MOMENTS OF THE FREE JACOBI PROCESS: A MATRIX APPROACH

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Abstract. We compute the large size limit of the moment formula derived in [14] for the Hermitian Jacobi process at fixed time. In particular, we identify the terms contributing to the limit and show they satisfy a double recurrence relation. We also determine explicitly some of them and revisit a special case relying on Carlitz summation identity for terminating 1-balanced ${}_4F_3$ functions taken at unity.

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1. Introduction and reminder

The Hermitian Jacobi process is a stationary matrix-valued process $(J_t)_{t\geq 0}$ whose marginal distribution converges weakly as $t \to +\infty$ to the distribution of the Jacobi unitary ensemble ([15]). It is defined as the radial part of the $m \times p$ upper-left corner of a $d \times d$ Brownian motion in the unitary group (see [7] and references therein for further details). The dynamics of its eigenvalues process as well as their β-extension were studied in [9], while their low and high temperature behaviours were determined in [26] and [25] respectively.

At the free probability theoretical side, there is still a considerable lack of explicit representations of the limiting moment sequence (the time normalisation below is only needed to obtain a non degenerate limit):

$$\lim_{m\to+\infty} \frac{1}{m}\mathbb{E}\left(\mathrm{tr}[(J_{t/d})^n]\right), \quad n \geq 1, \tag{1}$$

where $p = p(m) \to +\infty, d = d(m) \to +\infty$ are such that:

$$\lambda := \lim_{m\to+\infty} \frac{m}{p(m)} \in]0,1], \quad \theta := \lim_{m\to+\infty} \frac{p(m)}{d(m)} \in]0,1[.$$

For any fixed time $t > 0$, the moment sequence (1) determines the spectral distribution $\mu_t^{(\lambda,\theta)}$ of the large size limit of the Hermitian Jacobi process, the so-called free Jacobi process, in a non commutative probability

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space ([8]). So far and to the best of our knowledge, explicit expressions of these moments exist only for $\lambda = 1, \theta = 1/2$, and were obtained after solving a partial differential equation satisfied by the moment generating function of $\mu_t^{(1,1/2)}$ ([13]). More generally, the Lebesgue decomposition of $\mu_t^{(\lambda,\theta)}$ was determined in [18], yet the absolutely continuous part is encoded by the inverse (in composition's sense) of the solution of a radial Loewner equation (more precisely by the continuous extension of this inverse to the unit circle [19], see also [10, 11]).

Motivated by quantum information theory and especially by dynamical random quantum states, the moments of $\mu_t^{(1,1/k)}, k \geq 2$, has been further analysed in [12] relying on enumeration techniques. Though this analysis reveals a quite striking combinatorial structure illustrated by the Catalan triangle ([24])

$$\binom{2n}{n-h}, \quad 0 \leq h \leq n,$$

independently of k, it involves the moments of a non normal bounded operator which do not seem to have any simple form unless $k = 2$. Note in passing that unless $h = 0$, the presence of the above binomial coefficients in relation with the moments of the free Jacobi process remains till now unexplained. As to the binomial coefficient

$$\binom{2n}{n},$$

it arises in the moment sequence of the stationary (arcsine) distribution $\mu_\infty^{(1,1/2)}$ and counts the number of non crossing partitions of type B ([23]).

In order to get more insight into both the combinatorial and analytical structures of the moment sequence of $\mu_t^{(\lambda,\theta)}$, in particular into the occurrence of the above binomial coefficients, we pursued a matrix approach in [14] where we proved a formula for the moments $\mathbb{E}\left(\mathrm{tr}[(J_t)^n]\right), n \geq 1,$. Actually, the latter, recalled below, is given by a sign alternating double sum over hooks of fixed weight n and those contained in. Besides, we may express this double sum through terminating and 1-balanced ${}_4F_3$ hypergeometric functions taken at unity ([2]). However, computing the large size limit is still challenging due to the presence of alternating signs and to the complicated form of the summands as well.

At this stage, let us provide more details on the moment formula derived in [14]. To this end, recall from [14] the parameters

$$r := p - m, \quad q := d - p, \quad s := q - m,$$

and denote

$$M_n^{(r,s,m)}(t) := \mathbb{E}\left(\mathrm{tr}[(J_{t/d})^n]\right),$$

the n-th moment of $J_{t/d}$. Given a hook $\alpha = (n-k, 1^k)$, denote further $|\alpha| = n$ its weight and $l(\alpha) = k+1$ its length. Then, Theorem 1 in [14] reads:

$$M_n^{(r,s,m)}(t) = M_n^{(r,s,m)}(\infty) + \sum_{\substack{\alpha \text{ hook} \\ |\alpha|=n, l(\alpha)\leq m}} (-1)^{n-\alpha_1} \sum_{\substack{\tau\subseteq\alpha \\ \tau\neq\emptyset}} \frac{e^{-\nu_\tau(r,s,m)t/d}\, V_{\alpha_1,\tau_1}^{r,s,m}\, U_{l(\alpha),l(\tau)}^{r,s,m}}{(r+s+\tau_1+2m-l(\tau))(\tau_1+l(\tau)-1)}. \tag{2}$$

In (2), the notation $\tau \subseteq \alpha$ stands for the containment of the corresponding Young diagrams (so τ is a hook as well),

$$\nu_\tau(r,s,m) := \sum_{i=1}^{m} \tau_i(\tau_i + r + s + 1 + 2(m-i)), \tag{3}$$

$$V_{\alpha_1,\tau_1}^{(r,s,m)} := \frac{(d+2\tau_1-1)\Gamma(d+\tau_1)\Gamma(\alpha_1+m)\Gamma(p+\alpha_1)\Gamma(q+\tau_1)}{(\alpha_1-\tau_1)!\Gamma(d+\alpha_1+\tau_1)\Gamma(p+\tau_1)\Gamma(\tau_1)}, \tag{4}$$

$$U_{l(\alpha),l(\tau)}^{(r,s,m)} := \frac{(d+1-2l(\tau))\Gamma(d-l(\alpha)-l(\tau)+1)\Gamma(p-l(\tau)+1)}{(l(\alpha)-l(\tau))!(l(\tau)-1)!\Gamma(m-l(\alpha)+1)\Gamma(p-l(\alpha)+1)\Gamma(q-l(\tau)+1)\Gamma(d-l(\tau)+1)}, \tag{5}$$

and $M_n^{(r,s,m)}(\infty)$ is the n-th moment of the Jacobi unitary ensemble. We refer the reader to [20] for an expression of the latter and for its asymptotic analysis based on terminating ${}_4F_3$ hypergeometric functions taken at unity.

Since any hook $\tau \subseteq \alpha, \tau \neq \emptyset$ is a partition of the form

$$\tau := (h-j, 1^j), \quad 0 \leq j \leq h-1, \quad 1 \leq h \leq n,$$

then the double sum displayed in the RHS of (2) may be written as ([14], section 4.2):

$$(6) \quad \sum_{h=1}^{n}\sum_{j=0}^{h-1} \frac{(-1)^{n-h+j}e^{-\nu_{h,j(d)}t/d}}{h(n-h)!(h-j-1)!j!} \frac{(2m+2h-2j-1)(d-2j-1)\Gamma(p-j)\Gamma(q+h-j)\Gamma(d+h-j)}{(d+h-2j-1)\Gamma(q-j)\Gamma(d-j)}$$

$$\frac{\Gamma(h-j+m)\Gamma(d-n+h-2j-1)}{\Gamma(m+h-n-j)\Gamma(p+h-n-j)\Gamma(d+2h-2j)}{}_4F_3\left(\begin{matrix}-(n-h), m+h-j, p+h-j, d-n+h-2j-1; \\ m-n+h-j, p-n+h-j, d+2h-2j;\end{matrix}, 1\right),$$

where

$$(7) \qquad \nu_{h,j}(d) := \nu_{(h-j,1^j)}(r,s,m) = dh + (h-j)(h-j-1) - j(j+1) = dh + h(h-2j-1).$$

Here, ${}_4F_3$ is a terminating and 1-balanced (i.e. the sum of the bottom row parameters equals one plus the sum of the top row parameters) hypergeometric functions taken at unity. Various parameter transformations and reduction formulas exist for these functions ([2]). For instance, generalized hypergeometric series whose row and bottom parameters differ by integers admits simpler series representations (see [21] and references therein). Recently, a reduction formula proved in [6] (Theorem 2.1) reduces a 1-balanced terminating ${}_4F_3(1)$ to a terminating (non necessarily balanced) ${}_3F_2(1)$. In particular, it includes the celebrated Carlitz Summation formula for a special choice of the parameter set ([1]). As we shall see below, the latter allows to simplify considerably (6) when choosing $p(m) = m + (1/2), d(m) = 2m$, which in turn opens the way to retrieve the moment sequence of $\mu_t^{(1,1/2)}$ derived in [13]. Note that though this choice does not correspond to the Hermitian Jacobi process with integer parameters, the eigenvalue process still exists as the unique strong solution of a vector-valued stochastic differential equation with a singular drift ([17], Corollary 6.7.).

More generally, following another route inspired by the seminal paper [3], we shall prove the following theorem which is the main result of our paper:

Theorem 1. *Write $m = (\lambda\theta)d, p = \theta d$. Then for any $n \geq 1$, there exist real numbers $\left(c_{n,h,l}^{(\lambda,\theta)}\right)_{0\leq l\leq h-1\leq n-1}$ such that*

$$\lim_{d\to+\infty}\frac{1}{m}\sum_{\substack{\alpha \text{ hook} \\ |\alpha|=n, l(\alpha)\leq m}}\sum_{\substack{\tau\subseteq\alpha \\ \tau\neq\emptyset}} \frac{(-1)^{n-\alpha_1}e^{-\nu_\tau(r,s,m)t/d}\, V_{\alpha_1,\tau_1}^{r,s,m}\, U_{l(\alpha),l(\tau)}^{r,s,m}}{(r+s+\tau_1+2m-l(\tau))(\tau_1+l(\tau)-1)} = \sum_{h=1}^{n}(-1)^{h-1}\frac{e^{-ht}}{h}\sum_{l=0}^{h-1}\frac{(2ht)^l}{l!}c_{n,h,l}^{(\lambda,\theta)}.$$

Since

$$\lim_{d\to\infty}\frac{1}{m}M_n^{(r,s,m)}(\infty), \quad n \geq 1,$$

are the moments of the stationary distribution of the free Jacobi process (see [8], section 5), then Theorem 1 provides an expression for the moments of the free Jacobi process at any time $t > 0$. As to the description of $c_{n,h,l}^{(\lambda,\theta)}$, it is easily read off the end of the proof of Theorem 1. Loosely speaking, this proof uses a polynomial long division with respect to the variable d and only the constant term of the quotient polynomial contributes to the large d-limit. Besides, this term is shown to be polynomial in two variables, denoted (j,k) in the proof of Theorem 1, and $c_{n,h,l}^{(\lambda,\theta)}$ is the coefficient of $j^{h-l-1}k^{n-h}$ there.

Using the generating series of elementary symmetric and completely homogeneous polynomials, we shall represent the constant term of the quotient polynomial alluded to above as a doubly symmetric function in two alphabets. Actually, the latter are the roots of the dividend and of the divisor in the polynomial long division. However, though this representation is interesting in its own, we are not able to use it to obtain a relatively simple expression of $c_{n,h,l}^{(\lambda,\theta)}$. That is why we shall write down a finite and doubly-indexed recurrence equation whose last term is $c_{n,h,l}^{(\lambda,\theta)}$ which leads to explicit expressions of both $c_{n,n,l}^{(\lambda,\theta)}$ and $c_{n,h,h-1}^{(\lambda,\theta)}$. In particular, we get

$$c_{n,h,l}^{(1,1/2)} = \frac{(-1)^{h-l-1}}{2^{2n-1}}\binom{h}{l+1}\binom{2n}{n-h},$$

and recover the moments of $\mu_t^{(1,1/2)}$ already derived in [13].

The paper is organized as follows. In the next section, we use Carlitz summation formula to give another derivation of the moments of $\mu_t^{(1,1/2)}$. Section three is entirely devoted to the proof of Theorem 1. In the fourth section, we prove the representation of the constant term of the quotient polynomial through elementary symmetric and completely homogeneous polynomials. In the last section, we derive the expressions of $c_{n,n,l}^{(\lambda,\theta)}$ and $c_{n,h,h-1}^{(\lambda,\theta)}$ as well as the recurrence equation allowing to compute $c_{n,h,l}^{(\lambda,\theta)}$ for arbitrary parameters (λ, θ).

2. Carlitz summation formula

In [13], we proved the following result :

Theorem 2 ([13], Corollary 1)**.** *The moments of $\mu_t^{(1,1/2)}$ are given by:*

$$\int_0^1 x^n \mu_t^{(1,1/2)}(dx) = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{h=1}^{n}\binom{2n}{n-h}\frac{e^{-ht}}{h}L_{h-1}^{(1)}(2ht), \quad n \geq 1, \tag{8}$$

where $L_{h-1}^{(1)}$ is the $(h-1)$-th Laguerre polynomial of index one.

In this section, we supply another derivation of (8) starting from (6) and appealing to Carlitz summation formula ([1]). Of course, we shall restrict ourselves to the time-dependent part of the RHS of (8) since the stationary one is known to be the n-th moment of the arcsine distribution

$$\mu_\infty^{(1,1/2)} := \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x).$$

Another proof of (8)*.* Substitute $p(m) = m + (1/2), d = 2m$, in (6) and recall the Pochhammer symbol:

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},\ a > 0, \quad (-n)_k = (-1)^k\frac{n!}{(n-k)!},\ n \in \mathbb{N}.$$

Then Carlitz Summation formula entails (see e.g. [1], eq. (1)):

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix}-(n-h), m+h-j, m+h-j+1/2, 2m-n+h-2j-1;\\ m-n+h-j, m-n+h-j+1/2, 2m+2h-2j;\end{matrix}, 1\right) = \\
&\frac{(m-n+h-j-1/2)}{m-j-1/2}\frac{(-2n)_{n-h}}{(2m-2n+2h-2j-1)_{n-h}} = (-1)^{n-h}\frac{(m-n+h-j-1/2)}{m-j-1/2}\frac{(2n)!}{(n+h)!} \\
&\qquad\frac{\Gamma(2m-2n+2h-2j-1)}{\Gamma(2m-n+h-2j-1)}.
\end{aligned} \tag{9}$$

Taking into account the time change $t \mapsto t/d = t/(2m)$ in (1) and keeping in mind (6) and (7), we need to compute the limit as $m \to +\infty$ of:

$$\begin{aligned}
&\sum_{h=1}^{n}\frac{e^{-ht}}{h}\binom{2n}{n-h}\sum_{j=0}^{h-1}\frac{(-1)^j e^{-h(h-2j-1)t/(2m)}}{(h-j-1)!j!}\frac{(2m+2h-2j-1)(m-n+h-j-1/2)}{m(2m+h-2j-1)} \\
&\frac{\Gamma(m-j+1/2)\Gamma(m+h-j-1/2)\Gamma(m+h-j)\Gamma(2m-2n+2h-2j-1)\Gamma(2m+h-j)}{(m-j-1/2)\Gamma(m-j-1/2)\Gamma(2m+2h-2j)\Gamma(m+h-n-j+1/2)\Gamma(m+h-n-j)\Gamma(2m-j)}.
\end{aligned}$$

To this end, we use Legendre duplication formula to get:

$$\begin{aligned}
\frac{\Gamma(m+h-j-1/2)\Gamma(m+h-j)}{\Gamma(2m+2h-2j)} &= \frac{\sqrt{\pi}}{2^{2m+2h-2j-1}(m+h-j-1/2)}, \\
\frac{\Gamma(2m-2n+2h-2j-1)}{\Gamma(m+h-n-j+1/2)\Gamma(m+h-n-j)} &= \frac{2^{2m+2h-2j-2n-2}}{\sqrt{\pi}(m-n+h-j-1/2)}.
\end{aligned}$$

Consequently, we are led after some simplifications to compute:

$$\lim_{m\to+\infty}\sum_{h=1}^{n}\frac{e^{-h(h-1)t/(2m)}}{2^{2n}}\frac{e^{-ht}}{h}\binom{2n}{n-h}\sum_{j=0}^{h-1}\frac{(-1)^j e^{jht/m}}{(h-j-1)!j!}\frac{(2m-2j-1)}{m(2m+h-2j-1)}\frac{\Gamma(2m+h-j)}{\Gamma(2m-j)}=$$

$$\lim_{m\to+\infty}\sum_{h=1}^{n}\frac{e^{-h(h-1)t/(2m)}}{2^{2n}}e^{-ht}\binom{2n}{n-h}\sum_{j=0}^{h-1}e^{(2ht)j/(2m)}\binom{h-1}{j}\frac{(-1)^j(2m-2j-1)}{m(2m+h-2j-1)}\frac{(2m+h-j-1)!}{h!(2m-j-1)!}.$$

Up to the factor $(2m-2j-1)/(2m+h-2j-1)$ and to the time change $t\mapsto 2t$, the inner sum

$$\frac{1}{2m}\sum_{j=0}^{h-1}(-1)^j e^{(ht)j/(2m)}\binom{h-1}{j}\frac{(2m+h-j-1)!}{h!(2m-j-1)!},$$

was considered by Biane in [3] (see the proof of Lemma 3 there) who proved the convergence to

$$\frac{1}{h}L_{h-1}^{(1)}(ht).$$

Writing

$$\frac{2m-2j-1}{2m+h-2j-1}=1-\frac{h}{2m+h-2j-1}$$

we get

$$\lim_{m\to\infty}\frac{e^{-h(h-1)t/(2m)}}{2^{2n}}\sum_{h=1}^{n}e^{-ht}\binom{2n}{n-h}\sum_{j=0}^{h-1}(-1)^j e^{(2ht)j/(2m)}\binom{h-1}{j}\frac{2(2m-2j-1)}{(2m)(2m+h-2j-1)}\frac{(2m+h-j-1)!}{h!(2m-j-1)!}$$
$$=\frac{2}{2^{2n}}\sum_{h=1}^{n}e^{-ht}\binom{2n}{n-h}\frac{1}{h}L_{h-1}^{(1)}(2ht),$$

as desired. $\square$

A by-product of the previous proof is given in the following corollary:

Corollary 1. *For any $n\geq 1$, the n-th moment of J_t corresponding to the parameter set*

$$p=m+1/2,\quad d=2m\quad\Leftrightarrow\quad r=1/2,\quad s=-1/2,$$

reads:

$$M_n^{(1/2,-1/2,m)}(t)=\frac{1}{n!}\sum_{h=0}^{n-1}(-1)^h\binom{n-1}{h}\frac{(m-h)_n(3/2+m-h-1)_n}{(2m-i)_n}+$$
$$\sum_{h=1}^{n}\frac{e^{-h(h-1)t/(2m)}}{2^{2n}}e^{-ht}\binom{2n}{n-h}\sum_{j=0}^{h-1}e^{(2ht)j/(2m)}\binom{h-1}{j}\frac{(-1)^j(2m-2j-1)}{2m+h-2j-1}\frac{(2m+h-j-1)!}{h!(2m-j-1)!}.$$

Proof. The stationary distribution of the eigenvalues of the Hermitian Jacobi process is given by the Selberg weight:

$$\prod_{i=1}^{m}\lambda_i^r(1-\lambda_i)^s\prod_{1\leq i<v\leq m}(\lambda_i-\lambda_v)^2\mathbf{1}_{\{0<\lambda_m<\cdots<\lambda<1\}},$$

up to a normalizing constant. Its n-th moment was computed in [4], Corollary II. 3., and our sought expression follows after substituting there $a=3/2, b=1/2, N=m, k=n$. $\square$

Remark. *In* [6] *(Theorem 2.1), the following reduction formula was proved for terminating* 1*-balanced* ${}_4F_3$ *at unity:*

$${}_4F_3\left(\begin{matrix}-N, a-c+N, (c/2), (c+1)/2;\\ 1+a-e, (e/2), (e+1)/2;\end{matrix}, 1\right)=\frac{(1+a-c-e)_N(e-c)_N}{(1+a-e)_N(e)_N}{}_3F_2\left(\begin{matrix}-N, a-c+N, c;\\ c+e-a-N, e+N;\end{matrix}, 1\right).$$

Specializing it with

$$N = n - h, \quad p = p(m) = m + \frac{1}{2}, \quad c = 2(m+h-j), \quad e = 2(m+h-j-n),$$

we get

$$ {}_4F_3\left(\begin{matrix}-(n-h), m+h-j, m+h-j+1/2, d-n+h-2j-1; \\ m-n+h-j, m-n+h-j+1/2, d+2h-2j;\end{matrix}, 1\right) = \frac{(d-2m)_{n-h}(-2n)_{n-h}}{(d+2h-2j)_{n-h}(2m+2h-2j-2n)_{n-h}} \\ {}_3F_2\left(\begin{matrix}-(n-h), d-n+h-2j-1, 2m+2h-2j; \\ 2m-d+h+1-n, 2m-2j+h-n;\end{matrix}, 1\right).$$

The RHS of the last equation may be further written as ([6], Corollary 2.2):

$$\frac{(-2n)_{n-h}}{(2m+2h-2j-2n)_{n-h}} {}_3F_2\left(\begin{matrix}-(n-h), 2m-d+1, 2m+2h-2j; \\ 2m-2j+h-n, d+2h-2j;\end{matrix}, 1\right),$$

which reduces when $d = 2m$ to Cariltz summation formula (9):

$$\frac{(-2n)_{n-h}}{(2m+2h-2j-2n)_{n-h}} {}_2F_1\left(\begin{matrix}-(n-h), 1, \\ 2m-2j+h-n;\end{matrix}, 1\right) = \frac{(m-n+h-j-1/2)(-2n)_{n-h}}{(m-j-1/2)(2m-2n+2h-2j-1)_{n-h}},$$

by the virtue of the Gauss hypergeometric Theorem:

$${}_2F_1(-N, b; c, 1) = \frac{(c-b)_N}{(c)_N}.$$

3. The general case: proof of Theorem 1

We now proceed with the general case and prove Theorem 1. The proof is long and for ease of reading, we divide it into four steps:

(1) Express the summands of the second term in the RHS of (2) as a rational function in the variable d.
(2) Inspired by the proof of Lemma 2 in [3], we expand the exponential factor $e^{(2thj)/d}$ into powers of $1/d$, truncate the obtained series and perform there a polynomial long division.
(3) We obtain cancellations and determine exactly the terms having non zero contributions in the large d-limit. The expression displayed in Theorem 1 follows after straightforward computations.

3.1. **First step.** For any $n \geq 1$, denote

$$\tilde{M}_n^{(r,s,m)}(t) := \frac{1}{m} \sum_{\substack{\alpha \text{ hook} \\ |\alpha|=n, l(\alpha)\leq m}} (-1)^{n-\alpha_1} \sum_{\substack{\tau\subseteq\alpha \\ \tau\neq\emptyset}} \frac{e^{-\nu_\tau(r,s,m)t/d}\, V_{\alpha_1,\tau_1}^{(r,s,m)}\, U_{l(\alpha),l(\tau)}^{(r,s,m)}}{(r+s+\tau_1+2m-l(\tau))(\tau_1+l(\tau)-1)},$$

where we recall that $\nu_\tau(r,s,m)$, $V_{\alpha_1,\tau_1}^{(r,s,m)}$ and $U_{l(\alpha),l(\tau)}^{(r,s,m)}$ were defined in (3), (4) and (5) respectively. Recall also the parametrisation of the hooks $\tau \subseteq \alpha$:

$$\alpha_1 = n-k,\ \tau_1 = h-j, \quad l(\alpha) = k+1,\ l(\tau) = j+1.$$

Consequently, we may rewrite $\tilde{M}_n^{(r,s,m)}(t)$ as

$$\tilde{M}_n^{(r,s,m)}(t) = \frac{1}{m}\sum_{h=1}^{n}\sum_{j=0}^{h-1}\sum_{k=j}^{j+n-h}(-1)^k F(m,p,d,n,h,j,k)e^{-ht-\frac{ht}{d}(h-(2j+1))}$$

where

$$F(m,p,d,n,h,j,k) = \frac{\Gamma(d+h-j)\Gamma(n-k+m)\Gamma(d-j-k-1)\Gamma(p+n-k)\Gamma(q+h-j)}{(n-k-h+j)!(k-j)!j!\Gamma(d+n-k+h-j)\Gamma(p+h-j)\Gamma(h-j)} \\ \frac{(d+2h-2j-1)(d-2j-1)\Gamma(p-j)}{h(d+h-2j-1)\Gamma(m-k)\Gamma(p-k)\Gamma(q-j)\Gamma(d-j)}.$$

Performing the index changes $k \mapsto n-h+j-k$, we further obtain

$$(10)\quad \tilde{M}_n^{(r,s,m)}(t) = \sum_{h=1}^{n} \frac{e^{-ht-th(h-1)/d}}{n!}\binom{n}{h}\sum_{k=0}^{n-h}\sum_{j=0}^{h-1}(-1)^{n-h-k+j}\binom{n-h}{k}\binom{h-1}{j}\tilde{F}(m,p,d,n,h,j,k)e^{\frac{2thj}{d}}$$

where

$$\tilde{F}(m,p,d,n,h,j,k) = \frac{(d-2j-1)(d+2h-2j-1)(d-j)_h(d-p-j)_h(m+h-n-j+k)_n(p+h-n-j+k)_n}{m(d+h-2j-1)(p-j)_h(d-n+h-2j-1+k)_{n+h+1}}.$$

Write $m=(\lambda\theta)d, p=\theta d$, then

$$\begin{aligned}\tilde{F}(m,p,d,n,h,j,k) &= \tilde{F}((\lambda\theta)d,\theta d,d,n,h,j,k)\\ &= \frac{(d-2j-1)(d+2h-2j-1)(d-j)_h((1-\theta)d-j)_h((\lambda\theta)d+h-n-j+k)_n(\theta d+h-n-j+k)_n}{(\lambda\theta)d(d+h-2j-1)(\theta d-j)_h(d-n+h-2j-1+k)_{n+h+1}}.\end{aligned}$$

3.2. **Second step.** Since the Pochhammer symbol $(a)_k$ has exactly k factors, then a quick inspection shows that

$$(11)\qquad \tilde{F}((\lambda\theta)d,\theta d,d,n,h,j,k) \sim \frac{(1-\theta)^h(\lambda\theta^2)^n}{(\lambda\theta)\theta^h}d^{n-1}, \quad d=d(m)\to+\infty,$$

so that taking the limit as $m\to+\infty$ in (10) yields an indeterminate form unless $n=1$. Note that the situation was similar in [3] when dealing with the moments of the unitary Brownian motion. In that paper, the author succeeded to obtain cancellations making use of the following elementary, yet useful, lemma:

Lemma 1 ([3], Lemma 2)**.** *Let $0\le a\le b$ be integers. Then*

$$\sum_{j=0}^{b}(-1)^j j^b\binom{b}{j} = \begin{cases} 0 & \text{if} \quad a<b,\\ (-1)^a a! & \text{if} \quad a=b.\end{cases}$$

Mimicking the proof of Lemma 3 in [3], we expand the exponential term $e^{\frac{2thj}{d}}$ in (10):

$$\sum_{k=0}^{n-h}(-1)^{n-h-k}\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\binom{n-h}{k}\tilde{F}((\lambda\theta)d,\theta d,d,n,h,j,k)\left(\sum_{l=0}^{n-1}\frac{(2thj)^l}{l!d^l}+O(d^{-n})\right).$$

By the virtue of (11), we only need to analyse on the following triple sum:

$$\sum_{l=0}^{n-1}\frac{(-2th)^l}{l!}\sum_{k=0}^{n-h}\sum_{j=0}^{h-1}(-1)^{n-h-k+j}\binom{h-1}{j}j^l\binom{n-h}{k}\frac{\tilde{F}((\lambda\theta)d,\theta d,d,n,h,j,k)}{d^l}.$$

More precisely, we shall use Lemma 1 to obtain cancellations in the (j,k)-double sum. To this end, let[1]

$$P_{n,h}(j,k,d) := (d-2j-1)(d+2h-2j-1)(d-j)_h((1-\theta)d-j)_h((\lambda\theta)d+h-n-j+k)_n(\theta d+h-n-j+k)_n$$

denote the numerator of $\tilde{F}((\lambda\theta)d,\theta d,d,n,h,j,k)/d^l$, and for fixed $0\le l\le n-1$, let

$$D_{n,h,l}(j,k,d) := (\lambda\theta)d^{l+1}(d+h-2j-1)(\theta d-j)_h(d-n+h-2j-1+k)_{n+h+1},$$

be its denominator. Performing polynomial long division with respect to the variable d for fixed (n,h,l), there exists a polynomial (quotient) $Q_{n,h,l}(j,k,d)$ of degree $n-l-1$ such that

$$P_{n,h}(j,k,d) = D_{n,h,l}(j,k,d)Q_{n,h,l}(j,k,d) + \text{Remainder}.$$

Consequently, the asymptotic behaviour of $\tilde{M}_n^{(r,s,m)}(t)$ as $d\to+\infty$ is governed by:

$$(12)\qquad \sum_{l=0}^{n-1}\frac{(2th)^l}{l!}\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}j^l\sum_{k=0}^{n-h}(-1)^{n-h-k}\binom{n-h}{k}Q_{n,h,l}(j,k,d).$$

[1]For ease of notations, we shall omit the parameters (λ,θ).

3.3. **Third step.** Write

$$Q_{n,h,l}(j,k,d) := \sum_{i=0}^{n-l-1} q_{n-l-1-i}^{(n,h)}(j,k)d^i, \tag{13}$$

and note that its coefficients $(q_i^{(n,h)}(j,k))_{i=0}^{n-l-1}$ do not depend on the index l since the latter only appears in the factor d^l of $D_{n,h,l}(j,k,d)$. The following lemma describes the polynomial structure of these coefficients in the variables (j,k).

Lemma 2. *For any $0 \leq i \leq n-l-1$, the coefficient $q_i^{(n,h)}(j,k)$ is a polynomial of degree at most i in both variables j and k. Moreover, the coefficient of j^s there (resp. k^s) is a polynomial of degree at most $i-s$ in k (resp. j).*

Proof. Let $(a_i^{(n,h)})_{0\leq i\leq 2n+2h+2}$ and $(b_i^{(n,h)})_{0\leq i\leq n+2h+l+3}$ be respectively the coefficients of the polynomials $P_{n,h}(j,k,\cdot)$ and $D_{n,h,l}(j,k,\cdot)$ so that:

$$P_{n,h}(j,k,d) = \sum_{i=0}^{2n+2h+2} a_i^{(n,h)}(j,k)d^i \quad \text{and } D_{n,h,l}(j,k,d) = \sum_{i=l+1}^{n+2h+l+3} b_i^{(n,h)}(j,k)d^i.$$

Since the degree of the remainder of the polynomial long division is strictly less than B, then $(q_i^{(n,h)})_{i=0}^{n-l-1}$ satisfy the following relations:

$$q_i^{(n,h)}(j,k) = \tilde{a}_{2n+2h+2-i}^{(n,h)}(j,k) - \sum_{v=0}^{i-1} \tilde{b}_{n+2h+l+3-i+v}^{(n,h)}(j,k)q_v^{(n,h)}(j,k), \quad 0 \leq i \leq n-l-1 \tag{14}$$

where

$$\tilde{a}_{2n+2h+2-i}^{(n,h)}(j,k) = \frac{a_{2n+2h+2-i}^{(n,h)}(j,k)}{(\lambda\theta)\theta^h}, \quad \tilde{b}_{n+2h+l+3-i+v}^{(n,h)}(j,k) = \frac{b_{n+2h+l+3-i+v}^{(n,h)}(j,k)}{(\lambda\theta)\theta^h}.$$

Consequently, we readily infer from (14) by induction and from the product forms of $P_{n,h}(j,k,d)$ and $D_{n,h,l}(j,k,d)$ that $q_i^{(n,h)}(j,k)$ is a polynomial of degree at most i in both j and k. The last statements follows similarly from (14) by induction and a close (yet straightforward) inspection of the coefficients of j^s and of k^s in $q_i^{(n,h)}(j,k)$.

$\square$

Remark. *Later, we shall express the coefficients $(q_i^{(n,h)}(j,k))$ through elementary and completely homogeneous symmetric polynomials in the roots of $P_{n,h}(j,k,d)$ and of $D_{n,h}(j,k,d)$ respectively. These expressions may be used to write another proof of the previous lemma.*

With the help of Lemma 1 and Lemma 2, we are now able to obtain all the cancellations occurring in (12) and determine the terms having non zero contributions in the large d-limit:

Proposition 1. *All the terms in* (12) *vanish except those corresponding to the (k,j)-monomials $k^{n-h}j^{h-1-l}$ in the constant term coefficient $q_{n-l-1}^{(n,h)}(j,k)$ of the polynomial $Q_{n,h,l}(j,k,d)$.*

Proof. By the virtue of (12) and of (13), we need firstly to show that

$$\sum_{j=0}^{h-1}(-1)^j \binom{h-1}{j} j^l \sum_{k=0}^{n-h}(-1)^{n-h-k}\binom{n-h}{k} q_{n-l-1-i}^{(n,h)}(j,k)$$

is zero for any $1 \leq i \leq n-l-1$. According to Lemma 1 and since $q_{n-l-1-i}^{(n,h)}$ is a polynomial of degree at most $n-l-1-i$ in j and k by Lemma 2, we notice that the terms corresponding to $l > h-1$ in (12) give zero contribution. Indeed,

$$n-l-1-i = n-(l+1)-i < n-h-i < n-h$$

so that the inner sum over k is zero for all $1 \leq i \leq n-l-1$.

Now consider a j-monomial of degree s in $q_{n-l-1-i}^{(n,h)}$. If $s+l<h-1$ then the sum over j vanishes. Otherwise $s+l\geq h-1$ and the sum over k is zero for any $1\leq i\leq n-l-1$ since

$$n-l-1-i-s=n-1-i-(l+s)\leq n-1-i-(h-1)=n-h-i<n-h.$$

Secondly, let $i=0$ and recall from lemma 2 that the coefficient of j^s in $q_{n-l-1}^{(n,h)}$ is a polynomial in k of degree at most $n-l-1-s$. Reasoning as above, if $s+l<h-1$ then the sum over j vanishes while the sum over k does so if $s+l>h-1$ since

$$n-l-1-s=n-1-(l+s)<n-1-(h-1)=n-h.$$

Hence, we are only left with the monomials $k^v j^{h-1-l}, v\leq n-h$, in $q_{n-l-1}^{(n,h)}(j,k)$. But once more, the sum over k vanishes unless $v=n-h$, the proposition is proved. $\square$

The end of the proof of Theorem 1 goes as follows. The sum (12) reduces to

$$\sum_{l=0}^{h-1}\frac{(2th)^l}{l!}\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}j^l\sum_{k=0}^{n-h}(-1)^{n-h-k}\binom{n-h}{k}c_{n,h,l}^{(\lambda,\theta)}k^{n-h}j^{h-l-1}$$

where $c_{n,h,l}^{(\lambda,\theta)}$ is the coefficient of the monomial $k^{n-h}j^{h-l-1}$ in $q_{n-l-1}^{(n,h)}(j,k)$. Appealing again to Lemma 1, we may simplify further this triple sum to get

$$\sum_{l=0}^{h-1}\frac{(2th)^l}{l!}(-1)^{h-1}(h-1)!(n-h)!c_{n,h,l}^{(\lambda,\theta)},$$

so that

$$\begin{aligned}\lim_{d\to\infty}\tilde{M}_n^{(r,s,m)}(t)&=\sum_{h=1}^{n}\frac{e^{-ht}}{n!}\binom{n}{h}(-1)^{h-1}(n-h)!(h-1)!\sum_{l=0}^{h-1}\frac{(2th)^l}{l!}c_{n,h,l}^{(\lambda,\theta)}\\&=\sum_{h=1}^{n}(-1)^{h-1}\frac{e^{-ht}}{h}\sum_{l=0}^{h-1}\frac{(2ht)^l}{l!}c_{n,h,l}^{(\lambda,\theta)},\end{aligned}$$

as stated in Theorem 1.

4. Representations of the constant term coefficient $q_{n-l-1}(j,k)$

So far, we proved that the large size asymptotics of the moments of J_t are governed by the constant term coefficient $q_{n-l-1}(j,k)$ of the polynomial $Q_{n,h,l}(j,k,\cdot)$. Besides, this coefficient is the last term $i=n-l-1$ of the inductive relation (14). In this section, we derive another representation of $q_{n-l-1}(j,k)$ and of all the coefficients of $Q_{n,h,l}(j,k,\cdot)$ through elementary and complete homogeneous symmetric polynomials taken at the roots of $P_{n,h}(j,k,\cdot)$ and of $D_{n,h}(j,k,\cdot)$. Though we believe that this fact is very likely known, we were not able to find any reference to it and as such, we state and prove it below. Using a result of Goulden and Greene ([16]), the obtained representation may be further written as a complete symmetric polynomial in some set of variables (alphabet) formed out of the the roots of $P_{n,h}(j,k,\cdot)$ and of $D_{n,h}(j,k,\cdot)$.

For sake of simplicity, set

$$A:=2n+2h+2,\quad B:=n+2h+l+3,\quad C:=A-B=n-l-1,$$

for fixed n,h,l, and denote $(y_1,\dots,y_A)$ and $(z_1,\dots,z_B)$ the roots of the polynomials $P_{n,h}(j,k,\cdot)$ and $D_{n,h,l}(j,k,\cdot)$ respectively (which depend of course of (n,h,j,k)). Then,

Proposition 2. *The polynomial $Q_{n,h,l}(j,k,d)$ admits the following expansion:*

$$\begin{aligned}Q_{n,h,l}(j,k,d)&=d^{A-B}\frac{a_A}{b_B}\sum_{i=0}^{A-B}\frac{(-1)^i}{d^i}\sum_{v=0}^{i}(-1)^v e_{i-v}(y_1,\dots,y_A)h_v(z_1,\dots,z_B)\\&=\frac{a_A}{b_B}\sum_{i=0}^{C}d^i(-1)^{C-i}\sum_{v=0}^{C-i}(-1)^v e_{C-i-v}(y_1,\dots,y_A)h_v(z_1,\dots,z_B),\end{aligned}\tag{15}$$

where $e_i(y_1,\dots,y_A)$ *and* $h_i(z_1,\dots,z_B)$ *are the* i*-th elementary and complete homogeneous symmetric polynomials respectively. In particular, the constant term coefficient of* $Q_{n,h,l}(j,k,\cdot)$ *is given by*

$$q_C^{(n,h)}(j,k) = (-1)^C \frac{(1-\theta)^h(\lambda\theta^2)^n}{\theta^h(\lambda\theta)} \sum_{v=0}^{C} (-1)^v e_{C-v}(y_1,\dots,y_A)h_v(z_1,\dots,z_B). \tag{16}$$

Proof. One obviously has:

$$P_{n,h}(j,k,d) = a_A d^A \prod_{v=1}^{A}(1-y_v/d) = a^A d^A \sum_{v=0}^{\infty}(-1)^v e_v(y_1,\dots,y_A)\frac{1}{d^v},$$

where we set $e_v(y_1,\dots,y_A) = 0$ for any $v > A$, and (for large enough d):

$$\frac{1}{D_{n,h}(j,k,d)} = \frac{1}{b_B d^B}\prod_{v=1}^{B}(1-z_v/d) = \frac{1}{b_B d^B}\sum_{v=0}^{+\infty} h_v(z_1,\dots,z_B)\frac{1}{d^v}.$$

As a result,

$$\frac{P_{n,h}(j,k,d)}{D_{n,h}(j,k,d)} = d^{A-B}\frac{a_A}{b_B}\sum_{i=0}^{+\infty}\frac{(-1)^i}{d^i}\sum_{v=0}^{i}(-1)^v e_{i-v}(y_1,\dots,y_A)h_v(z_1,\dots,z_B).$$

But the quotient $Q_{n,h,l}(j,k,d)$ of the polynomial division in the LHS is the polynomial part of the RHS in the variable d whence the first expression of $Q_{n,h,l}(j,k,d)$ follows. Performing the index change $i \mapsto A-B-i = C-i$, one gets the second one.

Finally, the leading coefficients of $P_{n,h}(j,k,d)$ and $D_{n,h}(j,k,d)$ are given by

$$a_A = \theta^n(1-\theta)^h(\lambda\theta)^n, \quad b_B = (\lambda\theta)\theta^h,$$

respectively. Extracting the constant term coefficient in (15), the lemma is proved. □

Remark. *Recalling the representation* (13)

$$Q_{n,h,l}(j,k,d) = \sum_{i=0}^{n-l-1} q_{n-l-1-i}^{(n,h)}(j,k)d^i = \sum_{i=0}^{C} q_{C-i}^{(n,h)}(j,k)d^i,$$

we readily infer from (15) *that*

$$q_{C-i}^{(n,h)}(j,k) = (-1)^{C-i}\sum_{v=0}^{C-i}(-1)^v e_{C-i-v}(y_1,\dots,y_A)h_v(z_1,\dots,z_B), \quad 0 \le i \le C,$$

On the other hand, the roots $(y_1,\dots,y_A)$ *are given by:*

$$y_i = \begin{cases} 2j-2h+1, & i=1 \\ 2j+1, & i=2 \\ j-i+3, & 3 \le i \le h+2 \\ (j+h-i+3)/(1-\theta), & h+3 \le i \le 2h+2 \\ (j-k+n+h-i+3)/(\lambda\theta), & 2h+3 \le i \le n+2h+2 \\ (j-k+2n+h-i+3)/\theta, & n+2h+3 \le i \le 2n+2h+2 = A \end{cases}, \tag{17}$$

while

$$z_i = \begin{cases} 0, & 1 \le i \le l+1 \\ 2j-h+1, & i = l+2 \\ (j-i+l+3)/\theta, & l+3 \le i \le h+l+2 \\ 2j-k+n+l-i+4, & h+l+3 \le i \le n+2h+l+3 = B \end{cases}. \tag{18}$$

As a matter fact, the simple monomial j *is present in all the roots except* $z_i, 1 \le i \le l+1$. *Since* h_v *and* e_{C-i-v} *are homogenous polynomials (in their variables) of orders* v *and* $C-i-v$ *respectively, then* (15) *shows that* $q_{C-i}^{(n,h)}(j,k)$ *is a polynomial of degree at most* $C-i$ *in the variable* j. *As to the simple monomial* k, *it* *is present in more than C roots. Consequently, the same conclusion holds regarding the degree of $q_{C-i}^{(n,h)}(j,k)$ in the variable k and we get another proof of Lemma 2.*

The formula (16) is clearly symmetric in each of the alphabets $\mathcal{Y} := (y_1, \ldots, y_A)$ and $\mathcal{Z} := (z_1, \ldots, z_B)$. In the realm of algebraic combinatorics, a function enjoying this property is referred to as doubly symmetric. Moreover, q_C may be seen as a completely homogenous function in an another alphabet formed out of $\mathcal{Y}$ and $\mathcal{Z}$. Indeed, appealing again to the homogeneity of h_i, we write

$$\begin{aligned} q_C &= (-1)^C \frac{(1-\theta)^h(\lambda\theta^2)^n}{\theta^h(\lambda\theta)} \sum_{i=0}^{C} h_i(-z_1, \ldots, -z_B) e_{C-i}(y_1, \ldots, y_A) \\ &= (-1)^C \frac{(1-\theta)^h(\lambda\theta^2)^n}{\theta^h(\lambda\theta)} \sum_{i=0}^{C} h_i(-z_1, \ldots, -z_B, \underbrace{0, \ldots, 0}_{C\text{times}}) e_{C-i}(y_1, \ldots, y_A). \end{aligned}$$

Moreover, Lemma 2.1 and formula (18) in [16] entail:

Corollary 2. *The constant term coefficient q_C admits the following representation:*

$$q_C^{(n,h)}(j,k) = (-1)^C \frac{(1-\theta)^h(\lambda\theta^2)^n}{\theta^h(\lambda\theta)} \sum_{1 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_C \leq A} \prod_{i=1}^{C} (y_{\tau_i+i-1} - z_{\tau_i}),$$

where it is understood that $y_i = 0, i > A$ and $z_i = 0, i > B$.

5. On the coefficient $c_{n,h,l}^{(\lambda,\theta)}$

The limiting moment formula stated in Theorem 1 depends only on the coefficient $c_{n,h,l}^{(\lambda,\theta)}$ of the monomial $k^{n-h}j^{h-l-1}$ in the (j,k)-polynomial $q_{n-l-1}^{(n,h)}(j,k)$. The extraction of this coefficient may obtained from the above representations of $q_{n-l-1}^{(n,h)}(j,k)$ by successive differentiations with respect to (j,k) (viewed as real variables). In this section, we shall rather use (14) to compute inductively $c_{n,h,l}^{(\lambda,\theta)}$ for any (λ,θ). As a warm up, we firstly investigate the two special (univariate) cases corresponding to $n = h$ and to $l = h-1$, for which the coefficients of j^{n-l-1} and of k^{n-l-1} in $q_{n-l-1}^{(n,h)}(j,k)$ are constant (by the virtue of the second statement of Lemma 2). Doing so reveals further the role played by the special values $\lambda = 1$ and $\theta = 1/2$. For instance, we determine explicitly $c_{n,h,h-1}^{(\lambda,\theta)}$ whose expression considerably simplifies when $\lambda = 1$ without any restriction on θ. The same holds for $c_{n,n,l}^{(\lambda,1/2)}$ while $c_{n,n,l}^{(\lambda,\theta)}$ is expressed through complete ordinary Bell polynomials.

5.1. **The case $n = h$.** For any $0 \leq i \leq n-l-1$, let $X_i^{(n,\lambda,\theta)}$ denote the coefficient of j^i in $q_i^{(n,n)}(j,k)$, so that

$$c_{n,n,l}^{(\lambda,\theta)} = X_{n-l-1}^{(n,\lambda,\theta)}.$$

Then

Proposition 3. *We have, $X_0^{(n,\lambda,\theta)} = (\lambda\theta)^{n-1}(1-\theta)^n$ and for any $1 \leq i \leq n-l-1$, $X_i^{(n,\lambda,\theta)}$ satisfies*

$$X_i^{(n,\lambda,\theta)} = \alpha_i - \sum_{v=0}^{i-1} \beta_{i-v}^{(n,\theta)} X_v^{(n,\lambda,\theta)} \tag{19}$$

where

$$\alpha_i^{(n,\lambda,\theta)} = (-1)^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \binom{2}{i_1}\binom{n}{i_2}\binom{n}{i_3}\binom{n}{i_4}\binom{n}{i_5} 2^{i_1}(\lambda\theta)^{n-1-i_3}(1-\theta)^{n-i_4}\theta^{-i_5}$$

and

$$\beta_{i-v}^{(n,\theta)} = (-1)^{i-v} \sum_{i_1+i_2+i_3=i-v} \binom{1}{i_1}\binom{n}{i_2}\binom{2n+1}{i_3} 2^{i_1+i_3}\theta^{-i_2}, \quad 0 \leq v \leq i-1.$$

Proof. From Lemma 2, $X_0^{(n,\lambda,\theta)}$ is the highest degree coefficient $q_0^{(n,h)}$ in $Q_{n,h,l}(j.k,d)$ and we infer from (14) together with the definition of $P_{n,h,l}(j,k,d)$ that:

$$q_0^{(n,h)}(j,k) = \tilde{a}_{2n+2h+2}^{(n,h)}(j,k) = (\lambda\theta)^{n-1}\theta^{n-h}(1-\theta)^n$$

so that $X_0^{(n,\lambda,\theta)} = q_0^{(n,h)}(j,k) = (\lambda\theta)^{n-1}(1-\theta)^n$.

Appealing again to (14), we get (19) where $\alpha_i^{(n,\lambda,\theta)}$ and $\beta_{i-v}^{(n,\lambda,\theta)}, 0 \leq v \leq i-1$, are the coefficients of j^i and of j^{i-v} in $\tilde{a}_{2n+2h+l+2-i}^{(n,h)}(j,k)$ and in $\tilde{b}_{n+2h+l+3-i+v}^{(n,h)}(j,k)$ respectively. Indeed, the first statement of Lemma 2 shows that $q_v^{(n,h)}(j,k), 1 \leq v \leq i-1$, is a polynomial of degree at most v in the variable j. Moreover,

$$\tilde{a}_{2n+2h+2-i}^{(n,h)}(j,k) = (-1)^i(\lambda\theta)^{n-1}\theta^{n-h}(1-\theta)^h \sum_{1\leq j_1<\ldots j_i\leq 2n+2h+2} y_{j_1}\ldots y_{j_i}, \quad 1 \leq i \leq n-l-1, \tag{20}$$

and

$$\tilde{b}_{n+2h+l+3-i+v}^{(n,h)}(j,k) = (-1)^{i-v} \sum_{1\leq j_1<\cdots<j_{i-v}\leq n+2h+l+3} z_{j_1}\ldots z_{j_{i-v}}, \quad 0 \leq v \leq i-1, \tag{21}$$

are elementary symmetric polynomials in the roots $(y_1,\ldots,y_{2n+2h+1})$ and $(z_1,\ldots,z_{n+2h+l+3})$ respectively. As such, they are polynomials of degrees i and $i-v$ in each of the variables j and k.

Finally, the expressions of α_i and of β_{i-v} readily follow from (17) and (18), keeping in mind $n = h$. The proposition is proved. $\square$

5.2. **The case $\lambda = 1, \theta = 1/2$.** If $\theta = 1/2$ then the generalized Chu-Vandermonde identity entails:

$$\begin{aligned}
\beta_{i-v}^{(n,1/2)} &= (-1)^{i-v} \sum_{i_1+i_2+i_3=i-v} \binom{1}{i_1}\binom{n}{i_2}\binom{2n+1}{i_3} 2^{i_1+i_2+i_3} \\
&= (-2)^{i-v} \sum_{i_1+i_2+i_3=i-v} \binom{1}{i_1}\binom{n}{i_2}\binom{2n+1}{i_3} \\
&= (-2)^{i-v}\binom{3n+2}{i-v}.
\end{aligned}$$

Since $\beta_0 = 1$ then the recurrence relation (19) may be then written as:

$$\sum_{v=0}^{i}(-2)^{i-v}\binom{3n+2}{i-v}X_v^{(n,\lambda,1/2)} = \alpha_i^{(n,\lambda,1/2)},$$

and is inverted as:

$$X_i^{(n,\lambda,1/2)} = \sum_{v=0}^{i}\frac{(3n+2)_{i-v}}{(i-v)!}\alpha_i^{(n,\lambda,1/2)}, \quad 0 \leq i \leq n-l-1.$$

In particular,

$$X_{n-l-1}^{(n,\lambda,1/2)} = \sum_{v=0}^{n-l-1}\frac{(3n+2)_{n-l-1-v}}{(n-l-1-v)!}\alpha_v^{(n,\lambda,\theta)}.$$

If further $\lambda = 1$ then

$$\begin{aligned}
\alpha_i^{(n,1,1/2)} &= \frac{(-1)^i}{2^{2n-1}} \sum_{i_1+i_2+i_3+i_4+i_5=i} \binom{2}{i_1}\binom{n}{i_2}\binom{n}{i_3}\binom{n}{i_4}\binom{n}{i_5} 2^{i_1+i_3+i_4+i_5} \\
&= \frac{(-1)^i}{2^{2n-1}} \sum_{i_2=0}^{i}\binom{n}{i_2}2^{i-i_2} \sum_{i_1+i_3+i_4+i_5=i-i_2} \binom{2}{i_1}\binom{n}{i_3}\binom{n}{i_4}\binom{n}{i_5} \\
&= \frac{(-1)^i}{2^{2n-1}} \sum_{i_2=0}^{i}\binom{n}{i_2}\binom{3n+2}{i-i_2}2^{i-i_2}
\end{aligned}$$

where the last equality follows again from the generalized Chu-Vandermonde identity. In this case, (19) simplifies further to

$$\sum_{v=0}^{i}(-1)^v\binom{3n+2}{i-v}2^{i-v}X_v^{(n,1,1/2)} = \frac{1}{2^{2n-1}}\sum_{v=0}^{i}\binom{n}{v}\binom{3n+2}{i-i_2}2^{i-v}.$$

and the uniqueness of its solution (subject to the initial condition $X_0 = 1/2^{2n-1}$) clearly yields

$$X_v^{(n,1,1/2)} = \frac{(-1)^v}{2^{2n-1}}\binom{n}{v}.$$

As a result,

$$c_{n,n,l}^{(1,1/2)} = X_{n-l-1}^{(n,1,1/2)} = \frac{(-1)^{n-l-1}}{2^{2n-1}}\binom{n}{n-l-1} = \frac{(-1)^{n-l-1}}{2^{2n-1}}\binom{n}{l+1},$$

whence we deduce that the $h = n$ term of the sum displayed in the RHS of Theorem 1 reads

$$\frac{1}{2^{2n-1}}\frac{e^{-nt}}{n}\sum_{l=0}^{h-1}\frac{(-2nt)^l}{l!}\binom{n}{l+1} = \frac{e^{-nt}}{2^{2n-1}n}L_{n-1}^{(1)}(2nt),$$

which agrees with (8). More generally,

Proposition 4. *For all $0 \le l \le n-1$, we have*

$$c_{n,n,l}^{(\lambda,\theta)} = \sum_{v=0}^{n-l-1}\alpha_v^{(n,\lambda,\theta)}\gamma_{n-l-1-v}^{(n,\theta)}$$

where

$$\gamma_a^{(n,\theta)} = \sum_{\substack{v_1\ge 0,\dots,v_a\ge 0\\ v_1+2s_2\dots+av_a=a}}\frac{(v_1+\cdots+v_a)!}{v_1!\dots v_a!}(-\beta_1^{(n,\theta)})^{v_1}\dots(-\beta_a^{(n,\theta)})^{v_a}, \quad 0 \le a \le n-l-1.$$

Proof. The system of equations (19) may be written as

$$\begin{pmatrix}\alpha_0^{(n,\lambda,\theta)}\\ \alpha_1^{(n,\lambda,\theta)}\\ \vdots\\ \alpha_{n-l-1}^{(n,\lambda,\theta)}\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0 & \dots & 0\\ \beta_1^{(n,\theta)} & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{n-l-1}^{(n,\theta)} & \dots & \dots & 1\end{pmatrix}}_{\mathcal{B}}\begin{pmatrix}X_0^{(n,\lambda,\theta)}\\ X_1^{(n,\lambda,\theta)}\\ \vdots\\ X_{n-l-1}^{(n,\lambda,\theta)}\end{pmatrix}$$

where $\mathcal{B}$ is a lower triangular invertible Toeplitz matrix. Appealing to Corollary 1 in [22], we obtain the expression of the inverse matrix $\mathcal{B}^{-1}$ whence the formula for $c_{n,n,l}^{(\lambda,\theta)}$ follows. $\square$

Remark. *For any $1 \le a \le n-l-1$, $\gamma_a^{(n,\theta)}$ is a complete ordinary Bell polynomial in the variables $(-\beta_1^{(n,\theta)},\dots,-\beta_a^{(n,\theta)})$.*

5.3. **The case $l = h-1$.** This case seems a bit easier than the previous one and the proof of the following proposition is similar to that of Proposition 19.

Proposition 5. *For all $1 \le h \le n$, we have*

$$c_{n,h,h-1}^{(\lambda,\theta)} = (1-\theta)^h(\lambda\theta)^{n-1}(-\theta)^{n-h}\binom{2n}{n-h}\sum_{v=0}^{n-h}\frac{(h-n)_v(-n)_v}{(-2n)_v v!\theta^v}\,{}_2F_1(-v,-n,n-v+1;1/\lambda),$$

where ${}_2F_1$ is the Gauss hypergeometric function. In particular, for any $n \ge 1$, we have

$$c_{n,h,h-1}^{(1,\theta)} = (1-\theta)^n\theta^{n-1}\binom{2n}{n-h}.$$

Proof. Let $Y_i^{(n,h,\lambda,\theta)}$ denote the coefficient k^i in $q_i^{(n,h)}(j,k)$ so that $c_{n,h,h-1}^{(\lambda,\theta)} = Y_{n-h}^{(n,h,\lambda,\theta)}$. Then

$$Y_0 = (\lambda\theta)^{n-1}\theta^{n-h}(1-\theta)^h$$

and (14) shows that the sequence $(Y_v^{(n,h,\lambda,\theta)})_{0\le v\le i}$ satisfies

$$Y_i^{(n,h,\lambda,\theta)} = \alpha_i^{(n,h,\lambda,\theta)} - \sum_{v=0}^{i-1} Y_v^{(n,h,\lambda,\theta)}\beta_{i-v}^{(n,h)}, \quad 0\le i\le n-h. \tag{22}$$

where (by the virtue of (17), (18), (20) and (21)):

$$\begin{aligned}\alpha_i^{(n,h,\lambda,\theta)} &= (1-\theta)^h \sum_{v=0}^{i}\binom{n}{v}\binom{n}{i-v}(\lambda\theta)^{n-1-v}\theta^{n-h-(i-v)}\\ &= (1-\theta)^h(\lambda\theta)^{n-1}\theta^{n-h-i}\sum_{v=0}^{i}\binom{n}{v}\binom{n}{i-v}(\lambda)^{-v}\end{aligned}$$

and

$$\beta_i^{(n,h)} = \binom{h+n+1}{i}.$$

The recurrence relation (22) may be then written as:

$$\sum_{v=0}^{i}\binom{h+n+1}{i-v}Y_v = (1-\theta)^h(\lambda\theta)^{n-1}\theta^{n-h-i}\sum_{v=0}^{i}\binom{n}{v}\binom{n}{i-v}(\lambda)^{-v},$$

and is inverted as follows: for any $0\le i\le n-h$, one has:

$$Y_i^{(n,h,\lambda,\theta)} = (1-\theta)^h(\lambda\theta)^{n-1}\theta^{n-h}\sum_{v=0}^{i}(-1)^{i-v}\frac{(n+h+1)_{i-v}}{(i-v)!}\theta^{-v}\sum_{a=0}^{v}\binom{n}{a}\binom{n}{v-a}(\lambda)^{-a}.$$

Consequently,

$$\begin{aligned}Y_{n-h}^{(n,h,\lambda,\theta)} &= \frac{(1-\theta)^h(\lambda\theta)^{n-1}\theta^{n-h}}{(n+h)!}\sum_{v=0}^{n-h}(-1)^{n-h-v}\frac{(2n-v)!}{(n-h-v)!}\theta^{-v}\sum_{a=0}^{v}\binom{n}{a}\binom{n}{v-a}(\lambda)^{-a}\\ &= (1-\theta)^h(\lambda\theta)^{n-1}(-\theta)^{n-h}\binom{2n}{n-h}\sum_{v=0}^{n-h}(-1)^v\frac{(h-n)_v}{(-2n)_v}\frac{\theta^{-v}n!}{v!}\sum_{a=0}^{v}\binom{n}{a}\frac{v!}{(v-a)!(n-(v-a))!}(\lambda)^{-a}\\ &= (1-\theta)^h(\lambda\theta)^{n-1}(-\theta)^{n-h}\binom{2n}{n-h}\sum_{v=0}^{n-h}\frac{(h-n)_v}{(-2n)_v}\frac{\theta^{-v}(-n)_v}{v!}\sum_{a=0}^{v}\frac{(-n)_a(-v)_a}{a!(n-v+1)_a}(\lambda)^{-a}\\ &= (1-\theta)^h(\lambda\theta)^{n-1}(-\theta)^{n-h}\binom{2n}{n-h}\sum_{v=0}^{n-h}\frac{(h-n)_v(-n)_v}{(-2n)_v v!\theta^v}\,{}_2F_1(-v,-n,n-v+1;1/\lambda).\end{aligned}$$

Finally, if $\lambda=1$ then

$${}_2F_1(-v,-n,n-v+1;1) = \frac{(2n-v+1)_v}{(n-v+1)_v} = \frac{(2n!)(n-v)!}{n!(2n-v)!},$$

by the Gauss hypergeometric Theorem, whence

$$\begin{aligned}Y_{n-h}^{(n,h,1,\theta)} &= (1-\theta)^h(\theta)^{n-1}(-\theta)^{n-h}\binom{2n}{n-h}\sum_{v=0}^{n-h}\frac{(h-n)_v}{v!\theta^v}\\ &= (1-\theta)^h(\theta)^{n-1}(-\theta)^{n-h}\binom{2n}{n-h}\left(1-\frac{1}{\theta}\right)^{n-h}\\ &= (1-\theta)^n\theta^{n-1}\binom{2n}{n-h}.\end{aligned}$$

$\square$

5.4. **The general case.** Denote $Z_{a,b}^{(n,h,\lambda,\theta)}$ the coefficient of $j^a k^b$ in $q_{a+b}^{(n,h)}(j,k)$. Recall also that $c_{n,h,l}^{(\lambda,\theta)}$ is the coefficient of $j^{h-l-1}k^{n-h}$ in

$$q_{n-l-1}^{(n,h)}(j,k) = q_{(h-l-1)+(n-h)}^{(n,h)}(j,k),$$

in other words,

$$c_{n,h,l}^{(\lambda,\theta)} = Z_{h-l-1,n-h}^{(n,h,\lambda,\theta)}.$$

Appealing once more to (14), we prove the following:

Proposition 6. $Z_{0,0}^{(n,h,\lambda,\theta)} = (\lambda\theta)^{n-1}\theta^{n-h}(1-\theta)^h$ *and for any* $0 \leq a \leq h-l-1$ *and* $0 \leq b \leq n-h, (a,b) \neq (0,0)$*, one has:*

$$Z_{a,b}^{(n,h,\lambda,\theta)} = A_{a,b}^{(n,h,\lambda,\theta)} - \sum_{\substack{0\leq u\leq a,\ 0\leq v\leq b\\ 0\leq u+v\leq a+b-1}} Z_{u,v}\ B_{a-u,b-v}^{(n,h,\theta)} \tag{23}$$

where

$$A_{a,b}^{(n,h,\lambda,\theta)} = (-1)^a \sum_{i=0}^{b} \binom{n}{i}\binom{n}{b-i}$$
$$\sum_{i_1+i_2+i_3+i_4+i_5=a} \binom{2}{i_1}\binom{h}{i_2}\binom{h}{i_3}\binom{n-i}{i_4}\binom{n-(b-i)}{i_5} 2^{i_1}(1-\theta)^{h-i_3}(\theta\lambda)^{n-1-i-i_4}\theta^{(n-h)-(b-i)-i_5},$$

and

$$B_{a,b}^{(n,h,\theta)} = (-1)^a\binom{n+h+1}{b}\sum_{j_1+j_2+j_3=a}\binom{1}{j_1}\binom{h}{j_2}\binom{n+h+1-b}{j_3}\frac{2^{j_1+j_3}}{\theta^{j_2}}.$$

Proof. We follow the lines of the proof of Propositions 3 and (22): $Z_{0,0}^{(n,h,\lambda,\theta)}$ is the highest degree coefficient $q_0^{(n,h)}(j,k)$ in $Q_{n,h,l}(j.k,d)$ and its expression follows readily from (20). As to $A_{a,b}^{(n,h,\lambda,\theta)}$ and $B_{a,b}^{(n,h,\theta)}$ are nothing else but the coefficients of $j^a k^b$ in $\tilde{a}_{2n+2h+2-a-b}^{(n,h)}$ and $\tilde{b}_{n+2h+l+3-a-b}^{(n,h)}$ respectively, viewed as bivariate polynomials in (j,k). In this respect, note from (20) and (21) that their total degrees do not exceed $a+b$ since the roots $(y_1,\dots,y_{2n+2h+2})$ and $(z_1,\dots,z_{n+2h+l+3})$ are linear factors in (j,k). The double recurrence relation (23) follows then from (14) and it only remains to derive the expressions of $A_{a,b}^{(n,h,\lambda,\theta)}$ and $B_{a,b}^{(n,h,\theta)}$.

To derive the former, we consider the $2n = n+n$ roots

$$y_i = \begin{cases} (j-k+n+h-i+3)/(\lambda\theta), & 2h+3 \leq i \leq n+2h+2 \\ (j-k+2n+h-i+3)/\theta, & n+2h+3 \leq i \leq 2n+2h+2 \end{cases},$$

from which we choose b elements. Obviously, there are

$$\binom{n}{i}\binom{n}{b-i}, 0 \leq i \leq b,$$

possible choices, each contributing with a factor $(-1)^b k^b/[(\lambda\theta)^i\theta^{b-i}]$. Once a choice is fixed, we choose further a elements from the remaining $2n+2h+2-b$ roots in (17). The contribution of any choice is of the form

$$\frac{2^{i_1}1^{i_2}}{(1-\theta)^{i_3}(\lambda\theta)^{i_4}\theta^{i_5}}$$

where $i_1+i_2+i_3+i_4+i_5 = a$ and $0 \leq i_1 \leq 2$ and there are

$$\binom{2}{i_1}\binom{h}{i_2}\binom{h}{i_3}\binom{n-i}{i_4}\binom{n-(b-i)}{i_5}$$

possible choices. Keeping in mind (20), the expression of $A_{a,b}^{(n,h,\lambda,\theta)}$ follows. The derivation of the expression of $B_{a,b}^{(n,h,\theta)}$ is similar, we are done. $\square$

5.5. **Back to the case $\lambda = 1, \theta = 1/2$.** When $\lambda = 1, \theta = 1/2$, the expressions of $A_{a,b}^{(n,h,1,1/2)}$ and of $B_{a,b}^{(n,h,1/2)}$ reduce to:

$$\begin{aligned} A_{a,b}^{(n,h,1,1/2)} &= \frac{(-1)^a 2^b}{2^{2n-1}} \binom{2n}{b} \sum_{i=0}^{a} 2^{a-i} \binom{h}{i} \binom{2n+h+2-b}{a-i}, \\ B_{a,b}^{(n,h,1/2)} &= (-1)^a 2^a \binom{n+h+1}{b} \binom{n+2h+2-b}{a}. \end{aligned}$$

Consequently, (23) reads:

$$\sum_{\substack{0 \le u \le a,\ 0 \le v \le b \\ u+v \le a+b}} 2^{a-u} (-1)^u \binom{n+h+1}{b-v} \binom{n+2h+2-b+v}{a-u} Z_{u,v}^{(n,h,1,1/2)}$$

$$= \frac{2^b}{2^{2n-1}} \binom{2n}{b} \sum_{i=0}^{a} 2^{a-i} \binom{h}{i} \binom{2n+h+2-b}{a-i}.$$

Setting

$$Z_{u,v}^{(n,h,1,1/2)} = \frac{(-1)^u}{2^{2n-1}} \binom{h}{u} \tilde{Z}_{u,v}^{(n,h,1,1/2)},$$

the last equality may be written as:

$$\sum_{u=0}^{a} 2^{a-u} \binom{h}{u} \sum_{v=0}^{b} \binom{n+h+1}{b-v} \binom{n+2h+2-b+v}{a-u} \tilde{Z}_{u,v} = 2^b \binom{2n}{b} \sum_{u=0}^{a} 2^{a-u} \binom{h}{u} \binom{2n+h+2-b}{a-u}.$$

By uniqueness of the solution of (23) subject to the initial condition $Z_{0,0} = 1/2^{2n-1}$, we seek, for any fixed $0 \le u \le a$, a sequence $(\tilde{Z}_{u,v}^{(n,h,1,1/2)})_{0 \le v \le b}$ such that

$$\sum_{v=0}^{b} \binom{n+h+1}{b-v} \binom{n+2h+2-b+v}{a-u} \tilde{Z}_{u,v}^{(n,h,1,1/2)} = 2^b \binom{2n}{b} \binom{2n+h+2-b}{a-u}$$

Since we are mainly interested in $c_{n,h,l}^{(1,1/2)} = Z_{h-l-1,n-h}^{(n,h,1,1/2)}$, we shall take $u = a$ in the last equation which reduces to:

$$\sum_{v=0}^{b} \binom{n+h+1}{b-v} \tilde{Z}_{a,v}^{(n,h,1,1/2)} = 2^b \binom{2n}{b}, \quad 0 \le b \le n-h.$$

This equation may be inverted and yields:

$$\tilde{Z}_{a,b}^{(n,h,1,1/2)} = \sum_{v=0}^{b} 2^v \binom{2n}{v} (-1)^{b-v} \frac{(n+h+1)_{b-v}}{(b-v)!}.$$

Consequently,

$$\tilde{Z}_{a,n-h}^{(n,h,1,1/2)} = \sum_{v=0}^{n-h} 2^v \binom{2n}{v} (-1)^{n-h-v} \frac{\Gamma(2n+1-v)}{\Gamma(n+h+1)(b-v)!}$$

$$= \frac{(2n)!}{(n+h)!} \sum_{v=0}^{n-h} 2^v (-1)^{n-h-v} \frac{1}{v!(n-h-v)!} = \binom{2n}{n-h},$$

whence

$$Z_{a,n-h}^{(n,h,1,1/2)} = \frac{(-1)^a}{2^{2n-1}} \binom{h}{a} \binom{2n}{n-h},$$

and in turn

$$Z_{h-l-1,n-h}^{(n,h,1,1/2)} = \frac{(-1)^{h-l-1}}{2^{2n-1}} \binom{h}{l+1} \binom{2n}{n-h}.$$

Plugging the expression of $Z_{h-l-1,n-h}^{(n,h,1,1/2)}$ in the RHS of Theorem 1, we end up with:

$$\frac{1}{2^{2n-1}}\sum_{h=1}^{n}\binom{2n}{n-h}\frac{e^{-ht}}{h}\sum_{l=0}^{h-1}\frac{(-2ht)^l}{l!}\binom{h}{l+1} = \frac{1}{2^{2n-1}}\sum_{h=1}^{n}\binom{2n}{n-h}\frac{e^{-ht}}{h}L_{h-1}^{(1)}(2ht),$$

which agrees again with (8).

References

[1] *R. F. A. Abiodun.* An Application of Carlitz's Formula. *J. Math. Anal. Appl.* **70**, 114-119, (1979).

[2] *W. N. Bailey.* Generalized Hypergeometric Series. *Cambridge University Press, Cambridge*, 1935.

[3] *P. Biane.* Free Brownian motion, free stochastic calculus and random matrices. *Fields Inst. Commun.*, **12**. American Mathematical Society, Providence, RI, 1997, 1-19.

[4] *C. Carré, M. Deneufchatel, J. G. Luque, P. Vivo.* Asymptotics of Selberg-like integrals: the unitary case and Newton's interpolation formula. *J. Math. Phys.* **51**, (12), 19 (2010).

[5] *W. Chu.* Binomial convolutions and hypergeometric identities. *Rend. Del Cir. Mat. Di Pal, Ser. II.* **XLIII**, (1994), 333-360.

[6] *W. Chu.* Transformation formulae for terminating balanced 4F3-series and implications. *Hacet. J. Math. Stat.* **52** (2023), no. 2, 391-397.

[7] *L. Deleaval, N. Demni.* Moments of the Hermitian matrix Jacobi process. *J. Theoret. Probab.* **31**, (2018), no. 3, 1759-1778.

[8] *N. Demni.* Free Jacobi process. *J. Theor. Probab.* **Volume 21**, 118-143, (2008).

[9] *N. Demni.* β-Jacobi processes. *Adv. Pure Appl. Math.* **1**, (2010), no. 3, 325-344.

[10] *N. Demni.* Free Jacobi process associated with one projection: local inverse of the flow. *Complex Anal. Oper. Theory*, **10**, (2016), no. 3, 527?543.

[11] *N. Demni, T. Hamdi.* Inverse of the flow and moments of the free Jacobi process associated with one projection. *Random Matrices Theory Appl.* **7**, (2018), no. 2, 1850001, 19 pp.

[12] *N. Demni, T. Hamdi.* Summing free unitary Brownian motions with applications to quantum information. *Lett. Math. Phys.* **113**, (2023), no. 4, Paper No. 80, 39 pp.

[13] *N. Demni, T. Hamdi, T. Hmidi.* Spectral distribution of the free Jacobi process. *Indiana Univ. Math. J.* **61**, (2012), no. 3, 1351-1368.

[14] *N. Demni, T. Hamdi, A. Souaissi.* The Hermitian Jacobi process: a simplified formula for the moments and applications to optical fiber MIMO channels. *Funct. Anal. Appl.* **54** (2020), no. 4, 257-271.

[15] *P. J. Forrester.* Log-gases and random matrices. *London Math. Soc. Monogr. Ser.*, **34**. *Princeton University Press, Princeton, NJ,* (2010), xiv+791 pp.

[16] *I. Goulden, C. Greene.* A new tableau representation for supersymmetric Schur functions. *J. Algebra.* **170**, 687-703, (1994).

[17] *P. Graczyk, J. Malecki.* Strong solutions of non-colliding particle systems. *Electron. J. Probab.* **19** (2014), no. 119, 1-21.

[18] *T. Hamdi.* Spectral distribution of the free Jacobi process, revisited. *Anal. PDE*, **11** (8), 2137-2148 (2018).

[19] *T. Hamdi.* Free Mutual Information for Two Projections. *Complex Anal. Oper. Theory* **12** (7), 1697-1705 (2018).

[20] *C. Krattenthaler.* Asymptotic analysis of a Selberg-type integral via hypergeometrics. *Unpublished.* Available at https://www.mat.univie.ac.at/ kratt/artikel/asyselb.html.

[21] *A. R. Miller, R. B. Paris.* On a result related to transformations and summations of generalized hypergeometric series. *Math. Commun.* **17**, (2012), no. 1, 205-210.

[22] *M. Merca.* A generalisation of the symmetry between complete and elementary symmetric functions. *Indian J. Pure Appl. Math.* **45** (2014), no. 1, 75-89.

[23] *V. Reiner.* Non-crossing partitions for classical reflection groups, *Discrete Mathematics.* **177**, (1997), 195-222.

[24] *L. W. Shapiro.* A Catalan triangle. *Discrete Mathematics.* **Volume 14**, Issue 1, 1976, 83-90.

[25] *H. D. Trinh, K. D. Trinh.* Beta Jacobi ensembles and associated Jacobi polynomials. *J. Stat. Phys.* **185**, (2021), no. 1, Paper No. 4, 15 pp.

[26] *M. Voit.* Freezing limits for Calogero-Moser-Sutherland particle models. *Stud. Appl. Math.* **151**, (2023), no. 4, 1230-1281.

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