Invariant nature of Raychaudhuri equation in the presence of Inflaton fields

Arijit Panda,^{1,2,*} Debashis Gangopadhyay,^{3,†} and Goutam Manna^{2,4,‡}

Department of Physics, Raiganj University, Raiganj, Uttar Dinajpur-733 134, West Bengal, India.
 Department of Physics, Prabhat Kumar College, Contai, Purba Medinipur-721404, India
 Department of Physics, School of Natural Sciences, Sister Nivedita University,
 DG 1/2, Action Area 1 Newtown, Kolkata, India, 700156
 Institute of Astronomy Space and Earth Science, Kolkata 700054, India
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We show that the Raychaudhuri equation remains invariant for certain solutions of scalar fields ϕ whose Lagrangian is non-canonical and of the form $\mathcal{L}(X,\phi) = -V(\phi)F(X)$, with $X = \frac{1}{2}g_{\mu\nu}\nabla^{\mu}\phi\nabla^{\nu}\phi$, $V(\phi)$ the potential. Solutions exist for both homogeneous and inhomogeneous fields and are reminiscent of inflaton scenarios.

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Introduction: The Raychaudhuri equation [1–6], is an essential component of contemporary cosmology. The equation describes the investigation and evaluation of the timedependent characteristics of the congruence of geodesics, as witnessed by another nearby congruence of geodesics throughout its development. The evolution primarily describes the progression of congruence inside a specific spacetime framework, and this progression is produced by a curve that follows the tangent vectors (which can be either timelike or null). The equation describes the rate at which the expansion of the congruence (a measurement that is not reliant on coordinates) is related to the gravitational influ-The equation was derived by Raychaudhuri solely based on geometrical concepts, without any reliance on other gravitational theories. The Raychaudhuri equation is commonly utilized to illustrate the attractive nature of gravity, which exerts a force between two material things. Subsequently, it emerged as a crucial lemma for investigating the renowned Hawking-Penrose singularity theorem[7–10]. On a basic level, the Raychaudhuri equation takes into account the non-isotropy (regarding rotation) and non-homogeneity (regarding shear) of the cosmos [6]. The equation gives the expansion rates of two null or timelike geodesics, which can converge or diverge depending on the energy conditions provided by the Ricci tensor. Converging geodesics can be used to explain the focusing theorem [11, 12], while diverging geodesics can help to understand the non-focusing theorem due to the expansion of the universe [12, 13].

To explore many world possibilities, Das et al. [12] investigated the Raychaudhuri equation in K-essence geometry, a non-canonical theory. The authors showed that the modified Raychaudhuri equation allows for both the conditional singular and non-singular structure of the universe for cases where the spacetime is of the flat K-essence emergent Friedmann-Lemaitre-Robertson-Walker (FLRW) type. They also dealt with the development of conditional caustic universes and the Focusing theorem. An example of a non-canonical theory that investigates several avenues in cosmological research is the K-essence theory [12, 14–26]. There are different ways to write the Lagrangian of the K-essence

field. One way is $\mathcal{L}(X,\phi) = -V(\phi)F(X)$ [14, 15, 17–20], another is $\mathcal{L}(X,\phi) = F(X) - V(\phi)$ [24, 25], and a third is $\mathcal{L}(X,\phi) \equiv \mathcal{L}(X) = F(X)$ [23, 26]. Here, $F(X) \equiv \mathcal{L}(X) (\neq X)$ is the non-canonical kinetic part with $X = \frac{1}{2}g_{\mu\nu}\nabla^{\mu}\phi\nabla^{\nu}\phi$, $V(\phi)$ is the canonical potential part. This theory emphasizes that the kinetic element dominates the potential part and shows a dynamic behavior that assures late-time acceleration without fine-tuning. Another noteworthy part of the Kessence idea is its capacity to create dark energy with a sound speed (c_s) continuously lower than light. This property may lessen cosmic microwave background disruptions at large angular scales. In addition, according to Scherrer [23], this theory may be applied to both dark energy and dark matter to form a unified theory. The Dirac-Born-Infeld (DBI) type non-canonical action [27, 28] was utilized by Manna et al. [29–36], and they have showed that the K-essence theory can be applied to the models of dark energy [29, 30, 34] as well as in gravitational perspectives [31, 32, 35, 36], without considering the dark sectors of the universe. The results from the Planck collaborations [37–39] have investigated the empirical evidence that supports the idea of K-essence with a DBI-type non-canonical Lagrangian among other things. The dynamical solutions of the K-essence equation of motion distinguish K-essence theories with non-canonical kinetic terms from relativistic field theories with canonical terms. These demonstrate spontaneous Lorentz invariance breakdown and metric changes for perturbations. The K-essence field Lagrangian's theoretical formulation has non-canonical features that the metric demands. These features distribute perturbations across the evolving analogue spacetime, or curved spacetime. This can imitate the background metric, creating late-time acceleration at the right time frame in the universe's evolution, which answers the 'cosmic coincidence' problem.

This letter aims to demonstrate the invariance of the Raychaudhuri equation in the presence of both homogeneous and inhomogeneous scalar fields within the framework of K-essence geometry. The aforementioned fields exhibit conditional correspondences with inflaton fields, which persist over the whole of the universe's early to late phases.

K-essence: The action of the K-essence geometry can be written as [14–16, 20, 23]

$$S_k[\phi, g_{\mu\nu}] = \int d^4x \sqrt{-g} \mathcal{L}(X, \phi), \qquad (1)$$

where $\mathcal{L}(X,\phi)$ represents the non-canonical Lagrangian with

^{*} arijitpanda260195@gmail.com

[†] debashis.g@snuniv.ac.in

[‡] goutammanna.pkc@gmail.com; Corresponding author

the canonical kinetic term $X = \frac{1}{2}g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi$. Here, the standard gravitational metric $g_{\mu\nu}$ has minimal coupling with the K-essence scalar field ϕ .

The energy-momentum tensor is defined as [16, 20]:

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_k}{\delta g^{\mu\nu}} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}$$
$$= \mathcal{L}_X \nabla_{\mu} \phi \nabla_{\nu} \phi + g_{\mu\nu} \mathcal{L}, \tag{2}$$

where $\mathcal{L}_{X} = \frac{d\mathcal{L}}{dX} \neq 0$, $\mathcal{L}_{XX} = \frac{d^{2}\mathcal{L}}{dX^{2}}$, $\mathcal{L}_{\phi} = \frac{d\mathcal{L}}{d\phi}$ and ∇_{μ} is the covariant derivative defined with respect to the gravitational metric $g_{\mu\nu}$.

We may formulate a scalar field equation of motion (EOM) in accordance with [16, 20, 29, 30]

$$-\frac{1}{\sqrt{-g}}\frac{\delta S_k}{\delta \phi} = \tilde{G}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi + 2X\mathcal{L}_{X\phi} - \mathcal{L}_{\phi} = 0, \qquad (3)$$

where the effective metric is

$$\tilde{G}^{\mu\nu} \equiv \frac{c_s}{\mathcal{L}_X^2} [\mathcal{L}_X g^{\mu\nu} + \mathcal{L}_{XX} \nabla^\mu \phi \nabla^\nu \phi] \tag{4}$$

with $(1 + \frac{2X\mathcal{L}_{XX}}{\mathcal{L}_{X}}) > 0$ and $c_s^2(X, \phi) \equiv (1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_{X}})^{-1}$. After a conformal transformation [29–32] $\bar{G}_{\mu\nu} \equiv \frac{c_s}{\mathcal{L}_{X}} G_{\mu\nu}$,

After a conformal transformation [29–32] $G_{\mu\nu} \equiv \frac{c_s}{\mathcal{L}_X} G_{\mu\nu}$, we can write the inverse metric of Eq. (4) as

$$\bar{G}_{\mu\nu} = g_{\mu\nu} - \frac{\mathcal{L}_{XX}}{\mathcal{L}_{X} + 2X\mathcal{L}_{XX}} \nabla_{\mu}\phi \nabla_{\nu}\phi. \tag{5}$$

The corresponding geodesic equation can also be changed as [29-32]

$$\frac{d^2x^{\alpha}}{d\lambda^2} + \bar{\Gamma}^{\alpha}_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} = 0, \tag{6}$$

where λ is an affine parameter and the new connection coefficients $(\bar{\Gamma}^{\alpha}_{\mu\nu})$ can be expressed as

$$\bar{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \frac{1}{2(1-2X)} \Big[\delta^{\alpha}_{\mu} \partial_{\nu} + \delta^{\alpha}_{\nu} \partial_{\mu} \Big] X, \tag{7}$$

The covariant derivative associated with the K-essence geometry $(\bar{G}_{\mu\nu})$ is

$$D_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + \bar{\Gamma}^{\nu}_{\mu\lambda}A^{\lambda} \tag{8}$$

with the inverse emergent metric $\bar{G}^{\mu\nu}$ defined through $\bar{G}_{\mu\lambda}\bar{G}^{\lambda\nu}=\delta^{\nu}_{\mu}$.

Modified Raychaudhuri equation: In K-essence geometry, the Raychaudhuri equation may be expressed using the covariant derivative equation (8) for a time-like vector field v^{β} connected to an emergent metric tensor $\bar{G}_{\mu\nu}$ as [12]

$$\frac{d\tilde{\Theta}}{d\bar{s}} + (D_{\alpha}v^{\beta})(D_{\beta}v^{\alpha}) = -\bar{R}_{\gamma\beta}v^{\gamma}v^{\beta} \tag{9}$$

where \bar{s} is an affine parameter, $\tilde{\Theta} \equiv D_{\alpha}v^{\beta}$ is the scalar expansion, and $\bar{R}_{\gamma\beta}$ is the Ricci tensor of this geometry. The new geometry is torsion-free and hence the above equation (9) is generated from the commutation relationship of the covariant derivatives (8) as:

$$[D_{\alpha}, D_{\beta}]v^{\mu} = \bar{R}^{\mu}_{\gamma\alpha\beta}v^{\gamma}, \tag{10}$$

where $\bar{R}^{\mu}_{\gamma\alpha\beta}$ is the Riemann tensor associated with the K-essence emergent spacetime. Following [12], we rewrite the modified Raychaudhuri equation of the K-essence geometry

$$\frac{d\tilde{\Theta}}{d\bar{s}} + \left(2\sigma^2 - 2\omega^2 + \frac{1}{3}\theta^2\right) - \frac{\theta}{(1 - 2X)}v^{\mu}\partial_{\mu}X + \frac{7(v^{\mu}\partial_{\mu}X)^2}{4(1 - 2X)^2} = -\bar{R}_{\gamma\beta}v^{\gamma}v^{\beta} \tag{11}$$

and the corresponding scalar expansion $(\tilde{\Theta})$ is [11, 12]

$$\tilde{\Theta} = D_{\alpha} v^{\alpha} \equiv \frac{1}{\sqrt{-\bar{G}}} \, \partial_{\alpha} (\sqrt{-\bar{G}} \, v^{\alpha}) = 3 \frac{\dot{a}}{a} - \frac{v^{\mu} \partial_{\mu} X}{1 - 2X}. \quad (12)$$

where $\theta = \nabla_{\alpha} v^{\alpha}$ is the usual scalar expansion, symmetric shear $\sigma_{\alpha\beta} = \frac{1}{2} (\nabla_{\beta} v_{\alpha} + \nabla_{\alpha} v_{\beta}) - \frac{1}{3} \theta h_{\alpha\beta}$, antisymmetric rotation $\omega_{\alpha\beta} = \frac{1}{2} \Big(\nabla_{\beta} v_{\alpha} - \nabla_{\alpha} v_{\beta} \Big), \ 2\sigma^2 = \sigma_{\alpha\beta} \sigma^{\alpha\beta},$ $2\omega^2 = \omega_{\alpha\beta}\omega^{\alpha\beta}$ and $\nabla_{\beta}\dot{v_{\alpha}} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}$ with the three dimensional hypersurface metric $h_{\alpha\beta} = g_{\alpha\beta} - v_{\alpha}v_{\beta}$. It should also be noted that in [12], the authors employed a DBI type non-canonical Lagrangian represented as $\mathcal{L}(X,\phi) = 1 - V\sqrt{1-2X}$, where $V(\phi) \equiv V$ is a constant potential that is significantly smaller than the kinetic energy of the K-essence scalar field. Here, in Eq. (12), the background gravitational metric is assumed to be flat FLRW. Note that Eq. (11) reduces to the original Raychaudhuri equation in regular geometry when ϕ is absent. Moreover, it is evident from equations (11) and (12) that when $v^{\mu}\partial_{\mu}X = 0$, the original Raychaudhuri equation and the original scalar expansion are recovered [5, 6, 11]. When $v^{\mu}\partial_{\mu}X = 0$ we have intriguing physical possibilities involving specific solutions of the K-essence scalar field. We now show that $v^{\mu}\partial_{\mu}X = 0$ admits non-trivial solutions for ϕ that ensure the recovery of the original Raychaudhuri equation.

Solutions: Considering the K-essence scalar field as inhomogeneous and isotropic, we define $\phi(x^i) \equiv \phi(r,t)$ with a flat FLRW type background gravitational metric. The expression of X is given by

$$X = -\frac{1}{2}\phi_t^2 + \frac{1}{2a^2}\phi_r^2 \tag{13}$$

where a=a(t) represents the scale factor and the metric signature is (-,+,+,+). Taking v^{μ} as (1,1,0,0), we have

$$v^{\mu}\partial_{\mu}X = \phi_{t}\phi_{tt} - \frac{\dot{a}}{a}\frac{1}{a^{2}}\phi_{r}^{2} + \frac{\phi_{r}\phi_{rt}}{a^{2}} - \phi_{t}\phi_{tr} + \frac{1}{a^{2}}\phi_{r}\phi_{rr}.$$
(14)

where $\phi_t = \frac{\partial \phi}{\partial t}$, $\phi_r = \frac{\partial \phi}{\partial r}$, $\phi_{tr} = \frac{\partial^2 \phi}{\partial t \partial r} = \phi_{rt}$, $\phi_{tt} = \frac{\partial^2 \phi}{\partial t^2}$, $\phi_{rr} = \frac{\partial^2 \phi}{\partial r^2}$ and $\dot{a} = \frac{\partial a}{\partial t}$. It is important to note that the four-vector v^{μ} can be expressed as (1,1,0,0) for an inhomogeneous field. However, it is crucial to bear in mind that this selection of v^{μ} represents the four-vector at a particular location in spacetime, and it may not be consistent at other locations unless there is a specific situation where this specific choice is justified. Usually, in an inhomogeneous field, the components of the four-velocity will differ at different locations, and it is necessary to define their variations over the relevant region of spacetime. The vector $v^{\mu} = (1,1,0,0)$ might be seen as a specific instance in this situation, however,

it does not serve as the overall answer for an inhomogeneous field. Furthermore, it is essential to note that our study focused solely on the invariance property of the Raycahudhuri equation when scalar fields are present, without addressing the cosmological behavior of the universe. For solving the constraint equation $v^{\mu}\partial_{\mu}X=0$ we consider the K-essence scalar field as

$$\phi(r,t) = \phi_1(r) + \phi_2(t). \tag{15}$$

Then Eq. (14) becomes

$$a^{2}\dot{\phi}_{2}\ddot{\phi}_{2} + \left(\frac{\dot{a}}{a}\right)(\phi_{1}')^{2} - \phi_{1}'\phi_{1}'' = 0 \tag{16}$$

where 'dot' means time derivative and 'prime' means space derivative. In equation (16) there is a space and time derivative in the second term. To simplify this we take the scale factor as defined by [40–43]:

$$a(t) = e^{H_0 t}. (17)$$

Here, H_0 represents the present value of the Hubble parameter. This specific scale factor can be utilized to describe the acceleration of the universe in later periods, when it is primarily influenced by dark energy [40], as well as during inflation in the early stages of the universe [41–43]. Therefore, the equation (16) can be separated as

$$[H_0(\phi_1')^2 - \phi_1'\phi_1''] = -e^{2H_0t}\dot{\phi}_2\ddot{\phi}_2 = w^2$$
(18)

where w^2 is the separation constant. It's important to mention that the separation constant w^2 is either zero or a positive non-zero value.

Case-I $w^2=0$: From Eq. (18) we get the solutions as $(e^{2H_0t} \neq 0)$:

(i)
$$\phi(r,t) = A + D;$$
 (19)

or, (ii)
$$\phi(r,t) = Bt + C + D$$
 (20)

or, (iii)
$$\phi(r,t) = A + \frac{1}{H_0} e^{H_0 r + E};$$
 (21)

or, (iv)
$$\phi(r,t) = Bt + C + \frac{1}{H_0}e^{H_0r + E};$$
 (22)

where A, B C D, and E are the arbitrary integration constants. In the first example, $\phi(r,t)$ has four distinct kinds of solutions, and $v^{\mu}\partial_{\mu}X$ is zero for all of them. The modified Raychaudhuri equation (11) reduces to the standard Raychaudhuri equation for these four categories of scalar fields. The constant scalar field is the first of the four solutions (19), the homogenous field is the second (20), and the inhomogeneous scalar fields are the last two (21 and 22). Thus, it can be concluded that the Raychaudhuri equation (11) and the scalar expansion (12) remain invariant for both the homogeneous and inhomogeneous scalar field solutions given above. Note that, the fourth solution (22) is the general solution of

Eq.(18) for $w^2 = 0$, which incorporates both the space part (r) and the time part (t). For t = 0 one can achieve the third solution (21), for r = 0, one would get the second solution (20) and for both t = 0 and r = 0 one would get the first solution (19). Typically, scalar field models are employed to elucidate the inflationary phases of the universe, which refer to the rapid expansion of space within a short period. In our fourth solution (22), it is evident that the scalar field exhibits exponential dependence on space and linear dependence on time, suggesting inflationary behavior [43].

Case-II $w^2 \neq 0$: From the Eq. (18) the general solutions of the K-essence scalar field are:

$$\phi^{\pm}(r,t) = \mp \frac{1}{2H_0^{3/2}} \left[2\sqrt{w^2 + e^{2H_0(r+G_1)}} - w \ln\left(w + \sqrt{w^2 + e^{2H_0(r+G_1)}}\right) + w \ln\left(H_0\left(-w + \sqrt{w^2 + e^{2H_0(r+G_1)}}\right)\right) \right] + G_2$$

$$\mp \frac{1}{2H_0} \left[2\left(\sqrt{\frac{w^2}{H_0}}e^{-2H_0t} + F_1\right) - \sqrt{F_1} \tanh^{-1}\left(\frac{\sqrt{\frac{w^2}{H_0}}e^{-2H_0t} + F_1}{\sqrt{F_1}}\right)\right) \right] + F_2. \quad (23)$$

where G_1 , G_2 , F_1 and F_2 are the arbitrary integration constants. It's important to note that $\phi^+(r,t)$ corresponds to the positive root solution, while $\phi^-(r,t)$ corresponds to the negative root solution of Eq. (18) considering $w^2 \neq 0$.

Furthermore, we may evaluate specific solutions of Eq. (18) by setting $F_1 = 0$ and $\phi'_1(r) = \frac{w^2}{H_0}$, then the solutions can be written as

$$\phi^{\pm}(r,t) = \pm \sqrt{\frac{w^2}{H_0}} r + G_3 \mp \frac{1}{H_0} \sqrt{\frac{w^2}{H_0}} e^{-H_0 t} + F_2$$
 (24)

where G_3 is an integration constant. Both solutions (23) and (24) are inhomogeneous. Equation (11) and the scalar expansion (12) reduce to the standard Raychaudhuri equation for the aforementioned inhomogeneous solutions of $v^{\mu}\partial_{\mu}X = 0$. It should be mentioned that the constant solution of the field Eq. (19) and particular solution Eq. (24) are not viable in our instance, since these solutions give X (13) to be zero, which is not acceptable in our interacting model.

Matching to inflaton fields: Now let's examine the behaviors of the solutions of the fields concerning the existing inflationary fields, using a standard technique described in Mukhanov's well-known book [43].

Case-I: Homogeneous field: Following Mukhanov [43], the inflationary phase of the cosmos can be characterized by a scalar field known as "inflaton" in conventional theory. According to the author, the ultra-hard equation of state is characterized by the condition that the kinetic component of the scalar field $(\dot{\varphi}^2)$ exceeds the potential $(V(\varphi))$ of the field at a homogeneous cosmology. In this scenario, the equation of state can be expressed as $p \approx +\rho$, where ρ represents the standard energy density of the universe. In the ultra-hard

stage, the expression for the field can be expressed by

$$\varphi = const - \frac{1}{\sqrt{12\pi}} \ln t. \tag{25}$$

The aforementioned solution corresponds to $H^2 \simeq \frac{1}{9t^2}$, which directly follows $a \propto t^{1/3}$ and $\rho \propto a^{-6}$. This solution is exact for a massless scalar field. At this stage, the field's trajectory rapidly ascends and eventually reaches the attractor. This greatly expands the range of initial conditions that result in an inflationary phase. Once the trajectory intersects the attractor at a point when it is flat, denoted as $|\phi| >> 1$, the subsequent solution elucidates a phase of accelerated expansion. Then the corresponding scalar field can be written as

$$\varphi(t) \simeq \varphi_i - \frac{m}{\sqrt{12\pi}} (t - t_i) \simeq \frac{m}{\sqrt{12\pi}} (t_f - t)$$
 (26)

where t_i is the time when the expansion starts and t_f is the time when the scalar field dies out. It should be noted that the solution obtained in (20) in our particular setting exhibits a comparable linearly time-dependent solution of φ . Hence, it may be posited that the K-essence scalar field, which maintains the invariance of the Raychaudhuri equation, is similar to an "inflaton" field.

Case-II: Inhomogeneous field: Arguments for the inflationary scenario are well given in Mukhanov [43]. We just state that the perturbations that occur during inflation are the source of the inhomogeneities; for this reason, these perturbative fields (Φ) and $(\delta\varphi)$ are referred to as inflaton fields. In Ref. [43], they have used the same type of action (1), the gauge-invariant metric perturbation (Φ) and scalar field perturbation $(\delta\varphi)$ has been expressed as

$$\Phi = 4\pi C \dot{\varphi}_0 \sqrt{\frac{p_X}{c_s}} \exp\left(\pm ik \int \frac{c_s}{a} dt\right)$$
 (27)

$$\delta \varphi \cong C\sqrt{\frac{1}{c_s p_X}} (\pm i c_s \frac{k}{a} + H + ...) \ exp(\pm i k \int \frac{c_s}{a}) dt \ (28)$$

which oscillate in the short-wavelength regime. Here, C is an integration constant, $p(X) \equiv \mathcal{L}(X)$, $p_X \equiv \frac{\partial p}{\partial X} = \mathcal{L}_X$ and $\dot{\varphi}_0$ is the square root of the kinetic part of the homogeneous scalar field.

Now we consider DBI type Lagrangian as $p(X) = \mathcal{L}(X) = 1 - V\sqrt{1 - 2X}$ and $B(X) = 4\pi C\dot{\varphi}_0\sqrt{\frac{p_X}{c_s}}$. Then

$$B(X) \cong D\left(1 + X + \frac{3}{2}X^2\right) \tag{29}$$

where $D=4\pi C\sqrt{V}\dot{\varphi}_0$. Again, considering the positive solutions of the above Eq.s (27) and (28) and $A(t)=k\int \frac{c_s}{a}dt$ we can write exponential parts as

$$\exp(iA(t)) = 1 + iA(t) - \frac{A^2(t)}{2!} + \frac{(iA(t))^3}{3!} - \dots$$
 (30)

Therefore, for real solutions, we can recast the above inhomogeneous inflaton fields (27) and (28) as

$$\Phi \simeq D\left(1 + X + \frac{3}{2}X^2 + \frac{5}{2}X^3\right)\left(1 - \frac{A^2(t)}{2!} - \frac{A^4(t)}{4!}\right)$$
(31)

$$\delta\varphi \simeq C\sqrt{V} \left[-\frac{k}{a} \left(A(t) - XA(t) - \frac{X^2}{2} A(t) + \frac{A^3(t)}{3!} - X\frac{A^3(t)}{3!} - \frac{X^2 A^3(t)}{3!} \right) + H \left(1 - \frac{A^2(t)}{2!} - \frac{A^4(t)}{4!} \right) \right] (32)$$

Our general solution, (compare with (23)), should now be taken into consideration. The integration constants can be considered as zero for simplicity i.e., $G_1 = G_2 = F_1 = F_2 = 0$. We also assume $w \ll e^{2H_0r}$. Then, from (23) the positive solution becomes,

$$\phi(r,t) = -\frac{1}{2H_0^{3/2}} \Big[2e^{H_0r} + w \ln H_0 + 2we^{-H_0t} \Big].$$
 (33)

Using (33) and (13), we express the radial part (e^{H_0r}) of (33) in terms of X, we can recast the Eq. (33) as

$$\phi = \mathcal{D}\left[1 + \mathcal{A}(t)X - \mathcal{A}^2(t)X^2 + \dots + \frac{\ln H_0}{2} + \frac{1}{a^2}\right]$$
 (34)

where $\mathcal{D} = -\frac{w}{H_0^{3/2}}$ and $\mathcal{A}(t) = \frac{H_0 a^2}{w^2}$. Thus far, we have expressed our general solution of field (23) in terms of Xand A(t), while ensuring that the constraint $v^{\mu}\partial_{\mu}X=0$ is maintained to ensure the invariance of the Raychaudhuri equation (11) without considering the curvature effect, namely the Ricci tensor of gravity. The solution of the field ϕ (34) carries some resemblance to the inhomogeneous inflation fields (31) and (32) under certain conditions for the DBI type action mentioned earlier. It should be noted that the matching of our solution (34) with the inflaton fields Eq.s (31) and (32) requires a discussion of the conditions. However, we cannot explore this topic here, as the inhomogeneous solutions of the fields have different origins and depend on the specific model Lagrangian $\mathcal{L}(X)$ or p(X)that we have employed. Hence, the inhomogeneous fields we have described in Eq.s (23) or (34) can be referred to as inflaton fields, even though they do not originate from any cosmological analysis in the early universe. Therefore, the Raychaudhuri equation can be employed to elucidate the origins of the primordial inhomogeneities via fields associated with inflation.

Concluding remarks: In the presence of both homogeneous and inhomogeneous scalar fields, the modified Raychaudhuri equation (11) returns to its original form. The Raychaudhuri equation is invariant in this sense. This may be conceivable since the Raychaudhuri equation is based on geometry and not on any specific theory of gravitation. Based on our findings, the universe is likely inhomogeneous. Remember that the Raychaudhuri equation has a geometrical identity and should be dynamical if we incorporate the Ricci tensor component from any gravity theory. Thus, the modified Raychaudhuri equation in its standard form is invariant for all gravitational theories, regardless of scalar field homogeneity or inhomogeneity. The Guardian [44] and recent observations [45–47] indicate that the cosmos is inhomogeneous. This observational evidence is in sync with our results obtained from the equation regarding inhomogeneous scalar fields. It is vital to highlight that this research does not analyze the universe's cosmic dynamics. Such investigations of cosmic dynamics for homogeneous scalar fields in the light of the Raychaudhuri equation has been done in [12].

On the other hand, we consider the scale factor (17) for the solution of Eq. (16), which may be used for the late-time acceleration era or inflationary era of the universe [40–43]. The inflationary paradigm predicts large-scale spatial homogeneity and isotropy in the cosmos. This suggests that the cosmos is the same everywhere. Other scenarios employing an inhomogeneous scalar field driving inflation with spatially different properties are conceivable. Inhomogeneities in the early universe may have been caused by quantum fluctuations and other mechanisms. These inhomogeneities may affect inflation predictions and density fluctuations. Inhomogeneous inflaton fields are caused by quantum perturbations, which is the main source of primordial inhomogeneities [43]. According to Mukhanov [43], we can say that our case's homoge-

neous and inhomogeneous fields also resemble inflaton fields under specific circumstances. Thus, the Raychaudhuri equation may be utilized to study primordial inhomogeneities utilizing inflation-related factors.

As a final comment, in this work, we solely focused on the invariance property of the Raychaudhuri equation in the presence of scalar fields, which can be called inflaton fields. We did not consider the cosmological behaviors and other effects of the universe.

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