THE DISTRIBUTION ON PERMUTATIONS INDUCED BY A RANDOM PARKING FUNCTION

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Abstract. A parking function on $[n]$ creates a permutation in S_n via the order in which the n cars appear in the n parking spaces. Placing the uniform probability measure on the set of parking functions on $[n]$ induces a probability measure on S_n. We initiate a study of some properties of this distribution.

1. Introduction and Statement of Results

Consider a row of n parking spaces on a one-way street. A line of n cars, numbered from 1 to n, attempts to park, one at a time. The ith car's preferred space is spot number $\pi_i \in [n]$. If this space is already taken, then car i proceeds forward and parks in the first available space, if one exists. If the car is unable to park, it exits the street. A sequence $\pi = \{\pi_i\}_{i=1}^n$ is called a parking function on $[n]$ if all n cars are able to park. It is easy to see that π is a parking function if and only if $|\{i : \pi_i \le j\}| \ge j$, for all $j \in [n]$. Let $\mathcal{P}_n$ denote the set of parking functions. It is well-known that $|\mathcal{P}_n| = (n+1)^{n-1}$. There are a number of proofs of this result; a particularly elegant one due to Pollack can be found in [3]. There is a large literature on parking functions and their generalizations; see, for example, the survey [5].

We can consider a random parking function by placing the uniform probability measure on $\mathcal{P}_n$. Denote this probability measure by $P^{\mathcal{P}_n}$. A study of random parking functions was initiated by Diaconis and Hicks in [1]. Since each parking function yields a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, where σ_j is the number of the car that parked in space j, it follows that a random

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parking function induces a distribution on the set S_n of permutations of $[n]$. In this paper we initiate a study of this distribution.

We will use the notation P_n and E_n to denote the uniform probability measure and the corresponding expectation on S_n. We will denote by P_n^{park} the probability measure on S_n induced by a random parking function in $\mathcal{P}_n$. The corresponding expectation will be denoted by E_n^{park}. To be more precise concerning the definition of the induced probability measure, define $T_n : \mathcal{P}_n \to S_n$ by $T_n(\pi) = \sigma$, if when using the parking function π, σ_j is the number of the car that parked in space j, for $j \in [n]$. For example, if $n = 4$ and $\pi = 2213 \in \mathcal{P}_4$, then we have $T_4(2213) = 3124 \in S_4$. We define

$$P_n^{\text{park}}(\sigma) = P^{\mathcal{P}_n}\left(T_n^{-1}(\sigma)\right). \tag{1.1}$$

For $1 \le i \le n < \infty$ and $\sigma \in S_n$, define

$$l_{n,i}(\sigma) = \max\{l \in [i] : \sigma_i = \max(\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-l+1}\}.$$

For $n \in \mathbb{N}$, define

$$L_n(\sigma) = \prod_{i=1}^n l_{n,i}(\sigma),\ \sigma \in S_n. \tag{1.2}$$

For example, if $\sigma = 379218645 \in S_9$, then $l_{n,i}(\sigma) = 1$, for $i \in \{1,4,5,7,8\}$, $l_{n,i}(\sigma) = 2$, for $i \in \{2,9\}$ and $l_{n,i}(\sigma) = 3$, for $i \in \{3,6\}$. Thus, $L_9(\sigma) = 1^5 2^2 3^3 = 36$.

We have the following proposition.

Proposition 1.

$$P_n^{\text{park}}(\sigma) = \frac{L_n(\sigma)}{(n+1)^{n-1}},\ \sigma \in S_n. \tag{1.3}$$

The following corollary of Proposition 1 is immediate, where the asymptotic behavior follows from Stirling's formula.

Corollary 1. *The expected value of the random variable $L_n = L_n(\sigma)$ on (S_n, P_n) satisfies*

$$E_n L_n = \frac{1}{n!}\sum_{\sigma \in S_n} L_n(\sigma) = \frac{(n+1)^{n-1}}{n!} \sim \frac{e^{n+1}}{\sqrt{2\pi}\, n^{\frac{3}{2}}}. \tag{1.4}$$

We will prove the following weak convergence result for L_n.

Theorem 1. *For any $\epsilon \in (0,2]$, the random variable $L_n = L_n(\sigma)$ on (S_n, P_n) satisfies*

$$\lim_{n\to\infty} P_n((2-\epsilon)^n \le L_n \le (2+\epsilon)^n)) = 1 \tag{1.5}$$

The following corollary follows immediately from Proposition 1 and Theorem 1.

Corollary 2. *For any $\epsilon \in (0,2]$, the random variable $P_n^{\text{park}} = P_n^{\text{park}}(\sigma)$ on (S_n, P_n) satisfies*

$$\lim_{n\to\infty} P_n\left(\left\{\sigma \in S_n : P_n^{\text{park}}(\sigma) \in \left(\frac{(2-\epsilon)^n}{(n+1)^{n-1}}, \frac{(2+\epsilon)^n}{(n+1)^{n-1}}\right)\right\}\right) = 1. \tag{1.6}$$

And the following corollary follows immediately from Corollary 1 and Theorem 1.

Corollary 3. *The expectation of the random variable $P_n^{\text{park}} = P_n^{\text{park}}(\sigma)$ on (S_n, P_n) is given by*

$$E_n P_n^{\text{park}} = \frac{1}{n!} \sim \frac{e^n}{\sqrt{2\pi} n^{n+\frac{1}{2}}}. \tag{1.7}$$

Comparing (1.6) and (1.7), we see that for all but a P_n-negligible set of permutations in S_n, the P_n^{park}-probability of a permutation in S_n is approximately $\frac{2^n}{(n+1)^{n-1}}$, but the "average" P_n^{park}-probability of a permutation in S_n is exponentially larger, namely asymptotic to $\frac{e^n}{\sqrt{2\pi} n^{n+\frac{1}{2}}}$. There is also a P_n-negligible set of permutations in S_n each of whose elements has super-exponentially larger P_n^{park}-probability than this average (and this is where almost all the P_n^{park}-probability lies), and a P_n-negligible set of permutations in S_n for which the P_n^{park}-probability is exponentially smaller than this average probability. In particular, we have the following corollary.

Corollary 4. *The maximum value of $P_n^{\text{park}} = P_n^{\text{park}}(\sigma)$ is equal to $\frac{n!}{(n+1)^{n-1}} \sim \frac{\sqrt{2\pi}\, n^{\frac{3}{2}}}{e^{n+1}}$ and is attained uniquely at $\sigma = 1 \cdots n$. The minimum value of P_n^{park} is equal to $\frac{1}{(n+1)^{n-1}}$ and is attained uniquely at $\sigma = n \cdots 1$.*

Proof. The function $L_n = L_n(\sigma), \sigma \in S_n$, attains its minimum value 1 uniquely at $\sigma = n \cdots 1$ and attains its maximum value $n!$ uniquely at $\sigma = 1 \cdots n$. $\square$

From the definition of a parking function, it is obvious that

$$P_n^{\text{park}}(\sigma_j = 1) = P^{\mathcal{P}_n}(\pi_1 = j),\ j \in [n].$$

In [1], the following asymptotic behavior was proven for π_1 (or any π_k by symmetry):

(1.8)

For fixed $j, P^{\mathcal{P}_n}(\pi_1 = j) \sim \dfrac{1 + P(X \ge j)}{n}$;

For fixed $j, P^{\mathcal{P}_n}(\pi = n - j) \sim \dfrac{P(X \le j+1)}{n}$,

where X is a random variable satisfying $P(X = j) = e^{-j}\dfrac{j^{j-1}}{j!},\ j = 1, 2, \cdots$.

Thus, it follows that (1.8) also holds with $P^{\mathcal{P}_n}(\pi_1 = j)$ replaced by $P_n^{\text{park}}(\sigma_j = 1)$. It would be nice to obtain similar type asymptotics for $P_n^{\text{park}}(\sigma_j = k)$, for general j, k. It doesn't seem that our results in this paper can help here. Nor do they seem to be useful for obtaining information on the distributions of certain classical permutation statistics under P_n^{park}, such as the number of inversions, the number of cycles or the number of descents.

We will also prove the following results.

Proposition 2. *For any $m \in [n]$,*

$$P_n^{park}(\sigma_1 \cdots \sigma_m = [m]) = \left(\frac{m+1}{n+1}\right)^m. \tag{1.9}$$

We have the following immediately corollary.

Corollary 5. *For any $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} P_n^{park}(\sigma_1 \cdots \sigma_{n-m} = [n-m]) = e^{-m} \tag{1.10}$$

In fact, we also have the following result.

Proposition 3. *For any $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} P_n^{park}(\sigma_{n-m+1} \cdots \sigma_n = n - m + 1 \cdots n) = e^{-m}. \tag{1.11}$$

Remark. Note that (1.10) and (1.11) give

$$\lim_{n\to\infty} P_n^{\text{park}}(\sigma_{n-m+1} \cdots \sigma_n = n - m + 1 \cdots n | \sigma_1 \cdots \sigma_{n-m} = [n-m]) = 1.$$

The proof of Proposition 1 is given in section 2, the proof of Theorem 1 is given in section 3 and the proofs of Propositions 2 and 3 are given in section 4.

2. Proof of Proposition 1

Recall the definition of $l_{n,i}$ from the paragraph containing equation (1.2) which defines L_n. For the proof of the proposition, it will be convenient to define $\tilde{l}_{n,i}(\sigma) = l_{n,\sigma_i^{-1}}(\sigma)$. For example, if $\sigma = 379218645$, then $\tilde{l}_{n,i}(\sigma) = 1$, for $i \in \{1,2,3,4,6\}$, $\tilde{l}_{n,i} = 2$, for $i \in \{5,7\}$ and $\tilde{l}_{n,i} = 3$, for $i \in \{8,9\}$. Of course, we can express L in terms of the $\{\tilde{l}_{n,i}\}$:

$$L_n(\sigma) = \prod_{i=1}^{n} l_{n,i}(\sigma) = \prod_{i=1}^{n} \tilde{l}_{n,i}(\sigma).$$

The proposition will follow if we show that for each $\sigma \in S_n$, there are $L_n(\sigma)$ different parking functions $\pi \in PF_n$ such that $T_n(\pi) = \sigma$, where T_n is as in the paragraph containing equation (1.2). Before giving a formal proof of the proposition, we illustrate the proof with a concrete example, from which the general result should be clear. Consider the permutation $\sigma = 379218645 \in S_9$. We look for those $\pi \in PF_9$ that satisfy $T_9(\pi) = \sigma$. From the definition of the parking process and from the definition of T_n, we need $\pi_1 = 5$ in order to have $\sigma_5 = 1$, $\pi_2 = 4$ in order to have $\sigma_4 = 2$, $\pi_3 = 1$ in order to have $\sigma_1 = 3$ and $\pi_4 = 8$ in order to have $\sigma_8 = 4$. In order to have $\sigma_9 = 5$, we can either have $\pi_5 = 9$, in which case car number 5 parks in its preferred space 9, or alternatively, $\pi_5 = 8$, in which case car number 5 attempts to park in its preferred space 8 but fails, and then moves on to space 9 and parks. Then we need $\pi_6 = 7$ in order to have $\sigma_7 = 6$. Then similar to the explanation regarding π_5, we need π_7 to be either 1 or 2 in order to have $\sigma_2 = 7$. In order to have $\sigma_6 = 8$, we can have either $\pi_8 = 6$, in which car number 8 parks directly in its preferred space 6, or alternatively $\pi_8 = 5$, in which case car number 8 tries and fails to park in space number 5 and then parks in space number 6, or alternatively, $\pi_8 = 4$, it which case car number 8 tries and fails to park in space number 4 and then also in space number 5, before finally parking in space number 6. Similarly, we need π_9 to be equal to 1,2 or 3 in order to have $\sigma_9 = 3$. Thus, there are

$1\times1\times1\times1\times2\times1\times2\times3\times3=\prod_{i=1}^{9}\tilde l_{9,i}(\sigma)$ different parking functions $\pi\in PF_9$ that yield $T_9(\pi)=\sigma$.

To give a formal proof for the general case, fix $\sigma\in S_n$. In order to have $T_n(\pi)=\sigma$, first we need $\pi_1=\sigma_1^{-1}$. Thus there is just one choice for π_1, and note that $\tilde l_{n,1}(\sigma)=1$. Now let $k\in[n-1]$ and assume that we have chosen $\pi_1,\cdots\pi_k$ in such a way that car number i has parked in space σ_i^{-1}, for $i\in[k]$. We now want car number $k+1$ to park in space σ_{k+1}^{-1}. By construction, this space is vacant at this point, and so are the $\tilde l_{n,k}(\sigma)-1$ spaces immediately to the left of this space. However the space $\tilde l_{n,k}$ spaces to the left of this space is not vacant (or possibly this space doesn't exist–it would be the zeroth space). Thus, by the parking process, car number $k+1$ will park in space σ_{k+1}^{-1} if and only if π_{k+1} is equal to one of the $\tilde l_{n,k+1}(\sigma)$ numbers $\sigma_{k+1}^{-1},\sigma_{k+1}^{-1}-1,\cdots,\sigma_{k+1}^{-1}-\tilde l_{n,k+1}(\sigma)+1$. This shows that there are $L_n(\sigma)=\prod_{i=1}^{n}\tilde l_{n,i}(\sigma)$ different parking functions π satisfying $T_n(\pi)=\sigma$. $\square$

3. Proof of Theorem 1

We begin with several preliminary results. Recall that P_n is the uniform probability measure on S_n.

Lemma 1.

$$P_n(l_{n,i}=j)=\begin{cases}\frac{1}{j}-\frac{1}{j+1}=\frac{1}{j(j+1)},\ j=1,\cdots,i-1;\\ \frac{1}{i},\ j=i.\end{cases}. \tag{3.1}$$

Proof. Fix i and let $j\in[i]$. The event $\{l_{n,i}(\sigma)\ge j\}$ is the event $\big\{\sigma_i=\max\{\sigma_i,\sigma_{i-1},\cdots,\sigma_{i-j+1}\}\big\}$. Since P_n is the uniform distribution on S_n, we have

$$P_n(l_{n,i}\ge j)=\frac{1}{j},\ i\in[n],\ 1\le j\le i. \tag{3.2}$$

The lemma now follows. $\square$

We now write

$$\mathcal{S}_n:=\log L_n=\sum_{i=1}^{n}\log l_{n,i}. \tag{3.3}$$

From Lemma 1, we have

$$E_n\log l_{n,i}=\sum_{j=1}^{i-1}\frac{\log j}{j(j+1)}+\frac{\log i}{i}. \tag{3.4}$$

Note that $E_n \log l_{n,i}$ does not depend on n, but of course it is only defined for $1 \le i \le n$.

Lemma 2.

$$\lim_{n,i\to\infty} E_n \log l_{n,i} = \log 2. \tag{3.5}$$

Proof. Recall the Abel-type summation formula [4]:

$$\sum_{1<r\le x} a(r)f(r) = A(x)f(x) - A(1)f(1) - \int_1^x A(t)f'(t)dt, \text{ where } A(r) = \sum_{i=1}^r a_i.$$

We apply this formula with $a(r) = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$ and $f(r) = \log r$. We have $A(r) = 1 - \frac{1}{r+1} = \frac{r}{r+1}$. Recalling (3.4), we obtain

$$\lim_{n,i\to\infty} E_n \log l_{n,i} = \lim_{i\to\infty} \sum_{j=1}^{i-1} \frac{\log j}{j(j+1)} = \lim_{i\to\infty} \left(\frac{i}{i+1} \log i - \int_1^i \frac{t}{t+1}\frac{1}{t} dt \right) =$$

$$\lim_{i\to\infty} \left(\frac{i}{i+1} \log i - \log(i+1) + \log 2 \right) = \lim_{i\to\infty} \left(\log \frac{i}{i+1} - \frac{\log i}{i+1} + \log 2 \right) = \log 2.$$

□

From (3.3) and (3.5), we conclude that

$$\lim_{n\to\infty} \frac{E_n \mathcal{S}_n}{n} = \lim_{n\to\infty} \frac{1}{n} E_n \log L_n = \log 2. \tag{3.6}$$

We now consider $E_n \mathcal{S}_n^2$. We have

$$E_n \mathcal{S}_n^2 = E_n \left(\sum_{i=1}^n \log l_{n,i} \right)^2 = \sum_{i=1}^n E_n \log l_{n,i}^2 + 2 \sum_{1\le i<j\le n} E_n \log l_{n,i} \log l_{n,j}. \tag{3.7}$$

We have the following proposition.

Proposition 4. *For $1 \le i < j \le n$, the random variables $l_{n,i}$ and $l_{n,j}$ on (S_n, P_n) are negatively correlated; that is,*

$$P_n(l_{n,i} \ge k, l_{n,j} \ge k) \le P_n(l_{n,i} \ge k) P_n(l_{n,j} \ge l), \text{ for } k, l \ge 1. \tag{3.8}$$

Proof. Since P_n is the uniform probability measure on S_n, for any $k \le i$, the events $\{l_{n,i} \ge k\} = \{\sigma_i = \max(\sigma_i, \cdots, \sigma_{i-k+1})\}$ and $\{l_{n,j} \ge l\} = \{\sigma_j = \max(\sigma_j, \cdots, \sigma_{j-l+1})\}$ are independent if $l \le j - i$. Thus, (3.8) holds with equality in these cases.

Consider now the case $k \le i$ and $j-i+1 \le l \le j$. In this case

(3.9)
$$\{l_{n,i} \ge k, l_{n,j} \ge l\} = \big\{\sigma_j = \max(\sigma_j, \sigma_{j-1}, \cdots, \sigma_r)\big\} \cap \big\{\sigma_i = \max(\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-k+1})\big\},$$

where $r = \min(i-k+1, j-l+1)$.

We have

$$P_n\left(\sigma_j = \max(\sigma_j, \sigma_{j-1}, \cdots, \sigma_r)\right) = \frac{1}{\max(l, j-i+k)} \le \frac{1}{l}. \tag{3.10}$$

Also,

(3.11)
$$P_n\left(\sigma_i = \max(\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-k+1} | \sigma_j = \max(\sigma_j, \sigma_{j-1}, \cdots, \sigma_r)\right) = \frac{1}{k}.$$

The proposition follows from (3.9)–(3.11) and (3.2). □

We can now prove the theorem.

Proof of Theorem 1. Since $l_{n,i}$ and $l_{n,j}$ are negatively correlated, one has $E_n f(l_{n,i}) g(l_{n,j}) \le E_n f(l_{n,i}) E_n g(l_{n,j})$, if f and g are increasing functions on $[n]$. In particular then,

$$E_n \log l_{n,i} \log l_{n,j} \le E_n \log l_{n,i} E_n \log l_{n,j}. \tag{3.12}$$

Using (3.12), a standard straightforward calculation gives

$$\mathrm{Var}(\mathcal{S}_n) \le \sum_{i=1}^{n} \mathrm{Var}(\log l_{n,i}). \tag{3.13}$$

From (3.1), we have

$$E_n(\log l_{n,i})^2 = \sum_{j=1}^{i-1} \frac{(\log j)^2}{j(j+1)} + \frac{(\log i)^2}{i(i+1)}.$$

Using this with (3.5) and (3.13), we conclude that there exists a $C > 0$ such that

$$\mathrm{Var}(\mathcal{S}_n) \le Cn, \ n \in \mathbb{N}. \tag{3.14}$$

From (3.6) and(3.14), it follows from the second moment method that

$$\lim_{n\to\infty} P_n(\log 2 - \epsilon \le \frac{\mathcal{S}_n}{n} \le \log 2 + \epsilon) = 1, \text{ for all } \epsilon > 0. \tag{3.15}$$

Now (1.5) follows from (3.15) and (3.3). □

4. Proofs of Propositions 2 and 3

Proof of Proposition 2. Let $\sigma = T_n(\pi)$, where T_n is as in the paragraph containing equation (1.2). Then $\sigma_1 \cdots \sigma_m = [m]$ if and only if $\pi_1 \cdots \pi_m \in \mathcal{P}_m$. Thus, there are $(m+1)^{m-1}$ choices for $\pi_1 \cdots \pi_m$. Given $\pi_1 \cdots \pi_m \in \mathcal{P}_m$, we now consider how many sequences $\pi_{m+1} \cdots \pi_n$ there are so that the concatenated sequence $\pi_1 \cdots \pi_m \pi_{m+1} \cdots \pi_n$ belongs to $\mathcal{P}_n$. Of course, we start with the restriction $1 \le \pi_j \le n$, for all $j \in \{m+1, \cdots n\}$. It is easy to see that such a sequence $\pi_{m+1} \cdots \pi_n$ will be such that the above concatenated sequence belongs to $\mathcal{P}_n$ if and only if this sequence results in all $n-m$ cars being able to park in the following scenario: There is a one-way street with n spaces, but with the first m of them already taken up by a trailer. A sequence of $n-m$ cars enters, each with a preferred parking space between 1 and n. It is known that the number of such sequences resulting in all $n-m$ cars successfully parking is equal to $(m+1)(n+1)^{n-m-1}$ [2]. Thus, the number of parking functions $\pi \in \mathcal{P}_n$ such that $\sigma = T_n(\pi)$ satisfies $\sigma_1 \cdots \sigma_m = [m]$ is equal to $(m+1)^{m-1}(m+1)(n+1)^{n-m-1}$. Consequently,

$$P_n^{\text{park}}(\sigma_1 \cdots \sigma_m = [m]) = \frac{(m+1)^{m-1}(m+1)(n+1)^{n-m-1}}{(n+1)^{n-1}} = \left(\frac{m+1}{n+1}\right)^m.$$

□

Proof of Proposition 3. Let $\sigma = T_n(\pi)$, where T_n is as in the paragraph containing equation (1.2). In order to have $\sigma_{n-m+1} \cdots \sigma_n = n-m+1 \cdots n$, it is of course necessary to have $\sigma_1 \cdots \sigma_{n-m} = [n-m]$. As in the proof of Proposition 2, but with m replaced by $n-m$, the number of sequences $\pi_1 \cdots \pi_{n-m}$ such that $\sigma_1 \cdots \sigma_{n-m} = [n-m]$, is $(n-m+1)^{n-m-1}$, and for each such $\pi_1 \cdots \pi_{n-m}$, the number of sequences $\pi_{n-m+1} \cdots \pi_n$ such that the concatenation $\pi_1 \cdots \pi_{n-m} \pi_{n-m+1} \cdots \pi_n$ belongs to $\mathcal{P}_n$ is equal to $(n-m+1)(n+1)^{m-1}$. It is easy to see from the definition of the parking process that a sequence $\pi_{n-m+1} \cdots \pi_n$ from among these $(n-m+1)(n+1)^{m-1}$ sequences will be such that for the concatenation $\pi_1 \cdots \pi_{n-m} \pi_{n-m+1} \cdots \pi_n$, one has $\sigma_{n-m+1} \cdots \sigma_n = n-m+1 \cdots n$ if and only if $\pi_{n-m+i} \le n-m+i$, for $i \in [m]$. There are $\prod_{i=1}^m (n-m+i) = \frac{n!}{(n-m)!}$ such sequences. From this it follows that

$$P_n^{\text{park}}\left(\sigma_{n-m+1}\cdots\sigma_n = n-m+1\cdots n|\sigma_1\cdots\sigma_{n-m} = [n-m]\right) = \frac{\frac{n!}{(n-m)!}}{(n-m+1)(n+1)^{m-1}},$$

and consequently,

(4.1)

$$\lim_{n\to\infty} P_n^{\text{park}}\left(\sigma_{n-m+1}\cdots\sigma_n = n-m+1\cdots n|\sigma_1\cdots\sigma_{n-m} = [n-m]\right) = 1.$$

Now (1.11) follows from (4.1) and (1.10). □

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