FRACTIONAL-DIFFRACTION-OPTICS CAUCHY PROBLEM: RESOLVENT-FUNCTION SOLUTION OF THE MATRIX INTEGRAL EQUATION

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Abstract. The fractional diffraction optics theory has been elaborated using the Green function technique. The optics-fractional equation describing the diffraction X-ray scattering by imperfect crystals has been derived as the fractional matrix integral Fredholm–Volterra equation of the second kind. In the paper, to solve the Cauchy problems, the Liouville–Neumann-type series formalism has been used to build up the matrix Resolvent-function solution. In the case when the imperfect crystal-lattice elastic displacement field is the linear function $f(\mathbf{R}) = ax + b$, $a, b = const$, the explicit solution of the diffraction-optics Cauchy problem has been obtained and analyzed for arbitrary fractional-order-parameter α, $\alpha \in (0, 1]$.

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1 Introduction

Usually, the diffraction optics (DO) has been formulated and based on the known differential Takagi–Taupin (TT) equations when the fractional-order-parameter $\alpha = 1$ (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8] for details). At the same time, in the last decades, substantial progress has been achieved in mathematical physics using equations with fractional-order derivatives [9], [10], [11], [12], [13], [14], [15], [16]. Indeed, the different theory-physics models for solving the Cauchy problems using systems of fractional differential equations have been treated in [17], [18], [19], [20], [21], [22], [23], [24], [25].

Following this logic, one can push one step further in the DO now founded on the TT-type equations with the fractional derivatives of the arbitrary order $\alpha \in (0, 1]$ along the crystal depth.

In the paper, using Green's function technique, one derives the matrix Fredholm–Volterra integral equation of the second kind and builds up the matrix Resolvent-function solution of the DO Cauchy problem. A goal of working is to develop the integral formalism of the DO theory earlier proposed by authors [26] and to build up the matrix Resovent-function solution of the fractional Cauchy problem.

As an example, for an arbitrary fractional-order-parameter (FOP) α, $\alpha \in (0,1]$, when the crystal-lattice displacement field function $f(\mathbf{R})$ is a linear function, namely: $f(\mathbf{R}) = ax + b$, $a, b = const$, one finds out an explicit analytical solution of the DO Cauchy problem.

Accordingly, the original system of fractional DO equations has the form (cf. [26])

$$\begin{pmatrix} \partial_{0t}^{\alpha} - \frac{\partial}{\partial x} & 0 \\ 0 & \partial_{0t}^{\alpha} + \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix} =$$

$$= i \begin{pmatrix} \gamma & \sigma \exp[if(x,t)] \\ \sigma \exp[-if(x,t)] & \gamma \end{pmatrix} \begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix}, \qquad (1.1)$$

with the DO Cauchy problem's condition

$$\begin{pmatrix} E_0(x,0) \\ E_h(x,0) \end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_h(x) \end{pmatrix}, \quad -\infty < x < \infty, \qquad (1.2)$$

where $\varphi_0(x)$ and $\varphi_h(x)$ are the given real-valued functions.

2 Preliminaries

Following up to [13], the Gerasimov–Caputo fractional derivative beginning at point a is determined as (cf. [13])

$$\partial_{at}^{\nu} g(t) = \operatorname{sgn}^n(t-a) D_{at}^{\nu-n} \frac{d^n}{dt^n} g(t), \quad n-1 < \nu \leq n, \quad n \in \mathbb{N}, \qquad (2.1)$$

where D_{ay}^{ν} is the Riemann–Liouville fractional integro-differential operator of order ν is equal to

$$D_{ay}^{\nu} g(y) = \frac{\operatorname{sgn}(y-a)}{\Gamma(-\nu)} \int_a^y \frac{g(s)ds}{|y-s|^{\nu+1}}, \quad \nu < 0,$$

for $\nu \geq 0$ the operator D_{ay}^{ν} can be determined by recursive relation

$$D_{ay}^{\nu} g(y) = \operatorname{sgn}(y-a) \frac{d}{dy} D_{ay}^{\nu-1} g(y), \quad \nu \geq 0, \qquad (2.2)$$

$\Gamma(z)$ is the Euler gamma-function.

Note that in the limit case of the FOP $\alpha = 1$ the operator $\partial_{0t}^{\alpha} g(t)$ reduces to the standard derivative $\frac{d}{dt} g(t)$.

Formula for the composition of operators of fractional integration is valid as [13]

$$D_{0t}^{\nu} D_{0t}^{\delta} g(t) = D_{0t}^{\nu+\delta} g(t), \quad \nu < 0, \quad \delta < 0. \tag{2.3}$$

There is a formula for fractional integration by parts, namely [13]:

$$\int_0^t g(t,\xi) D_{0\xi}^{\nu} h(\xi) d\xi = \int_0^t h(\xi) D_{t\xi}^{\nu} g(t,\xi) d\xi, \quad \nu < 0. \tag{2.4}$$

Further, we denote the Fourier transform of the function $f(x)$ by $(f(x))_k$, the Laplace transform of the function $g(t)$ by $(g(t))_p$, and respectively, the double Fourier–Laplace transform of the function $h(x,t)$ by $(h(x,t))_{k,p}$.

By using the following formula [14, p. 98]

$$[\partial_{0t}^{\alpha} H(x,t)]_p = p^{\alpha} [H(x,t)]_p - p^{\alpha-1} H(x,0),$$

one can get

$$[O_{\pm}^{\alpha} H(x,t)]_{k,p} = (p^{\alpha} \pm ik) [H(x,t)]_{k,p} - p^{\alpha-1} [H(x,0)]_k, \tag{2.5}$$

The following series

$$\phi(\beta,\rho;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta k + \rho)}, \quad \beta > -1, \quad \rho \in \mathbb{C}$$

defines the Wright function [27] which depends on two parameters ρ and μ.

Accordingly, the following differentiation formula is valid [27]:

$$\frac{d}{dz} \phi(\beta,\rho;z) = \phi(\beta,\beta+\rho;z), \quad \beta > -1. \tag{2.6}$$

Let $\beta \in (0,1)$, and $\mu,\nu \in \mathbb{R}$, and the inequality takes place

$$\beta \in (0,1), \quad 0 \leq |\arg \lambda| < \frac{1-\beta}{2}\pi,$$

then the formula

$$D_{0y}^{\nu} y^{\mu-1} \phi(-\beta,\mu;-\lambda y^{-\beta}) = y^{\mu-\nu-1} \phi(-\beta,\mu-\nu;-\lambda y^{-\beta}) \tag{2.7}$$

is valid [28].

The following formula take place [29]

$$\left(y^{\delta-1} \phi(-\beta,\mu;-ty^{-\beta})\right)_p = p^{-\mu} e^{-p^{\beta} t}. \tag{2.8}$$

Correspondingly, the integrals

$$\int\limits_0^\infty \frac{\cos kx}{k^2+\rho^2}dk = \frac{\pi}{2\rho}e^{-\rho x}, \tag{2.9}$$

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} \frac{e^{-x\sqrt{p^2+\sigma^2}}e^{pt}dp}{\sqrt{p^2+\sigma^2}} = J_0\left(\sigma\sqrt{t^2-x^2}\right)\Theta(t-|x|), \tag{2.10}$$

are held on, where $\Theta(x)$ is the Heaviside function, and $J_0(x)$ is the zero-order Bessel function of real argument.

The properties-in-details of the function

$$G_{\alpha,\gamma}(x,t) = \frac{1}{2}\int\limits_{|x|}^\infty e^{i\gamma\tau}J_0\left(\sigma\sqrt{\tau^2-x^2}\right)\frac{1}{t}\phi\left(-\alpha,0;-\frac{\tau}{t^\alpha}\right)d\tau$$

have been analyzed in the [30].

Note, in limit of the FOP $\alpha=1$, the function $G_{\alpha,\gamma}(x,t)$ can be cast into the form

$$\lim_{\alpha\to 1} G_{\alpha,\gamma}(x,t) = e^{i\gamma|x|}J_0\left(\sigma\sqrt{t^2-x^2}\right)\Theta(t-|x|).$$

The estimate

$$\left|\frac{\partial^m}{\partial x^m}D_{0t}^\nu G_{\alpha,\gamma}(x,t)\right| \le C|x|^{-\theta}y^{\alpha(1-m+\theta)-\nu-1}, \quad \theta\ge 0 \tag{2.11}$$

holds on for arbitrary $m\in\mathbb{N}\cup\{0\}$ and $\nu\in\mathbb{R}$, where C is the positive constant [30].

Accordingly, the following estimate

$$\left|\frac{\partial^m}{\partial x^m}D_{0t}^\nu G_{\alpha,\gamma}(x,t)\right| \le C\exp\left(-\sigma_0|x|^\varepsilon t^{-\alpha\varepsilon}\right), \tag{2.12}$$

holds on as $|x|\to\infty$ for all $t<\infty$ [30], and for arbitrary $m\in\mathbb{N}\cup\{0\}$, $\nu\in\mathbb{R}$; here and further $\sigma_0<(1-\alpha)\alpha^{\alpha\varepsilon}$, $\varepsilon=\frac{1}{1-\alpha}$; C is the positive constant.

Lemma 2.1. [21, p. 172]. For $0<\mu,\nu<1$ the estimate

$$\int\limits_{x-\delta_1}^{x+\delta} |x-y|^{\mu-1}|y-\xi|^{\nu-1}dy \le C_\gamma|x-\xi|^{\gamma-1}, \tag{2.13}$$

holds on, where

$$\gamma=\min\{\mu,\nu\}, \quad C_\gamma=\left[\frac{2}{\gamma}+B(\mu,\nu)\right](\delta_1+\delta_2)^{\mu+\nu-\gamma}.$$

Lemma 2.2. [21, p. 176]. Let $0 < \eta < s < t < T$, $\sigma_2 < \sigma_3 < \sigma_1$, then the inequality

$$\int\limits_{x\pm\delta}^{\pm\infty} \exp\left[-\frac{\sigma_1|x-y|^{\varepsilon}}{(t-s)^{\alpha\varepsilon}}\right] \exp\left[-\frac{\sigma_2|y-\xi|^{\varepsilon}}{(s-\eta)^{\alpha\varepsilon}}\right] dy \leq C \exp\left[-\frac{\sigma_2|x-\xi|^{\varepsilon}}{(t-\eta)^{\alpha\varepsilon}}\right], \quad (2.14)$$

takes place, where

$$C = \frac{\sigma_1 - \sigma_3}{\varepsilon T^{\varepsilon\alpha}} \delta^{1-\varepsilon} \exp\left[-\frac{\sigma_1 - \sigma_3}{T^{\varepsilon\alpha}} \delta^{\varepsilon}\right].$$

Furthermore, one needs the following formulae (see [31, p. 201])

$$\int\limits_0^{\tau} \frac{\eta}{\sqrt{\tau^2 - \eta^2}} \cos\left(\rho\sqrt{\tau^2 - \eta^2}\right) J_0(\sigma\eta)\, d\eta = \frac{1}{k} \sin(k\tau), \quad (2.15)$$

$$\int\limits_0^{\tau} \frac{J_1(\sigma\eta)}{\sqrt{\tau^2 - \eta^2}} \cos\left(\rho\sqrt{\tau^2 - \eta^2}\right) d\eta = \frac{1}{\sigma\tau} \cos(\rho\tau) - \frac{1}{\sigma\tau} \cos(k\tau), \quad (2.16)$$

where $k = \sqrt{\sigma^2 + \rho^2}$.

Under formula (2.16), one can obtain

$$\int\limits_0^{\tau} \sin\left(\rho\sqrt{\tau^2 - \eta^2}\right) J_1(\sigma\eta)\, d\eta = \rho \int\limits_0^{\tau} \left(\xi \int\limits_0^{\xi} \frac{\cos\left(\rho\sqrt{\xi^2 - \eta^2}\right)}{\sqrt{\xi^2 - \eta^2}} J_1(\sigma\eta)\, d\eta\right) d\xi =$$

$$= \frac{\rho}{\sigma} \int\limits_0^{\tau} [\cos(\rho\xi) - \cos(k\xi)]\, d\xi = \frac{1}{\sigma} \sin(\rho\tau) - \frac{\rho}{\sigma k} \sin(k\tau). \quad (2.17)$$

The Stankovic's transformation integral (see [32, p. 84])

$$\int\limits_0^{\infty} \exp(\lambda\tau) t^{\nu-1} \phi\left(-\mu, \nu; -\tau t^{-\mu}\right) d\tau = t^{\mu+\nu-1} E_{\mu}\left(-\lambda t^{\mu}; \mu + \nu\right), \quad (2.18)$$

takes place for any $\lambda \in \mathbb{C}$, $\mu \in (0,1)$, $\nu \in \mathbb{R}$, where

$$E_{\rho}(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + \rho k)}$$

is the Mittag–Leffler-type function [33, c. 117].

3 Matrix integral formalism of the DO Cauchy problem

Let us convert the DO Cauchy problem in the 'differential form' (1.1)–(1.2) to the matrix Fredholm–Volterra-type integral equation of the second kind. The latter takes a special sense to build up the Resolvent-function solution of the DO Cauchy problem in terms of the Liouville–Neumann-type series, which in turn is very important to computer-aided modeling and recovering the crystal-lattice displacement field function $f(\mathbf{R})$ from the X-ray DO microtomography data.

The system of differential TT-type equations (1.1) may be rewritten into the form

$$\begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} = i\sigma\mathbf{K}\mathbf{E}, \tag{3.1}$$

where

$$O_+^\alpha = \partial_{0t}^\alpha + \frac{\partial}{\partial x}, \quad O_-^\alpha = \partial_{0t}^\alpha - \frac{\partial}{\partial x},$$

$$\mathbf{E} \equiv \mathbf{E}(x,t) = \begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix}, \quad \mathbf{K} \equiv \mathbf{K}(x,t) = \begin{pmatrix} 0 & e^{if(x,t)} \\ e^{-if(x,t)} & 0 \end{pmatrix}.$$

Acting onto both sides of (3.1) by the operator $diag(O_+^\alpha - i\gamma, O_-^\alpha - i\gamma)$, one obtains

$$\begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} =$$

$$= i\sigma \begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \mathbf{K}\mathbf{E} \tag{3.2}$$

and after some straightforward routine calculations, one finds out

$$\begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \mathbf{K}\mathbf{E} = \left[\partial_{0t}^\alpha + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} - i\gamma\right] (\mathbf{K}\mathbf{E}) = \partial_{0t}^\alpha(\mathbf{K}\mathbf{E}) +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\left(\frac{\partial}{\partial x}\mathbf{K}\right)\mathbf{E} + \mathbf{K}\left(\frac{\partial}{\partial x}\mathbf{E}\right)\right] + \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right) - \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right) - i\gamma\,\mathbf{K}\mathbf{E} =$$

$$= \mathbf{K}\begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} + \partial_{0t}^\alpha(\mathbf{K}\mathbf{E}) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{\partial}{\partial x}\mathbf{K}\right)\mathbf{E} - \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right) =$$

$$= \mathbf{K}\begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} + \left(\partial_{0t}^\alpha + if_x'\right)(\mathbf{K}\mathbf{E}) - \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right). \tag{3.3}$$

The column vector $\mathbf{E} = \mathbf{E}(x,t)$ is nothing else the solution of Eq. (3.1) and $\mathbf{K}^2$ is equal to Unit matrix, from Eq. (3.3) it directly follows

$$i\sigma \begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \mathbf{KE} = -\sigma^2 \mathbf{E} + i\sigma \left(\partial_{0t}^\alpha + if_x'\right)(\mathbf{KE}) - i\sigma \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right).$$

The last allows us to rewrite down Eq. (3.2) into the form

$$\begin{pmatrix} (O_+^\alpha - i\gamma)(O_-^\alpha - i\gamma) + \sigma^2 & 0 \\ 0 & (O_-^\alpha - i\gamma)(O_+^\alpha - i\gamma) + \sigma^2 \end{pmatrix} \mathbf{E} =$$

$$= i\sigma \left(\partial_{0t}^\alpha + if_x'\right)(\mathbf{KE}) - i\sigma \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right). \tag{3.4}$$

Keeping in mind Eq.(2.5), and applying the double Fourier–Laplace transform to Eq. (3.4), one can obtain

$$\left[(O_+^\alpha - i\gamma)(O_-^\alpha - i\gamma)E_0(x,t)\right]_{k,p} = (p^\alpha - i\gamma + ik)[(O_-^\alpha - i\gamma)E_0(x,t)]_{k,p} -$$

$$-p^{\alpha-1}\left\{[(O_-^\alpha - i\gamma)E_0(x,t)]_{t=0}\right\}_k = (p^\alpha - i\gamma + ik)(p^\alpha - i\gamma - ik)[E_0(x,t)]_{k,p} -$$

$$-p^{\alpha-1}\Big\{(p^\alpha - i\gamma + ik)[E_0(x,0)]_k + i\sigma[e^{if(x,0)}E_h(x,0)]_k\Big\}, \tag{3.5}$$

and

$$\left[(O_-^\alpha - i\gamma)(O_+^\alpha - i\gamma)E_h(x,t)\right]_{k,p} = (p^\alpha - i\gamma - ik)(p^\alpha - i\gamma + ik)[E_h(x,t)]_{k,p} -$$

$$-p^{\alpha-1}\Big\{(p^\alpha - i\gamma - ik)[E_h(x,0)]_k + i\sigma[e^{-if(x,0)}E_0(x,0)]_k\Big\}. \tag{3.6}$$

Here, one has used the relationships (*cf.* (3.1))

$$\left[(O_-^\alpha - i\gamma)E_0(x,t)\right]_{t=0} = i\sigma e^{if(x,0)}E_h(x,0),$$

$$\left[(O_+^\alpha - i\gamma)E_h(x,t)\right]_{t=0} = i\sigma e^{-if(x,0)}E_0(x,0).$$

Thus from Eqs.(3.4) – (3.6) it directly follows

$$\begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix}_{k,p} = \frac{p^{\alpha-1}}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} \times$$

$$\times \left\{ \begin{pmatrix} p^\alpha - i\gamma + ik & 0 \\ 0 & p^\alpha - i\gamma - ik \end{pmatrix} \begin{pmatrix} E_0(x,0) \\ E_h(x,0) \end{pmatrix}_k + i\sigma \begin{pmatrix} e^{if(x,0)}E_h(x,0) \\ e^{-if(x,0)}E_0(x,0) \end{pmatrix}_k \right\} +$$

$$+\frac{1}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} \left\{ \begin{pmatrix} \partial_{0t}^\alpha + if_x' & 0 \\ 0 & \partial_{0t}^\alpha + if_x' \end{pmatrix} \begin{pmatrix} i\sigma e^{if}E_h(x,t) \\ i\sigma e^{-if}E_0(x,t) \end{pmatrix} - \right.$$

$$-\left(\begin{array}{cc} 0 & i\sigma e^{if} \\ i\sigma e^{-if} & 0 \end{array}\right)\left(\begin{array}{c} \partial_{0t}^{\alpha} E_0(x,t) \\ \partial_{0t}^{\alpha} E_h(x,t) \end{array}\right)\Bigg\}_{k,p}. \tag{3.7}$$

Applying the Efros's theorem for operational calculus [34, p. 512], the equalities (2.6)–(2.10), and using the inverse double Fourier–Laplace transform, the following elations take place

$$\left(\frac{1}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2}\right)_{x,t} = \frac{1}{i(2\pi)^2}\int\limits_{-i\infty}^{i\infty} dp \int\limits_{-\infty}^{\infty} \frac{e^{pt+ikx}}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} dk =$$

$$= \frac{1}{2}\int\limits_{|x|}^{\infty} e^{i\gamma\tau} J_0\left(\sigma\sqrt{\tau^2 - x^2}\right)\frac{1}{t}\phi\left(-\alpha, 0; -\frac{\tau}{t^\alpha}\right) d\tau = G_{\alpha,\gamma}(x,t),$$

$$\left(\frac{p^{\alpha-1}}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2}\right)_{x,t} = \frac{1}{i(2\pi)^2}\int\limits_{-i\infty}^{i\infty} dp \int\limits_{-\infty}^{\infty} \frac{p^{\alpha-1}e^{pt+ikx}}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} dk =$$

$$= D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x,t) = \frac{1}{2}\int\limits_{|x|}^{\infty} e^{i\gamma\tau} J_0\left(\sigma\sqrt{\tau^2 - x^2}\right) t^{-\alpha}\phi\left(-\alpha, 1-\alpha; -\frac{\tau}{t^\alpha}\right) d\tau.$$

As a result, from (3.7) one obtains the DO integral matrix equation

$$\mathbf{E}(x,t) = (\mathbf{A}^{\alpha,\gamma}\mathbf{E}(x,t))(x,t) + (\mathbf{B}^{\alpha,\gamma}\mathbf{E}(x,0))(x,t), \tag{3.8}$$

where the following notations are introduced

$$(\mathbf{A}^{\alpha,\gamma}\mathbf{E}(x,t))(x,t) = -i\sigma\int\limits_0^t dv \int\limits_{-\infty}^{\infty} G_{\alpha,\gamma}(x-u, t-v)\cdot\Bigg\{ D_{0v}^{\alpha-1}\frac{\partial}{\partial v}[\mathbf{K}(u,v)\mathbf{E}(u,v)] +$$

$$+\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)\left[\frac{\partial}{\partial u}\mathbf{K}(u,v)\right]\mathbf{E}(u,v) - \mathbf{K}(u,v) D_{0v}^{\alpha-1}\frac{\partial}{\partial v}\mathbf{E}(u,v)\Bigg\} du,$$

$$(\mathbf{B}^{\alpha,\gamma}\mathbf{E}(x,0))(x,t) = \int\limits_0^t dv \int\limits_{-\infty}^{\infty} D_{tv}^{\alpha-1} G_{\alpha,\gamma}(x-u, t-v)\times$$

$$\times\left(\begin{array}{cc} O_+^\alpha - i\gamma & i\sigma e^{if(u,0)} \\ i\sigma e^{-if(u,0)} & O_-^\alpha - i\gamma \end{array}\right)\mathbf{E}(u,0)\delta(v) du,$$

where $\delta(v)$ is the Dirac delta-function.

Using the formula for integration by parts and its fractional analogue (2.4), the estimates (2.11) and (2.12), definition of the fractional Riemann–Liouville derivative, one finds out

$$(\mathbf{A}^{\alpha,\gamma}\mathbf{E}(x,t))(x,t) = -i\sigma \int\limits_{0}^{t} dv \int\limits_{-\infty}^{\infty} \mathbf{K}_1(x,t;u,v)\mathbf{E}(u,v)du-$$

$$-i\sigma \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t)\mathbf{K}(u,0)\mathbf{E}(u,0)du, \tag{3.9}$$

where

$$\mathbf{K}_1(x,t;u,v) = D_{vt}^{\alpha} G_{\alpha,\gamma}(x-u,t-v)\cdot \mathbf{K}(u,v)+$$
$$+if'_u(u,v)\cdot G_{\alpha,\gamma}(x-u,t-v)\mathbf{K}(u,v) + D_{vt}^{\alpha}\left[G_{\alpha,\gamma}(x-u,t-v)\mathbf{K}(u,v)\right].$$

In view of (2.3) the $(\mathbf{B}^{\alpha,\gamma}\mathbf{E}(x,0))(x,t)$ may be rewritten as

$$(\mathbf{B}^{\alpha,\gamma}\mathbf{E}(x,0))(x,t) = -\int\limits_{-\infty}^{\infty} D_{0t}^{2\alpha-1} G_{\alpha,\gamma}(x-u,t)\mathbf{E}(u,0)du-$$

$$-i\gamma \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t)\mathbf{E}(u,0)du+$$

$$+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t)\mathbf{E}'(u,0)du+$$

$$+i\sigma \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t)\mathbf{K}(u,0)\mathbf{E}(u,0)du. \tag{3.10}$$

Taking into account Eqs. (3.9), (3.10), the integral matrix equation (3.8) may be to reduce

$$\mathbf{E}(x,t) + i\sigma \int\limits_{0}^{t} dv \int\limits_{-\infty}^{\infty} \mathbf{K}_1(x,t;u,v)\mathbf{E}(u,v)du = \mathbf{F}(x,t), \tag{3.11}$$

where the following notations are introduced

$$\mathbf{F}(x,t) = (\mathbf{B}^{\alpha,\gamma}\mathbf{E}(x,0))(x,t) - i\sigma \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t)\mathbf{K}(u,0)\mathbf{E}(u,0)du.$$

4 Matrix Resolvent-function solution of the DO Cauchy problem

Let $A(x)$ be the matrix with entries $a_{ij}(x)$. By notation $|A(x)|_*$ one denotes a scalar function taking the maximum absolute value of entries $a_{ij}(x)$ of the matrix $A(x)$ for each x; i.e., $|A(x)|_* = \max\limits_{i,j} |a_{ij}(x)|$.

Let $f(x,t)$ and $f_t(x,t)$ be the continuous, bounded functions.

From the Eqs. (2.11) – (2.14) one obtains following estimates for the matrix kernel function $\mathbf{K}_1$

$$|\mathbf{K}_1(x,t;\xi,\eta)|_* \leq C|x-\xi|^{-\theta}(t-\eta)^{\beta-1}, \tag{4.1}$$

$$|\mathbf{K}_1(x,t;\xi,\eta)|_* \leq C(t-\eta)^{\beta-1}\exp\left[-\frac{\sigma_0|x-\xi|^{\varepsilon}}{(t-\eta)^{\alpha\varepsilon}}\right], \tag{4.2}$$

where $\beta = \alpha\theta$, $\theta \in (0,1)$, $\sigma_0 < (1-\alpha)\alpha^{\alpha\varepsilon}$, $\varepsilon = \frac{1}{1-\alpha}$.

Further, let us find out the corresponding estimates for iterative kernels functions, namely:

$$\mathbf{K}_n(x,t;\xi,\eta) = \int\limits_{\eta}^{t} dv \int\limits_{-\infty}^{\infty} \mathbf{K}_{n-1}(x,t;u,v)\mathbf{K}_1(u,v;\xi,\eta)du.$$

From estimates (4.1) and (4.2) it follows

$$|\mathbf{K}_2(x,t;\xi,\eta)|_* \leq$$

$$\leq \int\limits_{\eta}^{t} dv \left(\int\limits_{x-\delta_1}^{x+\delta_2} + \int\limits_{-\infty}^{x-\delta_1} + \int\limits_{x+\delta_2}^{\infty}\right) |\mathbf{K}_1(x,t;u,v)\mathbf{K}_1(u,v;\xi,\eta)|_* du \leq$$

$$\leq CC_{2,\beta,\theta}(t-\eta)^{2\beta-1}\left(|x-\xi|^{-\theta} + \exp\left[-\frac{\sigma_0|x-\xi|^{\varepsilon}}{(t-\eta)^{\alpha\varepsilon}}\right]\right), \tag{4.3}$$

where

$$C_{2,\beta,\theta} = \frac{\Gamma^2(\beta)C_{1-\theta}}{\Gamma(2\beta)}.$$

Accordingly, from (4.3), keeping in mind $0 < e^{-s} < 1$, for $s > 0$, one obtains

$$|\mathbf{K}_2(x,t;\xi,\eta)|_* \leq CC_{2,\beta,\theta}|x-\xi|^{-\theta}(t-\eta)^{2\beta-1}, \tag{4.4}$$

$$|\mathbf{K}_2(x,t;\xi,\eta)|_* \leq CC_{2,\beta,\theta}(t-\eta)^{2\beta-1}\exp\left[-\frac{\sigma_0|x-\xi|^{\varepsilon}}{(t-\eta)^{\alpha\varepsilon}}\right]. \tag{4.5}$$

Under the above, the following estimates take place

$$|\mathbf{K}_n(x,t;\xi,\eta)|_* \leq CC_{n,\beta,\theta}\Gamma(n\beta)|x-\xi|^{-\theta}(t-\eta)^{n\beta-1}, \tag{4.6}$$

$$|\mathbf{K}_n(x,t;\xi,\eta)|_* \leq CC_{n,\beta,\theta}(t-\eta)^{n\beta-1}\exp\left[-\frac{\sigma_0|x-\xi|^{\varepsilon}}{(t-\eta)^{\alpha\varepsilon}}\right]. \tag{4.7}$$

where

$$C_{n,\beta,\theta} = \frac{\Gamma^n(\beta)C_{1-\theta}^{n-1}}{\Gamma(n\beta)}.$$

The matrix Resolvent-function solution of the Eq. (3.11) takes the form

$$\mathbf{R}(x,t;\xi,\eta) = \sum_{n=1}^{\infty}\mathbf{K}_n(x,t;\xi,\eta). \tag{4.8}$$

From estimates (4.1)–(4.7), it follows the Liouville–Neumann-type series (4.8) converges and can be estimated as a value of

$$|\mathbf{R}(x,t;\xi,\eta)|_* \leq C\Gamma(\beta)(t-\eta)^{\beta-1}E_\beta\left[C_{1-\theta}\Gamma(\beta)(t-\eta)^{\beta};\beta\right]\times$$

$$\times\left(|x-\xi|^{-\theta}+\exp\left[-\frac{\sigma_0|x-\xi|^{\varepsilon}}{(t-\eta)^{\alpha\varepsilon}}\right]\right).$$

On the other hand, since the Mittag-Leffler-type function $E_\beta(s;\beta)$ is bounded on the interval $[0,T]$, the estimates of (4.1)–(4.2) are proved and valid also for the matrix Resolvent-function solution as a whole.

So, one can state the unique solution of the matrix integral Eq. (3.11) takes the form

$$\mathbf{E}(x,t) = \mathbf{F}(x,t) + \int_0^t\int_{-\infty}^{+\infty}\mathbf{R}(x,t;\xi,\eta)\mathbf{F}(\xi,\eta)d\xi d\eta.$$

5 The DO Cauchy problem. The FOP $\alpha = 1$

In the case when FOP $\alpha = 1$, the basic system (3.1) reduces to

$$\begin{pmatrix} \frac{\partial}{\partial t}-\frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t}+\frac{\partial}{\partial x}\end{pmatrix}\begin{pmatrix} E_0(x,t) \\ E_h(x,t)\end{pmatrix} =$$

$$= i\begin{pmatrix} \gamma & \sigma\exp[if(x,t)] \\ \sigma\exp[-if(x,t)] & \gamma\end{pmatrix}\begin{pmatrix} E_0(x,t) \\ E_h(x,t)\end{pmatrix}.$$

As was shown earlier [26], one can find out that for $\alpha \to 1$ integral equation (3.8) can be written down into the form

$$\mathbf{E}(x,t) = (\mathbf{A}^{1,\gamma}\mathbf{E}(x,t))(x,t) + (\mathbf{B}^{1,\gamma}\mathbf{E}(x,0))(x,t),$$

where

$$(\mathbf{A}^{1,\gamma}\mathbf{E}(x,t))(x,t) = i\sigma \int\limits_0^t dv \int\limits_{x-t+v}^{x+t-v} G_{1,\gamma}(x-u,t-v)\times$$

$$\times \begin{pmatrix} 0 & O_+\, e^{if(u,v)} \\ O_-\, e^{-if(u,v)} & 0 \end{pmatrix} \mathbf{E}(u,v)du,$$

$$(\mathbf{B}^{1,\gamma}\mathbf{E}(x,0))(x,t) = \int\limits_0^t dv \int\limits_{x-t+v}^{x+t-v} G_{1,\gamma}(x-u,t-v)\times$$

$$\times \begin{pmatrix} O_+ - i\gamma & i\sigma e^{if(u,0)} \\ i\sigma e^{-if(u,0)} & O_- - i\gamma \end{pmatrix} \delta(v)\mathbf{E}(u,0)du,$$

and respectively,

$$G_{1,\gamma}(x,t) = \frac{1}{2}e^{i\gamma t} J_0\left(\sigma\sqrt{t^2-x^2}\right)\Theta[t-|x|].$$

6 The DO Cauchy problem. The crystal-lattice displacement field function $f(\mathbf{R}) = ax + b$

Here we will build up a solution of the basic fractional DO Cauchy problem when the crystal-lattice displacement field function $f(\mathbf{R}) = ax + b$.

After trivial exponentional substitutions for the wave amplitudes $E_0(x,t)$, $E_h(x,t)$, the system (1.1) can written down as (for simplicity, further, the same notations for the wave amplitudes $E_0(x,t)$, $E_h(x,t)$, are to be saved)

$$\begin{aligned} \left(\partial^\alpha_{0t} - \frac{\partial}{\partial x}\right) E_0(x,t) &= i\gamma E_0(x,t) + i\sigma e^{iax} E_h(x,t), \\ \left(\partial^\alpha_{0t} + \frac{\partial}{\partial x}\right) E_h(x,t) &= i\sigma e^{-iax} E_0(x,t) + i\gamma E_h(x,t). \end{aligned} \tag{6.1}$$

Substituting the functions $E_0(x,t)$, $E_h(x,t)$ as

$$E_0(x,t) = \exp\left(i\frac{ax}{2}\right)\mathcal{E}_0(x,t), \quad E_h(x,t) = \exp\left(-i\frac{ax}{2}\right)\mathcal{E}_h(x,t),$$

one obtains

$$\begin{aligned}
&\left(\partial_{0t}^{\alpha} - \frac{\partial}{\partial x}\right)\mathcal{E}_0(x,t) = i\left(\gamma + \frac{a}{2}\right)\mathcal{E}_0(x,t) + i\sigma\mathcal{E}_h(x,t),\\
&\left(\partial_{0t}^{\alpha} + \frac{\partial}{\partial x}\right)\mathcal{E}_h(x,t) = i\sigma\mathcal{E}_0(x,t) + i\left(\gamma + \frac{a}{2}\right)\mathcal{E}_h(x,t),
\end{aligned} \tag{6.2}$$

and the initial condition

$$\left(\begin{array}{c} \mathcal{E}_0(x,0) \\ \mathcal{E}_h(x,0) \end{array}\right) = \left(\begin{array}{c} \psi_0(x) \\ \psi_h(x) \end{array}\right) = \left(\begin{array}{c} e^{-iax/2}\varphi_0(x) \\ e^{iax/2}\varphi_h(x) \end{array}\right), \quad -\infty < x < \infty. \tag{6.3}$$

Then, let us introduce the notations

$$\mathbf{\Gamma}(x,t) = \frac{1}{2}\int\limits_{|x|}^{\infty} g(t,\tau)\mathbf{Q}(x,\tau)d\tau + \mathbf{\Gamma}_0(x,t), \tag{6.4}$$

$$\mathbf{Q}(x,\tau) = \left[\begin{array}{cc} -\sigma\frac{\tau - x}{\sqrt{\tau^2 - x^2}}J_1(\sigma\sqrt{\tau^2 - x^2}) & i\sigma J_0(\sigma\sqrt{\tau^2 - x^2}) \\ i\sigma J_0(\sigma\sqrt{\tau^2 - x^2}) & -\sigma\frac{\tau + x}{\sqrt{\tau^2 - x^2}}J_1(\sigma\sqrt{\tau^2 - x^2}) \end{array}\right], \tag{6.5}$$

$$\mathbf{\Gamma}_0(x,t) = g(t,|x|)\left[\begin{array}{cc} \Theta(-x) & 0 \\ 0 & \Theta(x) \end{array}\right], \tag{6.6}$$

$$g(t,\tau) = \frac{e^{i(\gamma + a/2)\tau}}{t}\phi\left(-\alpha, 0; -\tau t^{-\alpha}\right),$$

$\Theta(x)$ is the Heaviside function, $J_m(z)$ is the m-order Bessel function of the argument z.

According to [25], the basic matrix solution of the DO Cauchy problem (6.2) – (6.3) has the form

$$\mathcal{E}(x,t) = \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1}\Gamma(x-\xi,t)\psi(\xi)d\xi, \tag{6.7}$$

in the class of function

$$\mathcal{E}(x,t) \in C(\overline{\Omega}), \quad \partial_{0t}^{\alpha}\mathcal{E}(x,t), \frac{\partial}{\partial x}\mathcal{E}(x,t) \in C(\Omega),$$

where $\psi(x) = [\psi_0(x), \psi_h(x)]^{tr} \in C(-\infty,\infty)$ is the function, which satisfies the Hölder condition, and the following relation as $|x| \to \infty$

$$\psi(x) = O(\exp(\rho|x|^{\varepsilon})) \quad \varepsilon = \frac{1}{1-\alpha}, \quad \rho < (1-\alpha)(\alpha T^{-1})^{\frac{\alpha}{1-\alpha}}.$$

Underline the solution to the Cauchy problem (6.2) – (6.3) is unique in the class of functions, which satisfy the condition

$$\mathcal{E}(x,t) = O(\exp(k|x|^{\varepsilon})), \quad \text{as} \quad |x| \to \infty,$$

for some $k > 0$.

From Eqs. (6.3) – (6.7) it follows the solution of the Cauchy problem (6.1), (1.2) can be cast into the form

$$\mathbf{E}(x,t) = \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1}\mathbf{G}(x,\xi,t)\varphi(\xi)d\xi, \tag{6.8}$$

where the following notations are introduced

$$\mathbf{G}(x,\xi,t) = \frac{1}{2}\int\limits_{|x-\xi|}^{\infty} g(t,\tau)\mathbf{S}(x,\xi,\tau)d\tau + \mathbf{G}_0(x,\xi,t), \tag{6.9}$$

$$\mathbf{S}(x,\xi,\tau) = \begin{bmatrix} -\sigma e^{iaX_1/2}\frac{\tau-X_1}{\sqrt{\tau^2-X_1^2}}h_1(\tau,X_1) & i\sigma e^{iaX_2/2}h_0(\tau,X_1) \\ i\sigma e^{-iaX_2/2}h_0(\tau,X_1) & -\sigma e^{-iaX_1/2}\frac{\tau+X_1}{\sqrt{\tau^2-X_1^2}}h_1(\tau,X_1) \end{bmatrix},$$

$$h_0(\tau,X_1) = J_0(\sigma\sqrt{\tau^2-X_1^2}), \quad h_1(\tau,X_1) = J_1(\sigma\sqrt{\tau^2-X_1^2})$$

$$\mathbf{G}_0(x,\xi,t) = \begin{bmatrix} e^{-i\gamma X_1}\Theta(-X_1) & 0 \\ 0 & e^{i\gamma X_1}\Theta(X_1) \end{bmatrix} \frac{1}{t}\phi\left(-\alpha,0;-|X_1|t^{-\alpha}\right), \tag{6.10}$$

$$X_1 = x-\xi, \quad X_2 = x+\xi.$$

7 Case of the function $f(\mathbf{R}) = ax+b$ and $E_0(x,0) = 1,\ E_h(x,0) = 0$

Let us consider the case when the initial conditions for the wavefield amplitudes $E_0(x,0)$ and $E_h(x,0)$ are constant and equal to

$$E_0(x,0) = \varphi_0(x) \equiv 1, \quad E_h(x,0) = \varphi_h(x) \equiv 0. \tag{7.1}$$

Then, the formulae (6.8)–(6.10) can be simplified and expressed in terms of the Mittag–Leffler-type functions.

Keeping in mind Eq. (7.1), Eq. (6.8) for $\mathbf{E}(x,t)$ can be written down as

$$\mathbf{E}(x,t) = \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1}\mathbf{G}(x,\xi,t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} d\xi =$$

$$
= \frac{1}{2} \int\limits_{-\infty}^{\infty} d\xi \int\limits_{|x-\xi|}^{\infty} D_{0t}^{\alpha-1} g(t,\tau) \mathbf{S}(x,\xi,\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\tau +
$$

$$
+ \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1} \mathbf{G}_0(x,\xi,t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\xi \equiv \mathbf{I}_1(x,t) + \mathbf{I}_2(x,t). \tag{7.2}
$$

By changing the integration order, let us evaluate the integral $\mathbf{I}_1(x,t)$

$$
\mathbf{I}_1(x,t) = \frac{1}{2} \int\limits_{0}^{\infty} D_{0t}^{\alpha-1} g(t,\tau) d\tau \int\limits_{x-\tau}^{x+\tau} \mathbf{S}(x,\xi,\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\xi =
$$

$$
= \frac{1}{2} \int\limits_{0}^{\infty} D_{0t}^{\alpha-1} g(t,\tau) d\tau \int\limits_{-\tau}^{\tau} \mathbf{S}(x,x-\eta,\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\eta. \tag{7.3}
$$

From the Eqs. (2.15)–(2.17), it directly follows up

$$
\int\limits_{-\tau}^{\tau} e^{\pm ia\eta/2} J_0(\sigma\sqrt{\tau^2-\eta^2}) d\eta = \frac{2}{k} \sin(k\tau), \tag{7.4}
$$

$$
\int\limits_{-\tau}^{\tau} e^{\pm ia\eta/2} \frac{\tau \mp \eta}{\sqrt{\tau^2-\eta^2}} J_1(\sigma\sqrt{\tau^2-\eta^2}) d\eta =
$$

$$
= -\frac{1}{\sigma} \left[2\cos(k\tau) - i\frac{a}{k} \sin(k\tau) - 2e^{-ia\tau/2} \right], \tag{7.5}
$$

where $k = \sqrt{a^2/4 + \sigma^2}$.

Accordingly, from Eqs. (7.4), (7.5) one obtains

$$
\int\limits_{-\tau}^{\tau} S_{11,22}(x,x-\eta,\tau) d\eta = -\sigma \int\limits_{-\tau}^{\tau} e^{\pm ia\eta/2} \frac{\tau \mp \eta}{\sqrt{\tau^2-\eta^2}} J_1(\sigma\sqrt{\tau^2-\eta^2}) d\eta =
$$

$$
= 2\cos(k\tau) - i\frac{a}{k} \sin(k\tau) - 2e^{-ia\tau/2}, \tag{7.6}
$$

$$
\int\limits_{-\tau}^{\tau} S_{12,21}(x,x-\eta,\tau) d\eta = i\sigma e^{\pm iax} \int\limits_{-\tau}^{\tau} e^{\mp ia\eta/2} J_0(\sigma\sqrt{\tau^2-\eta^2}) d\eta =
$$

$$
= i\sigma \frac{2}{k} e^{\pm iax} \sin(k\tau), \tag{7.7}
$$

where $S_{ij}(x, x-\eta, \tau)$ $(i,j=1,2)$ are the elements of matrix $\mathbf{S}(x, x-\eta, \tau)$.

Following equalities

$$e^{ia_1\tau}\left[2\cos(k\tau) - i\frac{a}{k}\sin(k\tau)\right] =$$

$$= \left(1-\frac{a}{2k}\right)e^{i(a_1+k)\tau} + \left(1+\frac{a}{2k}\right)e^{i(a_1-k)\tau}, \tag{7.8}$$

$$-2e^{ia_1\tau}e^{ia\tau/2} = -2e^{i\gamma\tau}, \tag{7.9}$$

$$e^{ia_1\tau}\sin(k\tau) = \frac{i}{2}e^{i(a_1-k)\tau} - \frac{i}{2}e^{i(a_1+k)\tau}, \tag{7.10}$$

hold, where

$$a_1 = \gamma + \frac{a}{2}, \quad k = \sqrt{a^2/4+\sigma^2}.$$

Next, exploiting calculations (7.6)–(7.10), from (7.3) one obtains

$$\mathbf{I}_1(x,t) = \int\limits_0^\infty t^{-\alpha}\phi\left(-\alpha, 1-\alpha; -\tau t^{-\alpha}\right)\mathbf{N}(x,\tau)\begin{pmatrix}1\\0\end{pmatrix}d\tau, \tag{7.11}$$

where

$$\mathbf{N}(x,\tau) = \mathbf{N}_1(x)e^{i(a_1+k)\tau} + \mathbf{N}_2(x)e^{i(a_1-k)\tau} - \begin{pmatrix}1&0\\0&1\end{pmatrix}e^{i\gamma\tau},$$

$$\mathbf{N}_{1,2}(x) = \frac{1}{2}\begin{pmatrix}1\mp\frac{a}{2k} & \pm\frac{\sigma}{k}e^{iax}\\ \pm\frac{\sigma}{k}e^{-iax} & 1\mp\frac{a}{2k}\end{pmatrix}.$$

From (6.10) one finds out

$$\mathbf{I}_2(x,t) = \int\limits_0^\infty t^{-\alpha}\phi\left(-\alpha, 1-\alpha; -\eta t^{-\alpha}\right)e^{i\gamma\eta}\begin{pmatrix}1\\0\end{pmatrix}d\eta. \tag{7.12}$$

Applying formula (2.18) to equalities (7.11) and (7.12), the total solution (7.2) can be cast into the form

$$\mathbf{E}(x,t) = \frac{1}{2}\begin{pmatrix}1-\frac{a}{2k} & 1+\frac{a}{2k}\\ \frac{\sigma}{k}e^{-iax} & -\frac{\sigma}{k}e^{-iax}\end{pmatrix}\begin{pmatrix}E_{\alpha,1}\left(i(a_1+k)t^\alpha\right)\\ E_{\alpha,1}\left(i(a_1-k)t^\alpha\right)\end{pmatrix}. \tag{7.13}$$

In the case when the FOP $\alpha \to 1$, the total solution (7.13) is reduced to

$$\mathbf{E}(x,t) = \frac{1}{2}\begin{pmatrix}1-\frac{a}{2k} & 1+\frac{a}{2k}\\ \frac{\sigma}{k}e^{-iax} & -\frac{\sigma}{k}e^{-iax}\end{pmatrix}\begin{pmatrix}e^{i(a_1+k)t}\\ e^{i(a_1-k)t}\end{pmatrix}. \tag{7.14}$$

Finally, putting on the parameters $\gamma = 0$ and $a = 0$, the solution (7.14) goes into the pendulum solutions of the canonic TT equations regarding the Bloch wavefield amplitudes in the perfect crystal (cf. [26])

$$E_0 = \cos\sigma t, \quad E_h = i\sin\sigma t.$$

8 Conclusion

In this paper, a goal of our study is to elaborate the novel mathematics framework for processing the DO data based on a concept of the matrix integral equation to solve the inverse Radon problem that arose in the computer microtomography (see, e.g., [5], [8]).

In the case when FOP $\alpha = 1$, the above results obtained can be applied to the standard X-ray DO using the computer diffraction microtjmography technique (cf. [26]).

In contrast to [7], [8], some advantage of our study is to open a window to develop the theory models of the X-ray and electron diffraction scattering by imperfect crystals. The fractional DO theory operates based on the matrix Fredholm–Volterra integral equation of the second kind. The matrix Resolvent-function solution of the Fredholm–Volterra integral equation of the second kind allows us to model the X-ray diffraction scattering by imperfect crystals. Besides, it can modify the optimizing procedure of the χ^2-target function in trial to solve the inverse Radon problem in the DO microtomography investigating the crystal-lattice defects structure.

The common approach of the DO imaging, the FOP $\alpha = 1$, has to be rethinked using the fractional derivatives formalism $\partial_{0t}^{\alpha} E$ in the sense of the Gerasimov–Caputo approach (2.1). Accordingly, in the fractional DO, it is also worth noting alternative interest due to the possibility of using the fractional Riemann–Liouville derivative $D_{0t}^{\alpha} E$ (2.2). The latter is a good topic for future work in the fractional DO.

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