Non-linear conductances of Galton–Watson trees and application to the (near) critical random cluster model

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Abstract

When considering statistical mechanics models on trees, such that the Ising model, percolation, or more generally the random cluster model, some concave tree recursions naturally emerge. Some of these recursions can be compared with non-linear conductances, or p-conductances, between the root and the leaves of the tree. In this article, we estimate the p-conductances of T_n, a supercritical Galton–Watson tree of depth n, for any $p > 1$ (for *quenched* realization of T_n). In particular, we find the sharp asymptotic behavior when n goes to infinity, which depends on whether the offspring distribution admits a finite moment of order q, where $q = \frac{p}{p-1}$ is the conjugate exponent of p. We then apply our results to the random cluster model on T_n (with wired boundary condition) and provide sharp estimates on the probability that the root is connected to the leaves. As an example, for the Ising model on T_n with plus boundary conditions on the leaves, we find that, at criticality, the *quenched* magnetization of the root decays like: (i) $n^{-1/2}$ times an explicit tree-dependent constant if the offspring distribution admits a finite moment of order 3; (ii) $n^{-1/(\alpha-1)}$ if the offspring distribution has a heavy tail with exponent $\alpha \in (1,3)$.

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1 Introduction and main results

We consider a super-critical branching process with reproduction law μ, and we denote by T the associated tree. We denote by Z a generic random variable with law μ and we let $m := \mathbf{E}[Z]$, that we assume to be finite. We assume for simplicity that $\mu(0) = 0$ so the tree is infinite (and has no leaves) and that $\mu(1) < 1$ so the tree is non-degenerate (and $m > 1$).

For $n \in \mathbb{N}$, we let T_n be the subtree of depth n, and we equip the tree T_n with a set of resistances $R(e)$ on its edges: if the edge is uv with u the parent of v (we will write $u \to v$), then we denote $R_v := R(e)$. Our main objective is to estimate the p-conductance (or p-capacity) of T_n, equipped with resistances $(R_v)_{v\in T_n}$. Such capacities arise naturally in the context of the random cluster model (including the percolation and Ising model) on a quenched Galton–Watson tree, we refer to Section 2 below for more details.

Let us mention that the usual effective resistance (or conductances) of random trees have already been considered in [1, 9], with a specific choice of (random) resistances $(R_v)_{v\in T_n}$ which corresponds to a *critical* case. Our work can therefore be seen as a generalization of [1, 9] to the case of p-resistances, with a wider range of (non-random) resistances; we improve some of their results and present some applications, in particular to the (near) critical percolation or Ising models on a quenched tree.

More generally, we study (concave) recursions on a Galton–Watson tree: these recursions may arise in the context of statistical mechanics models on tree-like graphs (see [35] for an overview), or more generally in belief propagation or population dynamics algorithm (see e.g. [26] for an overview). These tree-recursion are often considered in a distributional sense rather than with a fixed tree, which corresponds to an annealed setting for random graphs (*i.e.* the randomness of the tree is part of the recursion); then, one of the main question is that of the convergence to the fixed point distribution, and we refer for instance to [20, 31] for recent (and general) results. Our interest is here slightly different, since we study iterations on a *quenched* (*i.e.* fixed) Galton–Watson tree, focusing on the case where the distributional fixed point of the iteration is degenerate, equal to 0: our main objective is then to estimate precisely the decay rate of the recursion towards zero.

1.1 Non-linear (L^p) resistive networks

Let us consider a graph $G=(V,E)$ equipped with a set of (non-negative) resistances $(R(e))_{e\in E}$ on its edges. A general theory of non-linear resistances and capacities is by now well-developed, and are usually defined through discrete nonlinear potential theory, see for instance [34] for an overview. Here, we focus on a specific non-linear case, so-called "L^p resistive networks". We will give the definitions directly in terms of the L^p-Thomson's principle, since it is essentially the only tool we need for this article; we refer to [11] for a detailed review of L^presistances and conductances, see in particular [11, Thm. 2.13] for the L^p-Thomson's principle.

For A,Z two disjoint subsets of V, we consider a *flow* θ between A and Z, which is a function on ordered edges $\theta : E\to\mathbb{R}_+$ that verifies $\theta(-e)=-\theta(e)$ and Kirchoff's node law: for $x\in V\setminus(A\cup Z)$, $\sum_{y\sim x}\theta(x,y)=0$. The strength of the flow is then

$$\mathrm{Strength}(\theta)=\sum_{\substack{x\in A, y\notin A\\ x\sim y}}\theta(x,y)$$

and we say that θ is a flow from A to Z if $\mathrm{Strength}(\theta)\geqslant 0$; we also say that θ is a *unit* flow if $\mathrm{Strength}(\theta)=1$.

For $p>1$, we define the L^p-energy (or simply p-energy) of a flow θ from A to Z as

$$\mathcal{E}_p(\theta)=\sum_{e\in E}R(e)^{\frac{1}{p-1}}\theta(e)^{\frac{p}{p-1}}\,.$$

Then, the p-resistance and p-conductance (or p-capacity) between A and Z are defined, through Thomson's principle as follows:

$$\begin{aligned}\mathcal{R}_p(A\leftrightarrow Z) &:= \inf_{\substack{\theta:A\to Z\\ \mathrm{Strength}(\theta)=1}}\mathcal{E}_p(\theta)^{p-1}\\ \text{and}\quad \mathcal{C}_p(A\leftrightarrow Z) &:= \mathcal{R}_p(A\leftrightarrow Z)^{-1}\,.\end{aligned}\tag{1.1}$$

It is actually notationally convenient to introduce the conjugate exponent $q = \frac{p}{p-1}$ (*i.e.* such that $\frac{1}{p} + \frac{1}{q} = 1$), so that the Thomson's principle (1.1) can be rewritten as

$$\mathcal{R}_p(A \leftrightarrow Z)^s = \inf_{\substack{\theta: A \to Z \\ \text{Strength}(\theta)=1}} \sum_{e \in E} R(e)^s \theta(e)^q , \qquad \text{with} \;\; s := q - 1 = \frac{1}{p-1} . \tag{1.2}$$

Let us also stress that for $p = 2$ ($q = 2$), the p-resistance and p-conductance amount to the usual effective resistance and conductance.

Remark 1.1. From the definition (1.1), we can easily deduce the Series and Parallel laws for L^p electrical networks: we can formally write them as

$$\begin{aligned} \mathcal{R}_p\big(\,\underline{R_1}\;\underline{R_2}\,\big)^s &= \mathcal{R}_p\big(\,\underline{R_1}\,\big)^s + \mathcal{R}_p\big(\,\underline{R_2}\,\big)^s , \\ \mathcal{C}_p\Big(\,\substack{\underline{R_1} \\ \overline{R_2}}\,\Big) &= \mathcal{C}_p\big(\,\underline{R_1}\,\big) + \mathcal{C}_p\big(\,\underline{R_2}\,\big) . \end{aligned}$$

Remark 1.2. General non-linear networks can be defined by replacing the function $x \mapsto |x|^p$ by a strictly convex function $\varphi : x \mapsto \varphi(x)$ (in this case, Thomson's principle becomes harder to state); we refer to [12, 34] and references therein. Most of our results would hold also for these capacities, but we have chosen to restrict to the case of p-capacities for simplicity, also because the applications we have in mind do not require more generality.

1.2 Concave recursions on trees

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function, and consider the iteration on the rooted tree T_n

$$B_n(u) = \sum_{v \leftarrow u} \frac{R_u}{R_v} g\big(B_n(v)\big) , \tag{1.3}$$

where we recall that $v \leftarrow u$ means that v is a descendant of u in T_n. These types of recursions appear in some statistical mechanic models on trees, in particular for the Random Cluster Model (with parameter $\mathtt{q} \in (0, 2]$, $\mathtt{q} = 1$ corresponding to the percolation model and $\mathtt{q} = 2$ to the Ising model), see Section 2 for details. Let us now explain how they also appear when computing L^p capacities of trees seen as resistive networks.

For $p > 0$, and $u \in T_n \setminus \partial T_n$, let us consider the subtree $T_n(u)$ of all descendants of u inside T_n, that we equip with resistances $(R_u R_v)_{v \in T_n(u)}$; in other words, we simply multiply all original resistances inside $T_n(u)$ by R_u. We denote

$$C_n^{(p)}(u) := \mathcal{C}_p(u \leftrightarrow \partial T_n(u))$$

the p-conductance from the root to the leaves of this subtree $T_n(u)$, as in (1.1). Then, the Series and Parallel laws from Remark 1.1 yield the following relation

$$C_n^{(p)}(u) = \sum_{v \leftarrow u} \frac{R_u}{R_v} \frac{C_n^{(p)}(v)}{(1 + C_n^{(p)}(v)^s)^{1/s}} . \tag{1.4}$$

This is exactly the iteration (1.3) with the function $g(x) = \frac{x}{(1+x^s)^{1/s}}$ with $s := q - 1 = \frac{1}{p-1}$, and boundary condition $C_n^{(p)}(u) = 1$ if $u \in \partial T_n$.

Remark 1.3. The recursion (1.4) appears in [33, Lem. 3.1], but with $s = p - 1$ instead of $s = q - 1$. This is just a convention, which simply means that all results in [33] are stated with the conjugate exponent; in practice cap_p in [33] should refer to the $\frac{p}{p-1}$-conductance.

We can actually compare (1.4) with (1.3) with a general concave g if we make some assumptions on the function g. For instance, if we assume that g is bounded and that there is some $q > 1$ such that $x^{-q}(g(x) - x)$ remains bounded away from 0 and $+\infty$ as $x \downarrow 0$, then we get that there are some constants κ_1, κ_2 such that for all $x > 0$

$$\frac{x}{(1+\kappa_1 x^s)^{1/s}} \leqslant g(x) \leqslant \frac{x}{(1+\kappa_2 x^s)^{1/s}}\,, \tag{1.5}$$

with $s := q-1$. This is for instance verified in the random cluster model of parameter q (with $q = 2$ if $\mathsf{q} \in (0,2)$ and $q = 3$ for $\mathsf{q} = 2$), see Section 2 below.

One then gets the following proposition, similarly to Theorem 3.2 in [33].

Proposition 1.4. *Let the function g verify (1.5) for some $q > 1$, and define $B_n(u)$ iteratively as in (1.3) with boundary condition $B_n(u) \in [c, +\infty]$ for $u \in \partial T$, for some constant $c > 0$. Then, there are two constants κ_1' and κ_2' such that*

$$\kappa_2' C_n^{(p)}(u) \leqslant B_n(u) \leqslant \kappa_1' C_n^{(p)}(u)\,, \tag{1.6}$$

where $p := \frac{q}{q-1}$ is the conjugated exponent of q.

Let us also mention another interesting and easy lemma, which proves a monotonicity in p of the conductances, see Lemma 6.9 in [3].

Lemma 1.5. *For $1 < p \leqslant p'$, let q (resp. q') be the conjugated exponent of p (resp. p'). Then, for any $u \in T_n$ we have $C_n^{(p)}(u) \geqslant C_n^{(p')}(u)$. In particular, the p-conductance of a tree, $C_n^{(p)}(\rho) = \mathcal{C}_p(\rho \leftrightarrow \partial T_n)$ is non-increasing in p, or equivalently non-decreasing in q.*

1.3 Main results: p-conductances of Galton–Watson trees

From now on, we fix $p > 1$ so we will suppress it from notations and write $C_n(u)$ instead of $C_n^{(p)}(u)$. We wish to estimate the p-conductance of the Galton–Watson tree T_n, that is the p-conductance between the root and the leaves of T_n:

$$C_n := C_n(\rho) = \mathcal{C}_p(\rho \leftrightarrow \partial T_n)\,,$$

which is a random variable since it depend on the realization of the tree T. In the following, we focus on the case where the tree is equipped with resistances $(R_v)_{v\in T_n}$ that are of the form:

$$R_v = (R_n)^{-|v|}\,, \tag{1.7}$$

where $|v|$ is the distance between v and the root, and $R = R_n > 0$ is a number which is fixed along T_n, but that may depend on n. The choice (1.7) may seem rather restrictive but is the one that appears naturally in the statistical mechanics models that we are interested in; then R_n is related to the inverse temperature of the model (the fact that R_n may depend on n will allow us to study the *critical window* for the random cluster model, as will become clear in Section 2). It is also closely related to the choice of conductances considered in [9] (or [1]), where authors estimate the standard 2-conductance (or 2-resistance) with resistances $R_v = \xi_v\, m^{|v|}$, for i.i.d. random variables $(\xi_v)_{v\in T}$ — here we focus on the case where $\xi_v \equiv 1$ but we allow more generality in the growth rate of resistances in view of the applications we have in mind, see Section 2.

More generally, we consider a concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ verifying (1.5) and the recursion (1.3) with resistances $(R_v)_{v\in T_n}$ defined as in (1.7): with this choice of resistances, we have for all $u \in T_n \setminus \partial T_n$

$$B_n(u) = R_n \sum_{v \leftarrow u} g\big(B_n(v)\big)\,, \tag{1.8}$$

with boundary condition $B_n(u) \equiv b_0$ for $u \in \partial T_n$. We also denote $B_n := B_n(\rho)$; in particular, we have $B_n = C_n$ if $g(x) = x/(1+x^s)^{1/s}$ and $b_0 = 1$.

Let us stress once more that the resistances are $R_v = (R_n)^{-|v|}$, where R_n is a parameter that is fixed along the tree T_n and that might depend on n. In particular, except when $R_n \equiv R$, the recursion (1.8) that defines B_n is different from the one that defines B_{n+1}. In fact, one may think of B_n as belonging to a (tree-)triangular array of recursions $((B_n(u))_{u\in T_n}, n \geqslant 0)$.

Before we state our results, let us introduce the following normalizing sequence, which plays an important role:

$$a_n := \sum_{k=1}^{n} (mR_n)^{-ks} . \tag{1.9}$$

To anticipate a bit, let us notice that we have $a_n = f_n(mR_n)$, with $f_n(x) = x^{-s}\frac{x^{-ns}-1}{x^{-s}-1}$. In particular, we easily get that

$$\begin{array}{ll} \text{if } mR_n \in \big(0, 1-\frac{1}{n}\big] & a_n \asymp (1-(mR_n)^s)^{-1}(mR_n)^{-ns} , \\ \text{if } mR_n \in \big[1-\frac{1}{n}, 1+\frac{1}{n}\big] & a_n \asymp n , \\ \text{if } mR_n \in \big[1+\frac{1}{n}, 2\big] & a_n \asymp ((mR_n)^s - 1)^{-1} , \end{array} \tag{1.10}$$

where we have used the notation $a_n \asymp a'_n$ if the ratio a_n/a'_n is bounded from above and from below by two universal constants. Note that the first line also includes the case where $R_n \to 0$, but we will mostly focus on the case where $\liminf_{n\to\infty} R_n > 0$. We refer to the regime $mR_n \equiv 1$ as the *critical case* and one can interpret $|mR_n - 1| = O(\frac{1}{n})$ as being in the *critical window*.

We obtain different results depending on whether $Z \sim \mu$ admits or not a finite moment of order q; similarly to what happens for the critical behavior of the Ising model on random trees, see [17]. We start with the case where $\mathbf{E}[Z^q] < +\infty$, where our results are much sharper; we then turn to the case where $\mathbf{E}[Z^q] = +\infty$.

1.3.1 Case of a finite moment of order q

Our first result gives a general estimate on $\mathbf{E}[B_n]$; in particular, thanks to Proposition 1.4, we may focus on the p-conductance C_n of the tree T_n. We first give a result in the case where the offspring distribution admits a finite moment of order q.

Theorem 1.6. *Let $p > 1$, let $q = \frac{p}{p-1}$ be its conjugate exponent and let $s = q - 1 = \frac{1}{p-1}$. Assume that $Z \sim \mu$ admits a finite moment of order q. Then, there is a constant c_p (that depend only on the law μ and on p) such that*

$$c_p (a_n)^{-1/s} \leqslant \mathbf{E}[C_n] \leqslant (a_n)^{-1/s} ,$$

where a_n is defined in (1.9). Additionally, $(a_n^{1/s} C_n)_{n\geqslant 1}$ is tight in $(0, +\infty)$.

Remark 1.7. Let us note that when Z only admits a moment of order $r < q$, then Lemma 1.5 combined with Theorem 1.6 gives the bound

$$\mathbf{E}[\mathcal{C}_p(\rho \leftrightarrow \partial T_n)] \geqslant \mathbf{E}[\mathcal{C}_{p'}(\rho \leftrightarrow \partial T_n)] \geqslant (\tilde{a}_n)^{1/(r-1)} ,$$

with $p' = \frac{r}{r-1}$ the conjugate exponent of r and $\tilde{a}_n = \sum_{k=1}^{n} (mR_n)^{-k(r-1)}$. In particular, if $R_n = m^{-1}$, $\mathbf{E}[C_n] \geqslant n^{-1/(r-1)}$.

In view of (1.10), we have the following estimates:

$$\begin{array}{lll}
\text{if } mR_n \in \big(0, 1-\frac{1}{n}\big] & \mathbf{E}[B_n] \asymp (1-(mR_n)^s)^{1/s}(mR_n)^n\,, \\
\text{if } mR_n \in \big[1-\frac{1}{n}, 1+\frac{1}{n}\big] & \mathbf{E}[B_n] \asymp n^{-1/s} = n^{-(p-1)}\,, \\
\text{if } mR_n \in \big[1+\frac{1}{n}, 2\big] & \mathbf{E}[B_n] \asymp ((mR_n)^s - 1)^{1/s}\,.
\end{array} \tag{1.11}$$

To complete the picture, note that when $mR_n \geqslant 2$, $\mathbf{E}[B_n]$ is of order $(mR_n)^{-1}$.

Our next result shows the L^q convergence of the rescaled conductance (or of the rescaled B_n).

Theorem 1.8. *Let $p > 1$, let $q = \frac{p}{p-1}$ be its conjugate exponent and $s = q - 1$. Assume that $Z \sim \mu$ admits a finite moment of order q. Then, if $\limsup_{n\to\infty} mR_n \leqslant 1$ and $\inf_{n\geqslant 0} R_n > 0$, we have, as $n \to \infty$*

$$\frac{B_n}{\mathbf{E}[B_n]} \xrightarrow[n\to\infty]{L^p} W\,,$$

where $W := \lim_{n\to\infty} \frac{1}{m^n} Z_n$ is the a.s. limit of the usual martingale associated to the branching process. If $R_n \equiv R$ with $R \in (0, m^{-1}]$, then the convergence also holds almost surely.

Finally, we estimate precisely the expectation $\mathbf{E}[B_n]$, under some condition on the function g. We have the following result.

Proposition 1.9. *Let $p > 1$, let $q = \frac{p}{p-1}$ be its conjugate exponent and let $s = q - 1$. Assume that $Z \sim \mu$ admits a finite moment of order q and that the following limit exists $\kappa_g := \lim_{x\downarrow 0} \frac{1}{x^q}(x - g(x))$ in $(0, +\infty)$. Then, if $\lim_{n\to\infty} mR_n = \vartheta \in (0, 1]$, we have, as $n \to \infty$,*

$$\mathbf{E}[B_n] \sim \alpha_p(\vartheta)(a_n)^{-1/s}\,,$$

for some constant $\alpha_p(\vartheta)$, with $\alpha_p(1) := (s\kappa_g \mathbf{E}[W^q])^{-1/s}$ in the case where $\lim_{n\to\infty} mR_n = 1$.

Let us mention that Proposition 1.9 improves in particular the result [9, Thm. 1.2] which considers some additional source of randomness on the resistances, but treats only the case of linear ($p = q = 2$) conductances in the critical case $mR_n \equiv 1$ and requires a moment of order 3 (*i.e.* $q + 1$) for the branching process.

1.3.2 Case of an infinite moment of order q

In the case of an infinite moment of order q, we need assume some (either upper or lower) bounds on the truncated q-moment of $Z \sim \mu$. More precisely, we assume that there is some $\alpha \in (1, q]$ and some slowly varying function $L(\cdot)$ such that, for $x \geqslant 1$

$$\mathbf{E}[(Z \wedge x)^q] \geqslant c_1 L(x) x^{q-\alpha} \tag{1.12}$$

$$\mathbf{E}[(Z \wedge x)^q] \leqslant c_2 L(x) x^{q-\alpha} \tag{1.13}$$

for some constants c_1, c_2. We obtain a lower bound on $\mathbf{E}[B_n]$ assuming (1.13) and an upper bound on $\mathbf{E}[B_n]$ assuming (1.12).

Let h be a increasing function which verifies $h(x) \sim L(1/x)x^{\alpha-1}$ as $x \downarrow 0$. We denote by h^{-1} its inverse and let us stress that $h^{-1}(x) \sim \tilde{L}(x)x^{1/(\alpha-1)}$ as $x \downarrow 0$, for some slowly varying function $\tilde{L}$, see [8, §1.5.7]. Then, define

$$\begin{array}{lll}
\text{if } mR_n \in \big(0, 1-\frac{1}{n}\big] & \gamma_n = h^{-1}(1-mR_n)\,(mR_n)^n\,, \\
\text{if } mR_n \in \big[1-\frac{1}{n}, 1+\frac{1}{n}\big] & \gamma_n = h^{-1}(1/n)\,, \\
\text{if } mR_n \in \big[1+\frac{1}{n}, 2\big] & \gamma_n = h^{-1}(mR_n - 1)\,,
\end{array} \tag{1.14}$$

to be compared with (1.11), which corresponds to having $h(x) = x^{q-1}$. Let us also define $\tilde{\gamma}_n$ by setting $\tilde{\gamma}_n = \gamma_n$ if $mR_n \geqslant 1-\frac{1}{n}$ and $\tilde{\gamma}_n := h^{-1}(1/n)(n(1-mR_n))^{1/s}(mR_n)^n$ if $mR_n \leqslant 1-\frac{1}{n}$.

In practice, in the case where $R_n \equiv R$ and if $L(x) \equiv 1$ in (1.12)-(1.13), then we have: (i) $\gamma_n = \tilde{\gamma}_n \asymp (mR-1)^{1/(\alpha-1)}$ if $mR > 1$; (ii) $\gamma_n = \tilde{\gamma}_n \asymp n^{-1/(\alpha-1)}$ if $mR = 1$; (iii) $\gamma_n \asymp (mR)^n$ and $\tilde{\gamma}_n \asymp n^{\frac{1}{q-1}-\frac{1}{\alpha-1}}\gamma_n$ if $mR < 1$.

Theorem 1.10. *Assume that $0 < \inf_n mR_n \leqslant \sup_n mR_n \leqslant 2$ and let γ_n be as in (1.14). Then there are constants c, c' such that:*

- *If (1.12) holds, then $\mathbf{E}[B_n] \leqslant c\gamma_n$, and in particular*

$$\lim_{K\uparrow+\infty} \limsup_{n\to\infty} \mathbf{P}(B_n \geqslant K\gamma_n) = 0\,;$$

- *If (1.13) holds, then $\mathbf{E}[B_n] \geqslant c'\tilde{\gamma}_n$, or more precisely*

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \mathbf{P}(B_n \leqslant \varepsilon\tilde{\gamma}_n) = 0\,.$$

Notice that there is a tiny gap between γ_n and $\tilde{\gamma}_n$ in the case where $mR_n \leqslant 1-\frac{1}{n}$; this should be an artifact of the proof, and we believe that the correct decay is given by γ_n. However, γ_n and $\tilde{\gamma}_n$ only differ by a polynomial in $x_n := n(1-mR_n)$: indeed, by Potter's bound [8, Thm. 1.5.6] we have that $h^{-1}(1/n) \geqslant c_\delta (x_n)^{-\delta-\frac{1}{\alpha-1}} h^{-1}(1-mR_n)$. We therefore obtain that $\gamma_n \geqslant \tilde{\gamma}_n \geqslant c_\delta (x_n)^{a-\delta}\gamma_n$, with $a = \frac{1}{s} - \frac{1}{\alpha-1}$. Let us conclude by noting that in the case $mR_n \leqslant 1-1/n$, both $\gamma_n, \tilde{\gamma}_n$ have the same exponential decay in $(mR_n)^n$.

Remark 1.11. We mention that we consider an assumption on the truncated q-th moment since they are slightly more general than assumptions on the tail of Z as considered for instance in [17] (see Section 2.3 for some further comments); also, the truncated moments are the quantities naturally appearing in the proof. Consider for instance the following conditions:

$$\mathbf{P}(Z > x) \geqslant c_1' \hat{L}(x) x^{-\alpha}\,, \tag{1.15}$$
$$\mathbf{P}(Z > x) \leqslant c_2' \hat{L}(x) x^{-\alpha}\,, \tag{1.16}$$

for some slowly varying $\hat{L}$. Then, one can actually deduce (1.12) from (1.15) (resp. (1.13) from (1.16)), since we have $\mathbf{E}[(Z\wedge x)^q] = q\int_0^x t^{q-1}\mathbf{P}(Z>t)\mathrm{d}t$. We then have $L(x) = \hat{L}(x)$ in the case $\alpha < q$, whereas $L(x) = \int_1^x u^{-1}\hat{L}(u)\mathrm{d}u \gg \hat{L}(x)$ (see [8, Prop. 1.5.9.a]) in the case $\alpha = q$.

Let us also show that the conditions (1.12)-(1.13) are *weaker* than the tail conditions (1.15)-(1.16). Indeed, on one hand, we have $\mathbf{E}[(Z\wedge x)^q] \geqslant x^q \mathbf{P}(Z>x)$, so (1.13) implies the upper bound $\mathbf{P}(Z > x) \leqslant c_2 L(x) x^{-\alpha}$, but this is not optimal in the case $\alpha = q$. However, in the case $\alpha < q$, (1.13) is equivalent to (1.16) with $\hat{L} = L$. On the other hand, to obtain a lower bound on $\mathbf{P}(Z > x)$, one needs to have $\alpha < q$ and both (1.12)-(1.13); simply write $\mathbf{E}[(Z\wedge Ax)^q] \leqslant (Ax)^q\mathbf{P}(Z>x) + \mathbf{E}[(Z\wedge x)^q]$.

1.4 Organisation of the paper

In section 2 we introduce the main motivation for our results on non-linear conductances: the random cluster model (*RCM*) on Galton–Watson trees. In 2.2, we present the tree recursion arising in the model (we postpone the proof to Appendix A) and, in Section 2.3, we apply our main results on p-conductances in order to detail the critical behaviour of the *RCM* on a quenched GW tree.

The rest of the paper is divided as follows:

- In Section 3, we introduce a few preliminary (technical) results on branching processes.
- In Section 4 we focus on the estimates of the moments of C_n, in the case where $Z \sim \mu$ admits a finite moment of order q, and in particular we prove Theorem 1.6.
- In Section 5, we show the L^p convergence of the normalized conductance of Theorem 1.8 and we conclude the precise estimate of $\mathbf{E}[B_n]$ of Proposition 1.9.
- In Section 6, we prove 1.10, which treats the moments of C_n in the case $Z \sim \mu$ has an infinite moment of order p.
- In Appendix A we prove the RCM tree recursion presented in Section 2.2.
- In Appendix B we provide some technical proofs regarding branching processed with heavy tails, which are necessary for the estimates of Theorem 1.10.

2 The (critical) random cluster model on a quenched GW tree

The *Random Cluster Model* (RCM) or *FK-percolation*, introduced by Fortuin and Kasteleyn in [19], unifies a number of models in statistical physics, including the percolation model, the Ising or $\mathtt{q}$-states Potts models and the uniform spanning tree. Some of these models, when considered on trees, exhibit recursions as in (1.3), in particular the connection (sometimes called *survival*) probability of the RCM on a tree, see (2.2). We give a quick presentation of the model below, but we refer to [21] for a more complete overview of the RCM and its connection to other models; see also [18, Ch. 1].

Let us mention that one of the motivation for considering statistical mechanics models on trees is to apply the results to general random tree-like graphs, see e.g. [35] for an overview (see also [13, 16] in the context of the Ising model).

2.1 The random cluster model on a graph

For a finite graph $G = (V, E)$, the random cluster model is a Gibbs measure on percolation configurations $\omega \in \{0,1\}^E$, where in the configuration $\omega = (\omega_e)_{e \in E}$, an edge e is called *open* if $\omega_e = 1$ and *closed* if $\omega_e = 0$. We also denote $x \xleftrightarrow{\omega} y$ if x and y can be connected by an open path in ω. For two parameters $\mathtt{p} \in [0,1]$, $\mathtt{q} > 0$, the RCM measure $\mathbb{P}^G_{\mathtt{p},\mathtt{q}}$ is defined by

$$\mathbb{P}^G_{\mathtt{p},\mathtt{q}}(\omega) = \frac{1}{Z^G_{\mathtt{p},\mathtt{q}}} \mathtt{p}^{o(\omega)} (1-\mathtt{p})^{f(\omega)} \mathtt{q}^{k(\omega)} , \tag{2.1}$$

where $o(\omega) = \sum_{e \in E} \omega_e$ is the number of open edges, $f(\omega) = \sum_{e \in E} 1 - \omega_e$ is the number of closed edges and $k(\omega)$ is the number of *clusters* in ω, *i.e.* the number of connected components of the subgraph of G induced by ω. Here, $Z^G_{\mathtt{p},\mathtt{q}}$ is the partition function of the model, that is the constant that normalizes $\mathbb{P}_{\mathtt{p},\mathtt{q}}$ to a probability. Let us mention that when $\mathtt{q} \geqslant 1$, the model enjoys some monotonicity and correlation inequalities, see e.g [21, Ch. 3]; these are essential tools for many of the techniques developed for the study of the RCM, which is therefore often considered only for $\mathtt{q} \geqslant 1$.

Relation with percolation and Ising/Potts models. Notice that for $\mathtt{q} = 1$, one clearly recovers the percolation model with parameter $\mathtt{p}$. On the other hand, for $\mathtt{q} \in \{2,3,4,\dots\}$ the RCM can be coupled to the (ferromagnetic) $\mathtt{q}$-states Potts model with inverse temperature β verifying $e^{-\beta} = 1 - \mathtt{p}$, see [21, §1.4], defined by the Gibbs measure on configurations $\sigma \in \{1,2,\dots,\mathtt{q}\}^V$ by

$$\mu_{\beta,\mathtt{q}}(\sigma) = \frac{1}{\mathcal{Z}_{\beta,\mathtt{q}}} \exp\Big(\beta \sum_{(x,y)\in E} \mathbb{1}_{\{\sigma_x = \sigma_y\}}\Big) .$$

Notice that the case $\mathtt{q} = 2$ corresponds to the Ising model up to a change of parameter β, since then $\mathbb{1}_{\{\sigma_x=\sigma_y\}} = \frac{1}{2}(\sigma_x\sigma_y + 1)$.

Additionally, let us stress that in the coupling between the RCM and the $\mathtt{q}$-state Potts model, one has a direct relation between the two-point correlation functions for the Potts model and connection probabilities in the RCM, see [21, Thm. 1.16]:

$$\tau_{\beta,\mathtt{q}}(x,y) := \mu_{\beta,\mathtt{q}}(\sigma_x = \sigma_y) - \frac{1}{\mathtt{q}} = (1-\mathtt{q}^{-1})\mathbb{P}_{\mathtt{p},\mathtt{q}}\big(x \overset{\omega}{\longleftrightarrow} y\big)\,. \tag{2.2}$$

Notice here that in the definition of $\tau_{\beta,\mathtt{q}}(x,y)$, one compares the probability that two spins σ_x, σ_y are equal under the Potts measure $\mu_{\beta,\mathtt{q}}$ with the same probability if σ_x, σ_y were independent and uniform in $\{1,\dots,\mathtt{q}\}$.

Boundary conditions. Analogously to the Ising and Potts models, one may introduce a boundary condition when considering the random cluster model. Here, we will only focus on the *wired* boundary condition on the graph, which consists in contracting all vertices of the boundary ∂G into one single vertex denoted $\{\partial G\}$: we denote by $\bar{G}$ the resulting graph and $\bar{\mathbb{P}}^{\bar{G}}_{\mathtt{p},\mathtt{q}}$ the resulting RCM. In terms of the Potts model, this corresponds to putting all spins equal on the boundary; for the Ising model this is the model with plus (or minus) boundary condition.

2.2 The random cluster model on trees and its associated recursion

In the case of a tree T_n of depth n, the boundary ∂T_n corresponds to the leaves and $\bar{T}_n$ is the tree with all leaves identified to one vertex. In view of (2.2), the connection (or *survival*) probability

$$\pi_n = \pi_n^{\mathtt{p},\mathtt{q}}(\rho) := \bar{\mathbb{P}}^{\bar{T}_n}_{\mathtt{p},\mathtt{q}}(\rho \leftrightarrow \partial T_n)$$

is one of the key quantity of interest; note that it depends on the realization of the tree T and we will treat it for a quenched realization of T. In particular, in the case where $\mathtt{q} = 2$, one has that π_n is *equal* to the magnetization of the root in the Ising model on T_n with plus boundary condition on the leaves ∂T_n; a similar interpretation holds for π_n in the $\mathtt{q}$-Potts model if $\mathtt{q} \in \{2,3,\dots\}$, see (2.2). In particular, one is interested to know whether, depending on the parameters $\mathtt{p},\mathtt{q}$, the connection probability π_n vanishes or not as $n \to \infty$, and if it does, at which rate.

Let us mention that the RCM on d-ary (or Cayley) trees has been studied in several articles, see e.g. [21, §10.9-10] and references therein. In a nutshell, the results from [22] ([21] only treats the case $d = 2$) state that for every fixed $\mathtt{q} > 0$ a phase transition occurs at some $\mathtt{p}_c(q) \in (0,1)$ given by different formulas according to whether $\mathtt{q} \leqslant 2$ or $\mathtt{q} > 2$: if $\mathtt{q} \in (0,2]$, then $\mathtt{p}_c(\mathtt{q}) = \frac{\mathtt{q}}{d+\mathtt{q}-1}$ and the phase transition is continuous; if $\mathtt{q} > 2$, then $\mathtt{p}_q(\mathtt{q})$ is the unique value of $\mathtt{p}$ such that the polynomial $Q_{\mathtt{p},\mathtt{q}}(x) := (\mathtt{q}-1)x^{d+1} - (\mathtt{q}-1+\frac{\mathtt{p}}{1-\mathtt{p}})x^d + \frac{1}{1-\mathtt{p}}x - 1$ has a double root in $(0,1)$ (note that for $d = 2$, this gives $\mathtt{p}_c(q) = \frac{2\sqrt{\mathtt{q}-1}}{1+2\sqrt{q-1}}$), and the phase transition is discontinuous.

The random cluster model seems to have been considered on *d-regular* (random) graphs, see e.g. [4, 5, 23], which are locally d-ary trees. However, it does not seem to have been studied on more random graphs with no fixed degree, in particular, to the best of our knowledge, it has not been studied on random trees (for instance Galton–Watson trees). Our goal is to show that the connection probabilities verify some recursive relation on trees and apply our results of Section 1.3 to obtain information on the critical (or near and sub-critical) random cluster model.

The RCM iteration on trees. Let T_n be a tree of depth n with leaves only at generation n. We then define, for $u \in T_n$

$$\pi_n(u) = \pi_n^{\mathtt{p},\mathtt{q}}(u) := \bar{\mathbb{P}}_{\mathtt{p},\mathtt{q}}^{\bar{T}_n(u)}\big(u \leftrightarrow \partial T_n(u)\big)$$

the connection probability from the root to the leaves for the RCM inside the subtree $T_n(u)$, with *wired* boundary conditions. Then, we have the following recursion on the tree T_n, whose proof we present in Appendix A.

Proposition 2.1. *If $u \in \partial T_n$, we have $\pi_n(u) = 1$. If $u \notin \partial T_n$, then we have the following recursion:*

$$\varphi_{\mathtt{q}}(\pi_n(u)) = \prod_{y \leftarrow u} \varphi_{\mathtt{q}}\big(\gamma\, \pi_n(v)\big) \quad \textit{where} \quad \varphi_{\mathtt{q}}(x) := \frac{1-x}{1+(\mathtt{q}-1)x} \tag{2.3}$$

and $\gamma = \gamma_{\mathtt{p},\mathtt{q}} := \varphi_{\mathtt{q}}(1-\mathtt{p})$. Setting $B_n(u) := -\log \varphi_{\mathtt{q}}(\pi_n(u))$, we have $B_n(u) = +\infty$ if $u \in \partial T_n$ and, if $u \notin \partial T_n$ then we have the recursion: letting β be such that $1-\mathtt{p} = e^{-\beta}$,

$$B_n(u) = \sum_{v \leftarrow u} \psi_q^{-1}\big(\psi_{\mathtt{q}}(\beta)\psi_{\mathtt{q}}(B_n(v))\big)\,, \qquad \textit{where} \quad \psi_{\mathtt{q}}(x) := \frac{e^x-1}{e^x+\mathtt{q}-1} = \varphi_{\mathtt{q}}(e^{-x})\,. \tag{2.4}$$

Let us stress that when $\mathtt{q} = 2$, we have that $\psi_q(x) = \tanh(x)$, so one recovers the well-known Lyons' iteration for the Ising model, see [25] (see also [13, Lem. 2.3] for the specific form $\psi_q^{-1}\big(\psi_{\mathtt{q}}(\beta)\psi_{\mathtt{q}}(x)\big) = \tanh^{-1}(\tanh(\beta)\tanh(x))$ we have here).

Remark 2.2. As a side remark, let us note that the function $\varphi_{\mathtt{q}} : x \in [0,1] \mapsto \frac{1-x}{1+(\mathtt{q}-1)x}$ is an involution, *i.e.* $\varphi_{\mathtt{q}} \circ \varphi_{\mathtt{q}}(x) = x$ for any $x \in [0,1]$.

Properties of the recursion. Let us fix $\mathtt{q} > 0$ and, for every $\beta > 0$, let us introduce the function

$$g_\beta(x) = g_{\beta,\mathtt{q}}(x) := \frac{1}{\psi_{\mathtt{q}}(\beta)}\psi_{\mathtt{q}}^{-1}\big(\psi_{\mathtt{q}}(\beta)\psi_{\mathtt{q}}(x)\big)\,.$$

The normalization by $\psi_{\mathtt{q}}(\beta)$ is here to ensure that $g_\beta'(0) = 1$. With this notation, the recursion (2.4) becomes

$$B_n(u) = \psi_{\mathtt{q}}(\beta) \sum_{v \leftarrow u} g_\beta(B_n(v))\,, \tag{2.5}$$

and is similar to (1.3) with resistances $R_v := \psi_{\mathtt{q}}(\beta)^{-|v|}$, *i.e.* (1.8) with $R_n = \psi_{\mathtt{q}}(\beta)$.

Let us now stress that the function g_β is concave on $\mathbb{R}_+$ if and only if $\mathtt{q} \in (0,2]$; we will therefore focus on the case $\mathtt{q} \in (0,2]$. As a clue for this fact, notice that $g_\beta''(0) = \frac{\mathtt{q}-2}{\mathtt{q}}(1-\psi_{\mathtt{q}}(\beta))$, which is negative if $\mathtt{q} < 2$, equal to zero if $\mathtt{q} = 2$ and positive if $\mathtt{q} > 2$.

Notice also that g_β is bounded and that, as $x \downarrow 0$

$$\begin{aligned} \text{if } \mathtt{q} \in (0,2)\,, \qquad & g_\beta(x) = x - \frac{\mathtt{q}-2}{\mathtt{q}}(1-\psi_{\mathtt{q}}(\beta))x^2(1+o(1)) \\ \text{if } \mathtt{q} = 2\,, \qquad & g_\beta(x) = x - \frac{1}{3}(1-\psi_{\mathtt{q}}(\beta)^2)x^3(1+o(1)) \end{aligned} \tag{2.6}$$

so we indeed have (1.5) with $q = 2$ (and $s = 1$) if $\mathtt{q} \in (0,2)$ and with $q = 3$ (and $s = 3$) if $\mathtt{q} = 2$; this was noticed in [33] in the case of the Ising model (with plus boundary condition, which correspond to our wired boundary condition for the RCM).

Remark 2.3. Let us stress that a similar recursion also arises for the return probability of a random walk on a tree T_n equipped with resistances $(R_v)_{v\in T_n}$. Consider the "non-descending" probability

$$\pi_n(u) = \mathbf{P}_u\big(\tau_{\underline{u}} < \tau_{\partial T_n(u)}\big)\,,$$

where $\underline{u}$ is the parent of u and τ_A is the hitting time of the set A. The quantities $\pi_n(u)$ help encode the recurrence/transience of the graph and can be useful to estimate the Green function on the tree. Then, setting $b_n(u) = \frac{1-\pi_n(u)}{\pi_n(u)}$ (in particular $\pi_n(u) = (1+b_n)^{-1}$), one may verify that one gets the recursion

$$b_n(u) = \sum_{v\leftarrow u} \frac{R_v}{R_u} \frac{b_n(v)}{1+b_n(v)}\,,$$

so $b_n(v)$ is exactly the 2-conductance, $b_n(u) = \mathcal{C}_2(u \leftrightarrow \partial T_n(u))$.

2.3 Main results: critical point and critical behavior

2.3.1 The critical point and critical behavior at (and below) criticality

For any *fixed* β (recall $e^{-\beta} = 1-\mathtt{p}$), we have that (2.5) is exactly the recursion (1.8) with $R_n \equiv R = \psi_{\mathtt{q}}(\beta)$, and (1.5) holds with $q=2$ ($s=1$) if $\mathtt{q}\in(0,2)$ and with $q=3$ ($s=2$) if $\mathtt{q}=2$. First of all, Theorems 1.6-1.10 allows us to identify the critical point.

For any $\beta > \beta_c$ (*i.e.* $\mathtt{p} = 1-e^{-\beta} > \mathtt{p}_c$), define

$$\pi_n(\beta) = \pi_n^{\mathtt{q}}(\beta, T) := \bar{\mathbb{P}}^{\bar{T}_n}_{\mathtt{p}_n,\mathtt{q}}(\rho \leftrightarrow \partial T_n)\,, \qquad \pi(\beta) = \lim_{n\to\infty} \pi_n(\beta)\,, \tag{2.7}$$

where the limit exists by monotonicity.

Recall that $B_n = -\log \varphi_{\mathtt{q}}(\pi_n)$, so in particular π_n goes to 0 if and only if B_n goes to 0.

Theorem 2.4. *Let $\mathtt{q}\in(0,2]$, define $\pi_n = \bar{\mathbb{P}}^{\bar{T}_n}_{\mathtt{p},\mathtt{q}}(\rho\leftrightarrow\partial T_n)$ the connection (or survival) probability, and let $e^{-\beta} = 1-\mathtt{p}$. Let β_c be defined by the relation $m\psi_{\mathtt{q}}(\beta_c) = 1$, with $\psi_{\mathtt{q}}$ in (2.4). Then we have that, in $\mathbf{P}$-probability,*

$$\lim_{n\to\infty} \pi_n(\beta) = 0 \quad \text{if } \beta \leqslant \beta_c \qquad \text{and} \qquad \liminf_{n\to\infty} \pi_n(\beta) > 0 \quad \text{if } \beta > \beta_c\,.$$

In particular, $\pi(\beta) > 0$ if and only if $\beta > \beta_c$.

Remark 2.5. Using the definition of $\psi_{\mathtt{q}}(\beta)$ and using that $\varphi_{\mathtt{q}}$ is an involution, we can rewrite the relation $m\psi_{\mathtt{q}}(\beta_c) = 1$ as $e^{-\beta_c} = \varphi_{\mathtt{q}}(\frac{1}{m}) = \frac{m-1}{m+\mathtt{q}-1}$, or equivalently, $\mathtt{p}_c = 1-e^{-\beta_c} = \frac{\mathtt{q}}{\mathtt{q}+m-1}$.

Theorem 2.4 therefore identifies the critical point $\mathtt{p}_c = 1-e^{-\beta_c}$ for the random cluster model with $\mathtt{q}\in(0,2]$ on a (random) Galton–Watson tree. The case $\mathtt{q}>2$ is understood in the case of a d-ary tree but remains somehow mysterious on a random tree and let us briefly comment on it. In fact, when $\mathtt{q}>2$ the phase transition on a d-ary tree happens at a certain $\mathtt{p}_c(\mathtt{q})$ characterized by being the only (double) root in $(0,1)$ of a certain polynomial $Q^{(d)}_{\mathtt{p},\mathtt{q}}(x)$ (see 2.2). In terms of the recursion (2.5), this correspond to finding $\mathtt{p}$ such that $dg_{\beta,\mathtt{q}}$ (see (2.2)) intersects the identity function exactly once. For the RCM on a Galton–Watson tree, if $\mathtt{q}>2$, the *random* recursion (2.5) is still valid but the function g_β is then not concave anymore, and it is not clear whether a characterization of $p_c(\mathtt{q})$ in terms of a certain (averaged?) polynomial remains valid.

Additionally, our results of Section 1.3 directly give precise estimates for the critical case $\beta=\beta_c$. Let us stress that the survival probability on a quenched Galton–Watson tree has recently been studied for percolation (*i.e.* $\mathtt{q}=1$), see [2, 27], where the results are more precise in the case of a heavy-tail but the techniques used are very different. On the other hand, the result for the Ising model (or for the random cluster model with $\mathtt{q} \neq 1$) does not seem to be known.

The following result is a direct consequence of Theorem 1.8-Proposition 1.9 in the case where $Z \sim \mu$ admits a finite moment of order q and of Theorem 1.10 in the case where $\mathbf{E}[Z^q] = +\infty$. Notice that $-\log \varphi_{\mathtt{q}}(x) \sim \mathtt{q}x$ as $x \downarrow 0$, so $\pi_n \sim \mathtt{q}^{-1} B_n$ when B_n goes to 0. Additionally, note that from (2.6) we get that $\kappa_g := \lim_{x\downarrow 0} \frac{1}{x^2}(x - g_\beta(x)) = \frac{\mathtt{q}-2}{\mathtt{q}}(1-\psi_{\mathtt{q}}(\beta))$ if $\mathtt{q} \in (0,2)$ and $\kappa_g := \lim_{x\downarrow 0} \frac{1}{x^3}(x - g_\beta(x)) = \frac{1}{3}(1-\psi_{\mathtt{q}}(\beta)^2)$ if $\mathtt{q} = 2$.

Theorem 2.6 (Critical case). *Let $\beta = \beta_c$, with β_c defined by the relation $m\psi_{\mathtt{q}}(\beta_c) = 1$. We then have the following asymptotic results for π_n. Letting $q = 2$ $(s = 1)$ in the case $\mathtt{q} \in (0,2)$ and $q = 3$ $(s = 2)$ in the case $\mathtt{q} = 2$, we have the following:*

- *if $\mathbf{E}[Z^q] < +\infty$, we have, almost surely and in L^q,*

$$\lim_{n\to\infty} n^{\frac{1}{q-1}} \pi_n = \alpha_{\mathtt{q}} W \qquad \textit{with} \quad \alpha_{\mathtt{q}} := \begin{cases} \frac{\mathtt{q}}{\mathtt{q}-2}\frac{m}{m-1}\mathbf{E}[W^2]^{-1} & \textit{if } \mathtt{q} \in (0,2)\,, \\ \frac{\sqrt{3}}{\sqrt{2}}\frac{m}{\sqrt{m^2-1}}\mathbf{E}[W^3]^{-1/2} & \textit{if } \mathtt{q} = 2\,. \end{cases}$$

- *if $c_1 L(x) x^{q-\alpha} \leqslant \mathbf{E}[(Z \wedge x)^q] \leqslant c_2 L(x) x^{q-\alpha}$ for some $\alpha \in (1,q]$, some slowly varying function $L(\cdot)$ and constants $c_1, c_2 > 0$, then*

$$\left(\frac{\pi_n}{h^{-1}(1/n)}\right)_{n\geqslant 1} \quad \textit{is tight in } (0,+\infty),$$

  with $h^{-1}(\cdot)$ an asymptotic inverse of $L(1/x)x^{\alpha-1}$ as $x \downarrow 0$.

We complete the above result with the subcritical case $\beta < \beta_c$: our results imply a sharp estimate on the exponential decay rate of the survival probability. We can summarize our results as follows.

Theorem 2.7 (Sub-critical case). *Let $\mathtt{q} \in (0,2]$ and let β_c defined by the relation $m\psi_{\mathtt{q}}(\beta_c) = 1$. Then, if $\beta < \beta_c$, we have that $\lim_{n\to\infty} (\pi_n)^{1/n} = m\psi_{\mathtt{q}}(\beta) < 1$ in $\mathbf{P}$-probability.*

In fact, Section 1.3 gives much more precise results, that we describe informally as follows: letting $q = 2$ if $\mathtt{q} \in (0,2)$ and $q = 3$ if $\mathtt{q} = 2$, we have

- if $\mathbf{E}[Z^q] < +\infty$, we have $\lim_{n\to\infty} (m\psi_{\mathtt{q}}(\beta))^{-n}\pi_n = \alpha_{\mathtt{q}}(\beta) W$ for some constant $\alpha_{\mathtt{q}}(\beta)$;
- if $\mathbf{E}[(Z\wedge x)^q] = n^{c+o(1)}$ for some constant c, then $C \geqslant (m\psi_{\mathtt{q}}(\beta))^{-n}\pi_n \geqslant n^{-c'}$ as $n \to \infty$.

2.3.2 Near-supercritical connection probability

Once the critical point β_c is identified, it is also natural to consider the *near-critical regime*, *i.e.* take an inverse temperature that may depend on n, $\beta = \beta_n$, with $\lim_{n\to\infty} \beta_n = \beta_c$. This is why we allowed in (1.8) the resistance $R_n = \psi_{\mathtt{q}}(\beta_n)$ to also depend on n. In fact, Theorem 2.6 can be generalized to the whole near-critical regime. Let us however stress that in (2.5), the function $g_\beta = g_{\beta_n}$ also depends on n, but the bounds (1.5) and the expansion (2.6) are uniform in n (as soon as $\beta_n \to \beta_c$), so Theorem 2.6 can indeed be extended to the whole near-critical regime as a consequence of Theorems 1.8 and 1.10.

For simplicity of the exposition (and for later use), let us state only the case where $\lim_{n\to\infty} \beta_n = \beta_c$, but in the near-supercritical regime, that is when $m\psi_{\mathtt{q}}(\beta_n) - 1 \gg \frac{1}{n}$. Note that we have,

$$m\psi_{\mathtt{q}}(\beta_n) - 1 = m(\psi_{\mathtt{q}}(\beta_n) - \psi_{\mathtt{q}}(\beta_c)) \sim \frac{(m-1)(m+\mathtt{q}-1)}{m\mathtt{q}}(\beta_n - \beta_c)\,, \qquad \text{as } \beta_n \to \beta_c\,,$$

since $\psi'_{\mathtt{q}}(\beta_c) = \frac{\mathtt{q}\, e^{\beta_c}}{(e^{\beta_c}+\mathtt{q}-1)^2} = \frac{m-1}{m^2 \mathtt{q}}(m+\mathtt{q}-1)$, recalling also that $e^{\beta_c} = \frac{m+\mathtt{q}-1}{m-1}$, see Remark 2.5. Using also (1.10) when $m\psi_{\mathtt{q}}(\beta_n) - 1 \gg \frac{1}{n}$ we have

$$a_n := \sum_{k=1}^{n} (m\psi_{\mathtt{q}}(\beta_n))^{-ks} \sim \frac{1}{(m\psi_{\mathtt{q}}(\beta_n))^s - 1} \sim s^{-1}(m\psi_{\mathtt{q}}(\beta_n) - 1)^{-1} ,$$

we directly obtain the following result from Theorem 1.8-Proposition 1.9 and Theorem 1.10.

Theorem 2.8 (Near-supercritical)**.** *Let $(\beta_n)_{n\geqslant 0}$ be such that $\lim_{n\to\infty} \beta_n = \beta_c$, with also $\lim_{n\to\infty} n(\beta_n - \beta_c) = +\infty$. Define $\pi_n(\beta_n) := \bar{\mathbb{P}}^{T_n}_{\mathtt{p}_n,\mathtt{q}}(\rho \leftrightarrow \partial T_n)$, with $\mathtt{p}_n = 1 - e^{-\beta_n}$ (which goes to $\mathtt{p}_c$). Then, letting $q = 2$ if $\mathtt{q} \in (0,2)$ and $q = 3$ if $\mathtt{q} = 2$, we have the following:*

- *If $\mathbf{E}[Z^q] < +\infty$, then we have that, in L^q,*

$$\lim_{n\to\infty} \frac{\pi_n(\beta_n)}{(\beta_n - \beta_c)^{\frac{1}{q-1}}} = \tilde{\alpha}_{\mathtt{q}} W , \qquad \text{with } \tilde{\alpha}_{\mathtt{q}} := \begin{cases} \frac{m+\mathtt{q}-1}{\mathtt{q}-2} \mathbf{E}[W^2]^{-1} & \text{if } \mathtt{q} \in (0,2) , \\ \sqrt{\frac{3m}{2}}\, \mathbf{E}[W^3]^{-1/2} & \text{if } \mathtt{q} = 2 . \end{cases}$$

- *If $c_1 L(x) x^{q-\alpha} \leqslant \mathbf{E}[(Z \wedge x)^q] \leqslant c_2 L(x) x^{q-\alpha}$ for some $\alpha \in (1,q]$, some slowly varying function $L(\cdot)$ and constants $c_1, c_2 > 0$, then*

$$\Big(\frac{\pi_n}{h^{-1}(\beta_n - \beta_c)} \Big)_{n\geqslant 1} \quad \text{is tight in } (0,+\infty),$$

with $h^{-1}(\cdot)$ an asymptotic inverse of $L(1/x)x^{\alpha-1}$ as $x \downarrow 0$.

Notice that in Theorem 2.8, one can take $\beta_n \downarrow \beta_c$ arbitrarily slowly. We can therefore deduce the following critical behavior for the limiting survival probability $\pi(\beta)$ of the RCM on a quenched Galton–Watson tree.

Corollary 2.9. *Recall the definition (2.7) of $\pi(\beta) = \pi(\beta, T)$. Then, letting $q = 2$ if $\mathtt{q} \in (0,2)$ and $q = 3$ if $\mathtt{q} = 2$, we have the following critical behavior:*

- *If $\mathbf{E}[Z^q] < +\infty$, then $\pi(\beta) \sim \tilde{\alpha}_{\mathtt{q}} W\, (\beta - \beta_c)^{\frac{1}{q-1}}$, with $\tilde{\alpha}_{\mathtt{q}}$ as in Theorem 2.8.*
- *If $\mathbf{E}[(Z \wedge x)^q] \asymp L(x) x^{q-\alpha}$, then $\pi(\beta) \asymp h^{-1}(\beta - \beta_c)$, with $h^{-1}(\cdot)$ an asymptotic inverse of $L(1/x)x^{\alpha-1}$ as $x \downarrow 0$.*

Let us stress that Corollary 2.9 is related to existing results. For instance, [17] gives the critical behavior of the Ising model (with external field) on a tree-like graph. Our techniques are quite different, and we improve the results here by giving a sharp behavior in the case where $\mathbf{E}[Z^q] < +\infty$ (with the correct *random* constant); and treating a more general set-up in the heavy-tail case. Let us also note that [28] treats the survival probability of Bernoulli percolation on a GW tree and our Corollary 2.9 improves their main result in two directions: (i) we recover the first-order asymptotic with a weaker moment condition ($\mathbf{E}[Z^2] < +\infty$ instead of $\mathbf{E}[Z^{3+\eta}] < +\infty$); (ii) we treat the case of a GW with heavy tails.

3 Some useful preliminaries on Branching processes

In this section, we regroup some technical tools that will be used throughout the proofs. Some of them are standard, for instance some L^q inequalities for sums of independent random variables (see Section 3.1), but we recall them for convenience. Others are very natural, estimating the tail of (truncated) branching processes with heavy-tails (see Section 3.2), but we were not able to find them in the literature so we provide a proof in Appendix B.

3.1 Useful L^q inequalities for sums of independent random variables

Let us collect here some estimates on the L^q-norm of sums of independent random variables, which turn out to be extremely useful in the context of branching processes.

One of the main inequalities that we will use is Lemma 1.4 in [24], which mostly relies on the Marcinkiewicz–Zigmund inequality (see e.g. [10, Ch. 10.3]).

Lemma 3.1 (Lemma 1.4 in [24]). *Let $q > 1$ and let $(X_i)_{i\geqslant 1}$ be independent and* centered *random variables, with a finite moment of order $q > 1$. Then, for any $n \geqslant 1$, we have*

$$\mathbf{E}\Big[\Big|\sum_{i=1}^{n} X_i\Big|^q\Big] \leqslant (A_q)^q n^{\theta_q - 1} \sum_{i=1}^{n} \mathbf{E}\big[|X_i|^q\big]\,,$$

with $\theta_q := \max(1, \frac{q}{2}) < q$ and $A_q := 2\lceil \frac{q}{2}\rceil^{1/2}$ (in particular $A_q = 2$ if $q \in (1, 2]$).

Let us mention for completeness the following (simpler) result due to Neveu [30], in the case where $q \in (1, 2]$.

Lemma 3.2 ([30]). *Let $q \in (1, 2]$. For a* non-negative *r.v. X with a finite moment of order q, define $V_q(X) := \mathbf{E}[X^q] - \mathbf{E}[X]^q$. Then, if X, Y are independent non-negative r.v. with a finite moment of order q, we have that $V_q(X + Y) \leqslant V_q(X) + V_q(Y)$.*

In particular, if $(X_i)_{i\geqslant 1}$ are independent non-negative *random variables with a finite moment of order q, then for any $n \geqslant 1$ we have*

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{n} X_i\Big)^q\Big] \leqslant \mathbf{E}\Big[\sum_{i=1}^{n} X_i\Big]^q + \sum_{i=1}^{n} \mathbf{E}\big[X_i^q\big]\,.$$

As a direct consequence of Lemma 3.1, we state the following lemma that will be convenient in the context of branching processes.

Lemma 3.3. *Let $q > 1$, let X be a* non-negative *random variable and N be a $\mathbb{N}$-valued random variable, both with a finite moment of order $q > 1$. If $(X_i)_{i\geqslant 1}$ are i.i.d. random variables with the same distribution as X and independent of N, then we have,*

$$\Big\|\sum_{i=1}^{N} X_i\Big\|_q \leqslant A_q\big(\|N\|_{\theta_q}\big)^{\frac{\theta_q}{q}}\big(\|X - \mathbf{E}[X]\|_q\big) + \|N\|_q \mathbf{E}[X]\,,$$

with $\theta_q := \max(1, \frac{q}{2}) < q$ and $A_q := 2\lceil \frac{q}{2}\rceil^{1/2}$ as in Lemma 3.1.

Proof. First of all, letting $\bar{X}_i = X_i - \mathbf{E}[X]$, we have that

$$\Big\|\sum_{i=1}^{N} X_i\Big\|_q = \Big\|\sum_{i=1}^{N} \bar{X}_i + N\mathbf{E}[X]\Big\|_q \leqslant \Big\|\sum_{i=1}^{N} \bar{X}_i\Big\|_q + \|N\|_q \mathbf{E}[X]\,.$$

Now, using Lemma 3.1 conditionnally on N, we have that

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{N} \bar{X}_i\Big)^q\Big] \leqslant (A_q)^q \mathbf{E}\big[N^{\theta_q}\big]\mathbf{E}\big[|\bar{X}|^q\big]\,,$$

so that $\big\|\sum_{i=1}^{N} \bar{X}_i\big\|_q \leqslant A_q \mathbf{E}[N^{\theta_q}]^{1/q}\|X - \mathbf{E}[X]\|_q$. This concludes the proof. ∎

As a consequence of Lemma 3.3, we get the following bound for branching processes. Let $(Z_k)_{k\geqslant 0}$ be a branching process with offspring distribution $Z\sim\mu$ which admits a moment of order $q>1$, and mean denoted by $m:=\mathbf{E}[Z]$. Denote $W_k:=\frac{1}{m^k}Z_k$ the usual martingale and let $W:=\lim_{k\to\infty}W_k$ a.s., which is in L^q. Then, there are two constants $c_1(q):=A_q\mathbf{E}[W^{\theta_q}]^{1/q}$ and $c_2:=c_1+\|W\|_q$, such that the following holds: for any $k\geqslant 1$, if $(X_i)_{i\geqslant 0}$ are non-negative i.i.d. random variables independent of Z_k (with common distribution X) with a finite q-moment,

$$\Big\|\frac{1}{m^k}\sum_{i=1}^{Z_k}X_i\Big\|_q\leqslant c_1 m^{-k(1-\frac{\theta_q}{q})}\|X\|_q+c_2\mathbf{E}[X]\,. \tag{3.1}$$

Indeed, a simple application of Lemma 3.3 gives that

$$\Big\|\frac{1}{m^k}\sum_{i=1}^{Z_k}X_i\Big\|_q\leqslant A_q m^{-k}\big(m^k\|W_k\|_{\theta_q}\big)^{\frac{\theta_q}{q}}\|X-\mathbf{E}[X]\|_q+\|W_k\|_p\mathbf{E}[X]\,.$$

Using that $\|X-\mathbf{E}[X]\|_q\leqslant\|X\|_q+\mathbf{E}[X]$, this directly concludes (3.1), since $\sup_k\|W_k\|_\alpha=\|W\|_\alpha$ for any $\alpha\in[1,q]$ ($(W_k^\alpha)_{k\geqslant 0}$ is a submartingale), and also $m^{-k(1-\frac{1}{q}\theta_q)}\leqslant 1$ since $\theta_q<q$.

3.2 Supercritical branching processes with (truncated) heavy tails

Let μ be the reproduction law of a super-critical branching process, let $Z\sim\mu$ and note $m:=\mathbf{E}[Z]$ its mean. For $t>1$ some (large) fixed parameter, we define $\tilde Z:=Z\wedge t$ with distribution denoted $\tilde\mu$ and assume that t is large enough so that $\tilde m:=\mathbf{E}[\tilde Z]>1$. We define a *truncated* branching process with reproduction law *tilde*μ. As above, let $(Z_k)_{k\geqslant 0}$, resp. $\tilde Z_k$, be a branching process with offspring distribution $Z\sim\mu$, resp. $\tilde Z\sim\tilde\mu$, and denote $W_k:=\frac{1}{m^k}Z_k$ and $\tilde W_k:=\frac{1}{\tilde m^k}\tilde Z_k$ the corresponding martingales.

Notice that if Z verifies (1.13), then we have $\mathbf{E}[(\tilde Z)^q]\leqslant c_2L(t)t^{q-\alpha}$. We will actually work with the following bound on the tail of Z and $\tilde Z$ (which are equivalent to (1.13) if $\alpha<q$): for any $x\geqslant 1$

$$\mathbf{P}(Z>x)\leqslant c_2L(x)x^{-\alpha}\,,\qquad \mathbf{P}(\tilde Z>x)\leqslant c_2L(x)x^{-\alpha}\mathbb{1}_{\{x<t\}}\,. \tag{3.2}$$

We then have the following result, which is a variation of Theorem 1 (or Corollary 12) in [15].

Proposition 3.4. *Assume that* (3.2) *holds for the offspring distribution. Then, there are constants c,c' such that, uniformly in $\ell\geqslant 1$*

$$\mathbf{P}(W_\ell>x)\leqslant cL(x)x^{-\alpha}\,,$$

and, provided that t is large enough,

$$\mathbf{P}(\tilde W_\ell>x)\leqslant\begin{cases}cL(x)x^{-\alpha} & \textit{for all } x\geqslant 1\,,\\ t^{-c'x/t} & \textit{for all } x\geqslant t\,.\end{cases}$$

Let us mention that in [15] the authors provide a uniform upper and lower bound on the tail probability $\mathbf{P}(W_\ell>x)$, but with a slightly stronger assumption that the tail is *dominated varying*, *i.e.* $\mathbf{P}(Z>x/2)/\mathbf{P}(Z>x)$ remains bounded (it is for instance implied if we assume the counterpart lower bound $\mathbf{P}(Z>x)\geqslant c_1'L(x)x^{-\alpha}$); let us stress that the proof of Lemma 11-Corollary 12 in [15] actually only requires that the tail is upper bounded by a dominated varying function. On the other hand, the case of a *truncated* branching process does not seem to have been treated in the literature. We give full (self-contained) proof of both upper bounds in Proposition 3.4 in Appendix B.

4 Estimates on moments of C_n in the case $\mathbf{E}[Z^q] < +\infty$

4.1 Proof of Theorem 1.6

We decompose the proof of Theorem 1.6 into an upper bound, which requires only that $Z \sim \mu$ admits a finite first moment, and a lower bound, which requires a finite moment of order q for Z.

Let us denote $C_n(u)$ the conductance of the subtree $T_n(u)$, equipped with the same resistances. Then setting $\phi(u) = R_u C_n(u)$ we have the following recursion: for any $u \notin \partial T_n$,

$$\phi(u) = \sum_{v \leftarrow u} \frac{R_u}{R_v} \frac{\phi(v)}{(1+\phi(v)^s)^{1/s}} . \tag{4.1}$$

4.1.1 Upper bound on $\mathbf{E}[C_n]$

We start from the recursion (4.1). We proceed as in [3, §6.2.1] (see also [9, Lem. 3.1] in the case $p = q = 2$). Using the branching property and the fact that $R_u = R^{-|u|}$, we get that for any $u \notin \partial T_n$ and $v \leftarrow u$,

$$\mathbf{E}[\phi(u)] = mR_n \, \mathbf{E}\left[\frac{\phi(v)}{(1+\phi(v)^s)^{1/s}}\right] \leqslant mR_n \frac{\mathbf{E}[\phi_p(v)]}{(1+\mathbf{E}[\phi(v)]^s)^{1/s}} ,$$

where we have used that the function $x \mapsto \frac{x}{(1+x^s)^{1/s}}$ is concave for the last inequality. Denoting $w_k := \mathbf{E}[\phi(u)]^{-s}$ if $|u| = k$, we have the following recursive inequality:

$$w_k \geqslant (mR_n)^{-s}(1 + w_{k+1}).$$

Iterating, we finally get that

$$w_0 := \mathbf{E}[C_n]^{-s} \geqslant \sum_{k=1}^{n} (mR_n)^{-ks} + (mR_n)^{-ns} w_n ,$$

Recalling that we have defined $a_n := \sum_{k=1}^n (mR_n)^{-ks}$, we get that $\mathbf{E}[C_n]^{-s} \geqslant a_n$, that is $\mathbf{E}[C_n] \leqslant a_n^{-1/s}$.

In particular, using Markov's inequality, we have $\mathbf{P}(a_n^{1/s} C_n \geqslant K) \leqslant K^{-1}$, showing the first part of the tightness of $(a_n^{1/s} C_n)_{n \geqslant 0}$.

4.1.2 Lower bound on C_n, $\mathbf{E}[C_n]$

For the lower bound on $C_n = \mathcal{C}_p(\rho \leftrightarrow \partial T_n)$, we actually prove an upper bound for the p-resistance $\mathcal{R}_p(\rho \leftrightarrow \partial T_n)$; we proceed as in [3, §6.2.2]. Using Thomson's principle (1.2), we obtain an upper bound on the p-resistance simply by computing the energy of a well-chosen unit flow θ from ρ to ∂T_n. The uniform flow $\hat{\theta}$ on T is a natural choice (as in [32, Lem. 2.2], see also [9, Lem. 3.3]): define

$$\hat{\theta}(u,v) = \frac{Z_n(v)}{Z_n} ,$$

where $Z_n(v)$ is the number of descendants of v at generation n and $Z_n := Z_n(\rho)$. Then, $\hat{\theta}(u,v)$ is easily seen to be a unit flow from the root to the leaves of T_n, and we have

$$\mathcal{R}_p(\rho \leftrightarrow \partial T_n)^s \leqslant \mathcal{E}_p(\hat{\theta}) = \sum_{k=1}^{n} \sum_{|v|=k} (R_v)^s \Big(\frac{Z_n(v)}{Z_n}\Big)^q$$

Defining $W_n(v) := \frac{Z_n(v)}{\mathbf{E}[Z_n(v)]} = m^{-(n-k)} Z_n(v)$ for $|v| = k$ and $W_n := W_n(\rho)$, using that $R_u = R^{-|u|}$ and recalling that $s = q-1$, we can rewrite the upper bound as

$$\mathcal{R}_p(\rho \leftrightarrow \partial T_n)^s \leqslant \frac{1}{(W_n)^q} \sum_{k=1}^{n} (mR_n)^{-ks} \frac{1}{m^k} \sum_{|v|=k} W_n(v)^q \,. \tag{4.2}$$

Now, for any $\varepsilon > 0$, we can bound

$$\mathbf{P}\Big(\mathcal{R}_p(\rho \leftrightarrow \partial T_n)^s \geqslant \varepsilon^{-1} a_n\Big) \\ \leqslant \mathbf{P}\big(W_n \leqslant \varepsilon^{1/2q}\big) + \mathbf{P}\bigg(\sum_{k=1}^{n} (mR_n)^{-ks} \frac{1}{m^k} \sum_{|v|=k} W_n(v)^q \geqslant \varepsilon^{-1/2} a_n\bigg) \,.$$

Now, for the first term, since the martingale W_n converges a.s. to some non-degenerate random variable W with $\mathbf{P}(W > 0) = 1$, we get that $\mathbf{P}(W_n \leqslant \varepsilon^{1/2q}) \leqslant \delta_\varepsilon$ uniformly in n, for some $\delta_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$. For the second probability, using Markov's inequality (together with the branching property), we get that

$$\mathbf{P}\bigg(\sum_{k=1}^{n} (mR_n)^{-ks} \frac{1}{m^k} \sum_{|v|=k} W_n(v)^q \geqslant \varepsilon^{-1/2} a_n\bigg) \leqslant \frac{\varepsilon^{1/2}}{a_n} \sum_{k=1}^{n} (mR_n)^{-ks} \mathbf{E}\big[(W_{n-k})^q\big] \,.$$

Assuming that Z admits a finite moment of order q, we get that $(W_n)_{n\geqslant 1}$ is bounded in L^q (see [7]), so that there is a constant c_q such that $\mathbf{E}[(W_\ell)^q] \leqslant c_q$ uniformly in $\ell \geqslant 0$. This shows that the last probability is bounded by $c_q \varepsilon^{1/2}$, recalling the definition of a_n.

All together, and recalling that $C_n := \mathcal{C}_p(\rho \leftrightarrow \partial T_n) = \mathcal{R}_p(\rho \leftrightarrow \partial T_n)^{-1}$, we have obtained that for any $\varepsilon > 0$,

$$\mathbf{P}\big(C_n \leqslant \varepsilon^{1/s} (a_n)^{-1/s}\big) \leqslant \delta_\varepsilon + c_q \varepsilon^{1/2} \,, \tag{4.3}$$

which shows the scond part of the tightness of $(a_n^{1/s} C_n)_{n\geqslant 0}$.

For a lower bound on $\mathbf{E}[C_n]$, choose $\varepsilon := \varepsilon_q > 0$ sufficiently small so that $\delta_\varepsilon + c_q \varepsilon^{1/2} \leqslant 1/2$: we end up with

$$\mathbf{E}[C_n] \geqslant \frac{1}{2} \varepsilon_q^{1/s} (a_n)^{-1/s} \,,$$

which is the desired lower bound.

Remark 4.1. Note that more generally, the proof shows that, for any $u \in T_n$ with $|u| = k$,

$$c_p (a_{n-k})^{-1/s} \leqslant \varphi_k := \mathbf{E}[\phi(u)] \leqslant (a_{n-k})^{-1/s} \,.$$

From this, we get that there exist constants c_1, c_2 such that for any $1 \leqslant k \leqslant n$,

$$c_1 \leqslant \frac{\varphi_k}{\varphi_{k-1}} \leqslant c_2 \,.$$

Indeed, we have $a_{n-k} \leqslant a_{n-k+1} = (mR_n)^{-s} + (mR_n)^{-s} a_{n-k} \leqslant (1 + (mR_n)^{-s}) a_{n-k}$, so that $c_p \leqslant \varphi_k/\varphi_{k-1} \leqslant c_p^{-1} (1 + (mR_n)^{-s})^{1/s}$; for this, we need to assume that $\inf_{n\geqslant 0} R_n > 0$.

4.2 Control of the higher moments of C_n

We now give a technical result that control the moments of C_n, which is useful for the sequel.

Proposition 4.2. *Let $p > 1$ and let $q = \frac{p}{p-1}$ be its conjugate exponent. If $Z \sim \mu$ admits a moment of order $r \geqslant q$, then so does C_n. Additionally, if $\sup_n mR_n < m^{\max(\frac{r-1}{r}, \frac{1}{2})}$ and if $\inf_n R_n > 0$, then for any $r' \in (1, r]$, we have*

$$\sup_{n \geqslant 1} \frac{\mathbf{E}\big[(C_n)^{r'}\big]}{\mathbf{E}[C_n]^{r'}} < +\infty\,.$$

The proof would work similarly as in [9, Lem. 3.2] if r were an integer, but we aim at generalizing their result here (both to non-integer moments r, and to a p-conductance with $p \neq 2$). We take inspiration from [24, Prop. 1.3], which deals with the rate of convergence in L^q for the usual martingale W_n.

Proof. First of all, let us notice that we only have to prove the claim for $r' = r$, since then we can apply Jensen's inequality to get that $\mathbf{E}[C_n^{r'}] \leqslant \mathbf{E}[C_n^r]^{r'/r}$. Similarly, we only need to control the L^r norm of C_n, and show that

$$\|C_n\|_r \leqslant c_r \mathbf{E}[C_n]\,, \tag{4.4}$$

for some universal constant c_r.

Recall that we define $\phi(u) := R_u C_n(u) = (R_n)^{-|u|} C_n(u)$. Now, using (4.1), we have the general upper bound

$$0 \leqslant \phi(u) \leqslant R_n \sum_{v \leftarrow u} \phi(v)\,.$$

First of all, notice that we easily get by iteration that $\phi(v)$, hence $C_n(v)$, admits a finite moment of order r. If we iterate the above inequality for k generations, we have

$$C_n = \phi(\rho) \leqslant (R_n)^k \sum_{|v|=k} \phi(v) = (R_n)^k \sum_{i=1}^{Z_k} \phi_k^{(i)}\,,$$

where $(\phi_k^{(i)})_{i \geqslant 1}$ are i.i.d. copies of $\phi(v)$ for $|v| = k$ (independent of T_k, and in particular independent of Z_k).

Therefore, applying Lemma 3.3 (more precisely (3.1)) and denoting $\varphi_k = \mathbf{E}[\phi_k]$ (and $\varphi_0 = \mathbf{E}[C_n]$), we obtain that

$$\begin{aligned}\|C_n\|_r = \|\phi_0\|_r &\leqslant c_1 (mR_n)^k m^{-k(1-\frac{\theta_r}{r})} \|\phi_k\|_r + c_2 (mR_n)^k \varphi_k \\ &\leqslant c_1 (mR_n)^k m^{-k(1-\frac{\theta_r}{r})} \|\phi_k\|_r + c_2 2^k \varphi_k\,.\end{aligned}$$

Now, let us fix $k = k_r$ such that $\gamma_r := (c_1)^{1/k_r} m^{-(1-\frac{\theta_r}{r})} < 1$ with γ_r sufficiently close to $m^{-(1-\frac{\theta_r}{r})}$ so that $\hat{\gamma}_r := \sup_n \gamma_r m R_n < 1$ (recall that by assumption $\sup_n mR_n < m^{1-\frac{\theta_r}{r}}$). This way, we may write

$$\|C_n\|_r = \|\phi_0\|_r \leqslant (\gamma_r m R_n)^k \|\phi_k\|_r + c_r' \varphi_0 = (\hat{\gamma}_r)^k \|\phi_k\|_r + c_r' \varphi_0\,,$$

where we have also used that $\varphi_{k_r} \leqslant (c_2)^{k_r} \varphi_0$, see Remark 4.1. We can then iterate this inequality, applying it to $\|\phi_k\|_r, \|\phi_{2k}\|_r$, etc. Letting $n = k_r n_r + j_r$ with $0 \leqslant j_q \leqslant k_r - 1$, we have

$$\|C_n\|_r \leqslant c_r' \sum_{j=0}^{n_r} (\gamma_r m R_n)^{k_r j} \varphi_{k_r j} + c_r'' (\gamma_r m R_n)^{n_r k_r}\,, \tag{4.5}$$

with $c_r'' := c_r' \sup_{0 \leqslant j \leqslant k_r - 1} \|\phi_{n-j}\|_r$.

We can now estimate this last expression, using the estimate on $\varphi_{k_r j} \leqslant c(a_{n-jk_r})^{-1/s}$ from Remark 4.1, where we recall that $a_\ell := \sum_{i=1}^{\ell} (mR_n)^{-is}$. Hence, we need to control a_ℓ, depending on the value of mR_n.

• Let us start with the case $mR_n = 1$ for simplicity. In this case, $a_\ell = \ell$ for all ℓ, so we can bound the sum

$$\sum_{j=0}^{n_r} (\gamma_r m R_n)^{k_r j} \varphi_{jk_r} \leqslant c \sum_{j=0}^{n_r} (\hat{\gamma}_r)^{k_r j} (n - jk_r)^{-1/s} \leqslant c' n^{-1/s} .$$

Note also that $(\hat{\gamma}_r)^{k_r n_r} \leqslant C_r (\hat{\gamma}_r)^n \leqslant C'_r n^{-1/s}$, using that $\hat{\gamma}_r < 1$. From (4.5), and recalling that in this case we have $\mathbf{E}[C_n] \asymp n^{-1/s}$, see (1.11), we end up with

$$\|C_n\|_r \leqslant c_r \mathbf{E}[C_n] ,$$

as desired.

• Let us now deal with the case $mR_n > 1$. First of all, let us control a_ℓ for $1 \leqslant \ell \leqslant n$. As noticed before, $a_\ell = f_\ell(mR_n)$, with $f_\ell(x) = x^{-s} \frac{1-x^{-\ell s}}{1-x^{-s}}$, and we may use the following easy bounds: $f_\ell(x) \geqslant \ell$ if $x \in [1, 1 + \frac{1}{\ell}]$ and $f_\ell(x) \geqslant c(x-1)^{-1}$ if $x \in [1 + \frac{1}{\ell}, C]$. Hence, letting $\ell_0 := (mR_n - 1)^{-1}$, we have that

$$\varphi_{n-\ell} \leqslant c \begin{cases} \ell^{-1/s} & \text{if } \ell \leqslant \ell_0 \wedge n , \\ (mR_n - 1)^{1/s} & \text{if } \ell \geqslant \ell_0 \wedge n . \end{cases}$$

Let us consider two different cases.

First, if $mR_n \in (1, 1 + \frac{1}{n}]$, so that $\ell_0 \geqslant n$. Then we can bound the sum

$$\sum_{j=0}^{n_r} (\gamma_r m R_n)^{k_r j} \varphi_{k_r j} \leqslant c \sum_{j=0}^{n_r} (\hat{\gamma}_r)^{k_r j} (n - jk_r)^{-1/s} \leqslant c' n^{-1/s} ,$$

and we conclude as in the case $mR_n = 1$ that $\|C_n\|_r \leqslant c_r \mathbf{E}[C_n]$, since we also have $\mathbf{E}[C_n] \asymp n^{-1/s}$ in that case, see (1.11).

In the case where $mR_n \in (1 + \frac{1}{n}, C]$, we have $\ell_0 \leqslant n$, and we need to split the sum into two parts:

$$\begin{aligned} \sum_{j=0}^{n_r} (\gamma_r m R_n)^{k_r j} \varphi_{k_r j} &\leqslant c(mR_n - 1)^{1/s} \sum_{j=0}^{n_r - \ell_0/k_r} (\hat{\gamma}_r)^{k_r j} + \sum_{j=n_r - \ell_0/k_r}^{n_r} (\hat{\gamma}_r)^{k_r j} (n - jk_r)^{-1/s} \\ &\leqslant c' (mR_n - 1)^{1/s} + c'' \sum_{i=1}^{\ell_0} (\hat{\gamma}_r)^{n-i} i^{-1/s} \leqslant c''' (mR_n - 1)^{1/s} . \end{aligned}$$

Once more, this concludes the proof since $\mathbf{E}[C_n] \asymp (mR_n - 1)^{1/s}$, see (1.11), the last term in (4.5) being again negligible.

• Let us now deal with the case $mR_n < 1$. As above, using that $a_\ell = f_\ell(mR_n)$, with $f_\ell(x) = x^{-s} \frac{x^{-\ell s} - 1}{x^{-s} - 1}$, we may use the following easy bounds: $f_\ell(x) \geqslant \ell$ if $x \in [1 - \frac{1}{\ell}, 1)$ and $f_\ell(x) \geqslant c \, \frac{x^{-(\ell+1)s}}{1-x}$ if $x \in (0, 1 - \frac{1}{\ell}]$. Hence, letting $\ell_0 := (1 - mR_n)^{-1}$, we have that

$$\varphi_{n-\ell} \leqslant c \begin{cases} \ell^{-1/s} & \text{if } \ell \leqslant \ell_0 \wedge n , \\ (mR_n)^{\ell+1} (1 - mR_n)^{1/s} & \text{if } \ell \geqslant \ell_0 \wedge n . \end{cases}$$

Let us again consider two different cases. First, if $mR_n \in [1-\frac{1}{n}, 1)$, so that $\ell_0 \geqslant n$, then we conclude that $\|C_n\|_r \leqslant c_r \mathbf{E}[C_n]$ exactly as above.

In the case where $mR_n \in (0, 1-\frac{1}{n}]$, we have $\ell_0 \leqslant n$, and we again split the sum into two parts:

$$\sum_{j=0}^{n_r} (\gamma_r m R_n)^{k_r j} \varphi_{k_r j} \leqslant c m R_n (1-mR_n)^{1/s} \sum_{j=0}^{n_r - \ell_0/k_r} (\gamma_r m R_n)^{k_r j} (mR_n)^{n-k_r j} + \sum_{j=n_r-\ell_0/k_r}^{n_r} (\gamma_r m R_n)^{k_r j} (n - jk_r)^{-1/s} .$$

Now, the first sum is bounded by a constant times $(1-mR_n)^{1/s}(mR_n)^{n+1} \leqslant c\mathbf{E}[C_n]$, see (1.11) for the second inequality. The second term is bounded by a constant times

$$\sum_{i=1}^{\ell_0} (\gamma_r m R_n)^{n-i} i^{-1/s} \leqslant c(\gamma_r m R_n)^{n-\ell_0} (\ell_0)^{-1/s} = c(mR_n)^{n-\ell_0}(1-mR_n)^{1/s} .$$

Now, since $\inf_n mR_n > 0$, we get that $(mR_n)^{-\ell_0}$ remains bounded (recall $\ell_0 = (1-mR_n)^{-1}$), so this is bounded by a constant times $\mathbf{E}[C_n]$, recalling (1.11).

It remains to control the last term in (4.5). Again, this is bounded by a constant times $(\gamma_r)^n (mR_r)^{n_q k_q} \leqslant c(\gamma_q)^n (mR_n)^{n+1}$ since $\inf_n mR_n > 0$; now, this is negligible compared to $\mathbf{E}[C_n] \asymp (1-mR_n)^{1/s}(mR_n)^{n+1}$. □

4.3 Control of the ratios φ_k/φ_{k-1}

In view of Remark 4.1, we are able to bound the ratios φ_k/φ_{k-1}, where we recall that $\varphi_k := \mathbf{E}[\phi(u)]$ for $|u| = k$. Let us now give a more precise estimate of the ratio φ_k/φ_{k-1}.

Lemma 4.3. *With the same assumption as in Proposition 4.2, there exists a constant $c > 0$ such that $0 \leqslant \frac{\varphi_k}{\varphi_{k-1}} - (mR_n)^{-1} \leqslant \frac{c}{a_{n-k}}$. In particular, we have*

$$1 \leqslant \frac{(mR_n)^k \varphi_k}{\varphi_0} \leqslant \prod_{i=1}^{k} \Big(1 + \frac{c(mR_n)^i}{a_{n-i}}\Big) . \tag{4.6}$$

Proof. We start with the iteration (4.1) which defines $\phi(u)$, with $R_u = (R_n)^{-|u|}$ Notice that we can write the iteration as

$$\phi(u) = R_n \sum_{v \leftarrow u} \phi(v) - R_n \sum_{v \leftarrow u} f(\phi(v)) ,$$

with $f(x) = x - \frac{x}{(1+x^s)^{1/s}}$, which verifies $0 \leqslant f(x) \leqslant c \min(x^q, x)$.

Taking the expectation, we therefore get that if $|u| = k-1$,

$$\varphi_{k-1} = mR_n \varphi_k - mR_n \mathbf{E}[f(\phi(v))] .$$

All together, using Proposition 4.2 to get that $\mathbf{E}[f(\phi(v))] \leqslant C\varphi_k^q$, we get that

$$0 \leqslant mR_n \varphi_k - \varphi_{k-1} \leqslant c(mR_n)\varphi_k^q \leqslant cmR_n \frac{\varphi_k}{a_{n-k}} ,$$

where we have also used that $\varphi_k^{q-1} = \varphi_k^s \leqslant (a_{n-k})^{-1}$ from Remark 4.1. This concludes the first bound in Lemma 4.3. The bound (4.6) follows immediately by iteration. □

5 Convergence of the normalized conductance

We consider the recursion (1.8) on the tree of depth n, with resistances $R_v = (R_n)^{-|v|}$:

$$B_n(u) = R_n \sum_{v \leftarrow u} g\big(B_n(v)\big)\,,$$

with $g(\cdot)$ satisfying (1.5). We denote $B_n := B_n(\rho)$ and $b_n := \mathbf{E}[B_n]$. We now show the convergence of the normalized quantity $\hat{B}_n := \frac{1}{b_n} B_n$ assuming that $\mathbf{E}[Z^q] < +\infty$, *i.e.* we prove Theorem 1.8.

The method we use is analogous to that in [9, §5], with some modifications to obtain the convergence in L^q with only a finite q-moment assumption. We start by proving the L^1 convergence; then we upgrade it to a L^q convergence. We then conclude the section by proving the almost sure convergence in the case $R_n \equiv R$. We conclude the section by proving the sharp asymptotic of Proposition 1.9.

Preliminary observations and notation. For $|u| = k$, let us denote $b_{n-k} := \mathbf{E}[B_n(u)]$. Notice that, as $\kappa_1 C_n \leqslant B_n \leqslant \kappa_2 C_n$, we have that $b_{n-k} \asymp (a_{n-k})^{-1/s}$ with $a_\ell = \sum_{i=1}^{\ell} (mR_n)^{-is}$. Notice that Proposition 4.2 holds, so we have that $\mathbf{E}[B_n(u)^q] \leqslant c\mathbf{E}[B_n(u)]^q$ for some universal constant c; we will also make use of Lemma 4.3, which is also valid for b_{n-k} (the proof only uses that $0 \leqslant f(x) := x - g(x) \leqslant c \min(x^q, x)$).

5.1 Convergence in L^1

Let us write $f(x) = x - g(x)$, which is a non-negative function (by concavity of g), which verifies $f(x) \leqslant c \min(x^q, x)$ thanks to (1.5). Then, we can rewrite the above iteration as

$$B_n(u) = R_n \sum_{v \leftarrow u} B_n(v) - R_n \sum_{v \leftarrow u} f(B_n(v))\,,$$

so that iterating for the first k generations we get

$$B_n = (mR_n)^k \frac{1}{m^k} \sum_{|v|=k} B_n(v) - \Pi_{k,n}\,,$$

with

$$\Pi_{k,n} := \sum_{j=1}^{k} (mR_n)^j \frac{1}{m^j} \sum_{|v|=j} f(B_n(v)) \geqslant 0\,. \tag{5.1}$$

Normalizing by b_n, and denoting $\hat{B}_n(v) = \frac{1}{b_{n-k}} B_n(v)$ for $|v| = k$, we get that

$$\hat{B}_n = \frac{(mR_n)^k b_{n-k}}{b_n} \frac{1}{m^k} \sum_{|v|=k} \hat{B}_n(v) - \frac{1}{b_n}\Pi_{k,n} = W_k + I_{k,n} + J_{k,n} - \frac{1}{b_n}\Pi_{k,n}\,, \tag{5.2}$$

where $W_k = \frac{1}{m^k} Z_k$ is the usual martingale and we have set

$$I_{k,n} := \frac{1}{m^k} \sum_{|v|=k} \big(\hat{B}_n(v) - 1\big) \quad \text{and} \quad J_{k,n} := \Big(\frac{(mR_n)^k b_{n-k}}{b_n} - 1\Big) \frac{1}{m^k} \sum_{|v|=k} \hat{B}_n(v)\,.$$

We now treat the three terms in (5.2) separately. To anticipate on the L^q convergence, we bound the terms $I_{k,n}$, $J_{k,n}$ in L^q; we then control $\frac{1}{b_n}\Pi_{k,n}$ in L^1.

Control of $I_{k,n}$. Since $\hat{B}_n(v) - 1$ are i.i.d. centered random variables independent of T_k (hence of Z_k), we can apply Lemma 3.1 conditionally on Z_k to get that

$$\mathbf{E}\Big[\Big|\sum_{|v|=k}(\hat{B}_n(v)-1)\Big|^q\Big] \leqslant A_q^q \mathbf{E}\big[(Z_k)^{\theta_q}\big]\mathbf{E}\big[|\hat{B}_n(v)-1|^q\big]\,.$$

Therefore

$$\|I_{k,n}\|_q \leqslant A_q \frac{1}{m^k}\big(\|Z_k\|_{\theta_q}\big)^{\frac{\theta_q}{q}}\|\hat{B}_n(v)-1\|_q \leqslant A_q m^{-k(1-\frac{1}{q}\theta_q)}\big(\|W_k\|_{\theta_q}\big)^{\frac{\theta_q}{q}}\,\big(1+\|\hat{B}_n(v)\|_q\big)\,.$$

Now, since $\theta_q \leqslant q$, we have that $\sup_k \|W_k\|_{\theta_q} < +\infty$, and thanks to Proposition 4.2 we get that $\|\hat{B}_n(v)\|_q \leqslant c_q$ for some universal constant c_q. We therefore get that

$$\|I_{k,n}\|_q \leqslant c_q' m^{-k(1-\frac{1}{q}\theta_q)}\,, \tag{5.3}$$

which goes to 0 as $k \to \infty$ since $\theta_q < q$.
Control of $J_{k,n}$. First of all, we get that

$$\Big\|\frac{1}{m^k}\sum_{|v|=k}\hat{B}_n(v)\Big\|_q \leqslant \|W_k\|_q + \|I_{k,n}\|_q \leqslant c_q\,,$$

for some universal constant c_q. We therefore only have to focus on the term

$$\Big|\frac{(mR_n)^k b_{n-k}}{b_n} - 1\Big| \leqslant \prod_{i=1}^{k}\Big(1 + c\,\frac{(mR_n)^i}{a_{n-i}}\Big) - 1 \leqslant \exp\Big(c\sum_{i=1}^{k}\frac{(mR_n)^i}{a_{n-i}}\Big) - 1\,,$$

where we have used Lemma 4.3. We now show that the upper bound goes to 0 as $k \to \infty$.

Claim 5.1. *Define $k_n = \frac{1}{2}n$ if $mR_n \leqslant 1 + \frac{1}{n}$ and $k_n = (mR_n - 1)^{-1}$ if $mR_n \geqslant 1 + \frac{1}{n}$. Then, there is a constant c such that for all $k \leqslant k_n$, we have*

$$\sum_{i=1}^{k}\frac{(mR_n)^i}{a_{n-i}} \leqslant c\,\frac{a_k}{a_n} \qquad \textit{and} \qquad \sum_{i=1}^{k}\frac{1}{a_{n-i}} \leqslant c\,\frac{a_k}{a_n}\,.$$

In particular, for all $k \leqslant k_n$ we have that $\sum_{i=1}^{k_n}\frac{(mR_n)^i}{a_{n-i}}$ is bounded by a constant c: we then get that for any $k \leqslant k_n$

$$\|J_{k,n}\|_q \leqslant c_q\,\Big|\frac{(mR_n)^k b_{n-k}}{b_n} - 1\Big| \leqslant c'\sum_{i=1}^{k_n}\frac{(mR_n)^i}{a_{n-i}} \leqslant c''\frac{a_k}{a_n}\,, \tag{5.4}$$

which goes to 0 if $k \to \infty$ sufficiently slowly.

Proof of Claim 5.1. The case $mR_n = 1$ is trivial since then we have $a_k = k$ and $a_n = n$. A similar result holds when $|mR_n - 1| \leqslant n^{-1}$, since we also have $a_k \asymp k$ for all $k \leqslant n$ in that case.

In the case $mR_n < 1 - \frac{1}{n} \leqslant 1$, we can bound $(mR_n)^i \leqslant 1$ in the sum so we only need to control the second sum. Then, we can use the bound $a_{n-i} \geqslant c(1 - mR_n)^{-1}(mR_n)^{-(n-i)s}$ for all $i \leqslant n/2$, see (1.10). We therefore get

$$\sum_{i=1}^{k}\frac{1}{a_{n-i}} \leqslant c(1 - mR_n)(mR_n)^{-ns}\sum_{i=1}^{k}(mR_n)^{-is} \leqslant c\frac{a_k}{a_n}\,.$$

In the case $mR_n > 1 + \frac{1}{n} \geqslant 1$, we can bound $(mR_n)^i \geqslant 1$ so we only need to control the first sum. Using that $a_{n-i} \geqslant c(mR_n - 1)^{-1} \geqslant c a_n$ for all $i \leqslant n/2$, see (1.10), we get that

$$\sum_{i=1}^{k} \frac{(mR_n)^i}{a_{n-i}} \leqslant \frac{c}{a_n} \sum_{i=1}^{k} (mR_n)^i \leqslant \frac{ck}{a_n},$$

provided that $k \leqslant (mR_n - 1)^{-1}$, since $(mR_n)^i \leqslant C$ uniformly for $i \leqslant (mR_n - 1)^{-1}$. Note that we have $a_k \asymp k$ for $(mR_n - 1)^{-1}$, which concludes the proof. □

Control of $\frac{1}{b_n}\Pi_{k,n}$. Recalling the definition (5.1) of $\Pi_{k,n} \geqslant 0$ we get that

$$\frac{1}{b_n}\mathbf{E}[\Pi_{k,n}] = \frac{1}{b_n}\sum_{j=1}^{k}(mR_n)^j \mathbf{E}[f(B_n(v))] \leqslant c \sum_{j=1}^{k} \frac{(mR_n)^j b_{n-j}}{b_n} (b_{n-j})^s .$$

For the last inequality, we have used that $f(x) \leqslant c \min(x^q, x)$, so that for $|v| = j$ we have $\mathbf{E}[f(B_n(v))] \leqslant c\mathbf{E}[B_n(v)^q] \leqslant c'(b_{n-j})^q$, thanks to Proposition 4.2; recall also that $s = q - 1$.

Again, we can use Lemma 4.3 and Claim 5.1 to get that for all $j \leqslant k_n$

$$\frac{(mR_n)^j b_{n-j}}{b_n} \leqslant \prod_{i=1}^{j}\Big(1 + c\frac{(mR_n)^i}{a_{n-i}}\Big) \leqslant 1 + c''\frac{a_k}{a_n} \leqslant C .$$

Using also that $(b_{n-j})^s \leqslant c(a_{n-j})^{-1}$ by Remark 4.1, we therefore get that for

$$\frac{1}{b_n}\mathbf{E}[\Pi_{k,n}] \leqslant c' \sum_{j=1}^{k} \frac{1}{a_{n-j}} \leqslant c'' \frac{a_k}{a_n}, \tag{5.5}$$

using again Claim 5.1 for the last inequality.

Conclusion. Going back to (5.2) and collecting the bounds (5.3)-(5.4)-(5.5), we obtain that for all $k \leqslant k_n$ (with k_n defined in Claim 5.1), we have

$$\begin{aligned} \mathbf{E}\big[|\hat{B}_n - W|\big] &\leqslant \mathbf{E}\big[|W_k - W|\big] + c m^{-k(1-\frac{1}{q}\theta_q)} + c\frac{a_k}{a_n} \\ &\leqslant c' m^{-k(1-\frac{1}{q}\theta_q)} + c\frac{a_k}{a_n}, \end{aligned} \tag{5.6}$$

For the last inequality, we have used that $\mathbf{E}[|W - W_k|] \leqslant \|W - W_k\|_q \leqslant c m^{-k(1-\frac{1}{q}\theta_q)}$, see [24, Prop. 1.3].

Note that the assumption that $\limsup_{n\to\infty} mR_n \leqslant 1$ ensures that $k_n \to \infty$ and that $a_n \to +\infty$, see (1.10). Hence, we can choose $k = \hat{k}_n \leqslant k_n$ going to $+\infty$ sufficiently slowly so that the upper bound in (5.6) goes to zero. This concludes the proof that $(\hat{B}_n)_{n\geqslant 0}$ converges in L^1 to W. □

5.2 Convergence in L^q

Since we have the convergence $\hat{B}_n \to W$ in L^1, we also have the convergence in probability. To prove the convergence in L^q, we therefore simply need to show the uniform integrability of $(\hat{B}_n^q)_{n\geqslant 0}$.

But from (5.2), we have the upper bound $\hat{B}_n \leqslant W_k + I_{k,n} + J_{k,n}$, where we can choose $k = \hat{k}_n \leqslant k_n$ going to infinity slowly enough. Since $\hat{B}_n \geqslant 0$, we therefore get that

$$0 \leqslant \hat{B}_n^q \leqslant 3^q (W_{\hat{k}_n})^q + 3^q (I_{\hat{k}_n,n})^q + 3^q (J_{\hat{k}_n,n})^q ,$$

and we only have to prove the uniform integrability of the three terms on the right-hand side, which is easy.

First, (5.3) and (5.4) show that $(I_{\hat{k}_n,n})^q$ and $(J_{\hat{k}_n,n})^q$ converge to 0 in L^1; in particular they are uniformly integrable. Second, since μ admits a finite moment of order q, we have that $(W_k)_{k\geqslant 0}$ converges in L^q to W, so in particular $(W_k^q)_{k\geqslant 0}$ is uniformly integrable. This concludes the proof that $(\hat{B}_n^q)_{n\geqslant 0}$ is uniformly integrable, hence that $(\hat{B}_n)_{n\geqslant 0}$ converges in L^q to W. $\square$

5.3 Almost sure convergence

First of all, notice that if we take $k = \hat{k}_n = c\log n$ in (5.6), with a constant c sufficienly large, we obtain that

$$\mathbf{E}\Big[\big|\hat{B}_n - W\big|\Big] \leqslant c' n^{-2} + c\frac{a_{\hat{k}_n}}{a_n}\,. \tag{5.7}$$

Hence, if k_n from Claim 5.1 satisfies $k_n \geqslant c\log n = \hat{k}_n$ and if $a_{\hat{k}_n}/a_n$ is summable, we directly obtain the a.s. convergence $\lim_{n\to\infty} \hat{B}_n = W$.

This is for instance the case if we have $R_n \equiv R$ with $R \in (0, m^{-1}]$ since in that case $k_n = \frac{1}{2}n$ and $a_n \asymp (mR)^{-ns}$; more generally, it is verified if $mR_n \leqslant 1 - c\frac{\log n}{n}$ with some constant c large enough. This settles the a.s. convergence when $R_n \equiv R \in (0, m^{-1})$, and it remains to treat the critical case $R_n \equiv m^{-1}$.

Let $R_n \equiv m^{-1}$, so that $a_k = k$ for all $k \geqslant 1$, and in particular $\frac{a_{\hat{k}_n}}{a_n} \leqslant c\frac{\log n}{n}$. Consider the subsequence $(n^2)_{n\geqslant 0}$, so that the upper bound in (5.7) is summable along this subsequence. Then (5.7) gives the a.s. convergence $\lim_{n\to\infty} \hat{B}_{n^2} = W$. Hence, one simply needs to bridge the gaps between n^2 and $(n+1)^2$. Now, notice that $(B_n)_{n\geqslant 1}$ is non-increasing (this is where we use that $R_n \equiv R$ is fixed, see Remark 5.2 below), so we can write that, for all $n^2 \leqslant \ell < (n+1)^2$

$$\frac{b_{(n+1)^2}}{b_{n^2}}\hat{B}_{(n+1)^2} \leqslant \hat{B}_\ell \leqslant \frac{b_{n^2}}{b_{(n+1)^2}}\hat{B}_{n^2}\,,$$

and it only remains to show that $b_{(n+1)^2}/b_{n^2}$ goes to 1. But this simply comes from Lemma 4.3, which shows that

$$1 \leqslant \frac{b_{n^2+k}}{b_{n^2}} \leqslant \prod_{i=1}^{k}\Big(1 + \frac{C}{a_{n^2+i}}\Big) \leqslant \exp\Big(C\sum_{i=1}^{k}\frac{1}{a_{n^2+i}}\Big) \leqslant \exp\Big(C\frac{k}{n^2}\Big)\,,$$

using also that $a_{n^2+i} \geqslant n^2$ for all $i \geqslant 1$. Therefore, since $(n+1)^2 = n^2 + 2n + 1$, we get that $1 \leqslant \frac{b_{(n+1)^2}}{b_{n^2}} \leqslant \exp\big(C\frac{2n+1}{n^2}\big)$ and therefore goes to 1 as $n \to \infty$, concluding the proof that $\lim_{n\to\infty} \hat{B}_n = W$ almost surely. $\square$

Remark 5.2. One could try to use the idea of taking a subsequence to adapt the proof to a general sequence $(R_n)_{n\geqslant 0}$, but the difficulty is to compare B_n and B_{n+1}, since B_n uses resistances $(R_n)^{-|v|}$ inside T_n and B_{n+1} resistances $(R_{n+1})^{-|v|}$ inside T_{n+1}; the restriction to the case $R_n \equiv R$ allows for a comparison. Similarly, in the general case, there is no obvious relation between $b_n := \mathbf{E}[B_n]$ and $b_{n+1} := \mathbf{E}[B_{n+1}]$.

5.4 Precise asymptotic for $\mathbf{E}[B_n]$: proof of Proposition 1.9

Theorem 1.8 proves that $\hat{B}_n$ converges in L^q to W, so we get that

$$\lim_{n\to\infty} \mathbf{E}[(\hat{B}_n)^q] = \mathbf{E}[W^q]\,.$$

This shows in particular that $\mathbf{E}[B_n^q] \sim \mathbf{E}[W^q](b_n)^q$.

Note that, setting $f(x) = x - g(x)$, we have the relation $b_n = mR_n\big(b_{n-1} - \mathbf{E}[f(B_n(v))]\big)$ for $|v| = 1$, which can be rewritten as

$$b_n = mR_n b_{n-1}(1 - c_{n-1}) \,, \qquad \text{with } \; c_{n-1} := (b_{n-1})^{-1}\mathbf{E}[f(B_n(v))] \,.$$

Now, observe that since $f(x) \sim \kappa_g x^q$ as $x \downarrow 0$ and that $f(x) \leqslant cx^q$, we get by dominated convergence that $\mathbf{E}[f(B_n(v))] \sim \kappa_g \mathbf{E}[B_n(v)^q] \sim \kappa_g \mathbf{E}[W^p](b_{n-1})^p$, with $b_{n-1} \downarrow 0$ under the assumption of Proposition 1.9.

In particular, this gives that $c_{n-1} \sim \kappa_g \mathbf{E}[W^p](b_{n-1})^s$, with $s = q-1$. Setting $x_n = b_n^{-s}$, we get that

$$x_n = (mR_n)^{-s} x_{n-1}(1 + c'_{n-1}) = (mR_n)^{-s} x_{n-1} + (mR_n)^{-s} x_{n-1} c'_{n-1} \,,$$

with $c'_{n-1} = (1 - c_{n-1})^{-s} - 1 \sim s\kappa_g (b_{n-1})^s$. Iterating this relation, we get that

$$x_n = \sum_{k=1}^{n} (mR_n)^{-ks} x_{n-k} c'_{n-k} \,,$$

with $x_i c'_i \to s\kappa_g \mathbf{E}[W^q]$ as $i \to \infty$. Hence, we can choose ℓ_n going to $+\infty$ arbitrarily slowly and write that

$$\begin{aligned} x_n &= (1 + o(1)) s\kappa_g \mathbf{E}[W^p] \sum_{k=1}^{n-\ell_n} (mR_n)^{-ks} + (mR_n)^{-ns} \sum_{i=0}^{\ell_n} (mR_n)^{is} x_i c'_i \\ &= (1 + o(1)) s\kappa_g \mathbf{E}[W^p] a_{n-\ell_n} + (mR_n)^{-ns} \sum_{i=0}^{\ell_n} (mR_n)^{is} x_i c'_i \,. \end{aligned}$$

To conclude, let us notice that, in the case where $\lim_{n\to\infty} mR_n = \vartheta = 1$, then we can choose $\ell_n \to \infty$ so that $a_{n-\ell_n} \sim a_n$. We also find in that case that the second term is negligible compared to a_n provided that ℓ_n grows sufficiently slowly — this is clear if $mR_n \geqslant 1 - \frac{c}{n}$ and follows from the fact that $a_n \sim (1 - (mR_n)^s)^{-1}(mR_n)^{-(n+1)s}$ if $n(mR_n - 1) \to -\infty$. This shows that $x_n \sim s\kappa_g \mathbf{E}[W^p] a_n$ as $n \to \infty$ when $\vartheta = 1$.

On the other hand, if $\lim_{n\to\infty} mR_n = \vartheta \in (0,1)$, then $a_n \sim (1-\vartheta^s)^{-1}(mR_n)^{-ns}$, so for any $\ell_n \to \infty$ we have $a_{n-\ell_n} = o(a_n)$. We end up with $x_n \sim c_\vartheta (mR_n)^{-ns}$, with $c_\vartheta = \sum_{i=0}^{+\infty} \vartheta^{is} x_i c'_i$ which is a convergent sequence. This concludes the fact that $x_n \sim c_\vartheta(1 - \vartheta^s) a_n$ as $n \to \infty$.

All together, this gives the desired conclusion, since $x_n \sim (b_n)^{-s}$. □

6 Estimates on moments of C_n in the case $\mathbf{E}[Z^q] = +\infty$

In this section, we prove Theorem 1.10. Since we can bound B_n with C_n, see Proposition 1.4, we focus on estimates on C_n, $\mathbf{E}[C_n]$. Let us define, as in Section 4.1.1, $\phi(u) = R_u C_n(u)$ and recall that we have the following recursion (4.1):

$$\phi(u) = R_n \sum_{v \leftarrow u} g\big(\phi(v)\big) \qquad \text{with } g(x) = \frac{x}{(1 + x^s)^{1/s}} \,. \tag{6.1}$$

6.1 Upper bound on $\mathbf{E}[C_n]$

Let us assume that (1.12) holds and obtain the upper bound in Theorem 1.10. As in Section 4.2, let $\varphi_k = \mathbf{E}[\phi(u)]$ for $|u| = k$, so in particular $\varphi_n = 1$ and we need to estimate φ_0.

Using the recursion (6.1) we can write, again with the notation $f(x) = x - g(x)$,

$$\begin{aligned}\mathbf{E}[g(\phi(u))] &= mR_n\mathbf{E}[g(\phi(v))] - \mathbf{E}\Big[f\Big(R_n\sum_{v\leftarrow u} g(\phi(v))\Big)\Big] \\ &= mR_n\mathbf{E}[g(\phi(v))] - \mathbf{E}\Big[f\Big(R_n Z_u \mathbf{E}\big[g(\phi(v))\big]\Big)\Big] \\ &\qquad\qquad - \mathbf{E}\Big[f\Big(R_n\sum_{v\leftarrow u} g(\phi(v))\Big) - f\Big(R_n Z_u\mathbf{E}\big[g(\phi(v))\big]\Big)\Big]\,,\end{aligned} \tag{6.2}$$

where we have denoted Z_u the number of descendants of u. Notice that, for $v \in T_n$ such that $|v| = k+1$, we have $\varphi_k = mR_n\mathbf{E}[g(\phi(v))]$ and that, by convexity of f, the last term in (6.2) is non-positive: we end up with the following inequality: for all $1 \leqslant k \leqslant n$

$$\frac{1}{mR_n}\varphi_{k-1} \leqslant \varphi_k - \mathbf{E}\Big[f\Big(\frac{1}{m}Z\varphi_k\Big)\Big]\,.$$

Now, we can use the fact that $f(x) \geqslant c\min(x^q, x) \geqslant c(x\wedge 1)^q$, to get that

$$\mathbf{E}\Big[f\Big(\frac{1}{m}Z_u\varphi_k\Big)\Big] \geqslant c'(\varphi_k)^q\mathbf{E}\Big[\Big(Z\wedge\frac{m}{\varphi_k}\Big)^q\Big] \geqslant c'L(1/\varphi_k)(\varphi_k)^{-\alpha}\,.$$

where we have used assumption (1.12) for the last inequality.

All together, we end up with the following recursion: $\varphi_n = 1$ and for $0 \leqslant k \leqslant n$

$$\varphi_{k-1} \leqslant mR_n\varphi_k(1 - h(\varphi_k))\,, \tag{6.3}$$

where $h(x) \sim cL(1/x)x^{\alpha-1}$ as $x \downarrow 0$. Note that we can assume that both $x \mapsto h(x) \in [0,1)$ and $x \mapsto x(1-h(x))$ are increasing (by properties of regularly varying functions, we may assume that $h'(x) = c'L(1/x)x^{\alpha-2}$ so $1 - h(x) - xh'(x)$ remains positive).

We can therefore focus on the iteration

$$u_{k+1} = mR_n u_k(1 - h(u_k)) \qquad \text{with } h(x) \sim cL(1/x)x^{\alpha-1}\,, \tag{6.4}$$

started at $u_0 = 1$. We now have to obtain an upper bound on u_n.

Remark 6.1. Notice that the recursion (6.4) admits a non-zero fixed point u_* if $mR_n > 1$, which verifies $h(u_*) = \frac{mR_n - 1}{mR_n}$, but that if $mR_n \leqslant 1$ the only fixed point is $u_* = 0$.

• Let us start with the case where $mR_n \in [1 + \frac{1}{n}, 2]$, and let $\delta_n := \frac{mR_n-1}{mR_n}$. We let v_n be the fixed point of the equation (6.4), *i.e.* such that $h(v_n) = \delta_n$, and notice that it verifies $v_n = h^{-1}(\delta_n) \leqslant c\gamma_n$, where γ_n is defined in (1.14). Let assume that $v_n \leqslant u_0$, otherwise we have $u_k \leqslant v_n$ for all k and in particular $u_n \leqslant v_n \leqslant c\gamma_n$.

Now, by assumption we have that $(u_k)_{k\geqslant 0}$ is a decreasing sequence. Let us define $k_n := \min\{k : h(u_k) \leqslant C\delta_n\}$ for some (large) constant $C \geqslant 2$. Our goal is to show that if C is large enough then we have $k_n \leqslant n$, so in particular $u_n \leqslant u_{k_n}$ with $u_{k_n} \leqslant h^{-1}(C\delta_n) \leqslant c'\gamma_n$.

Now, for all $k < k_n$, we have that $mR_n(1 - h(u_k)) \leqslant 1 + \delta_n - h(u_k) \leqslant 1 - \frac12 h(u_k)$, so we end up with

$$u_{k+1} \leqslant u_k\big(1 - \tfrac12 h(u_k)\big)\,.$$

Now, let $H : (0,\infty) \to (0,\infty)$ be some decreasing function, with derivative given by $H'(x) = -(xh(x))^{-1}$; we also let $c > 0$ be a constant such that $H(x - t) \geqslant H(x) - ctH'(x)$ for all $x \in (0,1]$ and $t \in [0, x/2]$. We then have that for $k < k_n$,

$$H(u_{k+1}) \geqslant H\big(u_k - \tfrac12 u_k h(u_k)\big) \geqslant H(u_k) - cu_kH'(u_k)h(u_k) = H(u_k) + c\,.$$

All together, we get that for all $k \leqslant k_n$

$$H(u_k) - H(u_0) = \sum_{j=0}^{k-1} \big(H(u_{j+1}) - H(u_j)\big) \geqslant ck\,,$$

or, put otherwise $H(u_k) \geqslant ck$. Since $H(u) = \int_u^1 (th(t))^{-1}\mathrm{d}t$ and since $h(t)$ is regularly varying with index $\alpha - 1 > 0$, we get that $H(u) \sim \frac{1}{\alpha-1} h(u)^{-1}$ as $u \downarrow 0$, so we end up with the fact that, for all $k \leqslant k_n$

$$h(u_k)^{-1} \geqslant c'\, k \quad \text{ or } \quad h(u_k) \leqslant c''\, k^{-1}\,, \qquad \text{for all } k \leqslant k_n\,. \tag{6.5}$$

Now, applying this inequality with $k = k_n - 1$ and recalling the definition of k_n, we get that $k_n - 1 \leqslant c'' C^{-1}/\delta_n$. Since $\delta_n = (mR_n - 1)/R_n \geqslant 1/2n$ for $mR_n \in [1 + \frac{1}{n}, 2]$, we get that $k_n - 1 \leqslant 2c'' C^{-1} n$, which is smaller than n provided that C had been fixed large enough.

• The proof is analogous in the case where $mR_n \in [1 - \frac{1}{n}, 1 + \frac{1}{n}]$. Define $k_n := \min\{k, h(u_k) \leqslant 2/n\}$, so that as above we have $mR_n(1 - h(u_k)) \leqslant 1 - \frac{1}{2} h(u_k)$ for all $k < k_n$.

Then, similarly as in (6.5), we get that $h(u_k) \leqslant c'' k^{-1}$ for all $k \leqslant k_n$. Now, either we have $k_n \geqslant n$, in which case $h(u_n) \leqslant c''/n$, or we have $k_n < n$ in which case by definition of k_n we have $h(u_n) \leqslant h(u_{k_n}) \leqslant 2/n$. In any case we have that $u_n \leqslant h^{-1}(c/n)$, so that we obtain $u_n \leqslant c\gamma_n$, where γ_n is defined in (1.14), recalling also that $a_n \asymp 1/n$.

• Let us now treat the case where $mR_n \leqslant 1 - \frac{1}{n}$, with $\inf_n R_n > 0$. Let us set $\delta_n = 1 - mR_n \geqslant \frac{1}{n}$ and let us define define $k_n := \min\{k, h(u_k) \leqslant C\delta_n\}$ for some (large) constant C.

First, let us show that $k_n < n$, provided that C has been fixed large enough. For $k < k_n$ we use the inequality $u_{k+1} \leqslant u_k(1 - h(u_k))$, which thanks to (6.5) gives that $h(u_k) \leqslant c''/k$ for all $k \leqslant k_n$ and in particular $k_n - 1 \leqslant c' C^{-1}/\delta_n$. Since $\delta_n \geqslant 1/n$, this proves that $k_n < n$ provided that C is large.

Then, for $k \geqslant k_n$, we use the bound $u_{k+1} \leqslant mR_n u_k$ to get that

$$u_n \leqslant (mR_n)^{n-k_n} u_{k_n} \leqslant (mR_n)^n (1 - \delta_n)^{-k_n} h^{-1}(C\delta_n)\,,$$

where we have also used the definition of k_n for the last inequality. Now, since we have $k_n \leqslant c'/\delta_n$, the term $(1 - \delta_n)^{-k_n}$ remains bounded, while we have $h^{-1}(C\delta_n) \leqslant ch^{-1}(\delta_n)$. This concludes the proof of the upper bound in Theorem 1.10.

The last inequality simply comes from Markov's inequality, which gives $\mathbf{P}(C_n \geqslant K\gamma_n) \leqslant K^{-1}\gamma_n \mathbf{E}[C_n] \leqslant cK^{-1}$.

6.2 Lower bound on C_n, $\mathbf{E}[C_n]$

To obtain a lower bound on C_n, $\mathbf{E}[C_n]$, let us assume that (1.13) holds. We let $(t_n)_{n\geqslant 1}$ be a truncation sequence (to be optimized later on), and as in 3.2, we consider a *truncated* branching process $\tilde{T}$ with offspring distribution $\tilde{Z} := (Z \wedge t_n) \sim \tilde{\mu}$. Let also $\tilde{C}_n(v)$ be the p-capacities associated with the Galton–Watson tree $\tilde{T}_n$ of depth n with reproduction law $\tilde{\mu}$. Since we have truncated the offspring distribution there is a coupling for which $\tilde{T} \subset T$ and since the function $g(x) = \frac{x}{(1+x^s)^{1/s}}$ is increasing, we obtain that $\tilde{C}_n(v) \leqslant C_n(v)$ for all $v \in \tilde{T}$. In particular, we only need to obtain a lower bound on $\tilde{C}_n$, $\mathbf{E}[\tilde{C}_n]$.

We use the same method as in Section 4.1.2. By using the uniform flow and Thompson's principle, we have similarly to (4.2)

$$\tilde{\mathcal{R}}_p(\rho \leftrightarrow \partial \tilde{T}_n)^s \leqslant \frac{1}{(\tilde{W}_n)^q} \sum_{k=1}^{n} (\tilde{m} R_n)^{-ks} \frac{1}{\tilde{m}^k} \sum_{|v|=k} \tilde{W}_n(v)^q\,,$$

where $\tilde W_n(v) = \tilde m^{-(n-k)} \tilde Z_n(v)$ for $|v|=k$, $\tilde W_n = \tilde W_n(\rho)$. Then, exactly as in (4.3), we obtain the following bound for C_n:

$$\mathbf{P}\big(C_n \leqslant \varepsilon^{1/s} (\tilde a_n)^{-1/s}\big) \leqslant \delta_\varepsilon + \varepsilon^{1/2}\,,$$

where

$$\tilde a_n := \sum_{k=1}^n (\tilde m R_n)^{-ks} \mathbf{E}\big[(\tilde W_{n-k})^q\big]\,. \tag{6.6}$$

It only remains to show that $(\tilde a_n)^{-1/s} \geqslant c \tilde\gamma_n$, with $\tilde\gamma_n$ defined in (1.14). We now need to obtain an upper bound on (6.6). The lower bound on $\mathbf{E}[\tilde C_n] \geqslant c'\tilde\gamma_n$ then follows immediately.

Step 1. Estimate of $\mathbf{E}[(\tilde W_\ell)^p]$. Let us prove that, under the assumption (1.13), there is a constant c (independent of t_n) such that for all $\ell \geqslant 1$,

$$\mathbf{E}\big[(\tilde W_\ell)^q\big] \leqslant c L(t_n) t_n^{q-\alpha}\,. \tag{6.7}$$

Let us start with the case where $\alpha \in (1,q)$, which can be treated easily with Proposition 3.4. Indeed, recalling Remark 1.11, when $\alpha < q$, we have the bounds (3.2) on the tail of Z. Then provided that t_n is large enough, Proposition 3.4 shows that $\mathbf{P}(\tilde W_\ell > x) \leqslant cL(x)x^{-\alpha}$ for all $x \geqslant 1$ and $\mathbf{P}(\tilde W_\ell > x) \leqslant t_n^{-\alpha} e^{-c'x/2t_n}$ for $x \geqslant 2\alpha t_n/c'$ (assume also that $t_n \geqslant e$). Then, we obtain

$$\begin{aligned}\mathbf{E}\big[(\tilde W_\ell)^q\big] &= q\int_0^\infty x^{q-1}\mathbf{P}(\tilde W_\ell > x)\mathrm{d}x\\ &\leqslant c\int_0^{2\alpha t_n/c'} L(x)x^{q-\alpha-1}\mathrm{d}x + t_n^{-\alpha}\int_{2\alpha t_n/c'}^\infty x^{q-1} t_n^{\alpha - c'x/t_n}\mathrm{d}x\,.\end{aligned}$$

Then, by a simple change of variable $u = \frac{x}{t_n}$, we get

$$\mathbf{E}\big[(\tilde W_\ell)^q\big] \leqslant c'' L(t_n) t_n^{q-\alpha} + c t_n^{1-\alpha}\int_{\alpha/c'}^\infty u^{q-1} e^{-c'u/2}\mathrm{d}u \leqslant c''' L(t_n) t_n^{q-\alpha}\,,$$

using that the last integral is finite and $1-\alpha < q-\alpha$.

It remains to show (6.7) in the case where (1.13) holds with $\alpha = q$. We actually provide a proof that works as long as $\alpha \in (\frac q2, q]$. For any $\ell \geqslant 0$ and $k\geqslant 0$, let us write $\tilde W_{\ell+k} = \frac{1}{\tilde m^k}\sum_{i=1}^{\tilde Z_k} \tilde W_\ell^{(i)}$, where $(\tilde W_\ell^{(i)})$ are i.i.d. copies of $\tilde W_\ell$, independent of $\tilde Z_k$. Then, since $\mathbf{E}[\tilde W_\ell]=1$ for all ℓ, we can write

$$\tilde W_{\ell+k} - 1 = \frac{1}{\tilde m^k}\sum_{i=1}^{\tilde Z_k} (\tilde W_\ell^{(i)} - 1) + (\tilde W_k - 1)\,.$$

Then, by Lemma 3.1, we have that

$$\mathbf{E}\Big[\Big|\sum_{i=1}^{\tilde Z_k} (\tilde W_\ell^{(i)} - 1)\Big|^q \;\Big|\; \tilde Z_k\Big] \leqslant \frac{A_q^q}{\tilde m^{kq}} (\tilde Z_k)^{\theta_q}\mathbf{E}\Big[\big|\tilde W_\ell - 1\big|^q\Big]\,,$$

so that

$$\big\|\tilde W_{\ell+k} - 1\big\|_q \leqslant \frac{A_q}{\tilde m^k}\mathbf{E}\big[(\tilde Z_k)^{\theta_q}\big]^{1/q}\big\|\tilde W_\ell - 1\big\|_q + \|W_k - 1\|_q\,. \tag{6.8}$$

Note that if $\mathbf{E}[Z^{\theta_q}] < +\infty$, in particular if $\alpha > \theta_q = \max(1,\frac q2)$, then we have $\mathbf{E}[(\tilde Z_k)^{\theta_q}] \leqslant \mathbf{E}[(Z_k)^{\theta_q}] = m^{k\theta_q}\mathbf{E}[(W_k)^{\theta_q}]$ with $\mathbf{E}[(W_k)^{\theta_q}]$ bounded by a constant. All together, we obtain that

$$\big\|\tilde W_{\ell+k} - 1\big\|_q \leqslant c_q\Big(\frac{m^{\theta_q/q}}{\tilde m}\Big)^k \big\|\tilde W_\ell - 1\big\|_q + \|\tilde W_k - 1\|_q\,.$$

Now, since $\theta_q < q$, we can choose C large enough so that $m^{\theta_q/q} < \tilde{m}$ uniformly for $t_n \geqslant C$, and then we can fix k_q large enough so that $c_q(m^{\theta_q/q}/\tilde{m})^{k_q} < 1$. Then, iterating the above inequality, we obtain that there is a constant such that $\big\|\tilde{W}_{ik_q} - 1\big\|_q \leqslant C' \|\tilde{W}_{k_q} - 1\|_p$ for all $i \geqslant 0$. Since k_q is fixed, we can also apply (6.8) recursively to get that $\|\tilde{W}_{k_q} - 1\|_q \leqslant C'' \|\tilde{W}_1 - 1\|_q$. Since $\mathbf{E}[(\tilde{W}_j)^p]$ is non-decreasing, we therefore end up with

$$\sup_{\ell} \|\tilde{W}_\ell\|_q = \sup_i \|\tilde{W}_{ik_q}\|_q \leqslant C' + C''\big(\|\tilde{W}_1 - 1\|_q\big)\,,$$

so in particular $\mathbf{E}[(\tilde{W}_\ell)^q] \leqslant C + C' \mathbf{E}[\tilde{Z}^q]$ for all $\ell \geqslant 1$. Using (1.13), this concludes the proof of (6.7).

Step 2. Estimate of $\tilde{a}_n$: proof that $(\tilde{a}_n)^{-1/s} \geqslant c\tilde{\gamma}_n$. With (6.7) at hand and recalling the definition (6.6) of $\tilde{a}_n$, we have the following upper bound

$$\tilde{a}_n \leqslant cL(t_n) t_n^{q-\alpha} \sum_{k=1}^n (\tilde{m} R_n)^{-ks} = c t_n^s h(1/t_n) \sum_{k=1}^n (\tilde{m} R_n)^{-ks}\,, \tag{6.9}$$

with $h(x) \sim L(1/x) x^{\alpha-1}$ as $x \downarrow 0$ (recall $s = q-1$).

First of all, notice that we have the following identity for $\tilde{m}$:

$$\tilde{m} = \tilde{m}(t_n) = \mathbf{E}[Z \wedge t_n] = m - \mathbf{E}[(Z - t_n)\mathbb{1}_{\{Z > t_n\}}] \geqslant m - cL(t_n) t_n^{1-\alpha}\,,$$

where for the last inequality we have used that $\mathbf{P}(Z > x) \leqslant c_2 L(x) x^{-\alpha}$ (this is always valid, see Remark 1.11), so that $\mathbf{E}[(Z - t_n)\mathbb{1}_{\{Z > t_n\}}] = \int_{t_n}^\infty \mathbf{P}(Z > x)\mathrm{d}x \leqslant cL(t_n) t_n^{1-\alpha}$.

Let us now introduce the quantity $\tilde{\delta}_n = \tilde{\delta}_n(t_n)$ defined by $\tilde{m}(t_n) = (1 - \tilde{\delta}_n)\, m$ and notice that

$$\tilde{\delta}_n \leqslant cL(t_n) t_n^{1-\alpha} = c h(1/t_n).$$

• Let us start with the case $mR_n \in [1 - \frac{1}{n}, 1 + \frac{1}{n}]$. We then choose t_n such that $h(1/t_n) = 1/n$ so that $\tilde{\delta}_n \leqslant c/n$. We then get that $\tilde{m} R_n \geqslant (1 - \frac{1}{n})(1 - \delta_n) \geqslant 1 - \frac{c'}{n}$, and there exists a constant C such that $(\tilde{m} R_n)^{-ks} \leqslant C$ uniformly for $k \leqslant n$. All together, we get that

$$\tilde{a}_n \leqslant c\, t_n^s h(1/t_n) \sum_{k=1}^n C = cC\, t_n^s\,.$$

We conclude that $(\tilde{a}_n)^{-1/s} \geqslant c/t_n$ with $1/t_n = h^{-1}(1/n) = \gamma_n = \tilde{\gamma}_n$, recalling (1.14).

• We now treat the case $mR_n \in [1 + \frac{1}{n}, 2]$. Choose t_n such that $h(1/t_n) = c(mR_n - 1)$ with c small enough so that $\tilde{\delta}_n \leqslant \frac{1}{4}(mR_n - 1)$. Then, we get that $\tilde{m} R_n = mR_n(1 - \tilde{\delta}_n) \geqslant 1 + \frac{1}{2}(mR_n - 1)$, using that $(1 + x)(1 - \frac{1}{4}x) \geqslant 1 + \frac{1}{2}x$ for $x \in [0, 1]$. We end up with

$$\tilde{a}_n \leqslant c\, t_n^s h(1/t_n) \sum_{k=1}^n \Big(1 + \frac{mR_n - 1}{2}\Big)^{-ks} \leqslant c t_n^s (mR_n - 1) \frac{1}{1 - (1 + \frac{mR_n - 1}{2})^{-s}} \leqslant c' t_n^s\,.$$

We conclude that $(\tilde{a}_n)^{-1/s} \geqslant c/t_n$ with $1/t_n = h^{-1}(c(mR_n - 1)) \geqslant c\gamma_n = c\tilde{\gamma}_n$, recall (1.14).

• Finally, we treat the case $mR_n \in (0, 1 - \frac{1}{n}]$. Simply using that $\tilde{m} R_n < 1$, we have that

$$\tilde{a}_n \leqslant c\, t_n^s h(1/t_n) \frac{(\tilde{m} R_n)^{-(n+1)s} - 1}{(\tilde{m} R_n)^{-s} - 1} \leqslant c' t_n^s \frac{h(1/t_n)}{1 - mR_n} (mR_n)^{-ns} (1 - \tilde{\delta}_n)^{-ns}$$

where we have used that $(\tilde{m} R_n)^{-s} - 1 \geqslant (mR_n)^{-s} - 1 \geqslant c(1 - mR_n)$ and also that $\tilde{m} = (1 - \tilde{\delta}_n) m$. Since we have $\tilde{\delta}_n \geqslant ch(1/t_n)$, we get that $(1 - \tilde{\delta}_n)^{-ns} \leqslant \exp(cnh(1/t_n))$, which pushes us to choose t_n such that $h(1/t_n) = 1/n$. In particular, with this choice, $(1-\tilde{\delta}_n)^{-ns} \leqslant c$ and $\frac{h(1/t_n)}{1-mR_n} = (n(1-mR_n))^{-1}$.

Then, we have

$$(\tilde{a}_n)^{-1/s} \geqslant ct_n^{-1}(mR_n)^n \left(n(1-mR_n)\right)^{1/s}$$

so that $(\tilde{a}_n)^{-1/s} \geqslant \tilde{\gamma}_n$, recalling that $t_n^{-1} = h^{-1}(1/n)$ and the definition of $\tilde{\gamma}_n$.

This concludes the proof. $\square$

A About the iteration of the random cluster model on trees

In this section, we prove Proposition 2.1. For $u \in T_n$, let us introduce the events

$$A_u := \{\exists \text{ open path from } u \text{ to } \partial T_n(u) \text{ inside } T_n(u)\}.$$

Denoting $E_n(u)$ the set of edges in $T_n(u)$, we also introduce the following notation: for any $A \subset \{0,1\}^{E_n(u)}$, let

$$Z_u(A) = Z_{\mathtt{p},\mathtt{q}}^{T_n(u)}(A) := \sum_{\omega \in A} \mathtt{p}^{o(\omega)}(1-\mathtt{p})^{f(\omega)}\mathtt{q}^{k(\omega)} = \mathbb{E}_{\mathtt{p}}\big[\mathtt{q}^{K_u}\mathbb{1}_A\big] \tag{A.1}$$

be the partition function of the $(\mathtt{p},\mathtt{q})$-RCM model on $T_n(u)$ restricted to the event A; we also denote Z_u for the partition function with $A = \{0,1\}^{E_n(u)}$. Here, we have rewritten the partition function using $\mathbb{P}_{\mathtt{p}}$ the distribution of the usual percolation model with parameter $\mathtt{p}$, *i.e.* under $\mathbb{P}_{\mathtt{p}}$ the random variables $(\omega_e)_{e\in E_n}$ are i.i.d. Bern$(\mathtt{p})$, and K_u is the random variable that counts the number of percolation clusters in $\bar{T}_n(u)$.

We are now going to show the following relations between $Z_u(A_u), Z_u(A_u^c)$ and $Z_v(A_v), Z_v(A_v^c)$ for $v \leftarrow u$: denoting $d_u := |\{v, v \leftarrow u\}|$ the number of children of u, we have

$$Z_u(A_u^c) = \mathtt{q}^{2-d_u} \prod_{v\leftarrow u} \Big((1-\mathtt{p})Z_v + \mathtt{p}\mathtt{q}^{-1}Z_v(A_v^c)\Big)\,, \tag{A.2}$$

$$\mathtt{q}Z_u(A_u) + Z_u(A_u^c) = \mathtt{q}^{2-d_u} \prod_{v\leftarrow u} \Big(Z_v + \mathtt{p}(\mathtt{q}^{-1}-1)Z_v(A_v^c)\Big)\,. \tag{A.3}$$

The key observation is to write a relation between the number of clusters in $\bar{T}_n(u)$ and those in $(\bar{T}_n(v))_{v\leftarrow u}$: for every $\omega \in \{0,1\}^{E_n}$, we have

$$K_u = \sum_{v\leftarrow u} K_v - (d_u - 1) + 1 - \sum_{v\leftarrow u} \mathbb{1}_{\{\omega_{uv}=1\}}\mathbb{1}_{\{A_v^c\}} - \mathbb{1}_{A_u} \tag{A.4}$$

Indeed, counting first the clusters in $\bar{T}_n(u)$ if all edges uv, $v \leftarrow u$ are open, we get that the number of clusters in $\bar{T}_n(u)$ is the sum of the number of clusters in $(\bar{T}_n(v))_{v\leftrightarrow}$ minus $d_u - 1$, because the wired boundary condition contracts the d_u clusters attached to each $\partial\bar{T}_n(v)$ to a single cluster, plus 1 to include the cluster of the root u. Then, adding the edges uv for which $\omega_{uv} = 1$, this reduces the number of cluster by

- $\mathbb{1}_{A_v^c}$ each time that $\omega_{uv} = 1$, since having ω_{uv} connects two clusters that are not already connected through $\partial T_n(u)$ (which is wired);
- one (only once) on the event $A_u = \bigcup_{v\leftrightarrow u}\{\omega_{uv} = 1\} \cap A_v$, since then the root u is connected to $\partial T_n(u)$ so adding more than one open edge $\omega_{uv} = 1$ with A_v will not decrease further the number of clusters.

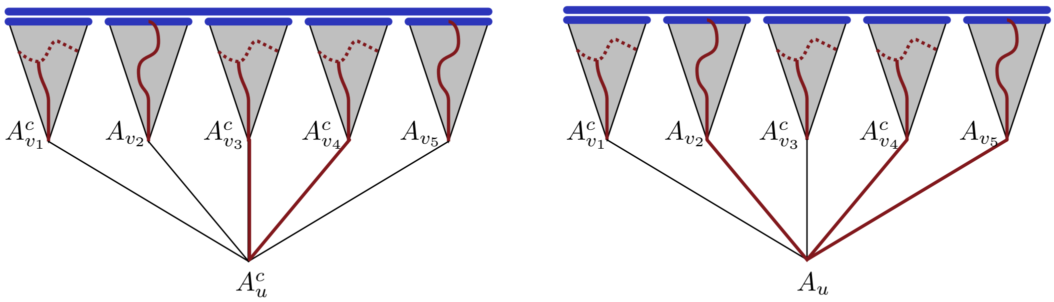

Figure 1: Illustration of the identity (A.4) that relates the numbers of clusters in $(\bar{T}_n(v))_{v\leftarrow u}$ to the number of clusters in $\bar{T}_n(u)$. We have illustrated two cases. On the left, the event A_u is not verified *i.e.* $u \not\leftrightarrow \partial T_n(u)$: there are open edges uv (in thick red) only connecting to subtrees $\bar{T}_n(v)$ where A_v is not verified, and each of these edges reduces the number of clusters by one. On the right, the event A_u is verified *i.e.* $u \leftrightarrow \partial T_n(u)$: there are open edges uv (in thick red) connecting to subtrees $\bar{T}_n(v)$ where A_v is verified and all these open edges reduce the global number of clusters only by one (other open edges connecting to subtrees $\bar{T}_n(v)$ where A_v is not verified still reduce the number of clusters each by one).

We refer to Figure 1 for an illustration of the relation (A.4).

Let us now prove (A.2)-(A.3). Starting from (A.1) and since the event A_u^c can be written as $\bigcap_{v\leftarrow u}\big(\{\omega_{uv}=0\}\cup(\{\omega_{uv}=1\}\cap A_v^c)\big)$, we have that

$$\begin{aligned} Z_u(A_u^c) &= \mathsf{q}^{2-d_u}\mathbb{E}_p\Big[\mathbb{1}_{A_u^c}\prod_{v\leftarrow u}\mathsf{q}^{K_v}\mathsf{q}^{-\mathbb{1}_{\{\omega_{uv}=1\}}\mathbb{1}_{A_v^c}}\Big] \\ &= \mathsf{q}^{2-d_u}\mathbb{E}_p\Big[\prod_{v\leftarrow u}\mathsf{q}^{K_v}\big(\mathbb{1}_{\{\omega_{uv}=0\}}+\mathsf{q}^{-1}\mathbb{1}_{\{\omega_{uv}=1\}}\mathbb{1}_{A_v^c}\big)\Big], \end{aligned}$$

where we have also used the relation (A.4) for the first identity. Using the independence of the ω_{uv} under $\mathbb{P}_\mathsf{p}$, we get (A.2).

On the other hand, starting again from (A.1) and the relation (A.4), we get that

$$\mathsf{q}Z_u(A_u) = \mathsf{q}^{2-d_u}\mathbb{E}_p\Big[\mathbb{1}_{A_u}\prod_{v\leftarrow u}\mathsf{q}^{K_v}\mathsf{q}^{-\mathbb{1}_{\{\omega_{uv}=1\}}\mathbb{1}_{A_v^c}}\Big].$$

Writing $\mathbb{1}_{A_u}=1-\mathbb{1}_{A_u^c}$, and recognizing the formula for $Z_u(A_u^c)$ from above, we get that

$$\begin{aligned} \mathsf{q}Z_u(A_u)+Z_u(A_u^c) &= \mathsf{q}^{2-d_u}\mathbb{E}_p\Big[\prod_{v\leftarrow u}\mathsf{q}^{K_v}\mathsf{q}^{-\mathbb{1}_{\{\omega_{uv}=1\}}\mathbb{1}_{A_v^c}}\Big] \\ &= \mathsf{q}^{2-d_u}\mathbb{E}_p\Big[\prod_{v\leftarrow u}\mathsf{q}^{K_v}\big(1+(\mathsf{q}^{-1}-1)\mathbb{1}_{\{\omega_{uv}=1\}}\mathbb{1}_{A_v^c}\big)\Big]. \end{aligned}$$

Again, using the independence of the ω_{uv} under $\mathbb{P}_\mathsf{p}$, we get (A.3).

Now, from (A.2)-(A.3), noticing that $Z_u(A_u)=\pi_n(u)Z_u$ and $Z_u(A_u^c)=(1-\pi_n(u))Z_u$, we get that

$$\begin{aligned} 1-\pi_n(u) &= \mathsf{q}^{2-d_u}\frac{\prod_{v\leftarrow u}Z_v}{Z_u}\times\prod_{v\leftarrow u}\Big(1-\mathsf{p}+\mathsf{p}\mathsf{q}^{-1}-\mathsf{p}\mathsf{q}^{-1}\pi_n(v)\Big) \\ 1+(\mathsf{q}-1)\pi_n(u) &= \mathsf{q}^{2-d_u}\frac{\prod_{v\leftarrow u}Z_v}{Z_u}\times\prod_{v\leftarrow u}\Big(1-\mathsf{p}+\mathsf{p}\mathsf{q}^{-1}+\mathsf{p}\mathsf{q}^{-1}(\mathsf{q}-1)\pi_n(v)\Big), \end{aligned}$$

so that dividing the first line by the second one we get that

$$\frac{1-\pi_n(u)}{1+(\mathsf{q}-1)\pi_n(u)} = \prod_{v\leftarrow u}\frac{1-\gamma_{\mathsf{p},\mathsf{q}}\pi_n(v)}{1+\gamma_{\mathsf{p},\mathsf{q}}(\mathsf{q}-1)\pi_n(v)},$$

with $\gamma_{\mathtt{p},\mathtt{q}} := \frac{\mathtt{p}\mathtt{q}^{-1}}{1-\mathtt{p}+\mathtt{p}\mathtt{q}^{-1}} = \frac{\mathtt{p}}{\mathtt{p}+\mathtt{q}(1-\mathtt{p})}$. This concludes the proof of (2.3).

Setting $b_n(u) := -\log \varphi_{\mathtt{q}}(\pi_n(u))$, inverting the relation (and noticing that $\varphi_{\mathtt{q}}^{-1} = \varphi_{\mathtt{q}}$) we easily get that $\pi_n(u) = \varphi_{\mathtt{q}}(\exp(-b_n(u))) = \psi_{\mathtt{q}}(b_n(u))$

$$b_n(u) = \sum_{v \leftarrow u} -\log \varphi_{\mathtt{q}}\big(\psi_{\mathtt{q}}(\beta)\psi_{\mathtt{q}}(b_n(v))\big)\,,$$

where we have also written $\gamma = \varphi_q(1-\mathtt{p}) = \psi_q(\beta)$. To conclude the proof of (2.4), it only remains to observe that $\psi_{\mathtt{q}}^{-1}(x) = -\log \varphi_{\mathtt{q}}(x)$, since $\psi_{\mathtt{q}}(t) = \varphi_{\mathtt{q}}(e^{-t})$ and $\varphi_{\mathtt{q}}^{-1} = \varphi_{\mathtt{q}}$.

B Branching processes with (truncated) heavy tails

This section is devoted to the proof of Proposition 3.4. Recall that we assume that

$$\mathbf{P}(Z > x) \leqslant L(x) x^{-\alpha}\,, \qquad \mathbf{P}(\tilde Z > x) \leqslant L(x) x^{-\alpha} \mathbb{1}_{\{x<t\}}\,,$$

where t is a fixed truncation parameter that we assume to be large.

We start with the proof of the first inequality in Proposition 3.4: there is a constant $c>0$

$$\mathbf{P}(W_\ell > x) \leqslant c L(x) x^{-\alpha} \qquad \text{for all } x \geqslant 1\,. \tag{B.1}$$

(We recall that this inequality could be deduced from the proof of [15], but we provide here a self-contained proof for completeness.) We start with a first general lemma — then, the proof relies on an inductive use of this lemma.

Lemma B.1. *Let $(X_i)_{i\geqslant 0}$ be independent non-negative random variables, with common mean $\mathbf{E}[X_i] = 1$ and which all satisfy*

$$\mathbf{P}(X_i > x) \leqslant \kappa_1 L(x) x^{-\alpha}\,, \qquad \textit{for } x \geqslant 1\,. \tag{B.2}$$

Let N be a $\mathbb{N}$-valued random variable with mean $\mu > 1$, independent of the X_i's. Then there is some constant $c>0$ and $\delta > 0$ (depending only on κ_1, α and $L(\cdot)$) such that, for $x \geqslant 1$,

$$\mathbf{P}\Big(\frac{1}{\mu}\sum_{i=1}^N X_i > x\Big) \leqslant \mathbf{P}\big(N > (1-\mu^{-\delta})\mu x\big) + c\mu^{-\frac12 \alpha}\mathbf{P}\big(N > \tfrac12 \mu x\big) + c\mu^{-\frac12(\alpha-1)} L(x) x^{-\alpha}\,.$$

Proof. Let $x \geqslant 1$ and let us set $\ell_1 := \frac12 \mu x$ and $\ell_2 = (1-\mu^{-\delta})\mu x$ with $\delta > 0$ fixed such that $\delta < \frac12 \wedge \frac{\alpha-1}{2\alpha}$. Then, we can split the probability as

$$\begin{aligned}\mathbf{P}\Big(\frac{1}{\mu}\sum_{i=1}^N X_i > x\Big) &= \Big(\sum_{\ell=1}^{\ell_1} + \sum_{\ell=\ell_1+1}^{\ell_2} + \sum_{\ell>\ell_2}\Big)\mathbf{P}(N=\ell)\mathbf{P}\Big(\sum_{i=1}^{\ell} X_i > \mu x\Big)\\ &=: T_1 \ + T_2 \ + T_3\,.\end{aligned}$$

Notice already that for the last term T_3, by definition of ℓ_2, we simply have

$$T_3 \leqslant \mathbf{P}\big(N > \ell_2\big) = \mathbf{P}\big(N > (1-\mu^{-\delta})\mu x\big)\,.$$

We now treat the remaining two terms. We use the following so-called one-big-jump behavior for sums of heavy tailed random variables, see e.g. [29, 14] (or [6, Thm. 5.1 & Eq. (5.1)] for a more convenient formulation, close to (B.3) below). We have the following statement:

assuming (B.2), there is a constant $c > 0$ such that, uniformly for ℓ such that $\mu x - \ell \geqslant \ell^{1-\delta}$, we have

$$\mathbf{P}\Big(\sum_{i=1}^{\ell} X_i > \mu x\Big) = \mathbf{P}\Big(\sum_{i=1}^{\ell}(X_i - \mathbf{E}[X_i]) > \mu x - \ell\Big) \leqslant c\ell\, \kappa_1 L(\mu x - \ell)(\mu x - \ell)^{-\alpha} . \tag{B.3}$$

Term T_1. Using (B.3), recalling that $\ell_1 = \frac{1}{2}\mu x$ so that in particular $\mu x - \ell \geqslant \frac{1}{2}\mu x$, the first term is bounded by

$$T_1 \leqslant cL(\mu x)(\mu x)^{-\alpha} \sum_{\ell=1}^{\frac{1}{2}\mu x} \ell \mathbf{P}(N = \ell) \leqslant c' \mu^{-\alpha+\delta} L(x) x^{-\alpha} \mathbf{E}[N] ,$$

using Potter's bound [8, Thm. 1.5.6]. Since $\mathbf{E}[N] = \mu$ and since we chose $\delta < \frac{1}{2}(\alpha - 1)$, we get that

$$T_1 \leqslant c\mu^{-\frac{1}{2}(\alpha-1)} L(x) x^{-\alpha} .$$

Term T_2. Using (B.3), recalling that $\ell_1 = \frac{1}{2}\mu x$ and $\ell_2 = (1 - \mu^{-\delta})\mu x$ (which verifies $\mu x - \ell_2 = \mu^{1-\delta} x \geqslant (\mu x)^{1-\delta} \geqslant \ell_2^{1-\delta}$), we get that

$$T_2 \leqslant \mathbf{P}\big(N > \ell_1\big) \mathbf{P}\Big(\sum_{i=1}^{\ell_2} X_i > \mu x\Big) \leqslant \mathbf{P}\big(N > \tfrac{1}{2}\mu x\big) \times c\kappa_1 L((\mu x)^{1-\delta})(\mu x)^{-(1-\delta)\alpha} .$$

All together, using again Potter's bound [8, Thm. 1.5.6] and since $\delta < 1/2$ and $x \geqslant 1$ we end up with

$$T_2 \leqslant c\mu^{-\alpha/2} \mathbf{P}\big(N > \tfrac{1}{2}\mu x\big) .$$

This concludes the proof. □

Let us now obtain (B.1) simply by iterating Lemma B.1; note that our assumption is that W_1 satisfies the tail condition (B.2). We write that $W_{\ell+1} = \frac{1}{m^\ell} \sum_{i=1}^{Z_\ell} W_1^{(i)}$, where $(W_1^{(i)})_{i \geqslant 1}$ are i.i.d. non-negative random variables with mean one, independent of Z_ℓ. Let $c_\ell := \prod_{i=1}^{\ell} (1 - m^{-\delta \ell})^{-1}$ so that we have $c_{\ell+1}(1 - m^{-\delta\ell}) = c_\ell$. Then, applying Lemma B.1 and noting that $\mu = \mathbf{E}[Z_\ell] = m^\ell$, we get that for any $x \geqslant 1$

$$\mathbf{P}\big(W_{\ell+1} \geqslant c_{\ell+1} x\big) \leqslant \mathbf{P}\big(W_\ell \geqslant c_\ell x\big) + cm^{-\frac{1}{2}(\alpha-1)\ell} L(x) x^{-\alpha} + cm^{-\alpha\ell/2} \mathbf{P}\big(W_\ell \geqslant \tfrac{1}{2} c_\ell x\big) , \tag{B.4}$$

where we also used that c_ℓ is bounded by a universal constant $c_\infty < +\infty$ to get the bound $L(c_{\ell+1}x)(c_{\ell+1}x)^{-\alpha} \leqslant cL(x)x^{-\alpha}$. Iterating (B.4), we get

$$\mathbf{P}\big(W_\ell \geqslant c_\ell x\big) \leqslant \kappa_\ell L(x) x^{-\alpha} ,$$

with $\kappa_{\ell+1} = \kappa_\ell + cm^{-\frac{1}{2}(\alpha-1)\ell} + c'\kappa_\ell m^{-\alpha\ell/2}$. All together, since $\kappa := \sup_{\ell \geqslant 1} \kappa_\ell < +\infty$, and $c_\infty := \sup_{\ell \geqslant 1} c_\ell < +\infty$, we get (B.1) (up to a change in the constants).

We now turn to the second inequality of Proposition 3.4, on the martingale $\tilde{W}_\ell$ associated with the truncated branching process. First, notice that we also have $\mathbf{P}(\tilde{Z} > x) \leqslant cL(x)x^{-\alpha}$ for all $x \geqslant 1$, so by the first part of the Proposition we have that $\mathbf{P}(\tilde{W}_\ell > x) \leqslant cL(x)x^{-\alpha}$, uniformly in $\ell \geqslant 1$. It therefore remains to prove the last inequality, *i.e.* that there is a constant $c' > 0$ such that, provided that t is large enough,

$$\mathbf{P}(\tilde{W}_\ell > x) \leqslant t^{-c'x/t} \qquad \text{for all } x \geqslant t , \tag{B.5}$$

uniformly in $\ell \geqslant 1$. For this, we use the following standard Chernov's bound: for any $\lambda > 0$, we have

$$\mathbf{P}(\tilde{W}_\ell > x) \leqslant e^{-\lambda x} \mathbf{E}\big[e^{\lambda \tilde{W}_\ell}\big] .$$

Then, our next task is to bound the Laplace transform of $\tilde{W}_\ell$, uniformly in ℓ. We prove that there exists some $c > 0$ such that, for all $\ell \geqslant 1$, provided that t is large enough,

$$\mathbf{E}\big[e^{c\frac{\log t}{t} \tilde{W}_\ell}\big] \leqslant 2 . \tag{B.6}$$

Plugged into the above and choosing $\lambda = \frac{c}{t}\log t$, we end up with $\mathbf{P}(\tilde{W}_\ell \geqslant x) \leqslant 2t^{-cx/t}$ for all $x \geqslant 1$, which proves (B.5). It remains to prove (B.6), and we rely on the following Lemma.

Lemma B.2. *Let $(X_i)_{i\geqslant 0}$ be independent non-negative random variables, with common mean $\mathbf{E}[X_i] = 1$. Let $y > 1$ and assume that $\mathbf{P}(X_i \in [0,t]) = 1$ and let $\sigma^2(t)$ be such that $\mathbf{E}[X_i^2] \leqslant \sigma^2(t)$ for all i. Let N be a $\mathbb{N}$-valued random variable with mean $\mu > 0$, independent of the X_i's. Then, for all $\lambda > 0$,*

$$\mathbf{E}\Big[\exp\Big(\frac{\lambda}{\mu}\sum_{i=1}^{N} X_i\Big)\Big] \leqslant \mathbf{E}\Big[\exp\Big(\Big(1 + \frac{\lambda}{\mu}\sigma^2(t)e^{t\lambda/\mu}\Big) \times \frac{\lambda}{\mu}N\Big)\Big] .$$

Proof. First of all, notice that taking first the conditional expectation with respect to N, we obtain

$$\mathbf{E}\Big[\exp\Big(\lambda\frac{1}{\mu}\sum_{i=1}^{N} X_i\Big)\Big] = \mathbf{E}\Big[\prod_{i=1}^{N}\mathbf{E}\Big[e^{\frac{\lambda}{\mu}X_i}\Big]\Big] .$$

We now control the term

$$\mathbf{E}\Big[e^{\frac{\lambda}{\mu}X_i}\Big] \leqslant 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}\mathbf{E}[X_i^2]e^{\frac{\lambda}{\mu}t} \leqslant \exp\Big(\frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}\mathbf{E}[X_1^2]e^{\lambda t/\mu}\Big) , \tag{B.7}$$

where we used that $e^x \leqslant 1 + x + x^2 e^{tx}$ for all $x \in [0,t]$ and the fact that $\mathbf{E}[X_i] = 1$. Bounding $\mathbf{E}[X_i^2] \leqslant \sigma^2(t)$ and combined with the previous identity, this gives the desired result. $\square$

We are now ready to conclude the proof of (B.6), using Lemma B.2 iteratively with the recurrence relation $\tilde{W}_{\ell+1} = \frac{1}{\tilde{m}^\ell}\sum_{i=1}^{\tilde{Z}_\ell} \tilde{W}_1^{(i)}$. Recalling that $\tilde{W}_1 \in [0,t]$ almost surely, Lemma B.2 then gives that, for any $\lambda > 0$,

$$\mathbf{E}\Big[e^{\lambda \tilde{W}_{\ell+1}}\Big] \leqslant \mathbf{E}\Big[e^{(1+\tilde{m}^{-\ell}\varepsilon_\lambda)\,\lambda \tilde{W}_\ell}\Big] , \tag{B.8}$$

with $\varepsilon_\lambda := \lambda\mathbf{E}[(\tilde{W}_1)^2]e^{\lambda t}$ (bounding also $e^{\lambda t/\tilde{m}^\ell} \leqslant e^{\lambda t}$).

Using the fact that we have $\mathbf{P}(\tilde{Z} > x) \leqslant c_2 L(x)x^{-\alpha}\mathbb{1}_{\{x<t\}}$, we obtain that: if $\alpha \in (1,2)$, then $\mathbf{E}[(\tilde{W}_1)^2] \leqslant c_2' L(t)t^{2-\alpha}$; if $\alpha \geqslant 2$, then $\mathbf{E}[\tilde{Z}] \leqslant \hat{L}(t_n)$ for $\alpha \geqslant 2$, for some slowly varying function (a constant if $\alpha > 2$). We can therefore write that

$$\varepsilon_\lambda \leqslant c\lambda\hat{L}(t)t_n^{2-\alpha\wedge 2}e^{\lambda t} ,$$

for some slowly varying function $\hat{L}(\cdot)$. Now, if we take $\lambda \leqslant c_\alpha \frac{\log t}{t}$ with a constant $c_\alpha := \frac{1}{2}(\alpha\wedge 2 - 1) > 0$, we get that $\varepsilon_\lambda \leqslant c_\alpha \log t\hat{L}(t)t^{-2c_\alpha}t^{c_\alpha}$, so in particular $\tilde{\varepsilon}_\lambda \leqslant 1$, provided that t is large enough.

Let us fix $\lambda = \delta\frac{\log t}{t}$ with $\delta := c_\alpha\prod_{i\geqslant 1}(1+\tilde{m}^{-i})^{-1}$, and let us define $(\lambda_k)_{1\leqslant k\leqslant \ell}$ by setting $\lambda_1 = \lambda$ and $\lambda_{k+1} := (1+\tilde{m}^{k-\ell})\lambda_k$ for $1 \leqslant k \leqslant \ell - 1$. Notice that $\lambda_k \leqslant \lambda_\ell \leqslant c_\alpha\frac{\log t}{t}$; in particular $\varepsilon_{\lambda_k} \leqslant 1$ for all $k \in \{1,\dots,\ell\}$. Applying (B.8) iteratively, we then get that

$$\mathbf{E}\Big[e^{\lambda\tilde{W}_\ell}\Big] = \mathbf{E}\Big[e^{\lambda_1\tilde{W}_\ell}\Big] \leqslant \mathbf{E}\Big[e^{\lambda_2\tilde{W}_{\ell-1}}\Big] \leqslant \cdots \leqslant \mathbf{E}\Big[e^{\lambda_\ell\tilde{W}_1}\Big] .$$

Similarly to (B.7), this last expression is bounded by

$$1 + \lambda_\ell + \lambda_\ell^2 \mathbf{E}[(\tilde{W}_1)^2] e^{\lambda_\ell t} \leqslant 1 + \lambda_\ell(1 + \varepsilon_{\lambda_\ell}) \leqslant 1 + 2\lambda_\ell \leqslant 2\,,$$

using again that $\varepsilon_{\lambda_\ell} \leqslant 1$ and then that $\lambda_\ell \leqslant c_\alpha \frac{\log t}{t} \leqslant 1$ if t is large enough. This concludes the proof of (B.6) and thus of the second part of Proposition 3.4. □

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