Steiner trees with infinitely many terminals on the sides of an angle

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Abstract

The Euclidean Steiner problem is the problem of finding a set $\mathcal{S}t$, with the shortest length, such that $\mathcal{S}t \cup \mathcal{A}$ is connected, where \mathcal{A} is a given set in a Euclidean space. The solutions $\mathcal{S}t$ to the Steiner problem will be called *Steiner sets* while the set \mathcal{A} will be called *input*. Since every Steiner set is acyclic we call it Steiner tree in the case when it is connected. We say that a Steiner tree is *indecomposable* if it does not contain any Steiner tree for a subset of the input.

We are interested in finding the Steiner set when the input consists of infinitely many points distributed on two lines. In particular we would like to find a configuration which gives an indecomposable Steiner tree.

It is natural to consider a self-similar input, namely the set $\mathcal{A}_{\alpha,\lambda}$ of points with coordinates $(\lambda^{k-1}\cos\alpha, \pm \lambda^{k-1}\sin\alpha)$, where $\lambda > 0$ and $\alpha > 0$ are small fixed values. These points are distributed on the two sides of an angle of size 2α in such a way that the distances from the points to the vertex of the angle are in a geometric progression.

To our surprise, we show that in this case the solutions to the Steiner problem for $\mathcal{A}_{\alpha,\lambda}$, when α and λ are small enough, are always decomposable trees. More precisely, any Steiner tree for $\mathcal{A}_{\alpha,\lambda}$ is a countable union of Steiner trees, each one connecting 5 points from the input. Each component of the decomposition can be mirrored with respect to the angle bisector providing $2^{\mathbb{N}}$ different solutions with the same length. By considering only a finite number of components we obtain many solutions to the Steiner problem for finite sets composed of 4k+1 points distributed on the two lines (2k+1) on a line and 2k on the other line. These solutions are very similar to the *ladders* of Chung and Graham.

We are able to obtain an indecomposable Steiner tree by adding, to the previous input, a single point strategically placed inside the angle. In this case the solution is in fact a self-similar tree (in the sense that it contains a homothetic copy of itself).

Finally, we show how the position of the Steiner points in the Steiner tree can be described by a discrete dynamical system which turns out to be equivalent to a 2-interval piecewise linear contraction.

1 Introduction

The finite Euclidean Steiner problem is the problem of finding a one-dimensional connected set St of minimal length containing a finite set of given points $A \subset \mathbb{R}^d$.

The history of the finite Euclidean Steiner problem is studied in the paper [2]. Brazil, Graham, Thomas and Zachariasen did a detailed research and discovered that the statement and basic results about the Steiner Problem were rediscovered (at least) three times: it was first stated by Gergonne in 1811, by Gauss in 1836 and by Jarník and Kössler [14] in 1934. The problem has become well-known as the "Steiner problem" after the great success of the book "What is Mathematics?" by Courant and Robbins [8].

In 1980-s and 1990-s, explicit solutions to the finite Steiner problem attracted the attention of several notable mathematicians. It is worth noting that Du, Hwang and Weng [9] completely solved the Steiner problem when \mathcal{A} is the set of vertices of a regular polygon. Rubinstein and Thomas [24] generalizes the result when the points of \mathcal{A} are uniformly enough distributed along a circle.

A setting which is similar to ours, can be found in the paper by Burkard and Dudás [3], who determined the Steiner trees for the vertex of a plane angle of size at least $\pi/6$ and all points in the sides at a distance $1, \ldots, n$. It turns out that apart from the points which are close to the vertex, the Steiner tree is composed of segments of length 1 joining the points on the sides of the angle. The results of our paper are also very similar to the results about ladders of Chung and Graham [7] and Burkard, Dudás and Maier [4]. A ladder is a collection of 2n lattice points placed on the vertices of n-1 adjacent unit squares, forming a rectangle of sides $1 \times (n-1)$. It turns out that when n is odd the Steiner tree is full, which means that terminal points are reached by a single edge of the tree. While if n is odd the Steiner tree is not full. By splitting the tree in the terminals of order two

one can decompose the Steiner tree into the union of small Steiner trees, each one containing only two or four terminal points. Notice that this is a case (relevant to our quest) where a full Steiner tree with an arbitrarily large number of points is presented. However, as the number of points is going to infinity also the length of the Steiner tree is going to infinity, hence this example is not suitable to construct a tree with infinitely many vertices.

Finding the Steiner tree for a finite set of points is a well-known problem in computational geometry. Garey, Graham and Johnson [11] proved that the Steiner problem is NP-hard. Rubinstein, Thomas and Wormald [25] proved that such a complexity persists even when the terminals are constrained to lay along two parallel lines and also in the case when the terminals are placed on the sides of an angle with size which is smaller than $2\pi/3$ (see Theorem 3 in [25]).

More recently, a generalized setting for the Steiner problem was given by Paolini and Stepanov [19]: the ambient space X can be any connected complete metric space with the Heine–Borel property (closed bounded sets are compact) and the given set of points can be any compact subset of the ambient space. In this setting there always exists a set St, with minimal 1-dimensional Hausdorff measure \mathcal{H}^1 , such that $St \cup A$ is connected. Such St is proven to be acyclic. If St itself is connected we call it $Steiner\ tree\ for\ A$. The set St is called an St in the Steiner problem. See Section 2.3 for precise definition and properties of Steiner trees.

Following the paper [19], the question of finding non trivial examples of Steiner trees for infinitely many points was raised. Of course, it is easy to find solutions with infinitely many points by splitting a finite tree into infinitely many pieces. For example given any compact set of points on the real line, $A \subset \mathbb{R}$, the solution to the Steiner problem with input A is clearly given by the set $St = [\min A, \max A] \setminus A$. If A is compact the solution St is composed of a countable number of disjoint open segments each with end points on ∂A . Much more difficult is to find an infinite Steiner tree which is *indecomposable* in the sense that it is not possible to split the tree St into two connected pieces St_1 and St_2 so that both St_1 and St_2 are Steiner trees over a subset of the set A of terminal points.

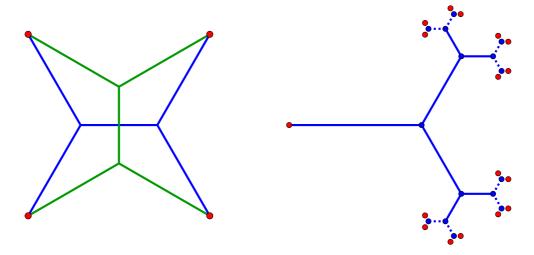


Figure 1: The left part contains two Steiner trees connecting the vertices of a square; the right part illustrates an example of a self-similar solution $\Sigma(\lambda)$.

The first example of an infinite, indecomposable Steiner tree was given in [21]. However it was not self-similar and its input set \mathcal{A} had zero Hausdorff dimension. The next natural question was to understand if the same result was true for the set of leaves of a self-similar binary tree $\Sigma(\lambda)$ composed of a segment of length 1 which splits into two segments of length λ^2 and so on... The splitting points are regular tripods meaning that there are three segments joining in the point forming equal angles of size $2\pi/3$ (see Figure 1, right-hand side picture). The input set $\mathcal{A}(\lambda)$ is composed of a root point (the first end-point of the unit segment) and all the accumulation points of the smaller and smaller segments. Note that, if $\lambda < \frac{1}{2}$ the set $\mathcal{A}(\lambda)$ turns out to be a self-similar, fractal set, with Hausdorff dimension $-\frac{\ln 2}{\ln \lambda} > 0$. Recently, in [6, 20], $\Sigma(\lambda)$ was proven to be the unique Steiner tree for the set $\mathcal{A}(\lambda)$ if λ is small enough.

Clearly every connected subset S of $\Sigma(\lambda)$ is the unique Steiner tree for the set of its endpoints and corner points (otherwise one can replace S with a better competitor in $\Sigma(\lambda)$).

So $\Sigma(\lambda)$ can be regarded as a universal graph for Steiner trees in the sense that it contains a Steiner subtree with any given possible finite combinatorics. This fact was recently used in [1] to show that the set of n-point configurations in \mathbb{R}^d with a unique Steiner tree is path-connected (as a subset of \mathbb{R}^{dn}).

In this paper we are interested in finding an explicit example of a Steiner tree connecting a countable number of points distributed along two lines forming an angle in the plane. The points will be equally distributed on the two sides of the angle in such a way that the distances of the points from the vertex of the angle are in a geometric progression.

A configuration like this was suggested in the Master's Thesis of Letizia Ulivi, where an existence theorem for Steiner trees was proven in the case when the input set was considered to be compact and countable, see [22].

Examples of this kind are interesting because the set of terminals is distributed on the boundary of a convex set. In this setting the Steiner tree is splitting the convex set into many *regions* and hence determines a *partition*. The Steiner Problem becomes a problem of *minimal partitions* and can hence be studied in the framework of geometric measure theory by using calibrations (see [18, 16, 5]).

To our surprise we have found that, under suitable assumptions (Theorem 1), the Steiner tree for such a set of points is decomposable into rescaled copies of a Steiner tree connecting only 5 points. However by adding a pair of terminal points in a precise position, outside the geometric progression, we are able to force the Steiner tree to be indecomposable, see Remark 1.

The main difficulty in the proof is the fact that the set of locally minimal trees may have a very complicated structure (see Remark 3) and our methods heavily use the global optimality of a solution.

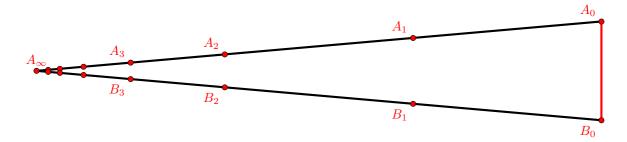


Figure 2: The input sets $\mathcal{A}_0 = \{A_\infty, A_1, B_1, A_2, B_2, \dots\}$, $\mathcal{A}_0 = \mathcal{A}_1 \cup [A_0 B_0]$. Here $\alpha = \frac{\pi}{36}$ and $\lambda = \frac{1}{2}$.

Our setup. Consider a positive $\lambda \leq \frac{1}{2}$ and an angle of size $2\alpha < \frac{\pi}{3}$ in the plane. Let A_{∞} be the vertex of the angle, and A_1, B_1 be points on distinct sides of the angle at a unit distance from A_{∞} (so that the middle point of the segment $[A_1B_1]$ has distance $\cos \alpha$ from A_{∞}). For $k \geq 1$, define $A_{k+1} \in [A_kA_{\infty}]$ and $B_{k+1} \in [B_kA_{\infty}]$ recursively by $|A_{k+1}A_{\infty}| = \lambda \cdot |A_kA_{\infty}|$ and $|B_{k+1}A_{\infty}| = \lambda \cdot |B_kA_{\infty}|$ (see Figure 2). Define A_0 , B_0 on the rays $[A_{\infty}A_1)$ and $[A_{\infty}B_1)$, respectively, in such a way that

$$|A_0 A_\infty| = |B_0 A_\infty| = \frac{1}{\lambda} - \frac{\tan(\alpha)}{\sqrt{3}\lambda}.$$

Let

$$\mathcal{A}_1 = \mathcal{A}_1(\alpha, \lambda) := \{A_\infty\} \cup \bigcup_{n=1}^{\infty} \{A_n, B_n\}$$

and

$$\mathcal{A}_0 = \mathcal{A}_0(\alpha, \lambda) := \mathcal{A}_1(\alpha, \lambda) \cup [A_0 B_0].$$

The following theorem is the main result of the paper.

Theorem 1. Suppose $0 < \alpha < \frac{\pi}{6}$ and $0 < \lambda \leq \frac{1}{2}$ are such that

$$\sqrt{\lambda} < \frac{\cos\left(\frac{\pi}{3} + \alpha\right)}{\cos\left(\frac{\pi}{3} - \alpha\right)}.\tag{1}$$

Then every solution St_1 to the Steiner problem for A_1 is the union of full trees on 5 terminals (see Fig. 3). The length of a solution has the following explicit formula

$$\mathcal{H}^{1}(\mathcal{S}t_{1}) = \left|\cos\alpha + \sqrt{3}\sin\alpha + \frac{2\lambda}{1-\lambda^{2}}e^{\frac{\pi i}{6}}\sin\alpha + \frac{2\lambda^{2}}{1-\lambda^{2}}e^{-\frac{\pi i}{6}}\sin\alpha\right|.$$

Note that every Steiner tree for \mathcal{A}_1 is not full and so can be decomposed into solutions for 5-terminals subset of \mathcal{A}_1 . Each of the solutions on 5 points can be mirrored with respect to the angle bisector, so we have $2^{\mathbb{N}}$ different solutions, all with the same length. Since every subtree of a Steiner tree is itself a Steiner tree with respect to its terminal points, we can state the following corollary, which holds in the framework of finite Steiner trees.

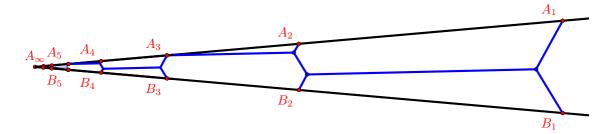


Figure 3: One of the solutions to the Steiner Problem for \mathcal{A}_1 . See Theorem 1. Here $\alpha = \frac{\pi}{36}$ and $\lambda = \frac{1}{2}$.

Corollary 1. Suppose α and λ are given with the assumptions of Theorem 1. Then the Steiner problem for the finite set $\{A_1, \ldots, A_{2k+1}, B_1, \ldots, B_{2k}\}$ has $2^k - 1$ different solutions each of which is the union of rescaled and mirrored copies of the Steiner tree on 5 terminals: $\{A_1, A_2, A_3, B_1, B_2\}$. The length also has an explicit formula.

We would like to slightly modify the input set A_1 to obtain an indecomposable solution. It turns out that a simple way is to consider the input A_0 , which is obtained from A_1 with the additional segment $[A_0B_0]$ (see definition above). We can then replace the segment with a single point x placed in a precise position on the segment $[A_0B_0]$ or with two points x_1, x_2 placed on the two sides of the angle in such a way that the Steiner tree has a triple point in the point x.

Theorem 2. Suppose that $\alpha < \frac{\pi}{6}$ and $\lambda \leq 1/2$ satisfy (1). Then the following statements hold true.

- (i) The Steiner problem for \mathcal{A}_0 has exactly 2 solutions $\mathcal{S}t_0^1$ and $\mathcal{S}t_0^2$ which are trees. The tree $\mathcal{S}t_0^2$ is the reflection of $\mathcal{S}t_0^1$ with respect to the angle bisector. Both solutions are indecomposable. The point $\{x\} = \mathcal{S}t_0^1 \cap [A_0B_0]$ (and the corresponding symmetric point $\mathcal{S}t_0^2 \cap [A_0B_0]$) has a distance $\frac{1}{\lambda+1}\sin\alpha$ from the midpoint of $[A_0B_0]$.
- (ii) The length of the solutions is

$$\mathcal{H}^{1}(\mathcal{S}t_{0}^{1}) = \mathcal{H}^{1}(\mathcal{S}t_{0}^{2}) = \frac{\cos\alpha}{\lambda} - \frac{\sin\alpha}{\sqrt{3}\lambda} + \frac{\sqrt{3}}{1-\lambda}\sin\alpha.$$

(iii) Each St_0^j , j = 1, 2, is a self-similar tree, in the sense that $f_2(St_0^j) \subset St_0^j$, where f_2 is the homothety with center A_{∞} and ratio λ^2 .

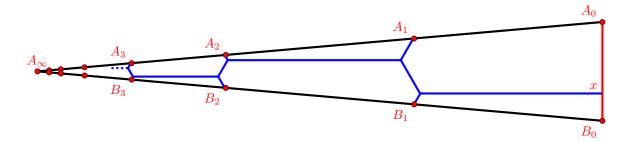


Figure 4: One of the two Steiner trees $\mathcal{S}t_0^1$, $\mathcal{S}t_0^2$ for the set \mathcal{A}_0 , see Theorem 2. The other solutions is the reflection with respect to the angle bisector. Here $\alpha = \frac{\pi}{36}$ and $\lambda = \frac{1}{2}$.

Remark 1. If we consider a Steiner tree St_0 for A_0 as stated in the previous theorem and insert a Steiner point in the vertex x with two additional edges $[xA'_0]$ and $[xB'_0]$ so that A'_0 and B'_0 are on the edges of the angle $[A_{\infty}, A_1)$ and $[A_{\infty}, B_1)$, respectively, we obtain a full Steiner tree for the set of terminals $A'_0 = A_1 \cup \{A'_0, B'_0\}$ placed on the sides of the triangle $\triangle A'_0 A_{\infty} B'_0$.

Structure of the paper. Section 2 contains the notation and some general results we will use in the proofs. These results are well-known for a finite input and we translate them to an infinite setup. For the sake of completeness, the proofs are collected in the Appendix. In Section 3 we describe the rough structure of a Steiner tree for both input sets A_0 and A_1 . The proofs of Theorem 1 and 2 are given in Sections 4 and 5, respectively. In Section 6 we present a formulation to find a "locally minimal" competitor by means of a dynamical system. Section 7 collects open questions. In the Appendix we have collected the proofs of well known results.

2 Notation and basics

2.1 Notation

If T is a subset of \mathbb{R}^d we denote by \overline{T} , ∂T and conv T, respectively the closure, the boundary and the convex hull of T. For $\rho > 0$ and $C \in \mathbb{R}^d$ we denote by $B_{\rho}(C)$ the open ball centered in C with radius ρ . If T is a set we denote by $B_{\rho}(T) = \bigcup_{C \in T} B_{\rho}(C)$ the open ρ -neighbourhood of T. We denote with $\mathcal{H}^1(T)$ the one-dimensional Hausdorff measure of the set T.

Given $B, C, D \in \mathbb{R}^d$ we denote by |BC| = |C - B| the Euclidean distance between B and C. Also, [BC] denotes the closed segment with endpoints at B and C, while |BC| denotes the open segment (without the endpoints). We denote with (BC) the line passing through B and C and with [BC) the half-line (ray) starting at B and passing through C. We denote with $\angle BCD$ the convex angle with vertex C and sides containing the points B and D and with $\triangle BCD$ the triangle with vertices B, C and D.

We let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive whole numbers.

2.2 Graphs and immersions

A graph Γ is a pair (V, E) where V is any set and E is a set such that every $e \in E$ is of the form $e = \{v, w\}$ with distinct $v, w \in V$. The elements of V are called *vertices* of Γ and the elements of E are called *edges*. If V is a vertex of Γ we define the order (or degree) of V as $\#\{e \in E : v \in e\}$ i.e. the number of edges joining in V.

If each vertex of Γ has finite order, we say that Γ is a locally-finite graph. If E and V are both finite, we say that Γ is a finite graph. A finite path γ is an alternating sequence of vertices and edges: $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that $e_k = \{v_{k-1}, v_k\}$. If $v_i \neq v_j$ for all (i, j) with i < j and $(i, j) \neq (0, n)$ we say that the path is simple. A (simple) path with $v_0 = v_n$ will be called a (simple) cycle. A path with no edges will be called trivial otherwise it will be called non-trivial. If for all pair of vertices $v, w \in V$ there is a finite path with first vertex v and last vertex v we say that the graph is connected. If there is no non-trivial cycle we say that the graph is a forest. A connected forest is called a tree.

Given $\varphi \colon V \to \mathbb{R}^d$ we say that the set

$$S_{\varphi} = \bigcup_{\{v,w\} \in E} [\varphi(v) \, \varphi(w)] \subset \mathbb{R}^d$$

is a geodesic immersion (or simply an immersion) of Γ in \mathbb{R}^d . The points $\varphi(v)$ are called immersed vertices while the segments $[\varphi(v)\varphi(w)]$ with $\{v,w\}\in E$ are called immersed edges. The length of an immersed graph is defined by

$$\ell(\varphi) = \sum_{\{v,w\} \in E} |\varphi(v)\,\varphi(w)|.$$

Clearly one has $\ell(\varphi) \geq \mathcal{H}^1(S_{\varphi})$.

If the graph is finite, the map ϕ is injective and the segments of the immersion have pairwise disjoint interior (i.e. the intersection of $]\varphi(v)\varphi(w)[$ and $]\varphi(v')\varphi(w')[$ is empty whenever $\{v,w\}\neq\{v',w'\}$) we say that the immersion is an embedding.

In the following we will often refer to immersed graphs, immersed vertices and immersed edges simply as graphs, vertices and edges.

We will say that an immersed graph S_{φ} is $full^*$ when the abstract graph is connected and all the convex angles between any two edges adjacent to the same vertex, are equal to $2\pi/3$. In particular in a $full^*$ tree all vertices have order at most three. The wind-rose of an immersed graph is the union of all lines passing through the origin and parallel to any edge of the graph. Clearly, in the planar case d=2, the wind-rose of a $full^*$ immersed graph comprises at most three different lines (so the corresponding directed segments have at most six directions).

We will say that an immersed graph S_{φ} is full if it is full* and there are no vertices of order two.

2.3 General Steiner trees

If S is a subset of a topological space, we say that S is *connected* if it is not the disjoint union of two non-empty closed sets. We say that S is *path-connected* if given any two points of S there is a continuous curve joining them in S. We say that S contains no loop if no subset of S is homeomorphic to the circle \mathbb{S}^1 . We say that S is a topological tree if S is connected and contains no loop.

Let $\mathcal{A} \subset \mathbb{R}^d$ and $\mathcal{S}t \subset \mathbb{R}^d$. We say that $\mathcal{S}t$ is a *Steiner set* for \mathcal{A} if $\mathcal{S}t \cup \mathcal{A}$ is connected and $\mathcal{H}^1(\mathcal{S}t) \leq \mathcal{H}^1(\mathcal{S})$ whenever $\mathcal{S} \subset \mathbb{R}^d$ and $\mathcal{S} \cup \mathcal{A}$ is connected. We say that $\mathcal{S}t$ is a *Steiner tree* if $\mathcal{S}t$ is a Steiner set and is itself connected.

The following theorem collects some of the general results from [19].

Theorem 3. If $A \subset \mathbb{R}^d$ is a compact set then there exists a Steiner set St for A. Moreover if St is a Steiner set for A and $H^1(St) < +\infty$ then

- 1. $St \cup A$ is compact;
- 2. St \ A has at most countable many connected components, and each of them has positive length;
- 3. $\overline{\mathcal{S}t}$ contains no loop (homeomorphic image of \mathbb{S}^1);
- 4. the closure of every connected component of St is a topological tree with endpoints on A (in particular it has at most countable number of branching points) and has at most one endpoint on each connected component of A;
- 5. if A is finite then $\overline{St} = St \cup A$ is an embedding of a finite tree;
- 6. for almost every $\varepsilon > 0$ the set $St \setminus B_{\varepsilon}(A)$ is an embedding of a finite graph.

We need to strengthen the result of the previous theorem to the case when \mathcal{A} is countable. In that case it is useful to prove that the Steiner set $\mathcal{S}t$ is locally finite also in the neighborhood of the isolated points of \mathcal{A} . The proof of the following statements can be found in the Appendix.

Lemma 1. Let $x_0 \in \mathcal{A}$ be an isolated point of the compact set \mathcal{A} . Let $\mathcal{S}t$ be a Steiner set for \mathcal{A} with $\mathcal{H}^1(\mathcal{S}t) < +\infty$. Then there exists $\rho > 0$ such that $\overline{\mathcal{S}t} \cap B_{\rho}(x_0)$ is an embedding of a finite graph.

Corollary 2. Let $A \subset \mathbb{R}^d$ be a compact set. Let A' be the set of accumulation points of A and $A_0 = A \setminus A'$ be the set of isolated points of A. Let St be a Steiner set for A with $\mathcal{H}^1(St) < +\infty$. Then, for almost every $\rho > 0$ the set $St \setminus B_{\rho}(A')$ is an embedding of a finite graph.

The previous corollary allows us to use the terminology of finite Steiner trees for a general Steiner tree $\mathcal{S}t$ with input \mathcal{A} . We will say that a point $x \in \mathcal{S}t \setminus \mathcal{A}$ is a Steiner point if there is $\rho > 0$ such that $\mathcal{S}t \setminus B_{\rho}(\mathcal{A}')$ is a finite embedded graph and x is a vertex of this graph. We will say that a segment $[x, y] \subset \mathcal{S}t \setminus \mathcal{A}'$ is an edge of the Steiner tree $\mathcal{S}t$ if there exists $\rho > 0$ such that [x, y] is an embedded edge of the finite embedded graph $\mathcal{S}t \setminus B_{\rho}(\mathcal{A}')$.

Notice also that if St is a Steiner set for A then $St \subset \text{conv } A$. In fact we know that the projection $\pi \colon \mathbb{R}^d \to \text{conv } A$ is a 1-Lipschitz map, which implies that $\mathcal{H}^1(X) \leq \mathcal{H}^1(\pi(X))$ for every measurable set $X \subset \mathbb{R}^d$. and hence by letting $St' = \pi(St)$ we have that $St' \cup A = \pi(St \cup A)$ and hence $St' \cup A$ is connected and $\mathcal{H}^1(St') \leq \mathcal{H}^1(St)$. This implies that St' is itself a Steiner set for A. But then St' = St, otherwise one would have $\mathcal{H}^1(St') < \mathcal{H}^1(St)$ which is a contradiction.

2.4 Maxwell's length formula

In the following lemma we identify the Euclidean plane with the complex plane \mathbb{C} . The proof of the following Lemma can be found in the Appendix.

Lemma 2 (Maxwell-type formula). Let S_{φ} be a full*, finite, immersed tree and let $p_1, p_2, \ldots, p_n \in \mathbb{C}$ be the vertices of order less than three. Suppose n > 1. If p_k is vertex of order one we define c_k as the unit complex number representing the outer direction of the unique edge of p_k . If p_k is a point of degree two then the two edges define an angle of $\frac{2}{3}\pi$ and we let c_k be the unit complex number with the direction, away from p_k , of the third implied edge which would complete a regular tripod in p_k .

Then one can write the length of the tree as

$$\ell(\phi) = \sum_{k=1}^{n} \bar{c_k} p_k. \tag{2}$$

In particular the right hand side of (2) is a real number.

Remark 2. If $\sum p_k$ converges absolutely then we can pass to the limit in (2). Clearly when $\{p_1, p_2, \dots\} = A_1$ this sum converges absolutely if we choose A_{∞} to be the origin of the complex plane.

2.5 Melzak algorithm

The following lemma is an immediate corollary of Ptolemy's theorem (see the Appendix for the proof).

Lemma 3 (Melzak's reduction, [17]). Let St be a Steiner tree for $A \subset \mathbb{R}^2$. If St contains a branching point q which is adjacent to two isolated points of A, say p_1 and p_2 , and the circle passing through p_1, p_2 and q contains no other point of St apart from the two edges $[p_1q]$ and $[p_2q]$, then the length $\mathcal{H}^1(St)$ is equal to the length of the tree St' obtained from St by removing the points p_1 and p_2 and the edges $[p_1q]$ and $[p_2q]$ and prolonging the remaining edge arriving in q up to a new vertex p such that pp_1p_2 is an equilateral triangle and (among the two possibilities) p and q are on opposite sides of the line (p_1, p_2) .

In the following we will call p the Melzak point for p_1 and p_2 .

2.6 A convexity argument

Let G = (V, E) be an abstract graph and φ an immersion of G in \mathbb{R}^d with $\ell(\varphi) < +\infty$. If $V_0 \subset V$ is fixed, we say that the immersion φ is *locally minimal* with V_0 fixed if there exists $\delta > 0$ such that $\ell(\varphi) \leq \ell(\psi)$ for every map $\psi \colon V \to \mathbb{R}^d$ with $\psi(v) = \varphi(v)$ when $v \in V_0$ and $|\psi(v) \psi(v)| < \delta$ for all $v \in V$.

The following theorem shows that the directions of the edges of a locally minimal immersion are uniquely determined. Often this is enough to deduce the uniqueness of locally, and hence globally minimal immersion of a given graph.

Theorem 4. Let $\Gamma = (V, E)$ be a graph and $\varphi, \psi \colon V \to \mathbb{R}^d$ be two locally minimal immersions of G into \mathbb{R}^d with fixed $V_0 \subset V$. Suppose that $\ell(\varphi) < +\infty$. Then $\ell(\psi) = \ell(\varphi)$ and one has that the edges of φ are parallel to the corresponding edges of ψ . More precisely, given any edge $\{v, w\} \in E$ either it has zero length in one of the immersions $(\varphi(v) = \varphi(w) \text{ or } \psi(v) = \psi(w))$ or there exists t > 0 such that $\varphi(v) - \varphi(w) = t(\psi(v) - \psi(w))$.

3 Preliminary lemmas

Lemma 4. Let \mathcal{A} be a compact subset of the Euclidean plane and let \mathcal{W} be a closed geometrical angle of size at least $\frac{2}{3}\pi$ such that $\mathcal{A} \cap \mathcal{W} = \emptyset$. Let $\mathcal{S}t$ be a Steiner tree for \mathcal{A} with $\mathcal{H}^1(\mathcal{S}t) < \infty$. Then $\mathcal{S}t \cap \mathcal{W}$ is either a segment or an empty set.

Proof. First of all we show that there are no Steiner points in \mathcal{W} . This is because in a Steiner point at least one of the three directions is contained in the angle V and hence in that direction another Steiner point should be found (because there are no points of \mathcal{A} in V), which is also inside the angle and further away from the vertex. These Steiner points should have an accumulation point on A which is not possible since $A \cap \mathcal{W} = \emptyset$.

Since there are no Steiner points in W we can state that $St \cap W$ is composed of line segments with end points on ∂W . Assume that there are two such segments $[A_1B_1]$ and $[A_2B_2]$, where A_1, A_2 belong to the same side of W while B_1, B_2 are on the other side. Let W be the vertex of the angle. Without loss of generality we assume that $|A_1B_1| \leq |A_2B_2|$. Then both $|A_1A_2|$ and $|B_1B_2|$ are shorter than $|A_2B_2|$, because the longest side of the triangle A_2B_2W is $[A_2B_2]$ and $[A_1A_2], [B_1B_2]$ are subsets of $[A_2W], [B_2W]$, respectively.

Since St is connected there is a path in $St \setminus W$ which joins the two segments $[A_1B_1]$ and $[A_2B_2]$. Hence if we remove the segment $[A_2B_2]$ and replace it with $[A_1B_1]$ or $[A_2B_2]$ we are able to obtain a connected set which is shorter than the original tree St but with the same terminals. This would contradict the minimality of St and conclude the proof.

Lemma 5. Suppose that $\alpha < \frac{\pi}{6}$ and λ satisfy (1). Let $\mathcal{S}t$ be a Steiner tree for the set \mathcal{A}_0 or \mathcal{A}_1 . Then for all $k \in \mathbb{N}$ there exist two disjoint convex sets \mathcal{C}_k and \mathcal{D}_k such that $A_j, B_j \in \mathcal{C}_k$ for all $j \leq k$, $A_j, B_j \in \mathcal{D}_k$ for all j > k, and $\mathcal{S}t \setminus (\mathcal{C}_k \cup \mathcal{D}_k)$ is a line segment with one end point on $\partial \mathcal{C}_k$ and one end point on $\partial \mathcal{D}_k$.

Proof. Let us show that the assumptions of the lemma imply, for all $k \in \mathbb{N}$, the existence of points $W_1 \in [B_k B_{k+1}]$ and $W_2 \in [A_k A_{k+1}]$ such that the two angles $\angle A_k W_1 A_{k+1}$ and $\angle B_k W_2 B_{k+1}$ have size at least $2\pi/3$ (see Fig. 5). Let W_1 be the point in $[B_k, B_{k+1}]$ such that $\triangle A_k A_\infty W_1$ is similar to $\triangle W_1 A_\infty A_{k+1}$, which is equivalent to say that $|A_\infty A_k| \cdot |A_\infty A_{k+1}| = |A_\infty W_1|^2$. So $|A_\infty W_1| = \lambda^{k-1/2}$. Let $2\gamma := \angle A_k W_1 A_{k+1}$. By the sine rule for $\triangle A_\infty A_k W_1$

$$\frac{\sqrt{\lambda}}{\sin \angle A_{\infty} A_k W_1} = \frac{1}{\sin \angle A_{\infty} W_1 A_k}.$$

Notice that since $\angle A_{\infty}A_kW_1 = \angle A_{\infty}W_1A_{k+1}$ we have $\pi = 2\alpha + 2\gamma + 2\angle A_{\infty}A_kW_1$ hence $\angle A_{\infty}A_kW_1 = \frac{\pi}{2} - \alpha - \gamma$ and $\angle A_{\infty}W_1A_k = 2\gamma + \angle A_{\infty}W_1A_{k+1} = 2\gamma + \frac{\pi}{2} - \alpha - \gamma = \frac{\pi}{2} - \alpha + \gamma$. Thus

$$\sqrt{\lambda} = \frac{\cos(\alpha + \gamma)}{\cos(\alpha - \gamma)}.$$

The right-hand side is decreasing in γ since

$$\frac{d}{d\gamma}\frac{\cos(\alpha+\gamma)}{\cos(\alpha-\gamma)} = -\frac{\sin 2\alpha}{\cos^2(\alpha-\gamma)}.$$

Hence (1) implies $\gamma > \frac{\pi}{3}$ and $\angle A_k W_1 A_{k+1} > \frac{2\pi}{3}$. Define W_2 as the reflection of W_1 with respect to the angle bisector.

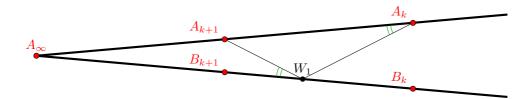


Figure 5: Definition of the point W_1 in Lemma 5

Now take W_1 and W_2 to be any angles of size $2\pi/3$ with vertices W_1 and W_2 contained in $\angle A_k W_1 A_{k+1}$ and $\angle B_k W_2 B_{k+1}$, respectively, so that W_1, W_2 have empty intersection with the terminal set. Lemma 4 assures that each of the two angles W_1 and W_2 contains a single line segment of St: denote this segments respectively by S_1 and S_2 . Since $St \subset \text{conv } \mathcal{A}$ and the segment $[W_1 W_2]$ splits $\text{conv } \mathcal{A}$ into two parts, both segments S_1 and S_2 must intersect $[W_1 W_2]$ and hence $S_1 \cup S_2$ must be itself a single line segment. So the statement is proven by taking \mathcal{C}_k and \mathcal{D}_k to be the connected components of $\mathbb{R}^2 \setminus (\mathcal{W}_1 \cup \mathcal{W}_2)$.

Lemma 6. Suppose that $\alpha < \frac{\pi}{6}$ and $\lambda \leq \frac{1}{2}$ satisfy (1).

- (i) Any full* Steiner tree St_1 for A_1 is uniquely determined by its wind-rose.
- (ii) Any Steiner set for A_0 is a tree.
- (iii) Any full* Steiner tree St_0 for A_0 is uniquely determined by $St_0 \cap [A_0B_0]$, which is always a point.
- (iv) Let $St = St_1$ or $St = St_0$. Then for every $k \in N$ there is a unique parallelogram \mathcal{P}_k whose sides are parallel to the wind-rose of St.
- (v) Every full finite component of a full* Steiner tree St_1 for A_1 contains an odd number of terminals.

Proof. Let us start with item (iv). In the case of the set $St = St_0$ notice that the edge of St_0 touching the segment $[A_0B_0]$ must be parallel to the bisector and hence cannot be used to construct a parallelogram \mathcal{P}_k . The parallelogram is hence uniquely defined by the other two directions of the wind rose. In the case of $St = St_1$ the proof is more involved. Let C_1 be the convex set given by Lemma 5 applied to St_1 with k = 1. The Lemma assures that A_1 and B_1 are the only terminals in C_1 and that $St_1 \cap \partial C_1$ is a point. Since $St_1 \subset \text{conv } A_1$ (see Section 2.3), points A_1 and B_1 have degree one in St_1 , so $St_1 \cap C_1$ is a tripod. Thus A_1 and B_1 are adjacent to the same Steiner point, let us call it S_1 . Consider the Melzak point C for A_1 and B_1 ; clearly C lies in the bisector. Put

$$\mathcal{A}' = \mathcal{A}_1 \cup \{C\} \setminus \{A_1, B_1\}.$$

By Lemma 3 applied with $p_1 = A_1$, $p_2 = B_1$ and q = C a full* tree St_1 corresponds to a full* tree St' for A' with the same wind-rose.

Since $\mathcal{S}t' \subset \operatorname{conv} \mathcal{A}'$, the segment of $\mathcal{S}t'$ incident to C belongs to the angle A_2CB_2 . Note that $\angle A_2CB_2 < 2\alpha = \angle A_2A_\infty B_2$ because $|A_\infty A_2| = |A_\infty B_2| < |A_2C| = |B_2C|$ since $\lambda \leq \frac{1}{2}$. Thus the wind-rose of $\mathcal{S}t'$ contains a direction defining an angle smaller than α with the bisector. This means that either A_k or B_k has only two directions from the wind-rose which go inside $\operatorname{conv} \mathcal{A}$ and thus \mathcal{P}_k is uniquely defined for $\mathcal{S}t_1$.

Denote the vertices of \mathcal{P}_k by $A_kV_kB_kU_k$ for $k \geq 1$, where $V_k \in \text{conv}\{A_{\infty}, A_k, B_k\}$. We claim that the parallelograms \mathcal{P}_k are disjoint. Call I the intersection of the segments $[A_1B_2]$ and $[A_2B_1]$. By symmetry I belongs to the bisector. Then (1) implies $\angle A_1IA_2 > \angle A_1W_1A_2 \geq 2\pi/3$, where W_1 is defined in the proof of Lemma 5. So $\angle A_1IB_1 = \angle A_2IB_2 = \pi - \angle A_1IA_2 < \pi/3$. Thus the domains $\{X \mid \angle A_1XB_1 \geq 2\pi/3\} \supset \mathcal{P}_1$ and $\{Y \mid \angle A_2YB_2 \geq 2\pi/3\} \supset \mathcal{P}_2$ are separated by the perpendicular to the bisector at I and hence \mathcal{P}_1 and \mathcal{P}_2 are disjoint. By similarity this implies that all parallelograms \mathcal{P}_k are disjoint.

Now we are going to prove that there is a single edge in St_0 which has a point on the segment $[A_0B_0]$ which implies item (ii). Assume the contrary, then $St_0 \setminus [A_0B_0]$ has at least 2 components. One of them contains A_{∞} , so its length is at least

$$\frac{\cos\alpha}{\lambda} - \frac{\sin\alpha}{\sqrt{3}\lambda}.$$

Also, the length of every component cannot be smaller than the distance between $[A_0B_0]$ and A_1 which is

$$\frac{\cos\alpha}{\lambda} - \frac{\sin\alpha}{\sqrt{3}\lambda} - \cos\alpha.$$

Summing up

$$\mathcal{H}^1(\mathcal{S}t_0) \ge 2\frac{\cos\alpha}{\lambda} - 2\frac{\sin\alpha}{\sqrt{3}\lambda} - \cos\alpha.$$

Under the assumption (1) this is greater than the length of the example we have¹, whose length is

$$\frac{\cos\alpha}{\lambda} - \frac{\sin\alpha}{\sqrt{3}\lambda} + \frac{\sqrt{3}}{1-\lambda}\sin\alpha;$$

this is equivalent to

$$\frac{\cos \alpha}{\lambda} - \frac{\sin \alpha}{\sqrt{3}\lambda} - \cos \alpha \ge \frac{\sqrt{3}}{1 - \lambda} \sin \alpha \tag{3}$$

which is shown in the end of the Appendix.

Let us prove items (i) and (iii). Recall that S_1 is the Steiner point adjacent to A_1 and B_1 in $\mathcal{S}t_1$, and also S_1 is a vertex of \mathcal{P}_1 (both for $\mathcal{S}t_0$ and $\mathcal{S}t_1$).

Now consider, in St_j , the simple path P_k from A_k to B_k for k > 1 (see Fig. 6). The number of connected components in $St_j \setminus P_k$ is equal to the number of branching points in P_k plus the number of terminals of degree 2 in $\{A_k, B_k\}$. By Lemma 5 the path P_k contains exactly two points from $St_j \setminus P_k$ (such points are either branching or terminal). We call these points S_k , T_k , in a way that $S_k \in [A_k V_k] \cup [V_k B_k]$ and $T_k \in [A_k U_k] \cup [U_k B_k]$. The corresponding edges have opposite directions in $St_j \setminus P_k$. Thus $P_k \subset P_k$.

Reasoning by induction on k the point S_k determines T_{k+1} and T_{k+1} determines S_{k+1} . This proves items (i) and (iii).

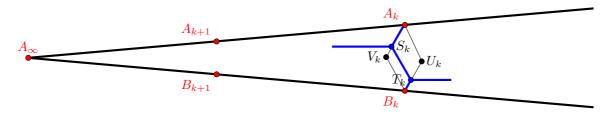


Figure 6: Behaviour of St_1 and St_0 around the terminals A_k and B_k .

Now we proceed with item (v). Note that in the construction we cannot have $\{S_k, T_k\} = \{A_k, B_k\}$, because otherwise we would have $P_k = [A_k B_k]$, but this edge has a direction which is not represented in the wind-rose. Thus the full component $\mathcal{S}t'_1$ of $\mathcal{S}t_1$ containing A_1 has an odd (or an infinite) number of terminals. Note that the terminal set of $\overline{\mathcal{S}t_1} \setminus \mathcal{S}t'_1$ is similar to \mathcal{A} with a scale factor λ^k , so a reduction implies item (iii).

4 Proof of Theorem 1

Lemma 7. If $\alpha < \frac{\pi}{6}$, and λ satisfy (1), then there exists a full* Steiner tree for A_1 .

Proof. Let St_1 be a solution to the Steiner problem for A_1 . Let St' be the full component of St_1 containing A_1 . If St' is infinite then it contains every segment between C_k and D_k , defined in Lemma 5, so every other full component should be a segment $[A_kB_k]$ which is impossible. This means $St' = St_1$ and we are done.

If $\mathcal{S}t'$ is finite, then by Lemma 6(v) $\mathcal{S}t'$ has an odd number of terminals and without loss of generality the largest index of a terminal of $\mathcal{S}t'$ is reached on A_{k+1} . Then $\mathcal{A}_{k+1} := \mathcal{A} \setminus \{A_1, \ldots, A_k\} \setminus \{B_1, \ldots, B_k\}$ is similar to \mathcal{A}_1 , i.e.

$$f_k(\mathcal{A}_1) = \mathcal{A}_{k+1},$$

where f_k is the homothety with center A_{∞} and ratio λ^k . So

$$\mathcal{H}^1(\mathcal{S}t_{k+1}) = \lambda^k \mathcal{H}^1(\mathcal{S}t_1),$$

where St_{k+1} is a Steiner tree for A_{k+1} and

$$\mathcal{H}^1(\mathcal{S}t') = (1 - \lambda^k)\mathcal{H}^1(\mathcal{S}t_1).$$

Let $\mathcal{S}t^*$ be the union of $f_k^j(\mathcal{S}t')$ over non-negative integers j. Note that $\mathcal{S}t^*$ connects \mathcal{A}_1 and

$$\mathcal{H}^1(\mathcal{S}t^*) = \sum_{j=0}^{\infty} \lambda^{kj} \mathcal{H}^1(\mathcal{S}t') = \frac{1}{1-\lambda^k} \mathcal{H}^1(\mathcal{S}t') = \mathcal{H}^1(\mathcal{S}t_1).$$

¹Formally we have a logical loop here, but the calculation of length in a given example is independent with any logical arguments.

Thus $\mathcal{S}t^*$ is a solution to the Steiner problem for \mathcal{A}_1 . By construction it is full*, so the lemma is proved.

Lemma 8. The length of a full* Steiner tree for A_1 is equal to

$$e^{-\beta i} \left(\cos \alpha + \sqrt{3} \sin \alpha + a e^{\frac{\pi i}{6}} + b e^{-\frac{\pi i}{6}} \right) = \left| \cos \alpha + \sqrt{3} \sin \alpha + a e^{\frac{\pi i}{6}} + b e^{-\frac{\pi i}{6}} \right|,$$

where $a = 2 \sin \alpha \sum_{j \in J} \lambda^j$, $b = 2 \sin \alpha \sum_{j \in \mathbb{N} \setminus J} \lambda^j$ for some $J \subset \mathbb{N}$.

Proof. Consider a complex coordinate system, determined by $A_{\infty}=0$, $A_1=\cos\alpha+i\sin\alpha$, $B_1=\cos\alpha-i\sin\alpha$. Let C be a Melzak point of A_1 and B_1 ; clearly, $C=\cos\alpha+\sqrt{3}\sin\alpha$ and

$$A_k = \lambda^{k-1}(\cos \alpha + i \sin \alpha), \qquad B_k = \lambda^{k-1}(\cos \alpha - i \sin \alpha).$$

Denote by β the angle smallest angle between an edge of the tree and the real axis; by the argument in the proof of Lemma 6(iv), one has $\beta \in [0, \alpha]$. Now we apply Lemma 2. By Melzak algorithm we can replace A_1 and B_1 with C. The unitary coefficient at C is $e^{-\beta i}$, at A_k is $e^{(-2\pi/3-\beta)i}$ or $e^{(-\pi/3-\beta)i}$ and at B_k is $e^{(\pi/3-\beta)i}$ or $e^{(2\pi/3-\beta)i}$, respectively. By the proof of Lemma 6(i) the sum of the coefficients at A_k and B_k is zero for every k > 1. So the summands with indices k are either equal to

$$2\sin\alpha e^{(-2\pi/3-\beta)i}\lambda^{k-1}i = 2\sin\alpha e^{(-\pi/6-\beta)i}\lambda^{k-1}$$

or to

$$2\sin\alpha e^{(-\pi/3-\beta)i}\lambda^{k-1}i = 2\sin\alpha e^{(\pi/6-\beta)i}\lambda^{k-1}.$$

Proper naming of a and b gives us the desired result.

Lemma 9. Suppose that $\lambda \leq \frac{1}{2}$ and $\alpha < \frac{\pi}{6}$ satisfy (1). If $\mathcal{H}^1(\mathcal{S}t_1^1) = \mathcal{H}^1(\mathcal{S}t_1^2)$ for distinct full* Steiner trees $\mathcal{S}t_1^1$ and $\mathcal{S}t_1^2$ for \mathcal{A}_1 , then $\mathcal{S}t_1^2$ is the reflection of $\mathcal{S}t_1^2$ with respect to the angle bisector.

Proof. Assume the contrary. Lemma 8 gives

$$\left|\cos \alpha + \sqrt{3}\sin \alpha + a_1 e^{\frac{\pi i}{6}} + b_1 e^{-\frac{\pi i}{6}}\right| = \left|\cos \alpha + \sqrt{3}\sin \alpha + a_2 e^{\frac{\pi i}{6}} + b_2 e^{-\frac{\pi i}{6}}\right|.$$

Put $c = \cos \alpha + \sqrt{3} \sin \alpha$ and $L_j = \cos \alpha + \sqrt{3} \sin \alpha + a_j e^{\frac{\pi i}{6}} + b_j e^{-\frac{\pi i}{6}}, j = 1, 2$. Then

$$L_j \cdot \bar{L_j} = c^2 + c(a_j + b_j)(e^{\frac{\pi i}{6}} + e^{-\frac{\pi i}{6}}) + a_i^2 + b_i^2 + a_j b_j(e^{\frac{\pi i}{3}} + e^{-\frac{\pi i}{3}}) = c^2 + \sqrt{3}c(a_j + b_j) + (a_j + b_j)^2 - a_j b_j.$$
(4)

Recall that $a_j + b_j = 2 \sin \alpha \cdot \frac{\lambda}{1-\lambda}$, so $L_1 \cdot \bar{L_1} = L_2 \cdot \bar{L_2}$ implies $a_1b_1 = a_2b_2$, hence we have either $a_1 = a_2$ and $b_1 = b_2$ or $a_1 = b_2$ and $a_2 = b_1$. Without loss of generality we may assume that we are in the first case.

Let a_1 and a_2 be the sums of λ^k over $k \in J_1$ and $k \in J_2$, respectively. Since $\lambda \leq \frac{1}{2}$ the sets of indices J_1 and J_2 coincide, say $J_1 = J_2 := J$. Define an abstract graph G = (V, E) with vertices

$$V := \{A_k, B_k, S_k, T_{k+1} \mid k \in \mathbb{N}\}$$

and edges

$$E := \{ (A_1, S_1), (S_1, B_1) \} \cup \{ (S_k, T_{k+1}) \mid k \in \mathbb{N} \} \cup \{ (A_{k+1}, S_{k+1}), (B_{k+1}, T_{k+1}) \mid k \in \mathbb{N} \setminus J \}.$$

Recall that St_1^1 and St_1^2 are Steiner trees, so they are global (and hence local) minima of $\ell_G(\cdot)$. By Theorem 4 for G and $V_0 = \mathcal{A}_1 \setminus \{A_\infty\}$ the trees St_1^1 and St_1^2 have a common wind-rose and by Lemma 6(i) they coincide. Thus in the case $a_1 = b_2$ and $a_2 = b_1$ the trees St_1^1 and St_1^2 are symmetric.

Now we are ready to prove Theorem 1.

Remark 3. Here we have shown that a full* Steiner tree for A_1 is determined (up to the reflection) by knowing which of the two possible directions of the wind-rose is used in each vertex A_k (this determines the set $J \subset \mathbb{N}$). We don't know which are the sets J that correspond to a locally minimal tree: the answer may depend on α and λ in some complicated way (we discuss it in Section 6). Fortunately, a relatively simple argument allows us to find a very strict condition on J which must hold for an absolute minimimer.

Consider the complex coordinate system from Lemma 8.

Let St_1 be a full* Steiner tree for A_1 and let \mathcal{P}_k be the parallelogram $A_kV_kB_kU_k$ defined in Lemma 6(iv). Without loss of generality suppose that $S_1 = V_1$ lies in the lower half-plane (otherwise consider the reflection of St_1). Thus every U_j lies in the upper half-plane and every V_j belongs to the lower half-plane. Clearly for every j the segments $[U_jA_j]$ lie in the upper half-plane and $[V_jB_j]$ lie in the lower half-plane. We need the following observations.

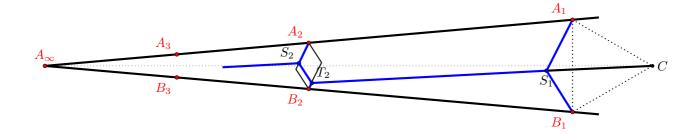


Figure 7: The construction used in the proof of Theorem 1.

- 1. One has $|V_j B_j| = |U_j A_j| < |U_j B_j| = |A_j V_j|$ for every j as U_j belongs to the upper half-plane and the inequality $|X A_j| < |X B_j|$ holds for every point X from the upper half-plane.
- 2. By Lemma 3 the point T_2 lies in the lower half-plane as T_2 belongs to the ray $[C, S_1)$, where C is the Melzak point of A_1 and B_1 (recall that C belongs to the real axis) and thus S_1 belongs to the lower half-plane. Then we have $T_2 \in [V_2B_2]$.
- 3. If T_j belongs to the upper half-plane then T_{j+1} belongs to the lower half-plane as the ray $[S_jT_{j+1})$ is co-directed with the ray $[S_1T_2) \subset [CT_2)$ and S_j belongs to $[V_jB_j]$ and hence to the lower half-plane.

Lemma 8 says that

$$\mathcal{H}^{1}(\mathcal{S}t_{1}) = e^{-\beta i} \left(\cos \alpha + \sqrt{3} \sin \alpha + ae^{\frac{\pi i}{6}} + be^{-\frac{\pi i}{6}} \right) = \left| \cos \alpha + \sqrt{3} \sin \alpha + ae^{\frac{\pi i}{6}} + be^{-\frac{\pi i}{6}} \right|,$$

where $a=2\sin\alpha\sum_{j\in J}\lambda^j$, $b=2\sin\alpha\sum_{j\in\mathbb{N}\setminus J}\lambda^j$ for some $J\subset\mathbb{N}$. Looking at equation (4) one notices that since a+b is fixed (independent of J) the total length is smaller when ab is larger, or, which is the same, when |a-b| is smaller. By observation 2, one has $1\in J$. Observation 3 says that j and j+1 cannot belong to $\mathbb{N}\setminus J$ simultaneously. So |a-b| is minimal for $J=\{2k-1\,|\,k\in\mathbb{N}\}$. The construction described in the proof of Lemma 7 started with a full tree for A_1,A_2,A_3,B_1,B_2 shows that such J is admissible. So the following lemma concludes the proof.

Lemma 10. Suppose that $\lambda \leq \frac{1}{2}$ and $\alpha < \pi/6$ satisfy the condition (1). Then every Steiner tree for $A_5 := \{A_1, A_2, A_3, B_1, B_2\}$ is full.

Proof. Assume the contrary, then a Steiner tree St_5 contains a terminal X of degree 2. Recall that the Steiner tree is contained in conv $A_5 = A_3A_1B_1B_2$ (see Figure 3). Clearly, $\angle A_3A_1B_1 = \angle A_1B_1B_2 = \pi/2 - \alpha < 2\pi/3$ and $\angle B_2A_3A_1 < \angle B_3A_3A_1 = \pi/2 + \alpha < 2\pi/3$. Thus X is either A_2 or B_2 . Repeating the argument of Lemma 5 we conclude that A_1 and B_1 are adjacent to the same Steiner point Y. Now if both A_2 and B_2 have degree 2, then St_5 contains segment $[A_2B_2]$, which is impossible since $\angle A_3A_2B_2$, $\angle A_3B_2A_2 \leq \pi/2 - \alpha$.

Now suppose that $\mathcal{S}t_5$ contains a full component which is an edge [XZ]. Clearly [XZ] cannot split conv \mathcal{A}_5 because $[XZ] \neq [A_2B_2]$, so $Z = A_3$. Since $|A_2A_3| < |B_2A_3|$, in this case $\mathcal{S}t_5$ is a union of $[A_2A_3]$ and a symmetric solution for $\{A_1, B_1, A_2, B_2\}$. But then $\angle A_3A_2W = 2\pi/3 - \alpha$, where W is the second Steiner point, a contradiction.

In the remaining case St_5 is a union of two similar tripods, which is longer by (4).

5 Proof of Theorem 2

Proof of item (i). Clearly \mathcal{A}_0 is a compact set, so the Steiner problem has a solution. By Lemma 6(ii) every Steiner set for \mathcal{A}_0 is a Steiner tree. Suppose that $\mathcal{S}t_0$ is a Steiner tree for \mathcal{A}_0 ; obviously $\mathcal{H}^1(\mathcal{S}t_0) < \infty$. Denote the point $\overline{\mathcal{S}t_0} \cap [A_0B_0]$ by x.

Lemma 11. St_0 is a full* tree.

Proof. Consider the full* subtree of $\mathcal{S}t_0$ containing x which is maximal with respect to inclusion. Call it $\mathcal{S}t'_0$. Since the segment of $\mathcal{S}t_0$ containing x is orthogonal to $[A_0B_0]$, the wind rose of $\mathcal{S}t'_0$ contains the direction of the bisector of the angle. Suppose, by contradiction that $\mathcal{S}t_0 \setminus \mathcal{S}t'_0$ is non-empty. Then $\mathcal{S}t'_0$ contains a terminal (without loss of generality suppose it is A_k) which has degree 2 in $\mathcal{S}t_0$ and degree 1 in $\mathcal{S}t'_0$.

Let $[A_k y]$, $[A_k z]$ be the edges incident to A_k , where z is a vertex of $\mathcal{S}t'_0$ and y is a vertex of $\mathcal{S}t_0$ but not of $\mathcal{S}t'_0$. Since $\mathcal{S}t'_0$ is maximal $\angle yA_k z > 2\pi/3$ and hence the segment $[A_k z]$ must be parallel to the bisector.

If y is a terminal point then by Lemma 5 $y \in \{A_{k+1}, B_{k+1}\}$. By Lemma 5 the edge $[A_k y]$ contains the segment of St_0 connecting C_k and D_k . Hence A_k and B_k belong to the same component of $St_0 \setminus [yA_k[$. By Lemma 5 the tree St_0 contains unique segment connecting C_{k-1} and D_{k-1} , so the path connecting A_k and A_k in St_0 has exactly one Steiner point, which should be z. This is impossible since $[A_k z]$ is parallel to the bisector.

Thus y is a branching point and so $[yB_k] \subset \mathcal{S}t_0$; recall that the angle yA_kz is strictly greater than $2\pi/3$. Let $\mathcal{S}t$ be the union of $\mathcal{S}t_0 \setminus \mathcal{S}t_0'$ and the reflection of $\mathcal{S}t_0'$. Let z' be the reflection of z, then B_kz' is also parallel to the bisector. Note that $\mathcal{H}^1(\mathcal{S}t) = \mathcal{H}^1(\mathcal{S}t_0)$, so $\mathcal{S}t$ is also a solution to the Steiner problem for \mathcal{A} . But

$$\angle yB_kz' = 2\pi - \angle zA_ky - \angle A_kyB_k < 2\pi/3,$$

so St is not a Steiner tree, contradiction.

Lemma 12. St_0 is a full tree.

Proof. By Lemma 11 $\mathcal{S}t_0$ is a full* tree. Let $\mathcal{S}t_0'$ be the full component containing x. Consider a terminal of $\mathcal{S}t_0'$ with degree 2 (without loss of generality it is A_k). Then the structure of the other full component $\mathcal{S}t$, containing A_k is uniquely determined S_k is the vertex of the rhombus \mathcal{R}_k . Thus $\mathcal{S}t$ is a Steiner tree for 4 terminals A_k , B_k , A_{k+1} , B_{k+1} (and its branching points are S_k , T_{k+1}). But

$$\angle A_{\infty}A_{k+1}T_{k+1} = \angle A_{\infty}B_{k+1}T_{k+1} = 2\pi/3 - \alpha$$

which means that both A_{k+1} and B_{k+1} cannot be a point of degree 2 in St_0 . This is a contradiction.

Lemma 13. Consecutive horizontal parts of St_0 lie on the opposite sides of the bisector of the angle.

Proof. We repeat the proof of Lemma 6(iii). Every \mathcal{P}_i is a rhombus with two vertices at the bisector of the angle. By Lemma 12 no segment of $\mathcal{S}t_0$ belongs to the bisector. Thus S_{i+1} and T_{i+1} always belong to the opposite sides of the bisector. Since S_iT_{i+1} is parallel to the bisector for every i, we are done.

Lemma 14. Consider a complex coordinate system, such that $A_{\infty} = 0$, $A_1 = \cos \alpha + i \sin \alpha$, $B_1 = \cos \alpha - i \sin \alpha$. Let x be the point where $\overline{\mathcal{S}t_0}$ touches $[A_0B_0]$ and suppose that $|A_0x| \leq |B_0x|$. Then

$$\mathcal{H}^1(\mathcal{S}t_0) = x + \left(\frac{\sqrt{3}}{1-\lambda} - \frac{i}{1+\lambda}\right) \sin \alpha$$

where x is the complex number representing the point x. In the case when $|A_0x| \leq |B_0x|$ we obtain the same formula with x replaced by \bar{x} .

Proof. We are going to use the length formula (2):

$$\mathcal{H}^{1}(\mathcal{S}t_{0}) = \sum_{j=1}^{\infty} \bar{c}_{j} A_{j} + \sum_{j=1}^{\infty} \bar{d}_{j} B_{j} + 1 \cdot x.$$
 (5)

By Lemma 12 every terminal has degree 1. If we suppose that $|A_0x| \leq |B_0x|$ the first horizontal edge of $\mathcal{S}t_0$ starting from x is going to hit the upper part of the first rhombus \mathcal{R}_1 . This means that the first Steiner point S_1 is connected to the upper terminal A_1 and has a direction $e^{2\pi i/3}$. The edge connected to the terminal B_1 has the opposite direction. By Lemma 13 the next horizontal edge is below the angle bisector and the direction of the edges on the terminals A_2 and B_2 are rotated by $-\pi/3$. The angles keep alternating between even and odd vertices, so, one has the following coefficients c_k , d_k in (5): $c_{2k-1} = e^{2\pi i/3}$, $c_{2k} = e^{\pi i/3}$, $d_{2k-1} = e^{-\pi i/3}$ and $d_{2k} = e^{-2\pi i/3}$, $k \in \mathbb{N}$. The coefficient for x is 1. Recall that $A_j = \lambda^j e^{i\alpha}$, $B_j = \lambda^j e^{-i\alpha}$. For odd j one has $\overline{c_j}A_j + \overline{d_j}B_j = 2\lambda^j \sin\alpha \cdot i \cdot e^{-2\pi i/3}$ while for even j one obtains $2\lambda^j \sin\alpha \cdot i \cdot e^{-\pi i/3}$. Thus (5) becomes:

$$\mathcal{H}^{1}(\mathcal{S}t_{0}) = x + 2(1 + \lambda^{2} + \lambda^{4} \dots)ie^{-2\pi i/3} \sin \alpha + 2\lambda(1 + \lambda^{2} + \lambda^{4} + \dots)ie^{-\pi i/3} \sin \alpha$$

$$= x + \left(\frac{2e^{-2\pi i/3}}{1 - \lambda^{2}} + \frac{2\lambda e^{-\pi i/3}}{1 - \lambda^{2}}\right) i \sin \alpha = x + \left(\frac{2e^{-\pi i/6}}{1 - \lambda^{2}} + \frac{2\lambda e^{\pi i/6}}{1 - \lambda^{2}}\right) \sin \alpha$$

$$= x + \left(\frac{\sqrt{3}}{1 - \lambda} - \frac{i}{1 + \lambda}\right) \sin \alpha.$$

Now we are ready to finish the proof of the theorem. By Lemma 12 St_0 is full*. Lemma 14 gives a formula for the length, which should be a real number. Thus

$$\Im x = \pm \frac{1}{1+\lambda} \sin \alpha.$$

This determines the point x proving item (i). Recall that $\Re x = \frac{\cos \alpha}{\lambda} - \frac{\sin \alpha}{\sqrt{3\lambda}}$, so

$$\mathcal{H}^{1}(\mathcal{S}t_{0}) = \frac{\cos\alpha}{\lambda} - \frac{\sin\alpha}{\sqrt{3}\lambda} + \frac{\sqrt{3}}{1-\lambda}\sin\alpha$$

and item (ii) is also proven.

We are left with the proof of item (iii). Clearly $\mathcal{S}t'_0 = \mathcal{S}t_0 \cap \operatorname{conv} f_2(\mathcal{A}_0)$ connects $f_2(\mathcal{A}_0)$. Note that $\mathcal{S}t'_0$ is a full tree so we may apply Lemma 2 to compute $\mathcal{H}^1(\mathcal{S}t'_0)$. This computation is essentially contained in the computation in the proof of Lemma 14; the only difference is that we omit two starting terms in geometric progressions. Thus we obtain

$$\mathcal{H}^1\left(\mathcal{S}t_0'\right) = \lambda^2 \mathcal{H}^1(\mathcal{S}t_0).$$

It means that $\mathcal{S}t_0 \cap f_2([A_0B_0])$ is a Steiner tree for $f_2(\mathcal{A}_0)$ and Lemma 13 gives $f_2(x) = \mathcal{S}t_0 \cap f_2([A_0B_0])$. Also $f_{-1}(\mathcal{P}_1) \cap [A_0B_0] = f_{-1}(V_1)$ so $f_{-1}(\mathcal{P}_1)$ does not intersect $\mathcal{S}t_0$. Thus $f_2(\mathcal{S}t_0)$ does not intersect \mathcal{P}_2 and so $f_2(\mathcal{S}t_0) \subset \mathcal{S}t_0$.

6 The dynamical system

Lemma 6(i) assures that any full* tree with terminals on the set \mathcal{A}_1 is uniquely determined by its wind-rose. If one knows the position of the line containing the long edge joining the first two parallelograms \mathcal{P}_1 and \mathcal{P}_2 then the whole tree is determined. Since the points of \mathcal{A}_1 are placed in geometric progression, each subsequent step of the tree construction is subject to the same recurrence equation with the appropriate rescaling.

It turns out that the resulting discrete dynamical system is conjugate to the inverse of a 2-interval piecewise contraction, whose law can be written as

$$F_{\lambda,\delta}: x \in [0,1] \to {\{\lambda x + \delta\}},$$

where $\{\cdot\}$ stands for the fractional part. See [15], [23], [10].

Suppose St is a full* tree satisfying the assumptions of Lemma 5. We know that the segments of St outside the parallelograms \mathcal{P}_i are all parallel to each other. Let e_1 be the unit vector parallel to these segments pointing from T_2 towards S_1 in Figure 7. Let e_2 and e_3 be the two unit vectors completing the wind-rose of St so that e_1 , e_2 , e_3 have zero sum and are counter-clockwise oriented (e_2 is pointing from T_2 towards S_2 and S_3 is pointing from S_3 towards S_4 . It is convenient to use a hexagonal (barycentric) coordinate system centered in S_3 and S_4 are all parallel to each other. We have S_4 are two pointing from S_4 towards S_4 and S_4 is pointing from S_4 towards S_4 and S_4 towards S_4 and S_4 is pointing from S_4 towards S_4 towards S_4 and S_4 is pointing from S_4 towards S_4 and S_4 is pointing from S_4 towards S_4 and S_4 is pointing from S_4 towards S_4

Using the notation of the proof Lemma 6 let $(l, \delta, -\delta)$ be the hexagonal coordinates of the point C_1 which is the center of the parallelogram \mathcal{P}_1 . In general we have $\delta \neq 0$ because the direction e_1 is not parallel to the angle bisector. Let $2a := |A_1V_1| = |U_1B_1|$ and $2b := |A_1U_1| = |V_1B_1|$. One has:

$$A_{1} = C_{1} + (0, b, -a) = (l, b + \delta, -a - \delta) = \left(l + \frac{a - b}{2}, \delta + \frac{a + b}{2}, -\delta - \frac{a + b}{2}\right),$$

$$B_{1} = C_{1} + (0, -b, a) = (l, \delta - b, a - \delta) = \left(l - \frac{a - b}{2}, \delta - \frac{a + b}{2}, \frac{a + b}{2} - \delta\right),$$

$$U_{1} = C_{1} + (0, -b, -a) = (l, \delta - b, -a - \delta) = \left(l + \frac{a + b}{2}, \delta + \frac{a - b}{2}, -\delta - \frac{a - b}{2}\right),$$

$$V_{1} = C_{1} + (0, b, a) = (l, \delta + b, a - \delta) = \left(l - \frac{a + b}{2}, \delta - \frac{a - b}{2}, \frac{a - b}{2} - \delta\right).$$

Then define $C_k = \lambda^{k-1}C_1, \ A_k = \lambda^{k-1}A_1, \ B_k = \lambda^{k-1}B_1, \ U_k = \lambda^{k-1}U_1 \ \text{and} \ V_k = \lambda^{k-1}V_1.$

The line segments $[S_{k-1}T_k]$ connecting the parallelograms \mathcal{P}_{k-1} and \mathcal{P}_k identified by Lemma 5 are parallel to the direction e_1 hence have coordinates of the form $(\cdot, \mu_{k-1}, -\mu_{k-1})$. We are going to find a recurrence relation between μ_{k-1} and μ_k . Lemma 6(i) gives a rule to obtain μ_k from μ_{k-1} , or to show that there is no corresponding μ_k . Since the prolongation of the segment between the parallelograms \mathcal{P}_{k-1} and \mathcal{P}_k must pass between the points A_k and B_k , one should have

$$\left(\delta - \frac{a+b}{2}\right)\lambda^{k-1} \le \mu_{k-1} \le \left(\delta + \frac{a+b}{2}\right)\lambda^{k-1}.$$

We consider a renormalized version of μ_k by defining

$$\nu_k := \frac{1}{2} + \frac{\lambda^{-k} \mu_k - \delta}{a + b}, \quad \text{i.e.} \quad \mu_k = \lambda^k \left((a + b) \left(\nu_k - \frac{1}{2} \right) + \delta \right)$$
 (6)

so that the previous conditions on μ_k become $\nu_k \in [0,1]$.

Suppose, without loss of generality, that b > a. Define

$$q^{\pm} := \frac{1}{2} + \frac{(1-\lambda)\delta - a}{\lambda(a+b)} \pm \frac{1}{2\lambda}$$
$$t_1 := \lambda(1-q^+), \quad t^* := \frac{a}{a+b}, \quad t_2 := -\lambda q^-.$$

Lemma 15. One has $\nu_{k+1} = g(\nu_k)$ with

$$g(t) = \begin{cases} \frac{t}{\lambda} + q^+ & \text{if } 0 \le t < t^*, \\ \frac{t}{\lambda} + q^- & \text{if } t^* < t \le 1. \end{cases}$$
 (7)

Moreover $g([0,t_1] \cup [t_2,1]) = [0,1]$, g(0) = g(1), g injective on $]0,1[\setminus \{t^*\}]$. If we restrict the values of g to the interval [0,1], the inverse function, $g^{-1}:[0,1]\setminus \{q^+\} \to [0,1]$ is uniquely defined and one has

$$g^{-1}(t) = \{\lambda t + t_2\}$$

where $\{x\} = x - |x|$ is the fractional part of x.

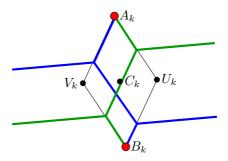


Figure 8: Possible behaviour of a locally minimal tree in the parallelogram \mathcal{P}_k .

Proof. We have only two possible behaviours at \mathcal{P}_k (they are drawn in blue and green in Fig. 8). When the input line coming from the right hand side hits the segment $[A_k, U_k]$ (green case) the line goes down, on the opposite side $[B_k, V_k]$. The output line is obtained adding the vector $2a\lambda^k e_3$ to the input. In hexagonal coordinates:

$$(t, \mu_k, -\mu_k) \mapsto (t, \mu_k, 2a\lambda^k - \mu_k) = (t - a\lambda^k, \mu_k - a\lambda^k, a\lambda^k - \mu_k).$$

In this case we obtain $\mu_{k+1} = \mu_k - a\lambda^k$. This happens when the line $(t, \mu_k, -\mu_k)$ passes above $U_k = \lambda^k U_1$, that is, when $\mu_k > \lambda^k \left(\delta + \frac{a-b}{2}\right)$.

Otherwise, if $\mu_k < \lambda^k \left(\delta + \frac{a-b}{2}\right)$, the input line hits the segment $[B_k, U_k]$ and it goes up, towards the opposite side $[A_k, V_k]$. In this case we add the vector $2b\lambda^k e_2$ obtaining:

$$(t, \mu_k, -\mu_k) \mapsto (t, \mu_k + 2b\lambda^k, -\mu_k) = (t - b\lambda^k, \mu_k + b\lambda^k, -\mu_k - b\lambda^k).$$

Hence $\mu_{k+1} = \mu_k + b\lambda^k$.

A line hitting the extremal points A_k or B_k will cause the tree to have a terminal of degree two (a finite full subtree). A line hitting the point U_k would split hitting both terminals A_k and B_k : by the argument of Lemma 6(v) this is not possible in a minimal Steiner tree. In conclusion, we have $\lambda^{-1-k}\mu_{k+1} = h(\lambda^{-k}\mu_k)$ with

$$h(x) = \begin{cases} \frac{x+b}{\lambda} & \text{if } -\frac{a+b}{2} < x - \delta < \frac{a-b}{2} \\ \frac{x-a}{\lambda} & \text{if } \frac{a-b}{2} < x - \delta < \frac{a+b}{2}. \end{cases}$$

To find the renormalized recurrence relation we put $\nu_k = \psi(\lambda^{-k}\mu_k)$ with

$$\psi(s) = \frac{1}{2} + \frac{s - \delta}{a + b}, \qquad \psi^{-1}(t) = (2t - 1)\frac{a + b}{2} + \delta$$

whence $\nu_{k+1} = g(\nu_k)$ with $g = \psi \circ h \circ \psi^{-1}$. With a straightforward computation one finds:

$$g(t) = \psi(h(\psi^{-1}(t))) = \frac{t}{\lambda} + \frac{1}{2} + \frac{(1-\lambda)\delta - a}{\lambda(a+b)} \pm \frac{1}{2\lambda} \quad \text{if } t \leq \frac{a}{a+b}.$$

This gives (7).

One has $g(t) = \frac{t}{\lambda} + q^{\pm}$. Hence the inverse function has the form $g^{-1}(s) = \lambda(s - q^{\pm})$. For $q^+ < s \le 1$ one has $g^{-1}(s) = \lambda(s - q^+)$. Since $0 < \lambda < 1$ it is clear that in this case $0 < \lambda(s - q^+) < 1$ hence $g^{-1}(s) = \{\lambda(s - q^+)\}$ for $q^+ < s \le 1$.

If $0 \le s < q^+$ one has $g^{-1}(s) = \lambda(s - q^-)$. But notice that $q^- = q^+ - \frac{1}{\lambda}$ hence $\lambda(s - q^-) = \lambda(s - q^+) + 1$. Notice that in this case $-1 < \lambda(s - qk) < 0$ and hence, also in this case, $g^{-1}(s) = \lambda(s - q^+) + 1 = \{\lambda(s - q^+)\}$.

Now consider the dynamical system ([0,1], g) which depends only on β . First, recall our proof scheme for $\mathcal{A} = \mathcal{A}_1$ (Theorem 1). We restrict the attention on full* candidates, then we show that the length is determined by J and the simplest cut of cases gave us that an optimal J is $2\mathbb{N}$.

Suppose now that one wants to check an existence of a competitor for a given J. Then β can be easily determined as the argument in Lemma 8. Lemma 15 explicitly defines the dynamics and it remains to check if the trajectory started in ν_1 is infinite. For a full* solution from Theorem 1 the corresponding trajectory is 2-periodic with the period ν_1 , 0 or ν_1 , 1. Note that this gives a way to finish the proof of Theorem 1 avoiding Lemma 10.

Also, let us note that the trajectory which corresponds to a locally minimal full tree for \mathcal{A}_1 is not periodic. One can force a full* tree to have a given β by addition a segment $[A_{\beta}B_{\beta}]$ with a proper slope to \mathcal{A}_1 . In the particular case of $\beta=0$ we add the segment $[A_0B_0]$ to obtain the input \mathcal{A}_0 ; other parameters are $a=b, \delta=0$ and $g^{-1}(t)=\{\lambda t+1-\lambda/2\}$. Steiner trees $\mathcal{S}t_0^1$ and $\mathcal{S}t_0^2$ from Theorem 2 again correspond to the 2-periodic points of ([0,1],g), namely $t=\frac{\lambda}{2(\lambda+1)}$ and $t=\frac{\lambda+2}{2(\lambda+1)}$. However for a general β there is no simple analogue of Lemma 13, which again makes the situation more intricate.

7 Open questions

Gilbert-Pollak conjecture on the Steiner ratio. A minimum spanning tree is a shortest connection of a given set A of points in the plane by segments with endpoints in A. It is well known that a minimum spanning tree can be found in a polynomial time.

The Steiner ratio is the infimum of the ratio of the total length of Steiner tree to the minimum spanning tree for a finite set of points in the Euclidean plane. The most known open problem on Steiner trees is whether the planar Steiner ratio is reached on an equilateral triangle (and so is equal to $\frac{\sqrt{3}}{2}$). This question is known as Gilbert–Pollak conjecture; despite the fact that some "proofs" are published this conjecture is open, see [13]. Graham [12] offered 1000\$ for the positive answer.

Analogues results in space. Steiner trees in a Euclidean d-dimensional space have the same local structure as in the plane, id est every branching point has degree 3 and moreover three adjacent segments have pairwise angles $2\pi/3$ and are coplanar. However a global structure is much more complicated. In particular a full (and full*) tree may have no wind-rose. Up to our knowledge all known series of explicit solutions are planar.

In particular, we are interested in a solution to the Steiner problem for an input which is a mix of a regular n-gon with the setup of this paper. Consider a regular 3-dimensional n-gon in the plane x = 1 centered at (1,0,0); let \mathcal{Q} be the set of its vertices. Then the input is the union of all $f^k(\mathcal{Q})$, $k \in \mathbb{N}$, where f is a homothety with the center in the origin and a small enough scale factor λ .

Some of our tools (in particular, Lemma 4) can be translated to the d-dimensional setup.

The number of solutions. It is known that the total number of full Steiner topologies with n terminals is

$$\frac{(2n-4)!}{2^{n-2}(n-2)!}.$$

How many of them may simultaneously give a shortest solution to the Steiner problem? In all known examples (for instance, ladders or Corollary 1) the number is at most exponential in n while the number of topologies has a factorial order of growth. Note that the answer may heavily depend on the dimension.

On the other hand, for $n \geq 4$ the set of planar *n*-point configurations with multiple solution to Steiner problem has Hausdorff dimension 2n-1 (as a subset of $(\mathbb{R}^2)^n$), see [1]. It is interesting to find the corresponding dimension in a *d*-dimensional Euclidean space.

Density questions. The following questions make sense both in two dimensions and in higher dimensions. Some of them seem not very difficult and can be considered as exercises. Let St be a Steiner tree and suppose $B_r(x)$ (or another good set) contain no terminal point of St. What is the largest possible number of branching points of St in $B_r(x)$? What is the maximum of $\mathcal{H}^1(St \cap B_r(x))/r$?

Notice that in this paper we proved that in the case of a Steiner tree with terminals on the boundary of a triangle we can have infinitely many Steiner points in the interior.

Let \mathcal{A} be a finite subset of $\partial B_1(0)$ (or of another good curve). What is the largest possible $\sharp(\mathcal{S}t \cap \partial B_{1/2}(0))$? What is the largest possible number of branching points of $\mathcal{S}t$ lying inside $B_{1/2}(0)$? What is the upper bound for $\mathcal{H}^1(\mathcal{S}t \cap B_{1/2}(0))$? Similar questions can be asked for any radius in place of $\frac{1}{2}$.

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Appendix: proofs of auxiliary and well-known results

Proof of Lemma 1. If $A = \{x_0\}$ the result is trivial. Without loss of generality, we can hence suppose that A contains at least two points.

Claim 1: there exists $\varepsilon > 0$ such that every connected component of $St \setminus A$ which intersect $\overline{B_{\varepsilon}(x_0)}$ contains x_0 . Let $\rho > 0$ be such that $\overline{B_{2\rho}(x_0)} \cap A = \{x_0\}$. If a connected component S of $St \setminus A$ touches $\overline{B_{\rho}(x_0)}$ but does not contain x_0 , then it must touch both $\partial B_{\rho}(x_0)$ and $\partial B_{2\rho}(x_0)$. Being S connected we would have $\mathcal{H}^1(S) \geq \rho$. Since $\mathcal{H}^1(St) < +\infty$, we conclude that only a finite number of connected components of $St \setminus A$ can possibly intersect $\overline{B_{\rho}(x_0)}$ without containing x_0 . Take ε smaller than the minimum distance of x_0 from each of these components to get the claim.

Claim 2: let S_0 be the connected component of \overline{St} which contains x_0 . Then $\overline{St \cap B_{\varepsilon}(x_0)} = S_0 \cap \overline{B_{\varepsilon}(x_0)}$. Suppose by contradiction that there exists $x \in \overline{St \cap B_{\varepsilon}(x_0)} \setminus S_0$. Let S be the connected component of \overline{St} which contains x and let S' be the connected component of $St \setminus A$ which contains x. Clearly $S' \subset S$ and $x \in S' \cap \overline{B_{\varepsilon}(x_0)}$. This is in contradiction with Claim 1.

Claim 3: $S_0 \cap \partial B_t(x_0) \neq \emptyset$ for all $t < \varepsilon$. Suppose by contradiction that $\partial B_t(x_0) \cap S_0 = \emptyset$ for some $t < \varepsilon$. Then $S_0 \subset B_t(x_0)$ because $x_0 \in S_0$ and S_0 is connected. This means that S_0 is also a connected component of $\mathcal{S}t \cup \mathcal{A}$ and this is not possible because $\mathcal{S}t \cup \mathcal{A}$ is connected by assumption and \mathcal{A} is assumed to have some other point outside $B_{\varepsilon}(x_0)$.

By Theorem 3, there exists $\rho < \varepsilon$ such that the set $\Sigma_{\rho} = \overline{\mathcal{S}t \setminus B_{\rho}(\mathcal{A})}$ is an embedding of a finite graph. Clearly $\Sigma_{\rho} \cap S_0 \neq \emptyset$ because $S_0 \subset \overline{\mathcal{S}t}$ touches $\partial B_{\rho}(x_0)$, $\Sigma_{\rho} \supset \mathcal{S}t \cap \partial B_{\rho}(x_0)$, $\overline{\mathcal{S}t} \setminus \mathcal{S}t \subset \mathcal{A}$ (Theorem 3) and $\partial B_{\rho} \cap \mathcal{A} = \emptyset$.

Hence $\Sigma = \Sigma_0 \cup S_0$ is connected and contains x_0 . Also $\Sigma \cap \partial B_{\rho}(x_0) = \Sigma_0 \cap \partial B_{\rho}(x_0)$ is finite (by Theorem 3). The set $S = \Sigma \cap \overline{B_{\rho}}$ is compact and has a finite number of connected components because each connected component has touches $\Sigma \cap \partial B_{\rho}(x_0)$, which is finite. Let C be a connected component of S. We claim that C is a Steiner set for $C \cap \partial B_{\rho}$, otherwise a Steiner set C' for $C \cap \partial B_{\rho}$ could be used to construct a competitor $St' = St \setminus C \cup C'$ and obtain the contradiction $\mathcal{H}^1(St') < \mathcal{H}^1(St)$. Since $C \cap \partial B_{\rho} \subset \Sigma \cap \partial B_{\rho}$ is finite, the set S is a finite embedded tree (Theorem 3) and hence S is an embedding of a finite graph.

Proof of Corollary 2. For almost every $\rho > 0$ Theorem 3 assures that $\mathcal{S}t \setminus B_{\rho}(\mathcal{A})$ is an embedding of a finite graph. The set $\mathcal{A} \setminus B_{\rho}(\mathcal{A}')$ is finite because it contains only isolated points without any accumulation point. For each x_0 in such finite set we can apply the previous lemma to find a neighborhood of x_0 where $\mathcal{S}t$ is an embedding of a finite graph. If ε smaller than the minimum of the radii of these neighborhoods, we can apply, again, Theorem 3 to find that also $\mathcal{S}t \setminus B_{\varepsilon}(\mathcal{A})$ is an embedding of a finite graph and hence deduce that $\mathcal{S}t \setminus B_{\rho}(\mathcal{A}')$ is an embedding of a finite graph.

Proof of Lemma 2. Let $v_1, \ldots, v_{n+t} \in \mathbb{C}$ be the vertices of St in the complex plane (they include the terminal points $v_k = p_k$ for $k = 1, \ldots, n$ and $t \leq n - 2$ branching points) and let E be the set of edges of St. Let $u(z) := \frac{z}{|z|} = e^{i \arg z}$ be the unit number representing the direction of the complex number z so that $|z| = z \cdot \overline{u(z)}$. One has

$$\mathcal{H}^{1}(\mathcal{S}t) = \sum_{(k,j)\in E} |v_{k} - v_{j}| = \sum_{(k,j)\in E} (v_{k} - v_{j}) \cdot \overline{u(v_{k} - v_{j})} = \sum_{(k,j)\in E} \left(v_{k} \cdot \overline{u(v_{k} - v_{j})} + v_{j} \cdot \overline{u(v_{j} - v_{k})} \right)$$

$$= \sum_{k=1}^{n+t} v_{k} \cdot \sum_{(k,j)\in E} \overline{u(v_{k} - v_{j})}.$$

If v_k is a branching point then the corresponding sum of directions is zero, so we have only a sum over the terminal points.

Proof of Lemma 3. Clearly, $\angle p_1pp_2 = \pi/3$, $\angle p_1qp_2 = 2\pi/3$ and the line (p_1, p_2) separates p and q. Hence p, p_1, p_2 and q are cocircular. Then $\angle p_1qp = \angle p_1p_2p = \pi/3$, so [pq] is a continuation of the remaining edge adjacent to q in St. By the Ptolemy's theorem

$$|pq| \cdot |p_1p_2| = |p_1q| \cdot |pp_2| + |p_2q| \cdot |pp_1|,$$

so
$$|pq| = |p_1q| + |p_2q|$$
 and $\mathcal{H}^1(\mathcal{S}t') = \mathcal{H}^1(\mathcal{S}t)$.

Proof of Theorem 4. We know that the function $x \mapsto |x|$ is a convex function defined on \mathbb{R}^d . Hence it turns out that the functional ℓ is convex on the vector space $(\mathbb{R}^d)^V$ of all immersions $\varphi \colon V \to \mathbb{R}^d$. So, the real function $F(t) = \ell(t\varphi + (1-t)\psi)$ is convex for $t \in [0,1]$ and if φ is a local minimum for ℓ it turns out that t=0 is a local minimum for F. Also, if $v \in V_0$, then $\varphi(v) = \psi(v) = (t\varphi + (1-t)\psi)$ (v), so ℓ is defined at $t\varphi + (1-t)\psi$. This easily implies that F is constant and hence

$$t\ell(\varphi) + (1-t)\ell(\psi) = \ell(t\varphi + (1-t)\psi) \tag{8}$$

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for every $t \in [0, 1]$. Thus $\ell(\varphi) = \ell(\psi)$.

But since the inequality (8) is true for subgraph of G, in particular for every summand in the formula, hence equality in (8) holds in each such summand, hence for all v and w which are joined by an edge of E one has

$$t|\varphi(v) - \varphi(w)| + (1-t)|\psi(v) - \psi(w)| = |t(\varphi(v) - \varphi(w)) + (1-t)(\psi(v) - \psi(w))| \qquad \forall t \in [0,1].$$

But we notice that the equality t|x| + (1-t)|x| = |tx + (1-t)y| holds for all $t \in [0,1]$ only when x is a positive multiple of y or when x = 0 or y = 0 since the function |x| is linear only on the half-lines through the origin. Letting $x = \varphi(v) - \varphi(w)$ and $y = \psi(v) - \psi(w)$ we obtain the claim of the theorem.

Proof of (3). We claimed that

$$\frac{\cos \alpha}{\lambda} - \frac{\sin \alpha}{\sqrt{3}\lambda} - \cos \alpha \ge \frac{\sqrt{3}}{1 - \lambda} \sin \alpha.$$

This is equivalent to

$$\cos \alpha \left(\frac{1}{\lambda} - 1\right) > \sin \alpha \left(\frac{1}{\sqrt{3}\lambda} + \frac{\sqrt{3}}{1 - \lambda}\right)$$

and

$$\cot \alpha > \frac{1+2\lambda}{\sqrt{3}(1-\lambda)^2}.$$

One has

$$\frac{d}{d\lambda} \frac{1+2\lambda}{\sqrt{3}(1-\lambda)^2} = \frac{2(2+\lambda)}{\sqrt{3}(1-\lambda)^3}$$

which is positive for $0 < \lambda \le 1/2$. Cotangent is decreasing, so in view of (1) it is enough to examine

$$\lambda = \left(\frac{\cos\left(\frac{\pi}{3} + \alpha\right)}{\cos\left(\frac{\pi}{3} - \alpha\right)}\right)^2 = \left(\frac{\cot\alpha - \sqrt{3}}{\cot\alpha + \sqrt{3}}\right)^2.$$

Put $t = \cot \alpha$, then the substitution gives that it is enough to show that

$$\frac{-\sqrt{3}t^4 + 44t^3 - 2\sqrt{3}t^2 - 12t - 9\sqrt{3}}{48t^2} > 0$$

for $t \in [\sqrt{3}, 3\sqrt{3} + 2\sqrt{6}]$, were the ends correspond to $\alpha = \pi/6$ and $\lambda = 1/2$, respectively. One can check that $-\sqrt{3}t^4 + 44t^3 - 2\sqrt{3}t^2 - 12t - 9\sqrt{3}$ is increasing on this interval and so the minimal value is $96\sqrt{3}$ at $t = \sqrt{3}$.

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