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# FROM TWO SIMPLE PROBLEMS TO THE CONNECTION OF SPECIAL POINTS

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**Abstract.** Both the USA TST 2008 [1] and the ELMO Shortlist 2013 [2] suggested two issues that are connected to fixed points. These problems provide a strong linkage between the various attributes of specific points in a triangle. In this article, we will first investigate various theorems concerning the fixed points that have been presented, and then we will demonstrate how those points are connected to a few triangle centers.

## 1. Introduction

The fixed-point problems have been seen in a great number of international mathematical olympiad competitions [4]. The generalization of these problems in this area, as well as the features of these problems, leave behind a plethora of notable results [10]. In this work, we talk about some of those problems and show the relationship to a few of different kinds of unique triangle centers.

In USA Team Selection Test 2008 [1], there is a fixed-point problem as following:

**Problem 1.** *Given triangle $ABC$, centroid $G$. Let $P$ be a variable point on segment $BC$. Let $Q, R$ be points on $AC, AB$ such that $PQ \parallel AB, PR \parallel AC$. Prove that $(AQR)$ passes through a fixed point $X$ such that $(AB, AG) = (AX, AC)$.*

**Proof.** Let $\omega_1$ be the circle which passes $A, B$ and tangents to $AC$. Let $\omega_2$ be the circle which passes $A, C$ and tangents to $AB$. Let $X$ be intersection of $\omega_1$ and $\omega_2$, so $X$ is fixed. Since two triangle $ABX$ and $AXC$ are homothetic, hence $\frac{AX}{BX} = \frac{AC}{AB}$. On the other hand, $\frac{AC}{AB} = \frac{PR}{BR} = \frac{AQ}{BR}$ so $\frac{AX}{BX} = \frac{AQ}{BR}$, also $(BR, BX) = (AX, AX)$ hence two triangle $BRX$ and $AQX$ are homothetic, therefore $(RB, RX) = (QA, QX)$, following that $A, Q, X, R$ are cyclic. Moreover, we have

$$\frac{\sin(AB, AX)}{\sin(CA, CX)} = \frac{\sin(AB, AX)}{\sin(BA, BX)} = \frac{BX}{AX} = \frac{AB}{AC}$$

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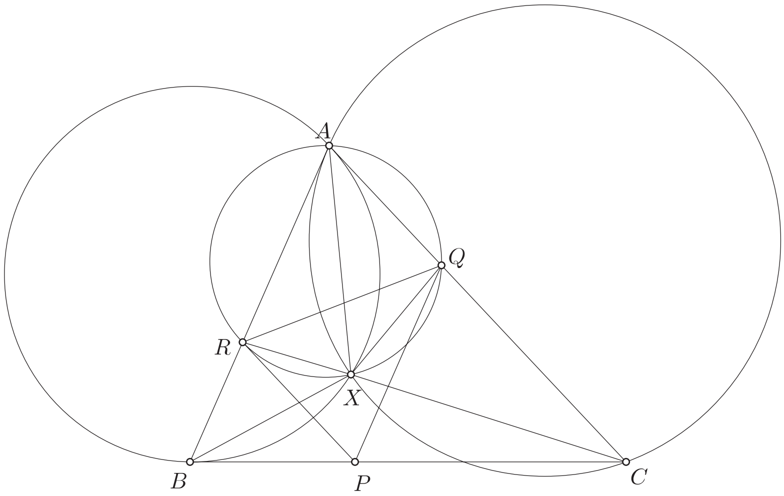


FIGURE 1. Proof of Problem 1

so $AX$ is the symmedian line of triangle $ABC$, which means $(AB, AG) = (AX, AC)$. The problem has been solved.

With this configuration, ELMO Shortlist 2013 [2] provide the following problem using the antiparallel lines instead of parallel ones:

**Problem 2.** *Given triangle $ABC$ and a centroid $G$. Let $P$ be a variable point on segment $BC$. Let $Q, R$ respectively be points on $AC, AB$ such that $PQ, PR$ are antiparallel lines of $AB, AC$. Prove that $(AQR)$ passes through a fixed point.*

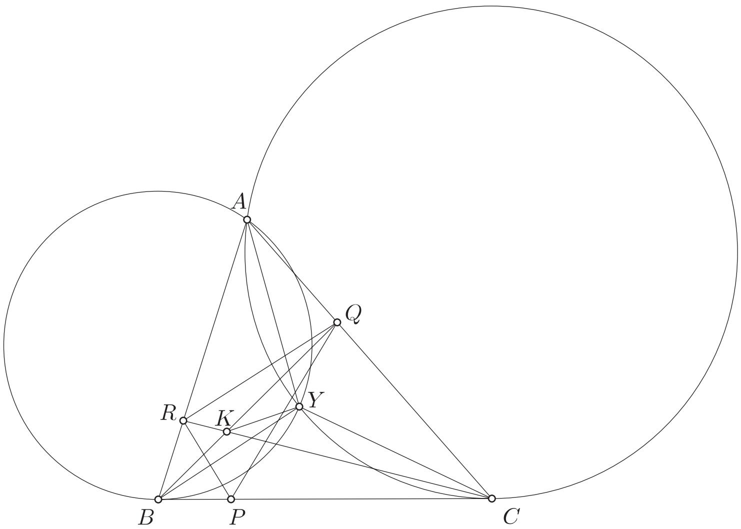


FIGURE 2. Proof of Problem 2

**Proof.** Let $\gamma_1$ be the circles passes $A, B$ and tangents to $BC$. Let $\gamma_2$ be the circle passes $A, C$ and tangents to $BC$. Let $Y$ be intersection of $\gamma_1$ and $\gamma_2$, so $Y$ is fixed. Let $K$ be intersection of $BQ$ and $CR$. We hav

$$(RK, RA) = (PC, PA) = (PB, PA) = (QK, QA) \pmod{\pi}.$$

Hence four points $A, R, K, Q$ are cyclic. Therefore:

$$\begin{aligned}(KB, KC) = (KR, KQ) = (AR, AQ) &= (AR, AY) + (AY, AQ)\\ &= (BY, BC) + (CB, CY)\\ &= (YB, YC) \pmod{\pi}.\end{aligned}$$

So four points $B, K, Y, C$ are cyclic. We have

$$(KR, KY) = (KY, KC) = (BY, BC) = (AR, AY) \pmod{\pi}.$$

Hence $Y$ is on $(ARK) \equiv (AQR)$. Moreover, by using the power of point's properties, we have the radical axis of $\gamma_1$ and $\gamma_2$ passes their common tangents, following that $AY$ passes $BC$ at its midpoint or $AY$ is the median line of triangle $ABC$.

The subsequent sections will go into the various qualities associated with the aforementioned pair of fixed points, which are closely linked to several notable triangle centers.

## 2. Properties related to the first problem

Two above problems appeared in many contests [1, 2] and journals [4] in the world, but it seems like there are just few properties about. This article will put forward some results about two fixed points mentioned above. We denotes $A_X, A_Y$ be that two points in the previous problems.

**Theorem 2.1.** *$A_X$ is projection point of the circumcenter of triangle $ABC$ on the $A-$symmedian line of that triangle.*

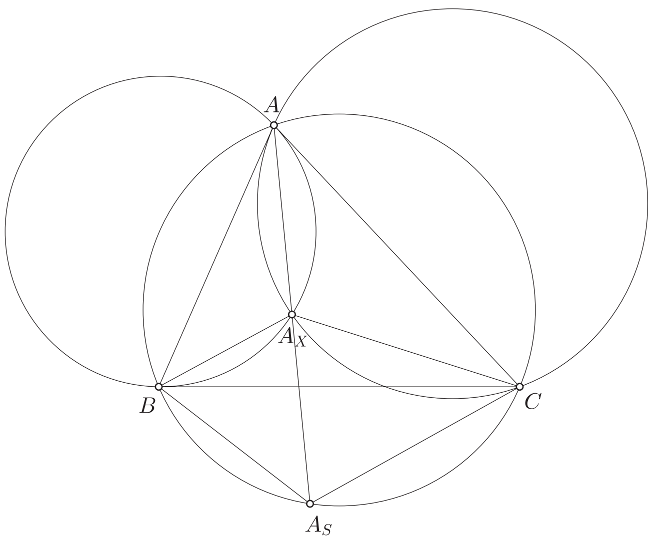


FIGURE 3. Proof of Theorem 2.1

**Proof.** Let $A_S$ be the second intersection of $AA_X$ and $(ABC)$. From Problem 1, we have two triangle $ABA_X$ and $AA_XC$ are homothetic hence $A_XA^2 = A_XB \cdot A_XC$. Moreover, we have

$$(A_XB, A_XA_S) = (A_XB, AA_X) = (AA_X, A_XC) = (A_XA_S, A_XC),$$

$$\begin{aligned}(A_SB, A_SA_X) = (CB, CA) &= (CB, CA_X) + (CA_X, CA)\\ &= (CB, CA_X) + (AA_S, AB)\\ &= (CB, CA_X) + (CA_S, CB)\\ &= (CA_S, CA_X).\end{aligned}$$

So two triangle $A_XBA_S$ and $A_XA_SC$ are homothetic. Therefore $A_XA_S^2 = A_XB \cdot A_XC$. Hence $A_XA = A_XA_S$ or $A_X$ is midpoint of $AA_S$, following that $A_X$ is projection point of the circumcenter of triangle $ABC$ on the $A$-symmedian line of that triangle.

**Theorem 2.2.** *Brocard circle [14] is the circle which passes two Brocard points $Z_1, Z_2$ [12, 13], Lemoine point $L$ [16] and circumcenter $O$ of triangle $ABC$. Let $A_X$ be the fixed point mentioned in Problem 1. The construction of $B_X$ and $C_X$ is performed in a similar manner, but with the utilization of vertices $B$ and $C$. Then $A_X$, $B_X$ and $C_X$ lie on Brocard circle [14] of triangle $ABC$.*

**Remark 2.1.** *$A_XB_XC_X$ is the second Brocard triangle of triangle $ABC$.*

**Lemma 2.1. (Mannheim's theorem)** *Given triangle $ABC$ and three points $L$, $M$, $N$ respectively be on $BC$, $CA$, $AB$. Let $A'$, $B'$, $C'$ respectively be three points that lie on $(AMN)$, $(BNL)$, $(CLM)$ such that $AA'$, $BB'$, $CC'$ are concurrent at $K$. Prove that four points $A'$, $B'$, $C'$, $K$ are cyclic.*

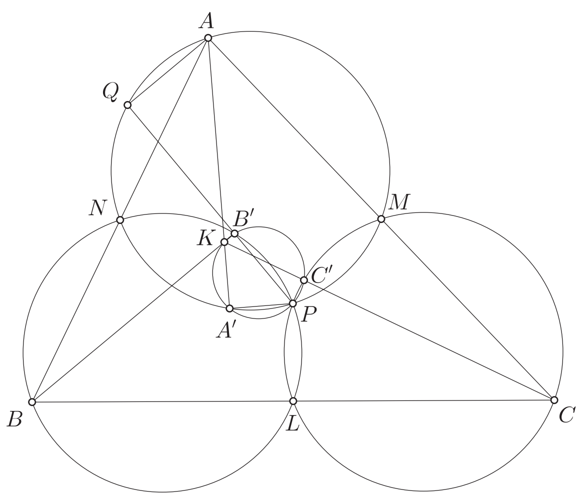


FIGURE 4. Proof of Lemma 2.1

**Proof.** Let $P$ be Miquel point of triangle $LMN$ with respect to triangle $ABC$. Let $Q$ be intersection of $PB'$ and $(AMN)$. Applying Reim's theorem to $(AMN)$ and $(BNL)$ we have $AQ \parallel BB'$. Through angle chasing:

$$(KA, KB') = (AA', AQ) = (PA', PB') \pmod{\pi}$$

Therefore $A', P, B', K$ are cyclic. Similarly, $B', P, C', K$ are cyclic. Hence four points $A'$, $B'$, $C'$, $K$ are cyclic.

**Lemma 2.2.** *Given triangle $ABC$ inscribed in $(O)$ and a point $P$ is not this triangle's vertex. Lines $AP$, $BP$, $CP$ respectively cut $(ABC)$ at $S$, $T$, $U$. Let $X$, $Y$, $Z$ be midpoints of segments $AS$, $BT$, $CU$, respectively. Prove that $X$, $Y$, $Z$ lie on $(OP)$ (a circle with center $O$ and radius $OP$).*

**Proof.** We have $X$, $Y$,$Z$ respectively be projection of $O$ on $AS$, $BT$, $CU$. Through angle chasing:

$$(XP, XO) = (YP, YO) = (ZP, ZO) = \frac{\pi}{2}.$$

Therefore $X, Y, Z$ lie on $(OP)$. The lemma has been proved.

Back to Theorem 2.2,

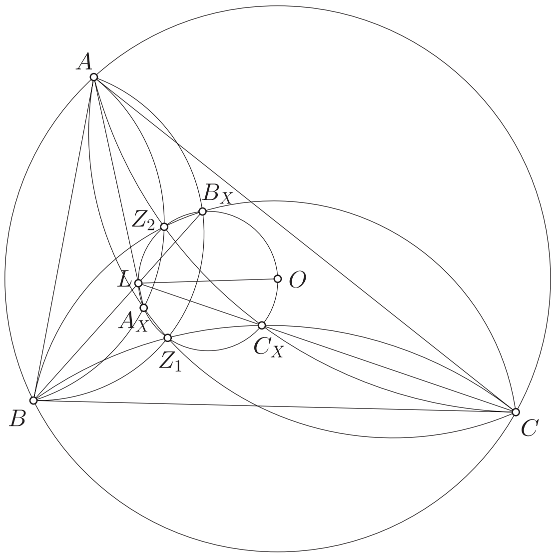


FIGURE 5. Proof of Theorem 2.2

**Proof.** Applying Lemma 2.1, to triangle $ABC$ with $AA_X, BB_X, CC_X$ are concurrent at $L$ and $Z_1$ is intersection of $(ABB_X), (BCC_X), (CAA_X)$ we have $L, Z_1$ lie on $(A_X B_X C_X)$. Similarly, $Z_2$ lies on $(A_X B_X C_X)$. From Lemma 2.2, we have $A_X, B_X, C_X$ lie on $OL$. The theorem has been proved.

**Theorem 2.3.** *Ray $AA_Y$ cuts $(ABC)$ at $A_M$. Prove that $BA_Y CA_M$ is parallelogram.*

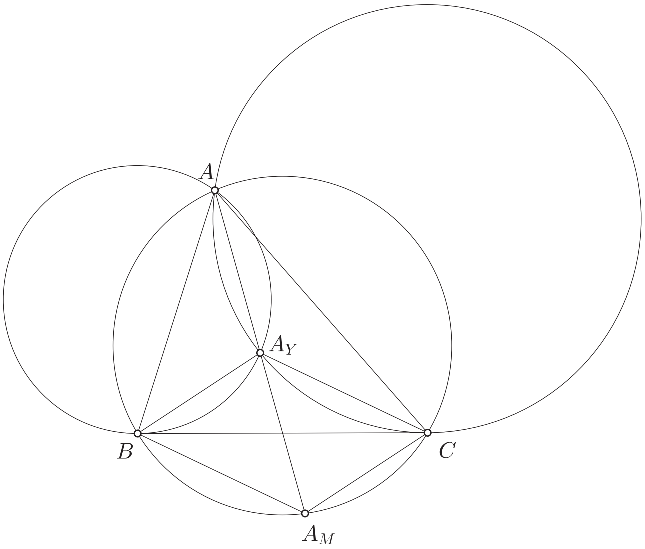


FIGURE 6. Proof of Theorem 2.3

**Proof.** Using the properties of $A_Y$ at Problem 2, we have

$$(BA_Y, BC) = (AB, AA_M) = (CB, CA_M).$$

Therefore $A_YB \parallel A_MC$. Similarly, $A_YC \parallel A_MC$ so $BA_YCA_M$ is a parallelogram. The problem has been solved.

**Corollary 2.1.** *$A_Y$ is symmetric with $A_S$ through $BC$.*

**Theorem 2.4.** *$A_X$ is the isogonal conjugate of $A_Y$ with respect to triangle $ABC$.*

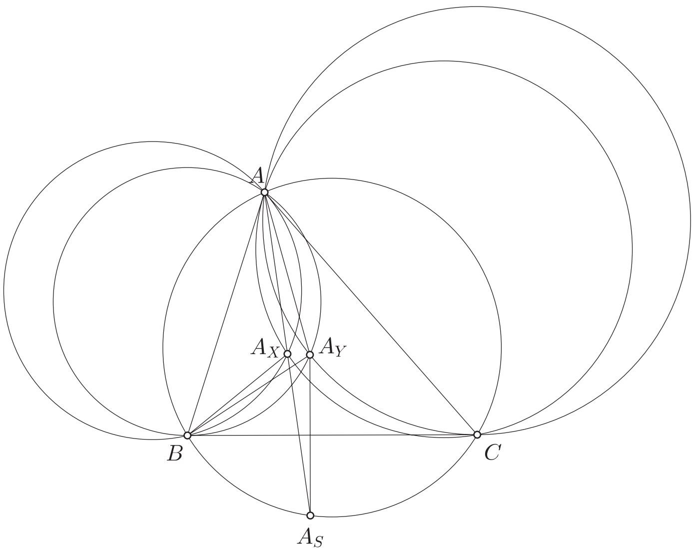


FIGURE 7. Proof of Theorem 2.4

**Proof.** Because $AA_X$ and $AA_Y$ respectively be symmedian line and median line at vertex $A$ are two isogonal conjugate lines so we have to prove that $BA_X$ and $BA_Y$ are two isogonal conjugate ones. Through angle chasing,

$$(BA_X, BA) = (AA_X, AC) = (AB, AA_Y) = (BC, BA_Y),$$

we have $BA_X$ and $BA_Y$ are isogonal conjugate, therefore $A_X$ is the isogonal conjugate of $A_Y$ with respect to triangle $ABC$.

**Theorem 2.5.** *Given triangle $ABC$, centroid $G$. Let $M_{AB}$, $M_{BC}$, $M_{CA}$ respectively be midpoints of $BC$, $CA$, $AB$. Let $A_G$ be midpoint of $AA_Y$. Prove that $A_G$ lies on $(M_{AB}M_{BC}M_{CA})$.*

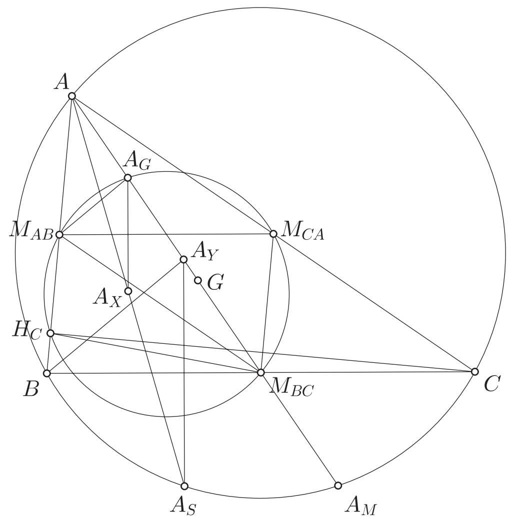


FIGURE 8. Proof of Theorem 2.5

**Proof.** Because $M_{AB}A_G$ is the midline of triangle $ABA_Y$ so we have:

$$(M_{AB}A, M_{AB}A_G) = (BA, BA_Y)$$

Draw altitude $CH_C$ of triangle $ABC$. Through angle chasing,

$$\begin{aligned}(M_{BC}H_C, M_{BC}A) &= (M_{BC}H_C, BA) + (BA, CB)\\ &\quad + (CB, CA) + (CA, M_{BC}A)\\ &= 2(BA, CB) + (CB, CA) + (CA, M_{BC}A)\\ &= (BA, CB) + (AB, AM_{BC})\\ &= (BA, BA_Y) + (BA_Y, CB) + (CB, CA_M)\\ &= (BA, BA_Y)\\ &= (M_{AB}A, M_{AB}A_G).\end{aligned}$$

Hence $A_G$ lies on $(M_{AB}H_CM_{BC}) \equiv (M_{AB}M_{BC}M_{CA})$. We have done.

**Theorem 2.6.** *Draw parallelogram $M_{AB}A_GM_{CA}A_{GY}$. Then $A_{GY}$ is midpoint of $AA_M$.*

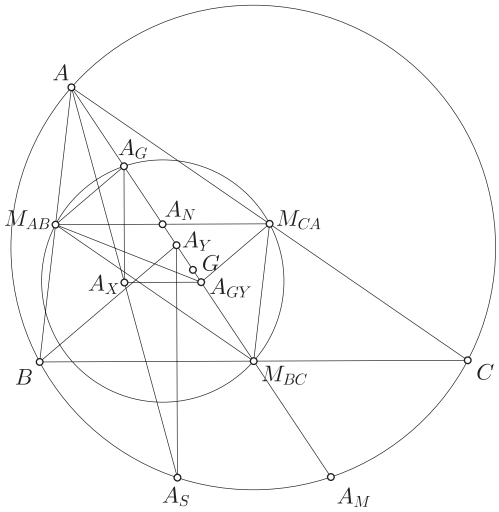


FIGURE 9. Proof of Theorem 2.6

**Proof.** Let $A_N$ be midpoint of $M_{AB}M_{CA}$. Because $M_{AB}A_GM_{CA}A_{GY}$ and $AM_{AB}M_{BC}M_{CA}$ are parallelogram, we can easily see that $A_N$ is midpoint of both $A_GA_{GY}$ and $AM_{BC}$. We have

$$\begin{aligned}\overline{AA_{GY}} &= \overline{AA_N} + \overline{A_NA_{GY}} = \overline{A_NM_{BC}} + \overline{A_GA_N} \\ &= \overline{A_GM_{BC}} = \overline{A_GA_Y} + \overline{A_YM_{BC}} \\ &= \frac{1}{2}\left(\overline{AA_Y} + \overline{A_YA_M}\right) = \frac{1}{2}\overline{AA_M}.\end{aligned}$$

Therefore $A_{GY}$ is midpoint of $AA_M$.

**Corollary 2.2.** *$A_XA_{GY} \parallel BC$.*

**Corollary 2.3.** *$A_X$ is symmetric with $A_G$ through $M_{AB}M_{CA}$.*

**Theorem 2.7.** *$A_Y$ is the Anticomplement point of $A_{GY}$ with respect to triangle $ABC$.*

**Proof.** From these above properties, we have:

$$\begin{aligned}\overline{GA_{GY}} &= \overline{GM_{BC}} - \overline{A_{GY}M_{BC}} = 2\overline{A_NG} - \overline{A_GA_Y} \\ &= \overline{2A_NG} - \overline{A_GA_N} - \overline{A_NA_Y} = \overline{A_NA_Y} + 2\overline{A_YG} - \overline{A_NA_{GY}} \\ &= \overline{A_{GY}A_Y} + 2\overline{A_YG} = \overline{A_{GY}G} + \overline{A_YG}.\end{aligned}$$

Hence $\overline{A_YG} = 2\overline{GA_{GY}}$ therefore $A_Y$ is the Anticomplement point of $A_{GY}$ with respect to triangle $ABC$.

**Theorem 2.8.** *The first Brocard triangle is a triangle which has three vertices respectively be intersection points of lines $AZ_1$, $BZ_1$, $CZ_1$, $AZ_2$, $BZ_2$, $CZ_2$. Let $Z_A, Z_B, Z_C$ be three vertices of this triangle. Prove that $Z_AZ_BZ_C$ is inscribed in $(A_XB_XC_X)$.*

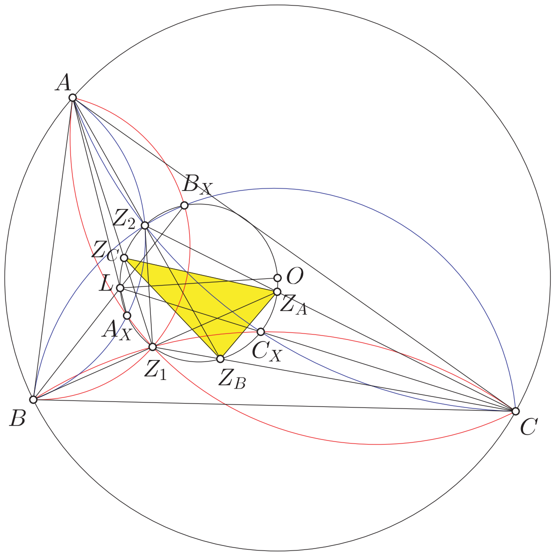


FIGURE 10. Proof of Theorem 2.8

**Proof.** Let $(AZ_1, AB) = (AC, AZ_2) = \alpha$. Through angle chasing,

$$\begin{aligned}(A_XZ_1, A_XZ_2) &= (A_XZ_1, A_XC) + (A_XC, A_XA) + (A_XA, A_XZ_2)\\ &= (AZ_1, AC) + (A_XA, A_XB) + (A_XA, Z_2A) + (Z_2A, A_XZ_2)\\ &= (AZ_1, AC) + (A_XA, A_XB) + (A_XA, Z_2A) + (BA, BA_X)\\ &= (AZ_1, AC) + (A_XA, A_XB) + (A_XA, Z_2A) + (AC, AA_X)\\ &= (AZ_1, AB) + (AB, AA_X) + (A_XA, A_XB) + (A_XA, Z_2A)\\ &= (AZ_1, AB) + (BA, A_XB) + (A_XA, Z_2A)\\ &= (AZ_1, AB) + (AC, AA_X) + (AA_X, AZ_2)\\ &= (AZ_1, AB) + (AC, AZ_2)\\ &= 2\alpha \pmod{\pi}.\end{aligned}$$

On the other hand, we have

$$(Z_AZ_1, Z_AZ_2) = (Z_AZ_1, CB) + (CB, Z_AZ_2) = \alpha + \alpha = 2\alpha.$$

Therefore $Z_A$ lies on $(Z_1A_XZ_2) \equiv (A_XB_XC_X)$. Similarly, $Z_B, Z_C$ lie on $(A_XB_XC_X)$. WE have done.

**Corollary 2.4.** *$Z_1$ is symmetric with $Z_2$ through $OL$ and $(OZ_1, OZ_2) = 2\alpha$.*

**Theorem 2.9.** *$OA_X$, $CZ_2$ cut $\omega_1$ at $O_W$, $C_W$. Then $O_WC_W \parallel OZ_A$.*

**Lemma 2.3.** *Let $BZ_1$ cut $\omega_2$ at $B_W$. Prove that $B_WC_W \parallel BC$.*

**Proof.** We have $(BB_W, BC) = (CA, CZ_1) = (B_WA, B_WZ_1)$ so $AB_W \parallel BC$. Similarly, $AC_W \parallel BC$ so $B_WC_W \parallel BC$. We have done.

Back to Theorem 2.9,

**Proof.** By Lemma 2.3 and $(BC, BZ_A) = (CZ_A, CB)$, we have $BCB_WC_W$ is an isosceles trapezoid or $OZ_A$ is perpendicular to $B_WC_W$. On the other

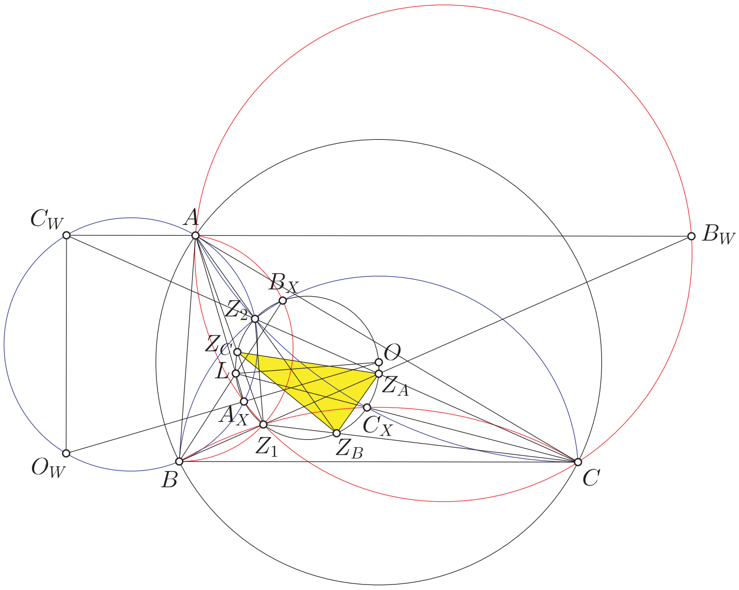


FIGURE 11. Proof of Theorem 2.9

hand, by Theorem 2.2, we have $(A_XA, A_XO_W) = (C_WA, C_WO_W) = \frac{\pi}{2}$. Therefore $O_WC_W$ is perpendicular to $B_WC_W$. So $O_WC_W \parallel OZ_A$.

**Theorem 2.10.** *$A_XZ_A$, $B_XZ_B$, $C_XZ_C$ are concurrent at $G$.*

**Lemma 2.4.** *Given triangle $ABC$. On the plane containing this triangle, choose $A'$, $B'$, $C'$ such that the triangles $A'CB$, $B'AC$, $C'BA$ are homothetic. We construct the parallelogram $BA'CD$. Prove that $A'CDB$ is also a parallelogram.*

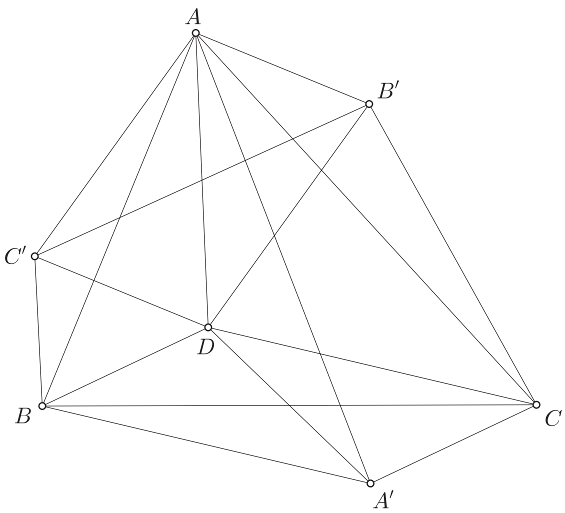


FIGURE 12. Proof of Lemma 2.4

**Proof.** We have
$$\frac{C'B}{C'D}=\frac{C'B}{AB'}=\frac{C'A}{B'C}=\frac{B'D}{B'C}$$
and
$$\begin{aligned}(C'B,C'D)&=(C'B,C'A)+(C'A,C'D)\\&=(B'A,B'C)+(B'D,B'A)=(B'D,B'C).\end{aligned}$$
Therefore triangle $C'BD$ and triangle $B'DC$ are homothetic. Hence
$$\frac{BD}{CD}=\frac{C'B}{B'D}=\frac{C'B}{C'A}=\frac{A'B}{A'C}.$$
On the other hand,
$$\begin{aligned}(DB,DC)&=(DB,DC')+(DC',DB')+(DB',DC)\\&=(DC,B'C)+(B'A,DB')+(DB',DC)\\&=(B'A,B'C)\\&=(A'C,AB).\end{aligned}$$
So triangle $DBC$ and triangle $A'CB$ are homothetic. Therefore, $BD\parallel A'C$ and $CD\parallel A'B$. We conclude that $A'CDB$ is a parallelogram.

**Lemma 2.5.** *$G$ is the centroid of triangle $Z_AZ_BZ_C$.*

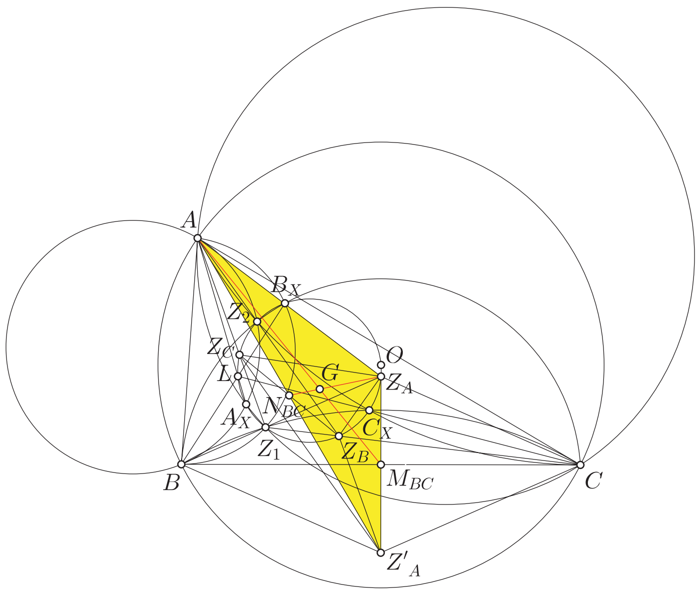


FIGURE 13. Proof of Lemma 2.5

**Proof.** Let $Z'_A$ be symmetric point with $Z_A$ through $BC$. Since triangle $BZ_AC$ is a isosceles triangle at $Z_A$, $BZ_ACZ'_A$ is a parallelogram. Applying Lemma 2.4 for triangle $ABC$, we have $AZ_CZ'_AZ_B$ is a parallelogram so $AZ'_A$ passes midpoint $N_{BC}$ of $Z_BZ_C$. In triangle $AZ_AZ'_A$, the two median lines $AM_{BC}$ and $Z_AN_{BC}$ intersect with each other at this triangle's centroid so that this point is also the centroid $G$ of triangle $ABC$ and triangle $Z_AZ_BZ_C$.

**Lemma 2.6.** *Given triangle $ABC$ and two disjoint arbitrary points $P$, $Q$. Let $X$, $Y$, $Z$ respectively be projection of $P$ on the lines which passes $Q$ and is perpendicular to $BC$, $CA$, $AB$. Let $X'$, $Y'$, $Z'$ respectively be projection of $Q$ on $AP$, $BP$, $CP$. Prove that $XX'$, $YY'$ and $ZZ'$ are concurrent.*

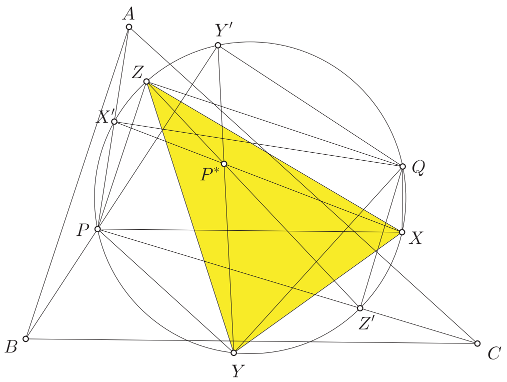


FIGURE 14. Proof of Lemma 2.6

**Proof.** Noted that $X$, $Y$, $Z$, $X'$, $Y'$, $Z'$ lie on a circle with diameter $PQ$ and $AB \parallel PZ$, $AC \parallel PY$ so we have

$$(AB, AC) = (ZP, PY) = (XY, XZ).$$

Similarly, $(BC, BA) = (YZ, YX)$ and $(CA, CB) = (ZX, ZY)$ so two triangles $XYZ$ and $ABC$ are homothetic. Let $P'$ be the image of $P$ through the homothety transformation which turns triangle $ABC$ into triangle $XYZ$. Let $P^*$ be the isogonal conjugate point of $P$ in triangle $XYZ$. We have

$$(XX', XZ) = (PX', PZ) = (AP, AB) = (XP^*, XZ).$$

Therefore $XX'$ passes $P^*$. Similarly, $YY'$ and $ZZ'$ passes $P^*$, which solves the lemma.

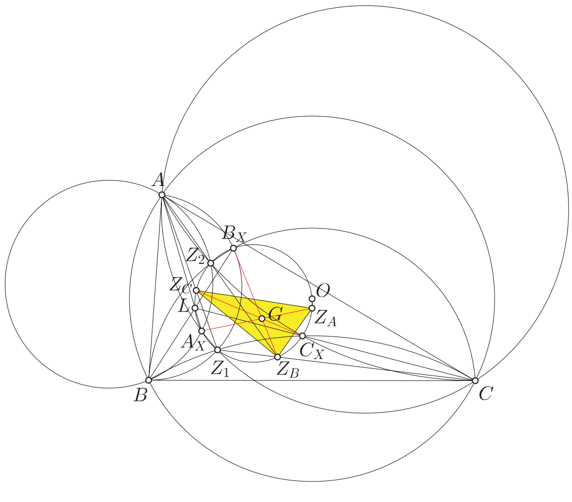


FIGURE 15. Proof of Theorem 2.10

Back to Theorem 2.10,

**Proof.** By applying Lemmas 2.5 and 2.6 for triangle $Z_AZ_BZ_C$, we solves the problem.

By Corollary 2.3, the Brocard circle is definitely the Hagge circle [5] of $G$ in triangle $M_{BC}M_{CA}M_{AB}$. We will discuss some properties related to this issue and use these ones to prove Theorem 2.10.

**Lemma 2.7** (**Hagge circle**). *Given triangle $ABC$, orthocenter $H$ and a point $P$ lies inside $(ABC)$. Rays $AP$, $BP$, $CP$ cuts $(ABC)$ at the second points $A_1$, $B_1$, $C_1$. Let $A_2$, $B_2$, $C_2$ respectively be the symmetry points of $A_1$, $B_1$, $C_1$ through $A$, $B$, $C$. Prove that $A_2$, $B_2$, $C_2$ and $H$ are cyclic.*

**Remark 2.2.** *The circle mentioned above called Hagge circle $\mathcal{H}_P$ of point $P$ with respect to triangle $ABC$.*

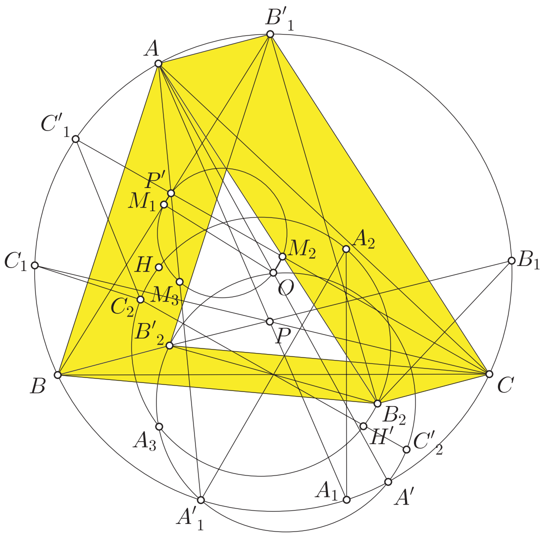


FIGURE 16. Proof of Lemma 2.7

**Proof.** Let $P'$ be the isogonal conjugate of $P$ in triangle $ABC$. $AP'$, $BP'$, $CP'$ cut $(ABC)$ respectively at $A'_1$, $B'_1$, $C'_1$. Let $B'_2$, $C'_2$ respectively be symmetry points of $B_2$, $C_2$ through midpoint of $BC$. Set $O$ is the center of $(ABC)$ with diameter $AA'$. It is easy to see that $A_2$, $B_2$, $C_2$ are symmetry points of $A'_1$, $B'_2$, $C'_2$ through midpoint of $BC$. We have $CB_2AB'_1$ and $CB_2BB'_2$ are parallelograms so the quadrilateral $ABB'_2B'_1$ is also a parallelogram. We conclude that midpoint $M_1$ of $AB'_2$ is the projection of $O$ on $BP'$. Similarly, midpoint $M_2$ of $AC'_2$ is the projection of $O$ on $CP'$. Let $M_3$ be midpoint of $AA'_1$. Then triangle $M_1M_2M_3$ is inscribed in a circle with diameter $OP'$. We use two geometric transformations respectively with the first one (1) is the homothety transformation $\mathbf{H}_A^2$ with center $A$ and ratio 2; and the later (2) is the symmetric one $\mathbf{S}_{M_{BC}}$ through midpoint $M_{BC}$ of $BC$, to points $M_3$, $M_1$, $M_2$, $O$ such that

$$\mathbf{S}_{M_{BC}} \circ \mathbf{H}_A^2 \colon M_3, M_1, M_2, O \mapsto A_2, B_2, C_2, H.$$

Since $M_1, M_2, M_3, O$ lie on a same circle so $A_2, B_2, C_2, H$ are also cyclic.

**Lemma 2.8.** *Given triangle $ABC$ and two points $S$ and $S'$ are isogonal conjugate. Let $X$, $X'$ respectively be intersection of $AS$, $AS'$ with $(ABC)$. Let $V$ be intersection of $AS'$ and $BC$. Prove that*

$$\frac{\overline{AS}}{\overline{SX}} = \frac{\overline{S'V}}{\overline{VX'}}.$$

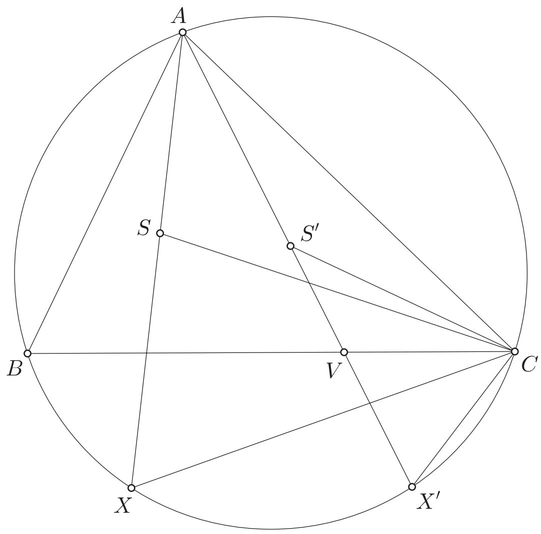


FIGURE 17. Proof of Lemma 2.8

**Proof.** We have

$$(CV, CX') = (AB, AX') = (AX, AC),$$

$$(X'V, X'C) = (XA, XC)$$

So two triangles $VCX'$ and $CAX$ are homothetic. On the other hand,

$$\begin{aligned}(S'X', S'C) &= (AS', AC) + (CA, CS')\\ &= (AB, AS) + (CS, CB)\\ &= (CB, CX) + (CS, CB)\\ &= (CS, CX),\end{aligned}$$

$$(X'S', X'C) = (XS, XC).$$

Therefore two triangles $CX'S'$ and $SXC$ are homothetic. So,

$$\overline{X'V} \cdot \overline{XA} = \overline{X'C} \cdot \overline{XC} = \overline{X'S'} \cdot \overline{XS}.$$

Hence, $\frac{\overline{XA}}{\overline{XS}} = \frac{\overline{X'S'}}{\overline{X'V}}$ or $\frac{\overline{AS}}{\overline{SX}} = \frac{\overline{S'V}}{\overline{VX'}}$.

**Lemma 2.9.** *Let $A_3$, $B_3$, $C_3$ respectively be intersection of $\mathcal{H}_P$ and $AH$, $BH$, $CH$. Prove that $A_2A_3$, $B_2B_3$, $C_2C_3$ are concurrent at $P$.*

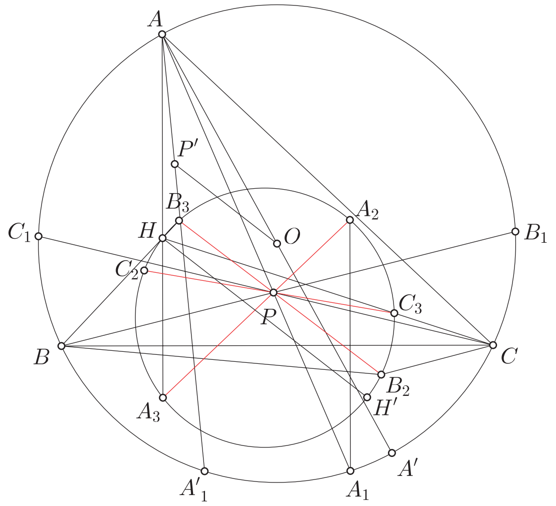


FIGURE 18. Proof of Lemma 2.9

**Proof.** We construct diameter $HH'$ of $\mathcal{H}_P$. From Lemma 2.7, we have

$$\mathbf{S}_{M_{BC}} \circ \mathbf{H}_A^2 \colon O, P' \mapsto H, H'.$$

Therefore $OP' \parallel HH'$ and $\overline{H'H} = 2\overline{OP'}$ so $H'$ is the Anticomplement point of $P'$ in triangle $ABC$. On the other hand, $A_3H' \parallel BC$ and $BC \perp AH$ hence $AA' = 2\mathrm{d}(P', BC)$. Using Lemma 2.8, we have

$$\frac{\overline{AP}}{\overline{PA_1}} = \frac{\mathrm{d}(P', BC)}{\mathrm{d}(A_1', BC)} = \frac{\mathrm{d}(P', BC)}{\mathrm{d}(A_1, BC)} = \frac{\overline{AA_3}}{\overline{A_2A_1}}.$$

This means $A_2A_3$ passes $P$. Similarly, $B_2B_3, C_2C_3$ passes $P$. We have solved the lemma.

Back to Theorem 2.10,

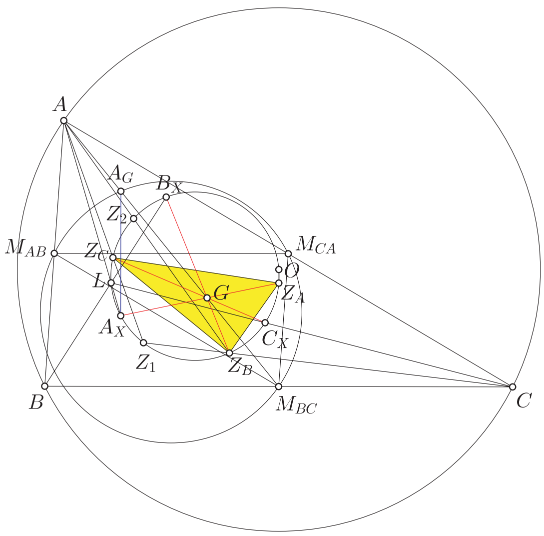


FIGURE 19. Proof of Theorem 2.10

**Proof.** By Corollary 2.3, the Brocard cirlce is the Hagge circle of $G$ with respect to triangle $M_{BC}M_{CA}M_{AB}$. By applying Lemma 2.7 and 2.9 for triangle $M_{BC}M_{CA}M_{AB}$, we have the result.

## 3. Properties related to the second problem

**Theorem 3.1.** *$A_Y$ is the projection of orthocenter $H$ on the $A-$median line of triangle $ABC$.*

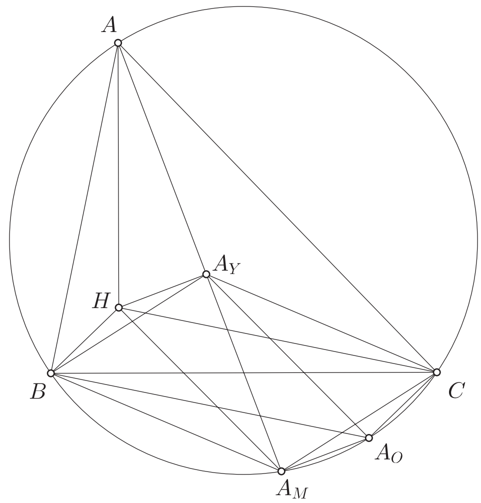


FIGURE 20. Proof of Theorem 3.1

**Proof.** Let $AA_O$ be diameter of $(ABC)$. Since $BHCA_O$ is a parallelogram, $M_{BC}$ is the midpoint of $A_OH$. On the other hand, $M_{BC}$ is also the mid point of $A_YA_M$. By Theorem 2.3, $HA_YA_OA_M$ is a parallelogram therefore $A_YH \parallel A_OA_M$. Moreover, $A_OA_M \perp AA_M$ hence $A_Y$ is the projection of $H$ on the $A$-median line.

**Corollary 3.1.** *Let $B_Y$ and $C_Y$ be constructed in a similar manner as $A_Y$, but with respect to $B$ and $C$ respectively. Let $H_AH_BH_C$ be the Orthocentroidal triangle with $H_A$, $H_B$, $H_C$ repsectively be projection of $G$ on $AH$, $BH$, $CH$. Then two triangles $A_YB_YC_Y$ and $H_AH_BH_C$ are inscribed in the circle with diameter $GH$ which called Orthocentroidal circle.*

**Corollary 3.2.** *The Hagge circle of the Lemoine point $L$ in triangle $ABC$ is the Orthocentroidal circle of this triangle.*

**Theorem 3.2.** *The radical line of $(A_XB_XC_X)$ and $(A_YB_YC_Y)$ passes midpoint $X(39)$ of two Brocard points.*

We can see the proof of this property here [9] using Barycentric coordinate.

**Theorem 3.3.** *$A_YH_A$, $B_YH_B$, $C_YH_C$ are concurrent at $L$.*

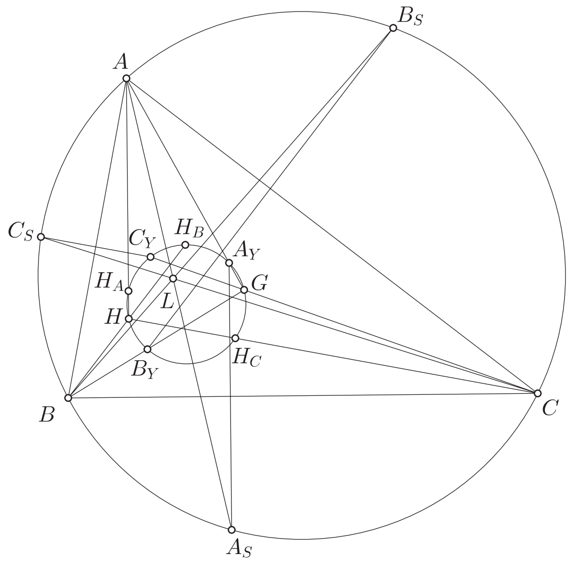


FIGURE 21. Demonstration of Corollary 3.2

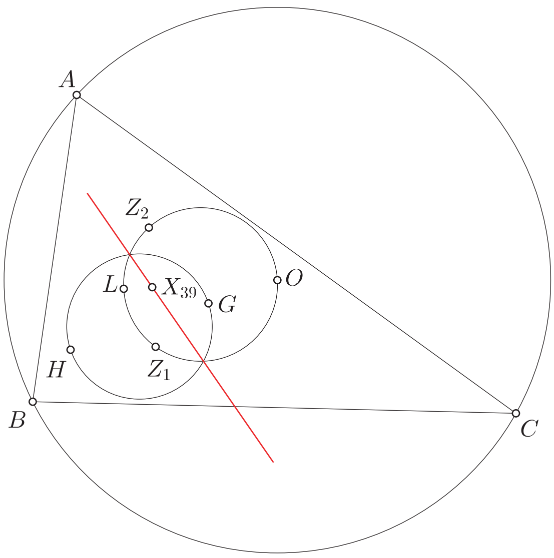


FIGURE 22. Demonstration of Theorem 3.2

**Lemma 3.1.** *$L$ is also the Lemoine point of triangle $H_AH_BH_C$.*

**Proof.** By Corollary 3.2, the Orthocentroidal circle of triangle $ABC$ is the Hagge circle of $L$ with respect to this triangle. Moreover, by Corollary 2.1, triangle $A_YB_YC_Y$ is the Hagge triangle of $L$, which means $A_Y$, $B_Y$, $C_Y$ are constructed in the same way of $A_2$, $B_2$, $C_2$ in Lemma 2.7, but with respect to $L$. We then apply Lemma 2.9 to completely solve this result.

Back to Theorem 3.3,

**Proof.** By applying Lemma 2.7, 2.9 and 3.1 for triangle $ABC$ with Lemoine point $L$ and the Orthocentroidal circle, we can prove the theorem.

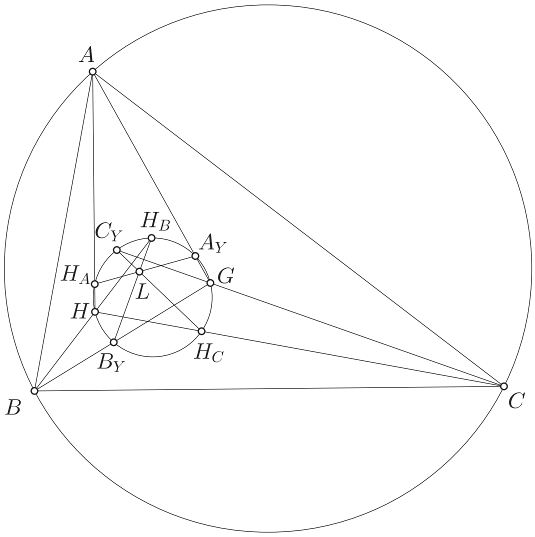


FIGURE 23. Proof of Theorem 3.3

**Theorem 3.4.** *The perpendicular bisector of* $AXA_{GY}$ *cuts* $AL$ *at* $L_N$*. Then* $AXA_Y \parallel L_N M_{BC}$*.*

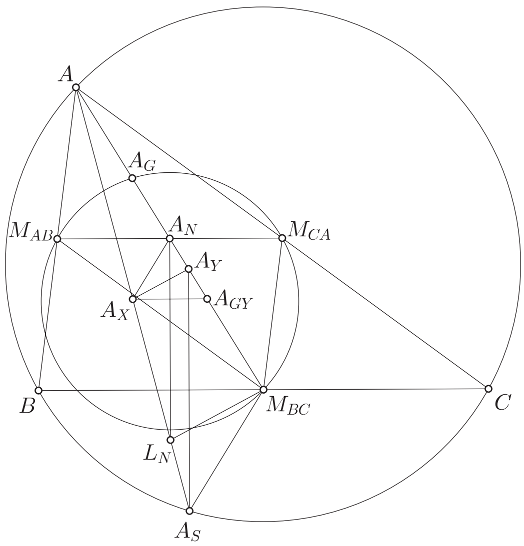


FIGURE 24. Proof of Theorem 3.4

**Proof.** By Corollary 2.2, $L_N$ also lies on the perpendicular bisector of $M_{AB}M_{CA}$ containing $A_N$ so $A_N L_N \parallel A_Y A_S$. Therefore

$$\overline{AL_N} \cdot \overline{AA_Y} = \overline{AA_N} \cdot \overline{AA_S} = \overline{AM_{BC}} \cdot \overline{AA_X}$$

so $\frac{\overline{AA_X}}{\overline{AL_N}} = \frac{\overline{AA_Y}}{\overline{AM_{BC}}}$. Hence, $A_X A_Y \parallel L_N M_{BC}$.

**Theorem 3.5.** $(A_X A_{GY} L_N)$ *is tangent to* $(BOC)$*.*

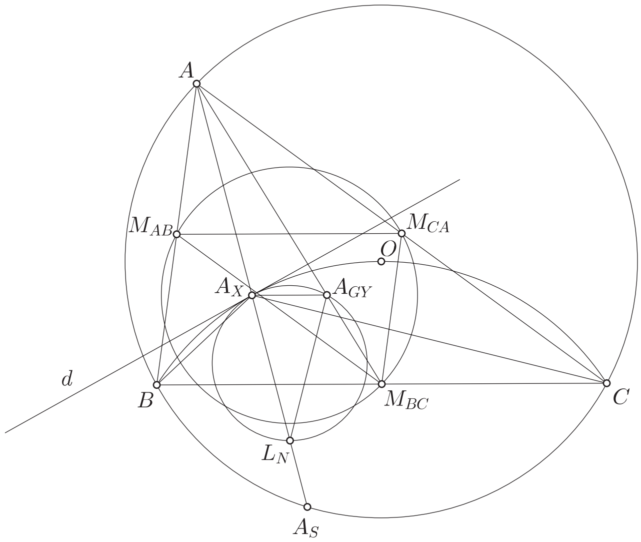


FIGURE 25. Proof of Theorem 3.5

**Proof.** By Theorem 2.1, $A_X$ lies on $(BOC)$. Let $Ad$ be a ray which belongs to tangent line at $A$ of $(A_XA_{GY}L_N)$ such that $Ad$ and $B$ are on the same side of the plane separated by $AL$. By Theorem 2.1, $A_XA_S$ is the $A_X$-bisector of triangle $A_XBC$. We have

$$\begin{aligned}(A_Xd, A_XB) &= (A_Xd, A_XA_S) + (A_XA_S, A_XB)\\ &= (A_XA_S, A_XA_{GY}) + (A_XC, A_XA_S)\\ &= (A_XC, A_XA_{GY})\\ &= (CB, CA_X).\end{aligned}$$

So $Ad$ is the tangent line of $(A_XBC) \equiv (BOC)$. We have done.

**Theorem 3.6.** *Let $A_{LX}$ be projection of $H$ on $AL$. Then $M_{BC}A_Y = M_{BC}A_{LX}$.*

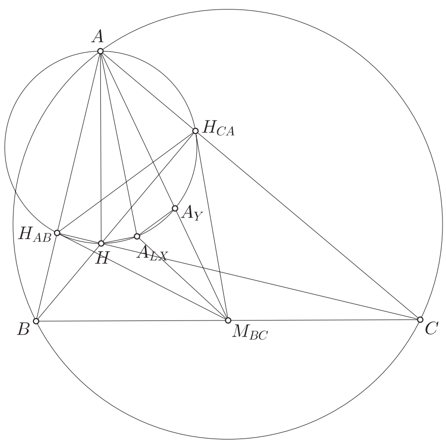


FIGURE 26. Proof of Theorem 3.6

**Proof.** Draw altitudes $BH_{CA}$ and $CH_{AB}$ of triangle $ABC$. It can be seen that $A, H, H_{CA}, H_{AB}, A_Y, A_{LX}$ lie on the circle with diameter $AH$. Since $M_{BC}$ is the intersection of two tangent lines at $H_{CA}$ and $H_{AB}$ of $(AH)$ so $AH_{AB}A_YH_{BC}$ is a harmonic quadrilateral. Since $AL$ is a median line of triangle $AH_{CA}H_{AB}$ hence $H_{CA}H_{AB} \parallel A_YA_{LX}$. Moreover, as $M$ lies on the perpendicular bisector of $H_{CA}H_{AB}$, which is also the perpendicular bisector of $A_YA_{LX}$, so that $M_{BC}A_Y = M_{BC}A_{LX}$.

**Corollary 3.3.** *$(A_YM_{BC}A_{LX})$ is tangent to the Euler circle of triangle $ABC$.*

**Theorem 3.7.** *Let $L_{BC}$ be intersection of $AL$ and $BC$. Then $A_YL_{BC}$ is a symmedian line of triangle $A_YBC$.*

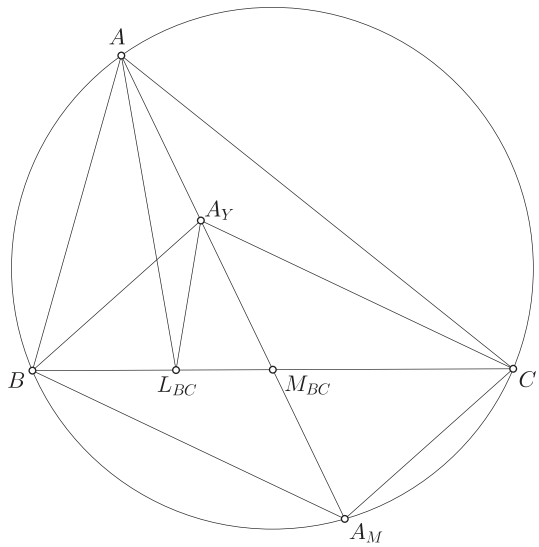


FIGURE 27. Proof of Theorem 3.7

**Proof.** Since $AL_{BC}$ is a symmedian line of triangle $ABC$ so

$$\frac{\overline{L_{BC}B}}{\overline{L_{BC}C}} = -\left(\frac{AB}{AC}\right)^2.$$

On the other hand,

$$\left(\frac{AB}{AC}\right)^2 = \left(\frac{AB}{M_{BC}B} \cdot \frac{M_{BC}C}{AC}\right)^2 = \left(\frac{A_MC}{A_MM_{BC}} \cdot \frac{A_MM_{BC}}{A_MB}\right)^2 = \left(\frac{A_YB}{A_YC}\right)^2$$

So

$$\frac{\overline{L_{BC}B}}{\overline{L_{BC}C}} = -\left(\frac{A_YB}{A_YC}\right)^2.$$

We conclude that $A_YL_{BC}$ is a symmedian line of triangle $A_YBC$.

**Theorem 3.8.** *$(A_YM_{BC}A_{LX})$ is tangent to $(BHC)$.*

**Lemma 3.2.** *Given triangle $ABC$. On segment $BC$, we choose $D$, $E$ such that $AD$, $AE$ are isogonal conjugate of each other in this triangle. Prove that $(ABC)$ is tangent to $(ADE)$.*

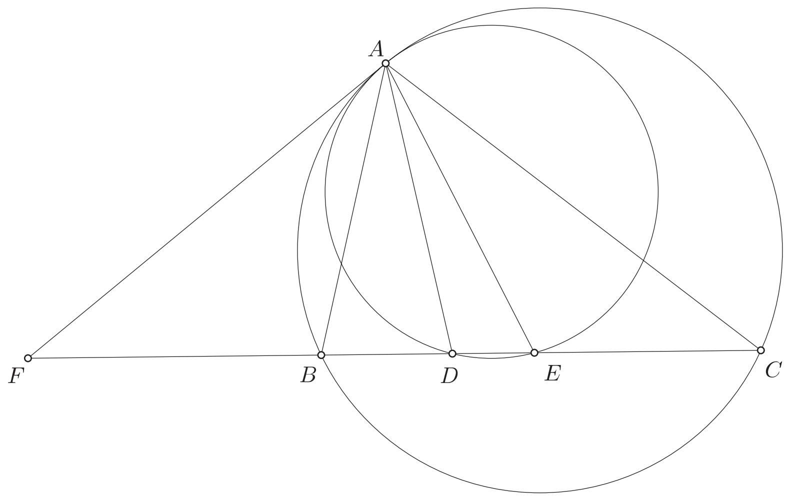


FIGURE 28. Proof of Lemma 3.2

**Proof.** The tangent line at $A$ of $(ABC)$ cuts line $BC$ at $F$. Then

$$(AF, AD) = (AF, AB) + (AB, AD) = (CA, CB) + (EA, CA) = (EA, CB)$$

so $AF$ is tangent to $(ADE)$. Thus, $(ABC)$ is tangent to $(ADE)$.

Back to Theorem 3.8,

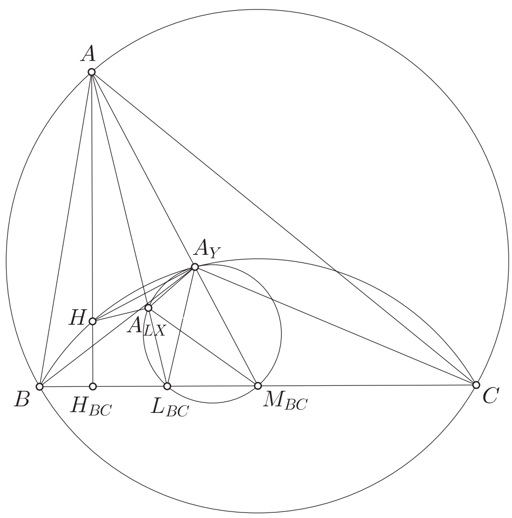


FIGURE 29. Proof of Theorem 3.8

**Proof.** Draw the altitude $AH_{BC}$ of triangle $ABC$. We have that $H_{BC}$ and $A_{LX}$ lie on the circle with diameter $HL_{BC}$. Therefore,

$$(L_{BC}A_{LX}, L_{BC}M_{BC}) = (HA_{LX}, HH_{BC}) = (A_YA_{LX}, A_YA).$$

So $A_Y, A_{LX}, L_{BC}, M_{BC}$ are cyclic. Using Theorem 3.7, $A_YL_{BC}$ is a symmedian line of triangle $A_YBC$. We use Lemma 3.2 to conclude that $(A_YM_{BC}A_{LX})$ touches $(BHC)$.

**Theorem 3.9.** *The rays $BA_Y$, $CA_Y$ cuts $(ABC)$ at $A_{BY}$, $A_{CY}$. Let $A'_{BY}, A'_{CY}$ be symmetry points of $A_{BY}, A_{CY}$ through $CA$, $AB$ respectively. Then $A'_{BY}$ and $A'_{CY}$ lie on $AL$.*

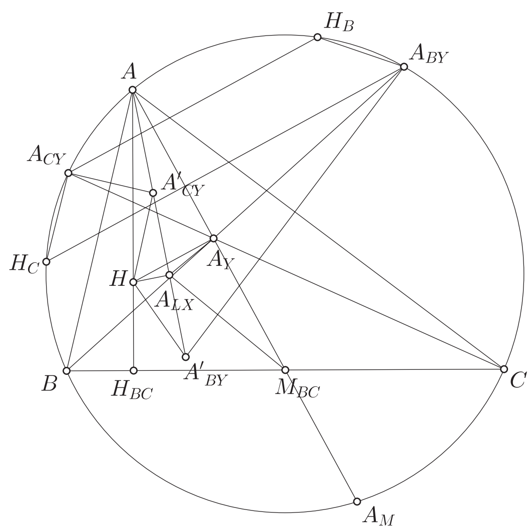


FIGURE 30. Proof of Theorem 3.9

**Proof.** We have

$$\begin{aligned}(AA'_{BY}, AC) &= (AC, AA_{BY}) = (BC, BA_{BY}) \\ &= (CB, CA_M) = (AB, AA_M) \\ &= (AL, AC).\end{aligned}$$

Thus $AL$ passes $A'_{BY}$. Similarly, $AL$ passes $A'_{CY}$.

**Theorem 3.10.** *$A_{LX}$ is midpoint of $A'_{BY}A'_{CY}$.*

**Proof.** Rays $BH$ and $CH$ cut $(ABC)$ respectively at $H_B$ and $H_C$. We have

$$(H_BA_{CY}, H_BB) = (CA_{CY}, CB) = (HA_Y, HH_B)$$

so $H_BA_{CY} \parallel HA_Y$, where $HA_Y \perp AG$. Simiarly $H_CA_{BY} \parallel HA_Y$, where $HA_Y \perp AG$ hence $H_BA_{CY}H_CA_{BY}$ is a isosceles trapezoid. We conclude that $H_CA_{CY} = H_BA_{BY}$. Using symmetry transformation through $AB$ and $AC$, we have $\mathbf{S}_{AB}\colon H, A_{CY} \mapsto H_C, A'_{CY}$ and $\mathbf{S}_{AC}\colon H, A_{BY} \mapsto H_B, A'_{BY}$. Therefore $HA'_{BY} = H_BA_{BY} = H_CA_{CY} = HA'_{CY}$, which means triangle $HA'_{BY}A'_{CY}$ is isosceles at $H$. Moreover, $A_{LX}$ is the projection of $H$ on $A'_{BY}A'_{CY}$ so $A_{LX}$ is midpoint of $A'_{BY}A'_{CY}$.

## 4. Conclusion

This article is a discussion on the properties associated with a pair of distinct triangle centers. Further investigations can be conducted in this domain to delve deeper into the subject matter and uncover additional distinctive characteristics pertaining to triangles.

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