We will show that  $\alpha$  has to be  $\pm 1$  for the *d*-simplex equations to hold. For the following proofs we will use the obvious adaptation of the notation introduced

in 3.2. We begin with the 3-simplex operator in (3.33).

$$\begin{aligned} R_{123}R_{145}R_{246}R_{356} \\ &= A_3A_5B_6\left(\frac{1}{\alpha}, +, +\right)_{123}\left(\frac{1}{\alpha}, +, +\right)_{145}\left(+, \alpha, \alpha\right)_{246} \\ &+ A_3B_5A_6\left(\frac{1}{\alpha}, +, +\right)_{123}\left(+, \alpha, \alpha\right)_{145}\left(\frac{1}{\alpha}, +, +\right)_{246} \\ &+ B_3A_5A_6\left(+, \alpha, \alpha\right)_{123}\left(\frac{1}{\alpha}, +, +\right)_{145}\left(\frac{1}{\alpha}, +, +\right)_{246} \\ &= \frac{1}{\alpha^2}R_{356}\left(+, \alpha, \alpha\right)_{123}\left(+, \alpha, \alpha\right)_{145}\left(+, \alpha, \alpha\right)_{246} \\ &= \frac{1}{\alpha^2}R_{356}\left[A_2A_4B_6\left(+, +, \alpha\right)_{123}\left(+, +, \alpha\right)_{145} \\ &+ \alpha A_2B_4A_6\left(+, +, \alpha\right)_{123}\left(\alpha, \alpha, \alpha^2\right)_{145} \\ &+ \alpha B_2A_4A_6\left(\alpha, \alpha, \alpha^2\right)_{123}\left(+, +, \alpha\right)_{145}\right] \\ &= \frac{1}{\alpha^2}R_{356}\left(+, \alpha^2, \alpha^2\right)_{246}\left(+, +, \alpha\right)_{123}\left(+, +, \alpha\right)_{145} \\ &= \frac{1}{\alpha^2}R_{356}\left(+, \alpha^2, \alpha^2\right)_{246}\left[A_1A_4B_5\left(+, +, +\right)_{123} \\ &+ A_1B_4A_5\left(+, +, +\right)_{123} \\ &+ \alpha B_1B_4A_5\left(\alpha, \alpha, \alpha\right)_{123}\right] \\ &= \frac{1}{\alpha^2}R_{356}\left(+, \alpha^2, \alpha^2\right)_{246}\left(+, +, \alpha^2\right)_{145}R_{123} \\ &= R_{356}R_{246}R_{145}R_{123}; \text{ iff } \alpha = \pm 1. \end{aligned}$$
(A.1)

Next we prove the same for the (4, 0) type 4-simplex operator in (3.46).

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = (+, \alpha^4)_{479,10} \left(+, \frac{1}{\alpha}\right)_{1234} \left(+, \frac{1}{\alpha}\right)_{1567} \left(+, \frac{1}{\alpha}\right)_{2589} \left(+, \frac{1}{\alpha}\right)_{368,10} \\ = (+, \alpha^4)_{479,10} \left(+, \alpha^2\right)_{368,10} \left(+, \frac{1}{\alpha^2}\right)_{1234} \left(+, \frac{1}{\alpha^2}\right)_{1567} \left(+, \frac{1}{\alpha^2}\right)_{2589} \\ = (+, \alpha^4)_{479,10} \left(+, \alpha^2\right)_{368,10} R_{2589} \left(+, \frac{1}{\alpha^3}\right)_{1234} \left(+, \frac{1}{\alpha^3}\right)_{1567} \\ = (+, \alpha^4)_{479,10} \left(+, \alpha^2\right)_{368,10} R_{2589} \left(+, \frac{1}{\alpha^2}\right)_{1567} \left(+, \frac{1}{\alpha^4}\right)_{1234} \\ = R_{479,10} R_{368,10} R_{2589} R_{1567} R_{1234} ; \text{ iff } \alpha = \pm 1.$$
(A.2)

We now turn to the (2, 2) type 4-simplex operator in (3.47).

$$\begin{aligned} R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} \\ &= \frac{1}{\alpha^2}R_{479,10}\left(+,+,+,\alpha,\alpha,\alpha\right)_{1234}\left(+,+,+,\alpha,\alpha,\alpha\right)_{1567} \\ &\times (+,+,+,\alpha,\alpha,\alpha)_{2589}\left(+,+,+,\alpha,\alpha,\alpha,\alpha\right)_{368,10} \\ &= \frac{1}{\alpha^2}R_{479,10}\left(\frac{1}{\alpha^2},\frac{1}{\alpha^2},\frac{1}{\alpha^2},+,+,+\right)_{368,10}\left(+,\alpha,\alpha,\alpha,\alpha,\alpha^2\right)_{1234} \\ &\times (+,\alpha,\alpha,\alpha,\alpha,\alpha^2)_{1567}\left(+,\alpha,\alpha,\alpha,\alpha,\alpha^2\right)_{2589} \\ &= \frac{1}{\alpha^2}R_{479,10}\left(\frac{1}{\alpha^2},\frac{1}{\alpha^2},\frac{1}{\alpha^2},+,+,+\right)_{368,10}\left(+,\alpha^2,\alpha^2,\alpha^2,\alpha^2,\alpha^4\right)_{2589} \\ &\times (+,+,\alpha,+,\alpha,\alpha)_{1234}\left(+,+,\alpha,+,\alpha,\alpha\right)_{1567} \\ &= \frac{1}{\alpha^2}R_{479,10}\left(\frac{1}{\alpha^2},\frac{1}{\alpha^2},\frac{1}{\alpha^2},+,+,+\right)_{368,10}\left(+,\alpha^2,\alpha^2,\alpha^2,\alpha^2,\alpha^4\right)_{2589} \\ &\times (+,+,\alpha^2,+,\alpha^2,\alpha^2)_{1567}R_{1234} \\ &= R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234} ; \text{ iff } \alpha = \pm 1. \end{aligned}$$

The above proofs go through for the more general 3- and 4-simplex operators in (3.38), (3.52) as well. Similar proofs can be worked out for higher d, which we leave to the interested reader.

### **B** The anti-*d*-simplex equations and its solutions

The *d*-simplex operators corresponding to the pairs (a, b) for odd *a* and *b* satisfy the anti-*d*-simplex equations. These are well defined for even *d*. In this appendix we will define these equations and prove that these pairs solve them.

The anti-d-simplex equations are essentially the same as the d-simplex equations except for an overall negative sign on the right hand sides of the latter. This is the case when d is even. The situation changes when d is odd, which will be discussed separately.

We begin with the even case. The anti-2-simplex equation or the anti-Yang-Baxter equation is given by

$$R_{12}R_{13}R_{23} = -R_{23}R_{13}R_{12}. (B.1)$$

Note the minus sign in the right hand side. This equation is solved by  $A_iB_j$ ,  $B_iA_j$  and their linear combination,

$$R_{ij} = A_i B_j + B_i A_j. \tag{B.2}$$

The proof, using the shorthand notation of Sec. 3.2, goes as

$$R_{12}R_{13}R_{23} = R_{23}(-,+)_{12}(+,-)_{13}$$
  
=  $R_{23}R_{13}(-,-)_{12}$   
=  $-R_{23}R_{13}R_{12}$ . (B.3)

The anti-4-simplex equation is given by

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = -R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}.$$
 (B.4)

This is solved by

$$R_{ijkl} = A_i A_j A_k B_l + A_i A_j B_k A_l + A_i B_j A_k A_l + B_i A_j A_k A_l, \tag{B.5}$$

corresponding to the pair (3, 1). The pair (1, 3) gives another solution with the A and B operators interchanged in (B.5). Using the shorthand notations introduced in Sec. 3.2 the proof goes as

$$\begin{aligned} R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} \\ &= R_{479,10}\left(-,+,+,+\right)_{1234}\left(-,+,+,+\right)_{1567}\left(-,+,+,+\right)_{2589}\left(+,-,-,-\right)_{368,10} \\ &= R_{479,10}R_{368,10}\left(-,-,+,+\right)_{1234}\left(-,-,+,+\right)_{1567}\left(-,-,+,+\right)_{2589} \\ &= R_{479,10}R_{368,10}R_{2589}\left(-,-,-,+\right)_{1234}\left(+,+,+,-\right)_{1567} \\ &= R_{479,10}R_{368,10}R_{2589}R_{1567}\left(-,-,-,-\right)_{1234} \\ &= -R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}. \end{aligned}$$
(B.6)

The extensions to other even d is straightforward. The linear structure in the space of solutions holds for these equations as well.

When d is odd the d-simplex operator corresponds to the pair (d, 0). For example when d = 3 this corresponds to

$$R_{ijk} = A_i A_j A_k + B_i B_j B_k. \tag{B.7}$$

This solves the anti-3-simplex equation or the anti-tetrahedron equation

$$R_{123}R_{145}R_{246}R_{356} = R_{356}^{(-)}R_{246}^{(-)}R_{145}^{(-)}R_{123}^{(-)}, \tag{B.8}$$

where

$$R_{ijk}^{(-)} = A_i A_j A_k - B_i B_j B_k.$$
(B.9)

The proof is as follows :

$$R_{123}R_{145}R_{246}R_{356} = R_{356}^{(-)}(+,-)_{123}(+,-)_{145}(+,-)_{246}$$
  
=  $R_{356}^{(-)}R_{246}^{(-)}(+,+)_{123}(+,+)_{145}$   
=  $R_{356}^{(-)}R_{246}^{(-)}R_{145}^{(-)}(+,-)_{123}$   
=  $R_{356}^{(-)}R_{246}^{(-)}R_{145}^{(-)}R_{123}^{(-)}$ . (B.10)

The extension to larger odd d values is obvious. It is worth noting that the anti-d-simplex equation, for odd d, is more natural when a spectral parameter is included into the d-simplex operator (B.7),

$$R_{ijk} = A_i A_j A_k + \mu_{ijk} \ B_i B_j B_k. \tag{B.11}$$

This satisfies a spectral parameter dependent version of (B.8)

$$R_{123}(\mu_{123}) R_{145}(\mu_{145}) R_{246}(\mu_{246}) R_{356}(\mu_{356})$$
  
=  $R_{356}(-\mu_{356}) R_{246}(-\mu_{246}) R_{145}(-\mu_{145}) R_{123}(-\mu_{123}).$  (B.12)

This easily generalises to higher values of odd d.

## C Solutions for other forms of the tetrahedron equation

In this section we will show that ansatzes constructed using Clifford algebras solve the edge form of the tetrahedron equation (2.3) and the quantized Yang-Baxter equation (2.4). For both cases we will use the ansatz in (3.33) which we repeat here :

$$R_{ijk} = A_i A_j B_k + A_i B_j A_k + B_i A_j A_k.$$
(C.1)

## **Proof that** (3.33) solves (2.3)

We invoke the shorthand notation introduced in Sec. 3.2 for this case.

$$R_{123}R_{124}R_{134}R_{234}$$

$$= R_{234}(-, -, +)_{123}(+, +, -)_{124}(+, +, -)_{134}$$

$$= R_{234}[A_1A_3B_4(+, -, -)_{123}(+, -, -)_{124}$$

$$+ A_1B_3A_4(-, +, +)_{123}(-, +, +)_{124}$$

$$- B_1A_3A_4(-, +, +)_{123}(+, -, -)_{124}]$$

$$= R_{234}R_{134}(+, -, -)_{123}(+, -, -)_{124}$$

$$= R_{234}R_{134}[A_1A_2B_4(+, +, +)_{123}$$

$$- A_1B_2A_4(-, -, -)_{123}]$$

$$= R_{234}R_{134}R_{124}R_{123}.$$
(C.2)

# **Proof that** (3.33) solves (2.4)

Using the shorthand notation in Sec. 3.2 the proof goes as :

$$R_{124}R_{135}R_{236}R_{456}$$

$$= R_{456}(-, +, +)_{124}(-, +, +)_{135}(+, -, -)_{236}$$

$$= R_{456}[A_2A_3B_6(-, -, +)_{124}(-, -, +)_{135}]$$

$$- A_2B_3A_6(-, -, +)_{124}(+, +, -)_{135}$$

$$- B_2A_3A_6(+, +, -)_{124}(-, -, +)_{135}]$$

$$= R_{456}R_{236}(-, -, +)_{124}(-, -, +)_{135}$$

$$= R_{456}R_{236}[-A_1A_3B_5(-, -, -)_{124}]$$

$$- A_1B_3A_5(-, -, -)_{124}$$

$$+ B_1A_3A_5(+, +, +)_{124}]$$

$$= R_{456}R_{236}R_{135}R_{124}.$$
(C.3)

# **D** Solutions of reflection equations

We now show that the operators generated from Clifford algebras can also solve the reflection equations [31]. This is shown for the d = 2 and d = 3 cases.

$$d = 2$$

The constant form of the reflection equation in the d = 2 case is given by

$$R_{12}K_2R_{21}K_1 = K_1R_{12}K_2R_{21}.$$
(D.1)

This is solved by the ansatz

$$R_{ij} = A_i A_j + B_i B_j ; \ K_j = A_j + B_j,$$
 (D.2)

with anticommuting A and B. Adapting the shorthand notation of Sec. 3.2 this can be proved as follows :

$$R_{12}K_{2}R_{21}K_{1} = K_{1}(+,-)_{12}(+,+)_{2}(+,-)_{21}$$
  
=  $K_{1}R_{12}(+,-)_{21}(+,-)_{2}$   
=  $K_{1}R_{12}K_{2}(+,+)_{21}$   
=  $K_{1}R_{12}K_{2}R_{21}$ . (D.3)

The ansatz in (D.2) can be generalised by including arbitrary linear combinations and more number of operators.

$$d = 3$$

The constant form of the d = 3 reflection equation is given by

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456}$$
  
=  $R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}.$  (D.4)

This equation is solved by the ansatz

$$R_{ijk} = A_i A_j B_k + A_i B_j A_k + B_i A_j A_k$$
  

$$K_{ijkl} = A_i A_j A_k A_l + B_i B_j B_k B_l.$$
(D.5)

Using the shorthand notation of Sec. 3.2 we can prove this as follows :

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456}$$

$$= R_{456}(-, -, +)_{689}(+, -)_{3579}(+, -, +)_{249}(+, -, +)_{258}(-, +)_{1478}(-, +)_{1236}$$

$$= R_{456}K_{1236}(+, +, +)_{689}(+, +)_{3579}(+, -, -)_{249}(+, -, -)_{258}(+, +)_{1478}$$

$$= R_{456}K_{1236}K_{1478}(+, -, +)_{689}(+, -)_{3579}(+, +, -)_{249}(-, -, -)_{258}$$

$$= R_{456}K_{1236}K_{1478}R_{258}(+, +, +)_{689}(+, +)_{3579}(+, +, +)_{249}$$

$$= R_{456}K_{1236}K_{1478}R_{258}R_{249}(+, -, -)_{689}(-, +)_{3579}$$

$$= R_{456}K_{1236}K_{1478}R_{258}R_{249}(+, -, -)_{689}(-, +)_{3579}$$

$$= R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}.$$
(D.6)

The ansatz in (D.5) generalises with the inclusion of arbitrary linear combinations and more number of operators from the sets in (3.8).

## **E** Non-Clifford solutions

Now we have a brief look at the possibility for a non-Clifford solution to the *d*-simplex equation. We consider the d = 2, d = 3 and d = 4 cases before we generalise to an arbitrary *d*.

d = 2: Consider the ansatz

$$R_{ij} = A_i B_j, \tag{E.1}$$

with the operators A and B satisfying the relation

$$AB = \alpha BA \; ; \; \alpha \in \mathbb{C}. \tag{E.2}$$

This satisfies the 2-simplex or the Yang-Baxter equation when  $\alpha = 1$ . This case is included in [39]. d = 3: Now we take the ansatz as

$$R_{ijk} = A_i B_j C_k, \tag{E.3}$$

with the operators A, B and C satisfying

$$AB = \alpha BA ; AC = \beta CA ; BC = \gamma CB ; \alpha, \beta, \gamma \in \mathbb{C}.$$
 (E.4)

This satisfies the 3-simplex or the tetrahedron equation when

$$\alpha\beta\gamma = 1. \tag{E.5}$$

A non-trivial choice for the parameters are the third roots of unity.

d = 4: This solution will help us identify the pattern for an arbitrary d. We introduce new notation to make things simpler. This is seen in the ansatz

$$R_{ijkl} = (A_1)_i (A_2)_j (A_3)_k (A_4)_l.$$
(E.6)

Here the indices i, j, k and l denote the local Hilbert spaces appearing in the d-simplex operators. The indices 1,2,3, and 4 index the operators of the algebra. We will assume that no confusion will arise to the reader due to this. The algebra satisfied by the  $A_p$  operators is

$$A_p A_q = \alpha_{pq} A_q A_p \; ; \; p < q \in \{1, 2, 3, 4\}.$$
(E.7)

After some simple computations we can verify that (E.6) satisfies the 4-simplex equation (3.44) when

$$\prod_{\substack{p,q=1\\p
(E.8)$$

A non-trivial solution is presented by the sixth roots of unity. This notation helps us to generalise to an arbitrary d.

**General** d: The ansatz for the *d*-simplex operator is given by

$$R_{i_1\cdots i_d} = \bigotimes_{p=1}^d \left(A_p\right)_{i_p},\tag{E.9}$$

with the  $A_p$  operators satisfying

$$A_p A_q = \alpha_{pq} A_q A_p \; ; \; p < q \in \{1, \cdots, d\}.$$
(E.10)

This satisfies the d-simplex equation when

$$\prod_{\substack{p,q=1\\p
(E.11)$$

A non-trivial solution is given by the  $\binom{d}{2}$  th roots of unity.

#### **F** Mathematica Codes

We consider the first two cases of Sec. 3.1 in what follows. This Appendix should be read along with the qubit solutions presented in Sec. 4.

Case 1 - 3-simplex operators : Below we show the Mathematica code to verify that the 3-simplex operator in (4.2) satisfies the spectral parameter dependent 3-simplex equation (3.37). The first(second) box is the input(output). The latter includes the choice of the randomly generated 15 parameters ( $\mu$ ) taken randomly from integers between -10 and 10. It also includes the Boolean value of the veracity of the 3-simplex equation. The *R*-matrices contain 6 parameters, the first three ( $\alpha$ ,  $\beta$ ,  $\gamma$ ) specify the indices on which it acts non-trivially and the remaining are chosen from the 15  $\mu$  parameters. Symbolic computations also confirms that the operator in (4.2) is a solution.

```
\begin{split} & \text{sigmaX} = \text{SparseArray}@PauliMatrix[1]; \\ & \text{sigmaZ} = \text{SparseArray}@PauliMatrix[3]; \\ & \mu = \text{Table}[\text{RandomInteger}[\{-10, 10\}], \{i, 1, 15\}] \\ & \text{R3d}[\alpha_{-}, \beta_{-}, \gamma_{-}, i_{-}, j_{-}, k_{-}] := \\ & \mu[[i]] * \text{KroneckerProduct}@@ReplacePart[ConstantArray[IdentityMatrix[2], 6], \\ & \{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaZ}\}] + \\ & \mu[[j]] * \text{KroneckerProduct}@@ReplacePart[ConstantArray[IdentityMatrix[2], 6], \\ & \{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaZ}, \gamma \rightarrow \text{sigmaX}\}] + \\ & \mu[[k]] * \text{KroneckerProduct}@@ReplacePart[ConstantArray[IdentityMatrix[2], 6], \\ & \{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaX}\}]; \\ & \text{isEqual} = \text{Simplify}[\text{R3d}[1, 2, 3, 8, 11, 12].\text{R3d}[1, 4, 5, 3, 7, 9]. \\ & \text{R3d}[2, 4, 6, 13, 10, 8].\text{R3d}[3, 5, 6, 14, 10, 12] == \\ & \text{R3d}[3, 5, 6, 14, 10, 12].\text{R3d}[2, 4, 6, 13, 10, 8].\text{R3d}[1, 4, 5, 3, 7, 9].\text{R3d}[1, 2, 3, 8, 11, 12]]; \\ & (*\text{Print result*}) \\ & \text{If[isEqual, Print["True"], Print["False"]]} \end{split}
```

 $\{7,2,6,4,8,-8,7,10,-7,-10,-3,6,8,9,-2\}$ 

True

4-simplex operators : The boxes below show the Mathematica code for the verification that a linear combination of the 4-simplex operators of the (4,0) and (2,2) types satisfy the spectral parameter dependent 4-simplex equation of (3.53). In the code each of the five 4simplex operators on each side of this equation depend on 12 parameters. The first four parameters  $(\alpha, \beta, \gamma, \delta)$  specify the position of the X and Z operators in the tensor product. The remaining 8 parameters are the coefficients appearing in the linear combination. These are picked randomly from integers between -10 and 10. The output shows these parameters and the Boolean value for the truth of the 4-simplex equation. The coefficients can take real or complex values as well. This requires different precision levels. sigmaX = SparseArray@PauliMatrix[1];

sigmaZ = SparseArray@PauliMatrix[3];

 $\mu = \text{Table}[\text{RandomInteger}[\{-10, 10\}], \{i, 1, 15\}]$ 

 $R4d[\alpha_{-},\beta_{-},\gamma_{-},\delta_{-},i_{-},j_{-},k_{-},l_{-},m_{-},n_{-},p_{-},q_{-}] :=$ 

 $\mu[[i]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \mathrm{sigmaX}, \beta \rightarrow \mathrm{sigmaX}, \gamma \rightarrow \mathrm{sigmaZ}, \delta \rightarrow \mathrm{sigmaZ}\}] +$ 

 $\mu[[j]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \operatorname{sigmaX}, \beta \rightarrow \operatorname{sigmaZ}, \gamma \rightarrow \operatorname{sigmaX}, \delta \rightarrow \operatorname{sigmaZ}\}] +$ 

 $\mu[[k]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \mathrm{sigmaZ}, \beta \rightarrow \mathrm{sigmaX}, \gamma \rightarrow \mathrm{sigmaX}, \delta \rightarrow \mathrm{sigmaZ}\}] +$ 

 $\mu[[l]]*KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \mathrm{sigmaX}, \beta \rightarrow \mathrm{sigmaZ}, \gamma \rightarrow \mathrm{sigmaZ}, \delta \rightarrow \mathrm{sigmaX}\}] +$ 

 $\mu[[m]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaZ}, \delta \rightarrow \text{sigmaX}\}]+$ 

 $\mu[[n]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \mathrm{sigmaZ}, \beta \rightarrow \mathrm{sigmaZ}, \gamma \rightarrow \mathrm{sigmaX}, \delta \rightarrow \mathrm{sigmaX}\}] +$ 

 $\mu[[p]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \mathrm{sigmaX}, \beta \rightarrow \mathrm{sigmaX}, \gamma \rightarrow \mathrm{sigmaX}, \delta \rightarrow \mathrm{sigmaX}\}] +$ 

 $\mu[[q]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{sigmaZ}, \gamma \rightarrow \text{sigmaZ}, \delta \rightarrow \text{sigmaZ}\}];$ 

Simplify [R4d[1,2,3,4,11,12,8,7,9,10,5,13]. R4d[1,5,6,7,11,15,2,5,6,7,8,14].

R4d[2, 5, 8, 9, 12, 15, 8, 13, 11, 15, 12, 10].R4d[3, 6, 8, 10, 13, 6, 1, 4, 14, 9, 5, 7].

R4d[4, 7, 9, 10, 14, 7, 2, 5, 8, 10, 11, 13] ==

R4d[4,7,9,10,14,7,2,5,8,10,11,13].R4d[3,6,8,10,13,6,1,4,14,9,5,7].

R4d[2, 5, 8, 9, 12, 15, 8, 13, 11, 15, 12, 10]. R4d[1, 5, 6, 7, 11, 15, 2, 5, 6, 7, 8, 14].

R4d[1, 2, 3, 4, 11, 12, 8, 7, 9, 10, 5, 13]]

 $\{9, -3, 7, -8, 8, 2, 6, -7, -6, 0, 7, -2, 7, -10, -7\}$ 

True

**5-simplex operators** : As an example we provide the Mathematica code for the (4, 1) type 5-simplex operator :

sigmaX = SparseArray@PauliMatrix[1];sigmaZ = SparseArray@PauliMatrix[3]; $\mu = \text{Table}[\text{RandomInteger}[\{-10, 10\}], \{i, 1, 15\}]$  $R5d1[\alpha_{-}, \beta_{-}, \gamma_{-}, \delta_{-}, \eta_{-}, i_{-}, j_{-}, k_{-}, l_{-}, m_{-}] :=$  $\mu[[i]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaX}, \delta \rightarrow \text{sigmaX}, \eta \rightarrow \text{sigmaZ}\}\}$  $\mu[[j]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaX}, \delta \rightarrow \text{sigmaZ}, \eta \rightarrow \text{sigmaX}\}\}$  $\mu[[k]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaZ}, \delta \rightarrow \text{sigmaX}, \eta \rightarrow \text{sigmaX}\}\}$  $\mu[[l]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \rightarrow \text{sigmaX}, \beta \rightarrow \text{sigmaZ}, \gamma \rightarrow \text{sigmaX}, \delta \rightarrow \text{sigmaX}, \eta \rightarrow \text{sigmaX}\}\}$  $\mu[[m]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{sigmaX}, \gamma \rightarrow \text{sigmaX}, \delta \rightarrow \text{sigmaX}, \eta \rightarrow \text{sigmaX}\}$ Simplify[R5d1[1, 2, 3, 4, 5, 7, 9, 10, 15, 12].R5d1[1, 6, 7, 8, 9, 13, 14, 12, 1, 2]. R5d1[2, 6, 10, 11, 12, 8, 9, 10, 15, 12].R5d1[3, 7, 10, 13, 14, 6, 4, 11, 14, 13]. R5d1[4, 8, 11, 13, 15, 2, 3, 9, 10, 11]. R5d1[5, 9, 12, 14, 15, 3, 6, 14, 11, 1] = =R5d1[5, 9, 12, 14, 15, 3, 6, 14, 11, 1].R5d1[4, 8, 11, 13, 15, 2, 3, 9, 10, 11]. R5d1[3, 7, 10, 13, 14, 6, 4, 11, 14, 13].R5d1[2, 6, 10, 11, 12, 8, 9, 10, 15, 12]. R5d1[1, 6, 7, 8, 9, 13, 14, 12, 1, 2].R5d1[1, 2, 3, 4, 5, 7, 9, 10, 15, 12]]

$$\{-5, -3, -4, -4, -6, -8, -2, 5, -5, -3, -4, 2, -3, 8, -9\}$$

True

**Case 2 - 3-simplex operators** : The boxes below show the Mathematica codes for verifying if the 3-simplex operators in (4.6) and (4.7) satisfy the spectral parameter dependent 3-simplex or tetrahedron equation. As mentioned in the Case 1 analysis, these operators depend on 3 parameters that are picked randomly from integers between -10 and 10. The output is shown in the second box. Ax = SparseArray@A;

sigmaZ = SparseArray@PauliMatrix[3];

 $\mu = \text{Table}[\text{RandomInteger}[\{-10, 10\}], \{i, 1, 15\}]$ 

 $R3dcase2[\alpha_{-},\beta_{-},\gamma_{-},i_{-},j_{-},k_{-}]:=$ 

 $\mu[[i]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 6],$ 

 $\{\alpha \rightarrow \mathrm{Ax}, \beta \rightarrow \mathrm{Ax}, \gamma \rightarrow \mathrm{sigmaZ}\}] +$ 

 $\mu[[j]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 6],$ 

 $\{\alpha \to \mathsf{Ax}, \beta \to \mathsf{sigmaZ}, \gamma \to \mathsf{Ax}\}] +$ 

 $\mu[[k]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 6],$ 

 $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{Ax}, \gamma \rightarrow \text{Ax}\}];$ 

 $R3dcase21[\alpha_{-},\beta_{-},\gamma_{-},i_{-},j_{-},k_{-}] {:=}$ 

 $\mu[[i]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 6],$ 

 $\{\alpha \rightarrow \mathrm{sigmaZ}, \beta \rightarrow \mathrm{sigmaZ}, \gamma \rightarrow \mathrm{Ax}\}] +$ 

 $\mu[[j]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 6],$ 

 $\{\alpha \rightarrow \mathrm{sigmaZ}, \beta \rightarrow \mathrm{Ax}, \gamma \rightarrow \mathrm{sigmaZ}\}] +$ 

 $\mu[[k]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 6],$ 

 $\{\alpha \rightarrow \mathrm{Ax}, \beta \rightarrow \mathrm{sigmaZ}, \gamma \rightarrow \mathrm{sigmaZ}\}];$ 

Simplify [R3dcase2[1,2,3,8,11,12]. R3dcase2[1,4,5,3,7,9]. R3dcase2[2,4,6,13,10,8].

R3dcase2[3, 5, 6, 14, 10, 12] = =

R3dcase2[3, 5, 6, 14, 10, 12].R3dcase2[2, 4, 6, 13, 10, 8].R3dcase2[1, 4, 5, 3, 7, 9].

R3dcase2[1, 2, 3, 8, 11, 12]]

Simplify[R3dcase21[1, 2, 3, 8, 11, 12].R3dcase21[1, 4, 5, 3, 7, 9].

R3dcase21[2, 4, 6, 13, 10, 8].R3dcase21[3, 5, 6, 14, 10, 12] = =

R3dcase21[3, 5, 6, 14, 10, 12]. R3dcase21[2, 4, 6, 13, 10, 8].

R3dcase21[1,4,5,3,7,9].R3dcase21[1,2,3,8,11,12]]

 $\{-2, 10, -4, -3, -2, -4, 0, -10, -5, -5, 7, 10, -5, 4, 9\}$ 

True

True

**4-simplex operators** : The Mathematica code for a linear combination of the (2, 2) and (4, 0) types 4-simplex operator in (4.8) and (4.9) is shown in the box below. For a random set of integer coefficients, this operator satisfies the spectral parameter dependent 4-simplex equation. This is seen in the output box below.

Ax = SparseArray@A;

sigmaZ = SparseArray@PauliMatrix[3];

 $\mu = \text{Table}[\text{RandomInteger}[\{-10, 10\}], \{i, 1, 15\}]$ 

R4d[ $\alpha_{-}, \beta_{-}, \gamma_{-}, \delta_{-}, i_{-}, j_{-}, k_{-}, l_{-}, m_{-}, n_{-}, p_{-}, q_{-}$ ]:=

 $\mu[[i]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \to Ax, \beta \to Ax, \gamma \to sigmaZ, \delta \to sigmaZ\}]+$ 

 $\mu[[j]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10], \\$ 

 $\{\alpha \rightarrow \mathrm{Ax}, \beta \rightarrow \mathrm{sigmaZ}, \gamma \rightarrow \mathrm{Ax}, \delta \rightarrow \mathrm{sigmaZ}\}] +$ 

- $$\begin{split} & \mu[[k]] * \text{KroneckerProduct} @@\text{ReplacePart}[\text{ConstantArray}[\text{IdentityMatrix}[2], 10], \\ & \{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{Ax}, \gamma \rightarrow \text{Ax}, \delta \rightarrow \text{sigmaZ}\}] + \end{split}$$
- $\mu[[l]]*KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$

 $\{\alpha \rightarrow \mathbf{A}\mathbf{x}, \beta \rightarrow \mathbf{sigmaZ}, \gamma \rightarrow \mathbf{sigmaZ}, \delta \rightarrow \mathbf{A}\mathbf{x}\}] +$ 

 $\mu[[m]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10], \\$ 

 $\{\alpha \rightarrow \operatorname{sigmaZ}, \beta \rightarrow \operatorname{Ax}, \gamma \rightarrow \operatorname{sigmaZ}, \delta \rightarrow \operatorname{Ax}\}] +$ 

 $\mu[[n]]*KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$ 

 $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{sigmaZ}, \gamma \rightarrow \text{Ax}, \delta \rightarrow \text{Ax}\}] +$ 

- $\mu[[p]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10], \\ \{\alpha \to Ax, \beta \to Ax, \gamma \to Ax, \delta \to Ax\}]+$
- $\mu[[q]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 10],$

 $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow \text{sigmaZ}, \gamma \rightarrow \text{sigmaZ}, \delta \rightarrow \text{sigmaZ}\}];$ 

Simplify[R4d[1, 2, 3, 4, 11, 12, 8, 7, 9, 10, 5, 13].R4d[1, 5, 6, 7, 11, 15, 2, 5, 6, 7, 8, 14]. R4d[2, 5, 8, 9, 12, 15, 8, 13, 11, 15, 12, 10].

R4d[3, 6, 8, 10, 13, 6, 1, 4, 14, 9, 5, 7].R4d[4, 7, 9, 10, 14, 7, 2, 5, 8, 10, 11, 13] = =

R4d[4, 7, 9, 10, 14, 7, 2, 5, 8, 10, 11, 13]. R4d[3, 6, 8, 10, 13, 6, 1, 4, 14, 9, 5, 7].

R4d[2, 5, 8, 9, 12, 15, 8, 13, 11, 15, 12, 10].

R4d[1, 5, 6, 7, 11, 15, 2, 5, 6, 7, 8, 14]. R4d[1, 2, 3, 4, 11, 12, 8, 7, 9, 10, 5, 13]]

 $\{-10, -7, 9, 10, 10, -3, -1, 10, 10, 5, 6, -7, 9, -5, 10\}$ 

True

**5-simplex operators** : The Mathematica code showing that the (4, 1) type 5-simplex operator satisfies the 5-simplex equation is shown below.

Ax = SparseArray@A;sigmaZ = SparseArray@PauliMatrix[3]; $\mu = \text{Table}[\text{RandomInteger}[\{-10, 10\}], \{i, 1, 15\}]$  $R5d2[\alpha_{-}, \beta_{-}, \gamma_{-}, \delta_{-}, \eta_{-}, i_{-}, j_{-}, k_{-}, l_{-}, m_{-}] :=$  $\mu[[i]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \to Ax, \beta \to Ax, \gamma \to Ax, \delta \to Ax, \eta \to sigmaZ\}$  $\mu[[j]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \to Ax, \beta \to Ax, \gamma \to Ax, \delta \to sigmaZ, \eta \to Ax\}$  $\mu[[k]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \to Ax, \beta \to Ax, \gamma \to sigmaZ, \delta \to Ax, \eta \to Ax\}$  $\mu[[l]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \to Ax, \beta \to sigmaZ, \gamma \to Ax, \delta \to Ax, \eta \to Ax\}$  $\mu[[m]] * KroneckerProduct@@ReplacePart[ConstantArray[IdentityMatrix[2], 15],$  $\{\alpha \rightarrow \text{sigmaZ}, \beta \rightarrow Ax, \gamma \rightarrow Ax, \delta \rightarrow Ax, \eta \rightarrow Ax\}$ Simplify[R5d2[1, 2, 3, 4, 5, 7, 9, 10, 15, 12].R5d2[1, 6, 7, 8, 9, 13, 14, 12, 1, 2]. R5d2[2, 6, 10, 11, 12, 8, 9, 10, 15, 12].R5d2[3, 7, 10, 13, 14, 6, 4, 11, 14, 13]. R5d2[4, 8, 11, 13, 15, 2, 3, 9, 10, 11]. R5d2[5, 9, 12, 14, 15, 3, 6, 14, 11, 1] = =R5d2[5, 9, 12, 14, 15, 3, 6, 14, 11, 1].R5d2[4, 8, 11, 13, 15, 2, 3, 9, 10, 11]. R5d2[3, 7, 10, 13, 14, 6, 4, 11, 14, 13].R5d2[2, 6, 10, 11, 12, 8, 9, 10, 15, 12]. R5d2[1, 6, 7, 8, 9, 13, 14, 12, 1, 2].R5d2[1, 2, 3, 4, 5, 7, 9, 10, 15, 12]]

 $\{4, -4, -5, -5, -3, -4, 6, -4, 1, -5, 4, -5, 1, 5, -6\}$ 

True

### References

- BAEZ, J. C., AND NEUCHL, M. Higher dimensional algebra: I. braided monoidal 2categories. Advances in Mathematics 121 (1995), 196–244.
- [2] BARDAKOV, V., CHUZINOV, B., EMEL'YANENKOV, I., IVANOV, M., KOZLOVSKAYA, T., AND LESHKOV, V. Set-theoretical solutions of simplex equations. arXiv:2206.08906 (2022).
- [3] BAXTER, R. On zamolodchikov's solution of the tetrahedron equations. *Communications in Mathematical Physics 88* (1983), 185–205.
- [4] BAXTER, R. J. Partition function of the eight-vertex lattice model. Annals of Physics 70, 1 (1972), 193–228.
- [5] BAXTER, R. J. Exactly solved models in statistical mechanics.
- [6] BAZHANOV, V. V., AND BAXTER, R. J. New solvable lattice models in three dimensions. Journal of Statistical Physics 69 (1992), 453–485.
- [7] BAZHANOV, V. V., AND BAXTER, R. J. Star-triangle relation for a three-dimensional model. *Journal of Statistical Physics* 71 (1993), 839–864.
- [8] BAZHANOV, V. V., MANGAZEEV, V. V., AND SERGEEV, S. M. Quantum geometry of 3-dimensional lattices and tetrahedron equation. arXiv: Mathematical Physics (2009), 23–44.
- [9] BAZHANOV, V. V., AND SERGEEV, S. M. Zamolodchikov's tetrahedron equation and hidden structure of quantum groups. *Journal of Physics A 39* (2005), 3295–3310.
- [10] BAZHANOV, V. V., AND STROGANOV, Y. G. Conditions of commutativity of transfer matrices on a multidimensional lattice. *Theoretical and Mathematical Physics* 52 (1982), 685–691.
- [11] BAZHANOV, V. V., AND STROGANOV, Y. G. Free fermions on a three-dimensional lattice and tetrahedron equations. *Nuclear Physics 230* (1984), 435–454.

- [12] CARTER, J. S., AND SAITO, M. On Formulations and Solutions of SIMPLEX Equations. International Journal of Modern Physics A 11, 24 (Jan. 1996), 4453–4463.
- [13] FRENKEL, I. B., AND MOORE, G. W. Simplex equations and their solutions. Communications in Mathematical Physics 138 (1991), 259–271.
- [14] GAVRYLENKO, P., SEMENYAKIN, M., AND ZENKEVICH, Y. Solution of tetrahedron equation and cluster algebras. *Journal of High Energy Physics 2021* (2020).
- [15] HIETARINTA, J. Labelling schemes for tetrahedron equations and dualities between them. Journal of Physics A: Mathematical and General 27, 17 (Sept. 1994), 5727–5748.
- [16] HIETARINTA, J. Permutation-type solutions to the yang baxter and other n-simplex equations. Journal of Physics A: Mathematical and General 30, 13 (July 1997), 4757–4771.
- [17] HIETARINTA, J., AND NIJHOFF, F. W. The eight tetrahedron equations. Journal of Mathematical Physics 38 (1997), 3603–3615.
- [18] INOUE, R., KUNIBA, A., AND TERASHIMA, Y. Tetrahedron equation and quantum cluster algebras. arXiv:2310.14529 (2024).
- [19] ISAEV, A. P., AND KULISH, P. P. Tetrahedron reflection equations. Modern Physics Letters A 12, 06 (Feb. 1997), 427–437.
- [20] KAPRANOV, M., AND VOEVODSKY, V. 2-categories and zamolodchikov tetrahedra equations.
- [21] KASHAEV, R. M., KOREPANOV, I. G., AND SERGEEV, S. M. Functional tetrahedron equation. *Theoretical and Mathematical Physics* 117, 3 (Dec. 1998), 1402–1413.
- [22] KAUFFMAN, L. H. Knots and Physics. WORLD SCIENTIFIC, 1991.
- [23] KHACHATRYAN, S. A., FERRAZ, A., KLÜMPER, A., AND SEDRAKYAN, A. Integrability in three dimensions: Algebraic bethe ansatz for anyonic models. *Nuclear Physics 899* (2015), 444–450.
- [24] KOREPANOV, I. Novel solutions to the tetrahedron equation. arXiv:1309.2931 (2013).
- [25] KOREPANOV, I. G. Tetrahedral zamolodchikov algebra and the two-layer flat model in statistical mechanics. *Modern Physics Letters B 03* (1989), 201–206.
- [26] KOREPANOV, I. G. Tetrahedral zamolodchikov algebras corresponding to baxter's *l*-operators. *Communications in Mathematical Physics* 154 (1993), 85–97.

- [27] KOREPANOV, I. G. The tetrahedron equation and algebraic geometry. Journal of Mathematical Sciences 83 (1994), 85–92.
- [28] KOREPIN, V. E., BOGOLIUBOV, N. M., AND IZERGIN, A. G. Quantum inverse scattering method and correlation functions. Cambridge university press.
- [29] KUNIBA, A. Combinatorial yang-baxter maps arising from the tetrahedron equation. Theoretical and Mathematical Physics 189 (2015), 1472–1485.
- [30] KUNIBA, A. Tetrahedron equation and quantum r matrices for q-oscillator representations mixing particles and holes. Symmetry, Integrability and Geometry: Methods and Applications (2018).
- [31] KUNIBA, A. Quantum groups in three-dimensional integrability. Springer Nature, 2022.
- [32] KUNIBA, A., AND OKADO, M. Tetrahedron and 3d reflection equations from quantized algebra of functions. *Journal of Physics A: Mathematical and Theoretical* 45 (2012).
- [33] KUNIBA, A., OKADO, M., AND SERGEEV, S. M. Tetrahedron equation and generalized quantum groups. *Journal of Physics A: Mathematical and Theoretical* 48 (2015).
- [34] KUNIBA, A., AND PASQUIER, V. Matrix product solutions to the reflection equation from three dimensional integrability. *Journal of Physics A: Mathematical and Theoretical* 51 (2018).
- [35] MAILLET, J. M., AND NIJHOFF, F. The tetrahedron equation and the four-simplex equation. *Physics Letters A 134*, 4 (1989), 221–228.
- [36] MAILLET, J. M., AND NIJHOFF, F. W. Integrability for multidimensional lattice models. *Physics Letters B* 224 (1989), 389–396.
- [37] MAJID, S. Foundations of Quantum Group Theory. Cambridge University Press, 1995.
- [38] MANGAZEEV, V. V., BAZHANOV, V. V., AND SERGEEV, S. M. An integrable 3d lattice model with positive boltzmann weights. *Journal of Physics A: Mathematical and Theoretical* 46 (2013).
- [39] PADMANABHAN, P., HAO, K., AND KOREPIN, V. Yang-baxter solutions from commuting operators. arXiv:2401.05662 (2024).
- [40] PADMANABHAN, P., SUGINO, F., AND TRANCANELLI, D. Braiding quantum gates from partition algebras. *Quantum* 4 (Aug. 2020), 311.
- [41] PADMANABHAN, P., SUGINO, F., AND TRANCANELLI, D. Local invariants of braiding quantum gates—associated link polynomials and entangling power. *Journal of Physics* A: Mathematical and Theoretical 54, 13 (Mar. 2021), 135301.

- [42] ROWELL, E. C., ZHANG, Y., WU, Y.-S., AND GE, M.-L. Extraspecial two-groups, generalized yang-baxter equations and braiding quantum gates. *Quantum Inf. Comput.* 10 (2007), 685–702.
- [43] SERGEEV, S., MANGAZEEV, V. V., AND STROGANOV, Y. G. The vertex formulation of the bazhanov-baxter model. *Journal of Statistical Physics 82* (1995), 31–49.
- [44] SHIROISHI, M., AND WADATI, M. Tetrahedral zamolodchikov algebra related to the six-vertex free-fermion model and a new solution of the yang-baxter equation. *Journal of* the Physical Society of Japan 64, 12 (1995), 4598–4608.
- [45] SHIROISHI, M., AND WADATI, M. Yang-baxter equation for the r-matrix of the onedimensional hubbard model. Journal of the Physical Society of Japan 64, 1 (1995), 57–63.
- [46] SKLYANIN, E. K., TAKHTADZHYAN, L. A., AND FADDEEV, L. D. Quantum inverse problem method. i. *Theoretical and Mathematical Physics* 40, 2 (1979), 688–706.
- [47] STROGANOV, Y. G. Tetrahedron equation and spin integrable models on a cubic lattice. Theoretical and Mathematical Physics 110 (1997), 141–167.
- [48] TAKHTADZHAN, L. A., AND FADDEEV, L. D. The quantum method of the inverse problem and the heisenberg xyz model. *Russian Mathematical Surveys* 34, 5 (oct 1979), 11.
- [49] TALALAEV, D. V. Zamolodchikov tetrahedral equation and higher hamiltonians of 2d quantum integrable systems. Symmetry Integrability and Geometry-methods and Applications 13 (2015), 031.
- [50] TURAEV, V. The yang-baxter equation and invariants of links. Inventiones mathematicae 92 (1988), 527–553.
- [51] YANG, C. N. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.* 19 (Dec 1967), 1312–1315.
- [52] ZAMOLODCHIKOV, A. B. Tetrahedra equations and integrable systems in threedimensional space. *Zh.Eksper.Teoret Fizika* 79 (1980).
- [53] ZAMOLODCHIKOV, A. B. Tetrahedron equations and the relativistic s-matrix of straightstrings in 2+1-dimensions. *Communications in Mathematical Physics 79* (1981), 489–505.
- [54] ZAMOLODCHIKOV, A. B., AND ZAMOLODCHIKOV, A. B. Factorized s-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. *Annals of Physics 120* (1979), 253–291.

[55] ZHANG, K., YU, K., HAO, K., AND KOREPIN, V. Optimal realization of yang-baxter gate on quantum computers. Advanced Quantum Technologies (Feb. 2024).