

Applicability of Eliashberg theory for systems with electron-phonon and electron-electron interaction: a comparative analysis

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We present a comparative analysis of the validity of Eliashberg theory for the cases of fermions interacting with an Einstein phonon and with soft nematic fluctuations near an Ising-nematic/Ising-ferromagnetic quantum-critical point (QCP). In both cases, Eliashberg theory is obtained by neglecting vertex corrections. For the phonon case, the reasoning to neglect vertex corrections is the Migdal “fast electron/slow boson” argument because the phonon velocity is much smaller than the Fermi velocity, v_F . The same argument allows one to compute the fermionic self-energy within Eliashberg theory perturbatively rather than self-consistently. For the nematic case, the velocity of a collective boson is comparable to v_F and this argument apparently does not work. Nonetheless, we argue that while two-loop vertex corrections near a nematic QCP are not small parametrically, they are small numerically. At the same time, perturbative calculation of the fermionic self-energy can be rigorously justified when the fermion-boson coupling is small compared to the Fermi energy. Furthermore, we argue that for the electron-phonon case Eliashberg theory breaks down at some distance from where the dressed Debye frequency would vanish, while for the nematic case it holds all the way to a QCP. From this perspective, Eliashberg theory for the nematic case actually works better than for the electron-phonon case.

I. INTRODUCTION

Migdal-Eliashberg (ME) theory has been developed to describe electron-phonon interaction in the normal state and phonon-induced superconductivity [1, 2]. Over the years, it has been successfully applied to numerous electron-phonon systems [3–8]. The theory, as formulated by Eliashberg [2], consists of a set of three coupled self-consistent one-loop equations for the normal and anomalous fermionic self-energies and the phonon polarization. These equations can be derived either diagrammatically or from the variational Luttinger-Ward functional [9], assuming that vertex corrections can be neglected. ME argued that vertex corrections are small in the ratio of the Debye frequency ω_D to the Fermi energy E_F . The physical reasoning, due to Migdal [1], is that in processes leading to vertex corrections, a fermion is forced to vibrate at the phonon frequency, which for small ω_D/E_F is far from its own resonance. In mathematical terms, the strength of vertex corrections is governed by the dimensionless Eliashberg parameter $\lambda_E = g^2/(\omega_D E_F)$. This parameter is different from the dimensionless coupling $\lambda = g^2/\omega_D^2$, which determines mass renormalization: $\lambda_E = \lambda(\omega_D/E_F) \ll \lambda$. By this reasoning, the one-loop theory remains under control not only at weak coupling, where $\lambda < 1$, but also at strong coupling, where $\lambda > 1$ but $\lambda_E < 1$ (i.e., at $\omega_D < g < (\omega_D E_F)^{1/2}$). Still, when the Debye frequency softens, λ_E eventually becomes large and the one-loop description breaks down. At larger λ_E , higher loop processes become relevant and eventually lead to polaronic description.

In recent years, a similar description has been applied to metals in which electron-electron interactions (screened Coulomb repulsion) give rise to long-range particle-hole order either in the spin or charge channel [10–41]. Near the onset of such an order, i.e., near a

quantum-critical point (QCP), it is natural to assume that the low-energy physics is described by an effective model with fermion-fermion interactions mediated by soft fluctuations of the bosonic order parameter that condenses at the QCP. The theory near a QCP is based on the same set of coupled equations that Eliashberg obtained for the electron-phonon case and bears his name.

The validity of Eliashberg theory for a system near a QCP has been questioned, however, on the grounds that the “fast fermion/slow boson” argument, used to justify the neglect of vertex corrections, is not applicable anymore because collective excitations are made out of fermions, and their velocity is of order v_F (see e.g., Ref. 42). The counter-argument [18] is that soft collective excitations are Landau overdamped and for this reason do behave as slow modes compared to fermions, i.e., Migdal’s reasoning is still valid, albeit for a different reason.

The aim of this communication is to settle the issue of applicability of the Eliashberg (one-loop) theory for a quantum-critical metal near a QCP. We compare the criteria for applicability of the Eliashberg theory for fermions interacting with a soft Einstein phonon with $\omega_D \ll E_F$, and with soft nematic fluctuations near an Ising-nematic/Ising-ferromagnetic QCP. We argue that the criteria for these two cases are somewhat similar, but not identical.

Below we restrict our analysis to the normal state at $T = 0$. This is a putative ground state as the true one is a superconductor in both cases. However, for spin-singlet pairing, the criterion for validity of the Eliashberg theory for a superconductor is actually weaker than that for the normal state as singular quantum corrections to the self-energy cancel in the gap equation like contributions from non-magnetic impurities [14, 43, 44]. For this reason, if ME theory is applicable in the putative normal state at

$T = 0$, it is also applicable to a superconductor. Restriction to $T = 0$ is essential because thermal fluctuations at a finite T do not fit into ME reasoning and may destroy ME theory faster than quantum fluctuations (see Refs. 8 and 45 for more detail).

We also do not discuss here logarithmic singularities in the fermionic self-energy in the Ising-nematic/Ising-ferromagnetic case that emerge at three-loop and higher-orders [46–49]. Most likely, these corrections generate an anomalous dimension for the fermionic propagator but the branch cut in the Green's function remains at the same place as in the Eliashberg theory.

Finally, for the electron-phonon case, we do not discuss here the proposals [50–53] that the transition to a ground state with polarons may be first order and happen already at $\lambda = O(1)$ due to singular behavior of n -loop contributions to the self-energy with $n \gg 1$ (Ref. [53]). In this communication we restrict with the analysis of the self-energy at the two-loop order.

A. Summary of the results

The summary of our results is the following. As said above, for the electron-phonon case, the smallness of two-loop vertex corrections is controlled by the Eliashberg parameter $\lambda_E = g^2/(\omega_D E_F) = \lambda \omega_D/E_F$. This parameter is small at weak coupling, when $\lambda < 1$, but remains small also at $\lambda > 1$, as long as $\lambda < E_F/\omega_D$. At larger λ vertex corrections become parametrically large and Eliashberg theory breaks down. This happens at a finite ω_D , unless one takes the double limit $E_F \rightarrow \infty$ and $\omega_D \rightarrow 0$ while maintaining $g^2 \ll \omega_D E_F$ [44, 54]. If E_F is kept finite, as we do in this paper, $\lambda_E = O(1)$ is reached at a finite ω_D .

For the Ising-nematic/Ising-ferromagnetic case, vertex corrections are small at weak coupling, when the corresponding λ^* , defined via $\lambda^* = \lim_{\omega_m \rightarrow 0} \partial \Sigma / \partial \omega_m$, is small. In the strong coupling regime $\lambda^* > 1$, vertex corrections are not small parametrically, however, they remain of order one and are small numerically (of order 10^{-2}). This behavior holds all the way up to the QCP, where vertex corrections remain numerically small.

There is more. The two Eliashberg equations for the dynamical fermionic self-energy and bosonic polarization, obtained from the Luttinger-Ward functional, are coupled self-consistent one-loop equations. The one for the bosonic polarization accounts for Landau damping of the bosons. For the electron-phonon case the *same* parameter $\lambda_E < 1$ that justifies the neglect of vertex corrections also allows one to simplify these equations in the following way: neglect the Landau damping and replace the self-consistent one-loop equation for the fermionic self-energy by the perturbative one-loop formula, in which the self-energy is expressed in terms of the bare fermionic propagator. This simplification holds because, even at strong coupling ($\lambda > 1$), typical fermionic energies ϵ_k are of order $\lambda_E E_F \ll E_F$, in which case in-

tegration over ϵ_k can be extended to infinite limits, and $\int d\epsilon_k / (i(\omega_m + \Sigma(\omega_m)) - \epsilon_k) = -i\pi \operatorname{sgn} \omega_m$, like for free fermions. The ability to replace the self-consistent equation for the self-energy by a perturbative one can be also expressed as a consequence of the smallness of typical fermionic momenta transverse to the Fermi surface compared to typical momenta along the Fermi surface ($\lambda_E k_F$ vs. k_F).

For the Ising-nematic/Ising-ferromagnetic case Landau damping is relevant but the self-consistent one-loop Eliashberg equations can be still reduced to the perturbative ones because, like in the phonon case, typical momenta transverse to the Fermi surface are parametrically smaller than typical momenta along the Fermi surface. The smallness holds in λ_E^* , which is the ratio of the fermion-boson coupling and the Fermi energy. This ratio must be small, as otherwise the low-energy description would not be valid. The value of λ_E^* does not depend on the distance to the QCP, and hence for the Ising-nematic/Ising-ferromagnetic case the one-loop perturbation theory remains valid even at the QCP.

The outcome of this analysis is that Eliashberg theory for the electron-phonon case is rigorously justified for $\lambda_E < 1$, even when the coupling $\lambda > 1$. By the same condition $\lambda_E < 1$, the fermionic self-energy can be computed in one-loop perturbation theory rather than self-consistently. Because λ_E is inversely proportional to ω_D , the theory is valid only at ω_D above some critical value.

For the Ising-nematic/Ising-ferromagnetic case, Eliashberg theory is not rigorously justified at strong coupling, but vertex corrections remain $O(1)$ and are small numerically. At the same time, there exists another small parameter λ_E^* (the ratio of the interaction to the Fermi energy), which justifies the computation of the fermionic self-energy in a perturbative one-loop analysis rather than self-consistently. This parameter remains small even at a QCP, i.e., Eliashberg theory can be applied to a system at a QCP, if one is willing to neglect numerically small vertex corrections.

From this last perspective, Eliashberg theory near a nematic QCP works better than for fermions interacting with a soft Einstein phonon, despite that the velocity of collective nematic fluctuations is of order v_F .

The structure of the paper is the following. In the next section we briefly review Eliashberg theory for electron-phonon interaction [2, 7, 55] and discuss the strength of one-loop vertex corrections. In Section III we discuss fermions near a nematic QCP. In Section IV we compare the two cases and present our conclusions.

II. ELIASHBERG THEORY FOR ELECTRON-PHONON SYSTEM

For definiteness we consider interaction with an Einstein phonon. Eliashberg theory for the normal state consists of the set of two equations for the fermionic self-

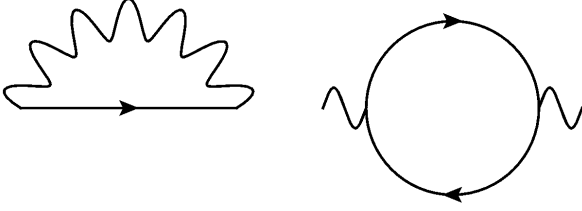


FIG. 1. Self-consistent one-loop electron self-energy (left panel) and polarization bubble for Einstein phonon (right panel). Solid lines indicate the renormalized electronic Green's function $G(k)$ and wavy lines the renormalized Einstein phonon propagator $\chi(q)$.

energy $\Sigma(k)$ and phonon polarization $\Pi(q)$, where $k \equiv (\mathbf{k}, \omega_m)$ and $q \equiv (\mathbf{q}, \Omega_n)$. The equations look simplest along the Matsubara axis, when $\omega = \omega_m = (2m+1)\pi T$ and $\Omega = \Omega_n = 2n\pi T$. The full fermionic Green's function and the full phonon propagator are related to Σ and Π via

$$G^{-1}(k) = G_0^{-1}(k) + i\Sigma(k), \quad \chi^{-1}(q) = \chi_0^{-1}(q) + \Pi(q), \quad (1)$$

where G_0 and χ_0 are electron and phonon propagators in the absence of interaction

$$G_0(k) = \frac{1}{i\omega_m - \epsilon_{\mathbf{k}}}, \quad \chi_0(q) = \frac{\chi_0}{\Omega_n^2 + \omega_D^2}. \quad (2)$$

To simplify calculations, we assume a parabolic dispersion, $\epsilon_{\mathbf{k}} = (\mathbf{k}^2 - k_F^2)/(2m)$. Note that we define the Matsubara self-energy as $i\Sigma(\omega_m)$ (the \mathbf{k} -dependent term in $\Sigma(k)$ is proportional to λ_E and thus negligible at small λ_E , see below), so that the inverse electron propagator can be written $G_0^{-1} = i(\omega_m + \Sigma(\omega_m)) - \epsilon_{\mathbf{k}}$. At $T = 0$, which we consider here, both ω_m and Ω_n are continuous variables.

The self-consistent equations for Σ and Π can be either derived diagrammatically, see Fig. 1, or obtained as stationary solutions of the Luttinger-Ward functional. In analytic form, they are

$$\Sigma(k) = i\tilde{g}^2 \int \frac{d^2\mathbf{q} d\Omega_n}{(2\pi)^3} G(k+q)\chi(q), \quad (3)$$

$$\Pi(q) = 2\tilde{g}^2 \int \frac{d^2\mathbf{k} d\omega_m}{(2\pi)^3} G(k+q)G(k) \quad (4)$$

where \tilde{g} is the electron-phonon coupling.

There are three energy scales in these equations: the Debye frequency ω_D , the Fermi energy $E_F = k_F^2/(2m)$, and the dimension-full effective interaction $g = (\tilde{g}^2 N_F \chi_0)^{1/2}$, where $N_F = k_F/(2\pi v_F)$ is the density of states at the Fermi level per spin component. Out of these one can introduce two dimensionless ratios

$$\lambda = \frac{g^2}{\omega_D^2}, \quad \lambda_E = \frac{g^2}{E_F \omega_D} = \lambda \frac{\omega_D}{E_F}. \quad (5)$$

Eliashberg theory is constructed under the assumptions that E_F is the largest energy scale and $\lambda_E \ll \lambda$. The

ME argument is that the strength of vertex corrections is determined by the smaller λ_E , while fermionic mass renormalization is determined by the larger λ . When both λ and λ_E are small, the theory falls into the weak coupling limit with G and χ close to their bare expressions. In the regime $\lambda > 1$, $\lambda_E < 1$, mass renormalization is large and the self-energy is larger than bare ω_m over a wide range of frequencies, yet vertex corrections are still small.

We will be chiefly interested in the strong coupling regime $\lambda > 1$, $\lambda_E < 1$. We go beyond previous work [8] and analyze vertex corrections in 2D for all phonon momenta. We also compute the two loop self-energy with vertex correction included. We show that the latter is small in λ_E , and becomes $O(1)$ when $\lambda_E \sim 1$.

ME found [1, 2] that the same small parameter λ_E allows one to simplify calculations within Eliashberg theory and obtain a simple expression for the self-energy, $\Sigma^{(E)}(\omega_m) = \lambda \omega_D \arctan(\omega_m/\omega_D)$ (Eq. (12) below), which depends only on frequency. Here we analyze the corrections to this expression, both analytically and numerically. We show that Eq. (12) can be rigorously justified only for frequencies $\omega_m \ll E_F$. At $\omega_m \sim E_F$, the expression is more complex (Eq. (13) below). In particular, $\Sigma(k_F, \omega_m)$ becomes complex on the Matsubara axis¹, similar to the self-energy in SYK-type models [56, 57].

We also show that the momentum-dependent part of the self-energy remains small as long as $\lambda_E < 1$.

We discuss the solution of the Eliashberg equations first and then use it to analyze the strength of vertex corrections.

A. Solution of Eliashberg equations

The equations for $\Sigma(k)$ and $\Pi(q)$ are coupled and in principle have to be solved together. We argue, however, that for $\lambda_E \ll 1$, the two equations can be solved independently. To demonstrate this, we make two assumptions and verify both a posteriori. First, we assume that $\partial\Sigma/\partial\omega_m$ scales as λ and is large at strong coupling, while $\partial\Sigma(k)/\partial\epsilon_{\mathbf{k}}$ scales as λ_E and is small when $\lambda_E < 1$. Accordingly, we approximate $\Sigma(\mathbf{k}, \omega_m)$ by $\Sigma(\omega_m)$. Second, we assume that $\Sigma(\omega_m)$ is parametrically smaller than E_F .

Consider Eq. (4) for $\Pi(q)$ first. We express $\Pi(q) = \Pi(\mathbf{q}, \Omega_n)$ as a sum of static and dynamic pieces $\Pi(\mathbf{q}, 0) + \delta\Pi(\mathbf{q}, \Omega_n)$. The static part $\Pi(\mathbf{q}, 0)$ generally comes from fermions with energies of order E_F . It renormalizes ω_D and χ_0 , and may also give rise to a momentum dependence of the phonon propagator. For the purposes of this study we assume that this renormalization is already included into $\chi_0(q)$ and that $\chi_0(q)$ is given by Eq. (2) (for more discussions on this issue see Refs. [8 and 58]). The

¹ This does not violate Kramers-Kronig relations, but when $\Sigma(k_F, \omega_m)$ is complex, both $\Sigma'(k_F, \omega)$ and $\Sigma''(k_F, \omega)$ on the real axis have even and odd components in ω .

second, dynamical, term comes from fermions with low

energies. For this term we obtain, after integrating over the angle between phonon \mathbf{q} and fermionic \mathbf{k} :

$$\delta\Pi(q) = -2i\tilde{g}^2 \int \frac{d\omega_m}{2\pi} \int \frac{kdk}{2\pi} \frac{1}{i\tilde{\Sigma}(\omega_m) - \epsilon_k} \left[\frac{\text{sgn}(\omega_m + \Omega_n)}{\sqrt{\frac{(kq)^2}{m^2} + \left(\tilde{\Sigma}(\omega_m + \Omega_n) + i\left(\epsilon_k + \frac{q^2}{2m}\right)\right)^2}} - \text{the same at } \Omega_n = 0 \right], \quad (6)$$

where for brevity we have defined $\tilde{\Sigma}(\omega_m) = \omega_m + \Sigma(\omega_m)$. For $\omega_m, \Omega_n \ll E_F$ and a generic $q \sim k_F$, the terms $\tilde{\Sigma}(\omega_m + \Omega_n)$ and ϵ_k under the square-root in Eq. (6) are much smaller than E_F and can be neglected compared to the two q -dependent terms, which are of order E_F . Equation (6) then simplifies to

$$\delta\Pi(q) = -2i\tilde{g}^2 N_F \int \frac{d\omega_m}{2\pi} \int \frac{d\epsilon_k}{i\tilde{\Sigma}(\omega_m) - \epsilon_k} \frac{\text{sgn}(\omega_m + \Omega_n) - \text{sgn}(\omega_m)}{v_F q \sqrt{1 - \left(\frac{q}{2k_F}\right)^2}} \quad (7)$$

where N_F is the density of states at the Fermi level. Integration over ϵ_k can now be extended to infinite limits (up to terms of order $\tilde{\Sigma}(\omega_m)/E_F \ll 1$) and yields $\int d\epsilon_k/(i\tilde{\Sigma}(\omega_m) - \epsilon_k) = -i\pi \text{sgn}(\omega_m)$. Integrating then over frequency, we obtain

$$\delta\Pi(q) = 2\tilde{g}^2 N_F \frac{\Omega_n}{v_F q \sqrt{1 - \left(\frac{q}{2k_F}\right)^2}}. \quad (8)$$

This expression is valid for $q < 2k_F$, which is the largest momentum transfer on the Fermi surface. At small q , it reduces to conventional Landau damping. Substituting $\delta\Pi$ into Eq. (1) and using Eq. (2), we obtain

$$\chi(\mathbf{q}, \Omega_n) = \frac{\chi_0}{\Omega_m^2 + \omega_D^2 + \gamma \frac{|\Omega_n|}{q \sqrt{1 - \left(\frac{q}{2k_F}\right)^2}}}, \quad (9)$$

where $\gamma = 2g^2/v_F$ and $g^2 = \tilde{g}^2 N_F \chi_0$.

It is convenient to measure q in units of $2k_F$ and Ω_n in units of ω_D . Introducing $\bar{q} = q/(2k_F)$ and $\bar{\Omega}_m = \Omega_n/\omega_D$, we re-express Eq. (9) as

$$\chi(\mathbf{q}, \Omega_n) = \frac{\chi_0}{\omega_D^2} \frac{1}{1 + \bar{\Omega}_m^2 + \frac{\lambda_E}{2} \frac{|\bar{\Omega}_m|}{\bar{q} \sqrt{1 - \bar{q}^2}}} \quad (10)$$

We see that the Landau damping term contains λ_E in the prefactor and is therefore small compared to one of the two other terms for all values of $\bar{\Omega}_m$ provided \bar{q} is not close to either zero or one.

We now substitute $\chi(\mathbf{q}, \Omega_n)$ from Eq. (10) into Eq. (3) for the self-energy. We compute separately $\Sigma(\mathbf{k}_F, \omega_m)$ and $\Sigma(\mathbf{k}, 0)$. The calculation of $\Sigma(\mathbf{k}_F, \omega_m)$ parallels the one for $\delta\Pi$: we first integrate over the angle between \mathbf{k}_F and \mathbf{q} and obtain

$$\Sigma(\omega_m) = \frac{g^2}{N_F} \int \frac{d\Omega_n}{2\pi} \int_0^{2k_F} \frac{q dq}{2\pi} \frac{1}{\Omega_m^2 + \omega_D^2 + \gamma \frac{|\Omega_n|}{q \sqrt{1 - (q/2k_F)^2}}} \frac{\text{sgn}(\omega_m + \Omega_n)}{\sqrt{(v_F q)^2 + \left(\tilde{\Sigma}(\omega_m + \Omega_n) + i\frac{q^2}{2m}\right)^2}}. \quad (11)$$

We set $\omega_m \ll E_F$ and assume and verify a posteriori that (i) typical q are of order k_F and typical Ω_n are of order ω_m and (ii) $\tilde{\Sigma}(\omega_m + \Omega_n)$ is parametrically smaller than E_F . We then evaluate the integrals over q and Ω_q neglecting the Landau damping term and $\tilde{\Sigma}(\omega_m + \Omega_n)$ under the square root. The calculation is elementary, and we obtain

$$\Sigma^{(E)}(\omega_m) = \lambda \omega_D \arctan\left(\frac{\omega_m}{\omega_D}\right), \quad (12)$$

where the index E denotes that this is the leading self-energy in Eliashberg theory. The self-energy behaves as $\lambda \omega_m$ at small frequencies and saturates at $(\pi/2)\lambda \omega_D$ at higher frequencies. We emphasize that the right-hand side of Eq. (12) does not depend on the self-energy and has the same form as if we used the Green's function for free fermions. The same holds for $\delta\Pi(q)$. In other words, to this accuracy, the one-loop self-consistent theory becomes equivalent to the one-loop perturbation the-

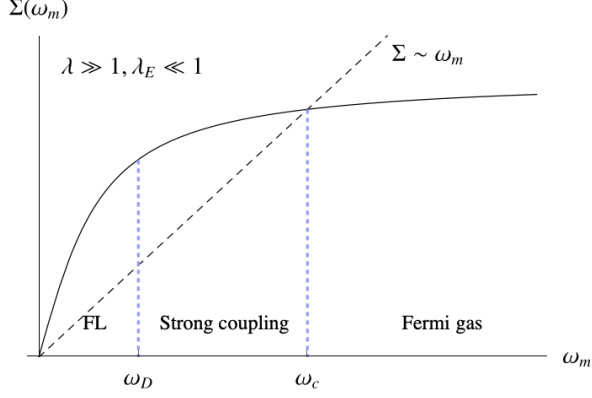


FIG. 2. (Color online) One-loop perturbative Eliashberg self-energy $\Sigma^{(E)}(\omega_m)$ at $\lambda \gg 1$, but $\lambda_E \ll 1$. The self-energy is linear in ω_m for $\omega_m < \omega_D$ and saturates at larger ω_m . It is larger than the bare ω_m for $\omega_m < \omega_c = (2/\pi)E_F\lambda_E$. For $\omega_m < \omega_D$, the system is in the Fermi liquid regime. In between ω_D and ω_c , it displays strong coupling, non-Fermi liquid behavior. At larger ω_m the self-energy is smaller than ω_m and the system behaves as a Fermi gas.

ory. We plot $\Sigma^{(E)}(\omega_m)$ in Fig. 2.

We next check the accuracy of approximations used to obtain Eq. (12). The typical q for the momentum integral are of order k_F and typical Ω_n for the frequency integral are of order ω_m , as we assumed. Further, the prefactor $\lambda\omega_D$ in Eq. (12) can be equivalently expressed as $\lambda_E E_F$. We see that for small λ_E , $\tilde{\Sigma}(\omega_m + \Omega_n) \ll E_F$. This justifies neglecting the self-energy term in Eq. (11) and in the calculation of $\delta\Pi(q)$. Also, the self-energy $\Sigma^{(E)}(\omega_m)$ exceeds the bare ω_m at small ω_m , up to $\omega_c = (\pi/2)\lambda\omega_D = (\pi/2)\lambda_E E_F$. As long as λ_E is small, ω_c is parametrically smaller than E_F .

B. Subleading terms in the self-energy

We now go beyond the leading term and estimate the subleading terms in the self-energy. They are the cor-

rections to the dynamic piece of the self-energy, arising from (i) keeping either the Landau damping term or (ii) keeping $\tilde{\Sigma}$ in right-hand-side of Eq. (11), and (iii) the correction from the static piece of the self-energy $\Sigma(\mathbf{k}, 0)$.

Keeping the Landau damping term in Eq. (11) leads to a relative correction $(1 + X(\omega_m))$, which is the largest at $\omega_m \sim \omega_D$, where $X \sim \lambda_E \log^2 \lambda_E$. This is a small correction when $\lambda_E \ll 1$. The leading correction from keeping $\tilde{\Sigma}(\omega_m + \Omega_n)$ in the integrand in Eq. (11) comes from $q \approx 2k_F$. This correction, which we label as $\delta\Sigma(\omega_m)$, is a bit tricky. First, there is a non-zero $\delta\Sigma(0)$. This contribution is purely imaginary in our notation of Matsubara self-energy as $i\Sigma(\omega_m)$, i.e., it is the real part of the actual self-energy. This term is often absorbed into the renormalization of the bare chemical potential. This is acceptable for the low-energy description, but we emphasize that $\text{Im}[\delta\Sigma(\omega_m)]$ generally depends on frequency. For this reason we don't subtract $\delta\Sigma(0)$ from $\delta\Sigma(\omega_m)$. Second, at $|\omega_m| < \omega_D$, the real part of $\delta\Sigma(\omega_m)$ is linear in ω_m , like $\Sigma^{(E)}(\omega_m)$, but with the prefactor $\lambda\sqrt{\lambda_E}$, which is smaller than that in Eq. (12) by $\sqrt{\lambda_E}$. For $\omega_D < |\omega_m| < E_F$, this correction becomes

$$\delta\Sigma(\omega_m) = -\omega_c \left(\frac{|\tilde{\Sigma}^{(E)}(\omega_m)|}{E_F} \right)^{1/2} \frac{1 + i \text{sgn } \omega_m}{\pi}. \quad (13)$$

Substituting $|\tilde{\Sigma}^{(E)}(\omega_m)| = |\omega_m| + \omega_c$, we find that for $|\omega_m| < \omega_c$ $\delta\Sigma(\omega_m)$ is smaller than $\Sigma^{(E)}$ by $(\omega_c/E_F)^{1/2} \sim (\lambda_E)^{1/2}$. At larger frequencies, $\delta\Sigma$ increases and at $\omega_m \sim E_F$ becomes of the same order as $\Sigma^{(E)}(\omega_m)$. We emphasize that this holds even when λ_E is small. At even larger $|\omega_m| > E_F$, the full self-energy has to be re-evaluated. We will not analyze this frequency range here.

We note that $\delta\Sigma(\omega_m)$ is a complex function of Matsubara frequency. Similar behavior of $\Sigma(\omega_m)$ has been previously reported in SYK-type systems [56, 57]. In our case the appearance of a complex $\Sigma(\omega_m)$ reflects the absence of particle-hole symmetry for a parabolic dispersion.

The computation of the static piece $\Sigma(\mathbf{k}, 0)$ proceeds in the same way. After angular integration, the left-hand side of Eq. (3) becomes

$$\Sigma(\mathbf{k}, 0) = \frac{g^2}{N_F} \int \frac{d\Omega_n}{2\pi} \int_0^{2k_F} \frac{qdq}{2\pi} \frac{1}{\Omega_m^2 + \omega_D^2 + \gamma \frac{|\Omega_n|}{q\sqrt{1-(q/2k_F)^2}}} \frac{\text{sgn}(\Omega_n)}{\sqrt{(v_F q)^2 + \left(\tilde{\Sigma}(\Omega_n) + i \left(\epsilon_k + \frac{q^2}{2m} \right) \right)^2}}. \quad (14)$$

Expanding in ϵ_k , we obtain

$$i\Sigma(\mathbf{k}, 0) = \epsilon_k \frac{2g^2}{N_F} \int \frac{d\Omega_n}{2\pi} \int_0^{2k_F} \frac{qdq}{2\pi} \frac{1}{\Omega_m^2 + \omega_D^2 + \gamma \frac{|\Omega_n|}{q\sqrt{1-(q/2k_F)^2}}} \frac{|\tilde{\Sigma}(\Omega_n)|}{\left(\tilde{\Sigma}^2(\Omega_n) + (v_F q)^2 (1 - (q/(2k_F)^2)) \right)^{3/2}}. \quad (15)$$

The largest contributions to the momentum integral

comes from small q and from $q \approx 2k_F$. Evaluating the

contributions from these two regions and performing the subsequent integration over frequency, we obtain

$$i\Sigma(\mathbf{k}, 0) = \lambda_E \epsilon_k. \quad (16)$$

As a consequence, the ϵ_k term in the bare Green's function gets multiplied by the factor $1 - \lambda_E$. This renormalization is small when λ_E is small.

Summarizing, we find that the condition $\lambda_E \ll 1$ allows us to (i) neglect the Landau damping term in the bosonic propagator, (ii) neglect the momentum dependence of the self-energy, and (iii) approximate $\Sigma(\omega_m)$ by the one-loop perturbative result Eq. (12). For the latter, the corrections, which make $\Sigma(\omega_m)$ complex, are parametrically small as long as ω_m remains below $\omega_c \sim \lambda_E E_F$. At larger frequencies, the corrections get stronger, and at $\omega_m \sim E_F$ become of the same order as

$\Sigma^{(E)}(\omega_m)$. To the best of our knowledge, this last result has not been presented earlier.

The smallness of corrections to the one-loop perturbative $\Sigma^{(E)}(\omega_m)$ can be understood by comparing relative energy scales and invoking the argument about slow phonons and fast electrons, which typically is reserved for vertex corrections. Indeed, the argument implies that for the same frequency, fermionic momenta are far smaller than phonon momenta. In our case, typical phonon momenta are of order k_F , while typical fermionic momenta are of order $\tilde{\Sigma}(\omega_m)/v_F$. For $|\omega_m| < \omega_c$, where Eq. (12) for $\Sigma^{(E)}(\omega_m)$ is rigorously justified, typical fermionic momenta are smaller by λ_E . In practical terms, this separation allows one to approximate $\int d^2k$ in the Eliashberg formula for the self-energy for a generic $\chi(q)$ by $2\pi N_F \int d\epsilon_{k'} d\theta$, where $\mathbf{k}' = \mathbf{k} + \mathbf{q}$ and θ is the angle between \mathbf{k} and \mathbf{k}' , with both set on the Fermi surface. Eq. (3) then reduces to

$$\Sigma(\mathbf{k}, \omega_m) = \Sigma(\omega_m) = i\tilde{g}^2 N_F \int \frac{d\Omega_n}{2\pi} \chi_L(\Omega_n) \int d\epsilon_{\mathbf{k}'} G(\mathbf{k}', \omega_m + \Omega_n) = g^2 \int_0^{\omega_m} d\Omega_n \chi_L(\Omega_n), \quad (17)$$

where we have defined $\chi_L(\Omega_m) = (1/2\pi) \int d\theta \chi(\Omega_n, \theta)$. For our choice of phonon propagator $\chi(q)$ this reduces to the Eliashberg result Eq. (12). On the real frequency axis, the same consideration yields $\Sigma''(\omega) = -g^2 \int_0^\omega \chi_L''(\Omega) d\Omega$. In the literature, $g^2 \chi_L''(\Omega)$ is often denoted $\alpha^2 F(\Omega)$ [7, 59].

This analysis is supported by numerical solution of the Eliashberg equations. The numerical results are obtained by solving the self-consistent equations iteratively, where the momentum integrations are carried out without making extra approximations. We found Fermi liquid behavior of $\Sigma(\omega_m)$ at frequencies below ω_D . The self-energy then saturates for an intermediate frequency range up to ω_c , and eventually decreases at larger frequencies. We see that the imaginary part of $\Sigma(\omega_m)$ remains weakly frequency dependent up to $\omega_m \sim E_F$. The implication is that in the whole range $|\omega_m| \leq E_F$ it can be absorbed into the renormalization of the chemical potential. The difference between $\text{Re}[\Sigma(\omega_m)]$ and the $\Sigma^{(E)}(\omega_m)$ is shown in Fig. 3 (b). It matches $\delta\Sigma(\omega_m)$ from Eq. Eq. (13). In particular, we verified in Fig. 3 (c) that for $\omega_m > \omega_D$, $\text{Re}[\Sigma(\omega_m)] - \Sigma^{(E)}(\omega_m)$ scales as $\lambda_E^{1/2}$, in agreement with Eq. (13).

C. Vertex corrections

The one-loop vertex correction is shown graphically in Fig. 4 (a) and the associated two-loop self-energy with vertex renormalization is shown in Fig. 4 (b). The relative strength of the two-loop self energy compared to the one-loop one determines the validity of the Eliash-

berg theory. The one-loop vertex correction and the two-loop self-energy in 2D have been discussed previously in Ref. [8]. Here we reproduce these earlier results and add additional details.

For external fermionic $k = (\mathbf{k}_F, \omega_m)$ and phononic $q = (\mathbf{q}, \Omega_n)$, the analytical expression for the one-loop $\Delta g/g$ is

$$\begin{aligned} \frac{\Delta g(k, q)}{g} &= -i \frac{g^2}{2\pi} \int \frac{d\omega'_m}{(\omega_m - \omega'_m)^2 + \omega_D^2} \int \frac{d\epsilon_{k'}}{i\tilde{\Sigma}(\omega'_m) - \epsilon_{k'}} \\ &\times \frac{\text{sgn}(\omega'_m + \Omega_n)}{\sqrt{(v_F |\mathbf{q}|)^2 + \left(\tilde{\Sigma}(\omega'_m + \Omega_n) + i(\epsilon_{k'} + \mathbf{q}^2/2m)\right)^2}}. \end{aligned} \quad (18)$$

For $\mathbf{q} = 0$, the vertex correction is *not small* for any λ_E as it must satisfy the Ward identity associated with the conservation of the total density:

$$\frac{g_{\text{full}}(k, q)}{g} = \frac{g_{\text{full}}(\omega_m, \Omega_m, \mathbf{q} = 0)}{g} = \frac{\tilde{\Sigma}(\omega_m + \Omega_n) - \tilde{\Sigma}(\omega_m)}{\Omega_n}. \quad (19)$$

At small ω_m , $\tilde{\Sigma}(\omega_m) \approx \omega_m(1 + \lambda)$, hence by the Ward identity $g_{\text{full}}(k, \Omega_m, \mathbf{q} = 0)/g = \lambda \gg 1$.

Putting $\mathbf{q} = 0$ in Eq. (18) and setting $\Omega_n > 0$ for definiteness, we perform the $\epsilon_{k'}$ integration to obtain the one-loop vertex correction

$$\begin{aligned} \frac{\Delta g(\omega_m, \Omega_m, \mathbf{q} = 0)}{g} &= g^2 \int_{-\Omega_n}^0 \frac{d\omega'_m}{(\omega_m - \omega'_m)^2 + \omega_D^2} \\ &\times \frac{1}{\tilde{\Sigma}(\omega'_m + \Omega_n) - \tilde{\Sigma}(\omega'_m)}. \end{aligned} \quad (20)$$

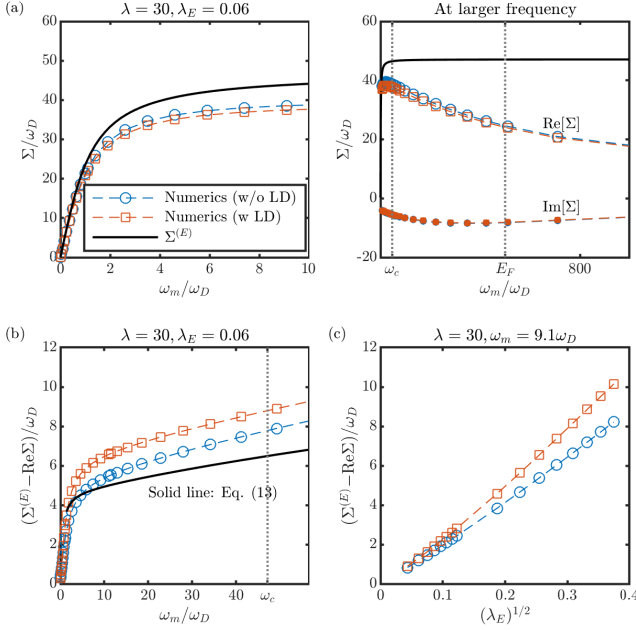


FIG. 3. (Color online) One-loop electron self-energy. We set $E_F = 500\omega_D$, $\lambda_E \simeq 0.06$ and $\lambda \simeq 30$. (a) Numerical solution of the self-consistent equation Eq. (3) v.s. the perturbative expression, $\Sigma^{(E)}(\omega_m)$, Eq. (12). The numerical results obtained with and without Landau damping are red and blue curves, respectively. The right panel in (a) displays the self-energy at larger frequencies, comparable to E_F . Both $\text{Re}[\Sigma(\omega_m)]$ and $\text{Im}[\Sigma(\omega_m)]$ evolve with frequency ($\text{Im}[\Sigma(\omega_m)]$ is a constant at smaller ω_m and is not shown in the left panel). (b) A comparison between $\Sigma^{(E)}(\omega_m) - \Sigma(\omega_m)$ and $\delta\Sigma(\omega_m)$ from Eq. (13). (c) The verification of the scaling relation $(\Sigma^{(E)}(\omega_m) - \Sigma(\omega_m))/\omega_D \propto \sqrt{\lambda_E}$ for a given $\omega_m/\omega_D \simeq 9$.

For small $\omega_m, \Omega_n < \omega_D$, this gives $\Delta g(k, \Omega_m, \mathbf{q} = 0)/g = \lambda/(1 + \lambda)$, which approaches 1 in the strong coupling limit $\lambda \gg 1$. Because $\Delta g(k, \Omega_m, \mathbf{q} = 0)/g \approx 1$, we need to include higher-order vertex corrections. A simple experimentation shows that at large λ , relevant diagrams for the full vertex $g_{\text{full}}(k, \Omega_m, \mathbf{q} = 0)$ form ladder series, shown in Fig. 4. The corresponding integral equation at arbitrary ω_m and Ω_n has the form

$$\frac{g_{\text{full}}(\omega_m, \Omega_m, \mathbf{q} = 0)}{g} = 1 + g^2 \int_{-\Omega_n}^0 \frac{d\omega'_m}{(\omega_m - \omega'_m)^2 + \omega_D^2} \times \frac{g_{\text{full}}(\omega'_m, \omega_m, \mathbf{q} = 0)/g}{\tilde{\Sigma}(\omega'_m + \Omega_n) - \tilde{\Sigma}(\omega'_m)}. \quad (21)$$

One can straightforwardly check that the solution of this equation is given by Eq. (19), in agreement with the Ward identity associated with the conservation of the total density.

For $q \sim k_F$ the result is different. Integrating over $\epsilon_{k'}$ in Eq. (18) in infinite limits we now obtain

$$\begin{aligned} \frac{\Delta g(k, q)}{g} &= \frac{g^2}{2} \int d\omega'_m \frac{1 - \text{sgn}(\omega'_m + \Omega_n) \text{sgn}(\omega'_m)}{(\omega_m - \omega'_m)^2 + \omega_D^2} \frac{1}{\sqrt{(v_F q)^2 + \left(\tilde{\Sigma}(\omega'_m + \Omega_n) - \tilde{\Sigma}(\omega'_m) + iq^2/(2m)\right)^2}} \\ &\approx \frac{g^2}{4E_F \bar{q} \sqrt{1 - \bar{q}^2}} \int d\omega'_m \frac{1 - \text{sgn}(\omega'_m + \Omega_n) \text{sgn}(\omega'_m)}{(\omega_m - \omega'_m)^2 + \omega_D^2} \end{aligned} \quad (22)$$

where $\bar{q} = |\mathbf{q}|/(2k_F)$. Evaluating the remaining frequency integral, we find

$$\frac{\Delta g(k, q)}{g} = \frac{\lambda_E}{4\bar{q}} \frac{1}{\sqrt{1 - \bar{q}^2}} \left(\arctan\left(\frac{\omega_m + \Omega_n}{\omega_D}\right) - \arctan\left(\frac{\omega_m}{\omega_D}\right) \right). \quad (23)$$

We see that for generic $\omega_m \sim \Omega_n \sim \omega_D$, $\Delta g(k, q)/g \sim \lambda_E$, as long as \bar{q} is not too close to either 0 or 1.

There is a caveat here: the vertex correction in Eq. (23) decreases at $\omega_m, \Omega_n > \omega_D$. On a more careful look, we found that this is an artifact of integrating over $\epsilon_{k'}$ in Eq. (18) in infinite limits instead of introducing the lower cutoff at $-E_F$. Repeating the integration with this lower

cutoff we find that even at $\omega_m, \Omega_n > \omega_D$, the vertex correction remains of the form

$$\frac{\Delta g(k, q)}{g} = C(q) \frac{\lambda_E}{4} \frac{1}{\bar{q} \sqrt{1 - \bar{q}^2}}, \quad (24)$$

where $C(q) = O(1)$.

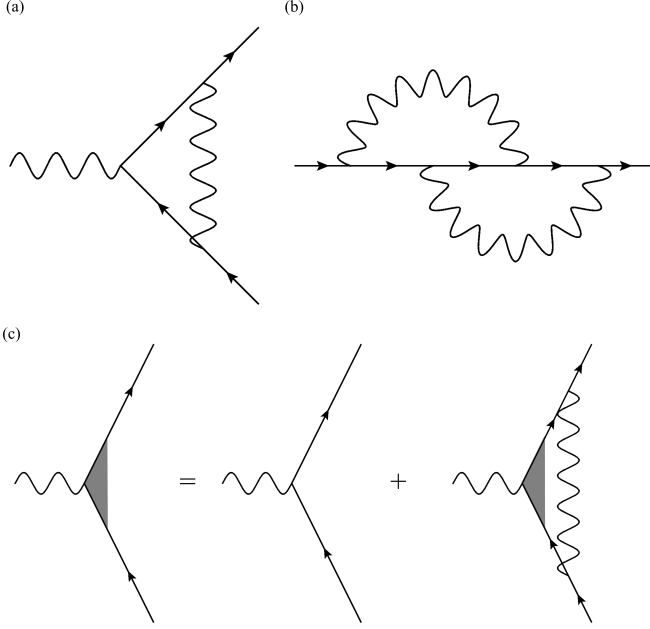


FIG. 4. Relevant diagrams: (a) one-loop vertex correction, (b) two-loop electron's self-energy with vertex correction included, (c) ladder series for the vertex.

We plot the one-loop vertex correction as a function of q and $\Omega_n \sim \omega_D$ in Fig. 5. We show the results obtained using both an infinite energy cutoff ($-\infty < \epsilon_{\mathbf{k}} < \infty$) and a finite energy cutoff ($-E_F < \epsilon_{\mathbf{k}} < \infty$). The two results for $\Delta g/g$ do differ, most notably at $|\mathbf{q}| \sim k_F$, but both remain of order λ_E for a generic q .

We next substitute $\Delta g(k, q)$ into the two-loop diagram for the self-energy, Fig. 4 (b). We label this contribution as $\Sigma^{(2)}(\omega_m)$. Analyzing the integral over phononic q , we find that it is singular at q close to 0 and $2k_F$, where the vertex correction is enhanced, but the singularity is only logarithmic. Evaluating the full integral, we find, in agreement with Ref. [8],

$$\Sigma^{(2)}(\omega_m) \sim (\lambda_E |\log \lambda_E|) \Sigma^{(E)}(\omega_m) \quad (25)$$

We see that as long as λ_E is small, the two-loop $\Sigma^{(2)}(\omega_m)$ is parametrically smaller than the Eliashberg self-energy, despite that the integral comes from q where the vertex correction is singular. This singularity only accounts for $|\log \lambda_E|$ in the prefactor.

In Fig. 6 we present the result of numerical evaluation of $\Sigma^{(2)}(\omega_m)$ using an infinite cutoff in the integration over $\epsilon_{\mathbf{k}}$. We see that the ratio $\Sigma^{(2)}/\Sigma^{(E)}$ is approximately constant at $\omega_m \leq \omega_D$ and as a function of λ_E does scale as $\lambda_E |\log \lambda_E|$. The drop of the ratio $\Sigma^{(2)}/\Sigma^{(E)}$ at larger frequencies is likely an artifact of an infinite cutoff. In any case, the ratio $\Sigma^{(2)}/\Sigma^{(E)}$ is small at small λ_E . For completeness we computed this ratio at $\lambda_E = 1$ and found that it still remains numerically small over the wide range of $\omega_D < \omega_m < E_F$ ($\Sigma^{(2)}/\Sigma^{(E)} \simeq 0.062$ at $\omega_m = \omega_D$).

For comparison with the Ising-nematic/Ising-ferromagnetic case below, we analyze how $\Delta g/g$

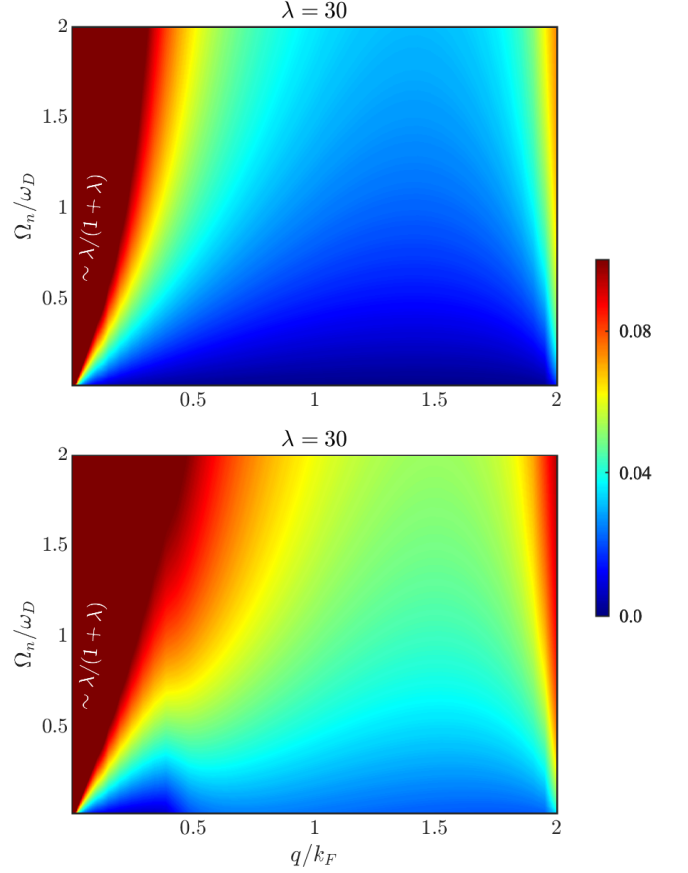


FIG. 5. (Color online) Numerical results for $\Delta g/g$ as a function of bosonic frequency Ω_n and momentum q for $\lambda = 30$, $\lambda_E = 0.06$, and external fermionic frequency $\omega_m = 0.1\omega_D$. Panels (a) and (b): the results of the calculations using infinite cutoff in the integration over dispersion, $-\infty < \epsilon_{\mathbf{k}} < \infty$, and a finite cutoff $-E_F < \epsilon_{\mathbf{k}} < \infty$, respectively.

depends on characteristic momenta and frequencies. For definiteness we set $\mathbf{k} = \mathbf{k}_F$, $\Omega_n \sim \omega_D$ and vary ω_m and q_{\perp} and q_{\parallel} , which are typical momentum components transverse and along the Fermi surface. At $\omega_D < \omega_m < \omega_c = (\pi/2)\lambda_E E_F$, where the self-energy $\Sigma(\omega_m)$ exceeds bare ω_m , which are typical momentum components transverse and along the Fermi surface. After a simple experimentation we found after a simple experimentation

$$\frac{\Delta g(k, q)}{g} \sim \frac{\omega_c}{\omega_c + v_F q_{\perp} + \frac{q_{\parallel}^2}{2m}}. \quad (26)$$

Typical $v_F q_{\perp}$ are of order ω_c , typical $q_{\parallel}^2/(2m)$ are of order E_F . For these momenta,

$$\frac{\Delta g(k, q)}{g} \sim \frac{v_F q_{\perp}^{\text{typ}}}{v_F q_{\perp}^{\text{typ}} + \frac{(q_{\parallel}^{\text{typ}})^2}{2m}}. \quad (27)$$

The “fast electrons/slow bosons” criterion requires typical q_{\perp} to be much smaller than typical q_{\parallel} . This holds for

$\lambda_E \ll 1$ because typical $q_{\perp}^{\text{typ}} \sim \lambda_E k_F$, while $q_{\parallel}^{\text{typ}} \sim k_F$. The strength of the vertex correction in Eq. (27) is, however, determined by the ratio of typical energies $v_F q_{\perp}^{\text{typ}} / ((q_{\parallel}^{\text{typ}})^2 / 2m)$ rather than typical momenta. In general, the ratios of typical energies and typical momenta are not the same, but in the electron-phonon case, $v_F q_{\perp}^{\text{typ}} / ((q_{\parallel}^{\text{typ}})^2 / 2m) \sim \lambda_E$ is the same as $q_{\perp}^{\text{typ}} / q_{\parallel}^{\text{typ}}$ because $q_{\parallel}^{\text{typ}} \sim k_F$. As a consequence, the single parameter λ_E allows one to simplify the Eliashberg equations and keep vertex corrections small.

For larger ω_m , we found

$$\frac{\Delta g(k, q)}{g} \sim \frac{\omega_c}{\omega_m + v_F q_{\perp}^{\text{typ}} + \frac{(q_{\parallel}^{\text{typ}})^2}{2m}}. \quad (28)$$

Now typical $v_F q_{\perp}$ are of order ω_m , while typical $q_{\parallel}^2 / (2m)$ are still of order E_F . For these momenta,

$$\frac{\Delta g(k, q)}{g} \sim \frac{\omega_c}{\omega_m + E_F} \sim \lambda_E \frac{1}{1 + \frac{\omega_m}{E_F}} \quad (29)$$

We see that the vertex correction remains of order λ_E .

D. Summary of Section II

There are three energy scales in the electron-phonon problem: the bosonic energy ω_D , the dimension-full electron-phonon coupling g and the Fermi energy E_F . This allows one to introduce two dimensionless ratios $\lambda = g^2 / \omega_D^2$ and $\lambda_E = g^2 / (E_F \omega_D) = \lambda \omega_D / E_F$. The latter is a small parameter for the Eliashberg theory. The strong coupling regime occurs at $\lambda \gg 1$, $\lambda_E \ll 1$. In this regime, the system displays Fermi liquid behavior at $\omega < \omega_D$, non-Fermi liquid behavior with $\Sigma(\omega_m) \approx (\pi/2)\lambda\omega_D = (\pi/2)\lambda_E E_F$ at $\omega_D < \omega_m < \omega_c$, where $\omega_c \sim \lambda\omega_D \sim E_F \lambda_E$, and Fermi-gas behavior at larger frequencies.

The three key results for electron-phonon system, which we will later use in comparison with the behavior near a nematic QCP, are the following.

First, Eliashberg theory is rigorously justified even at strong coupling $\lambda \gg 1$, as long as the Eliashberg parameter $\lambda_E \ll 1$. Two-loop vertex corrections to self-consistent one-loop Eliashberg theory are small in λ_E for all frequencies ω_m .

Second, the same small parameter λ_E also simplifies the calculations within Eliashberg theory: the Landau damping of phonons can be neglected in the calculation of the self-energy, and the self-energy $\Sigma(\mathbf{k}, \omega_m)$ can be approximated by the local $\Sigma(\omega_m)$ and computed perturbatively rather than self-consistently. This holds for $\omega_m < E_F$. At larger ω_m , “fast electron/slow boson” criterion is not valid, and the Eliashberg equation (3) has to be re-analyzed.

Third, vertex corrections to Eliashberg theory become $O(1)$ at $\lambda_E = O(1)$, and become parametrically large at

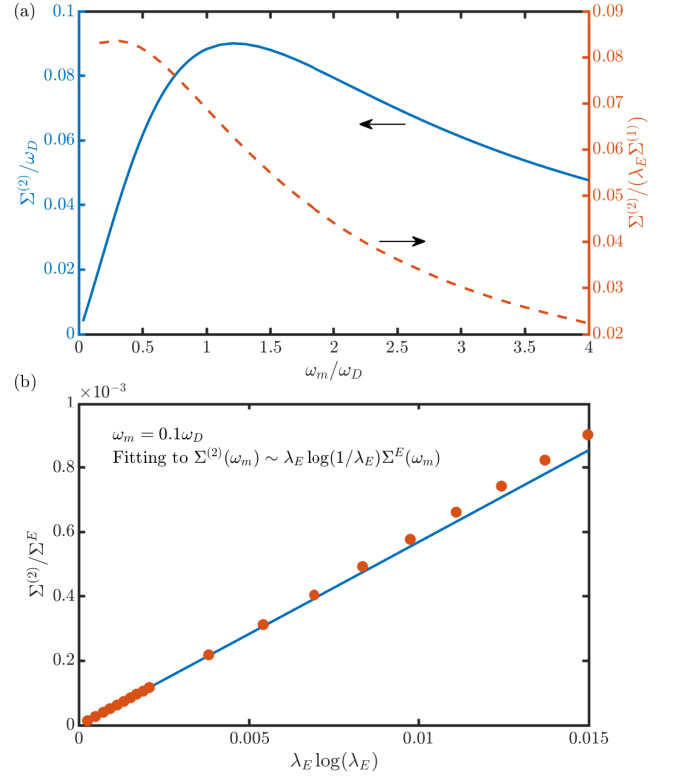


FIG. 6. (a) Blue line – the two-loop electron self-energy, obtained using infinite cutoff for integration over fermionic dispersion. The parameters are the same as in Fig. 3: $\lambda = 30$ and $\lambda_E = 0.06$. Brown line – the ratio $\Sigma^{(2)} / (\lambda_E \Sigma^{(E)})$. The ratio approaches a constant at $\omega_m \rightarrow 0$ and evolves at $\omega_m \sim \omega_D$. The drop of the ratio at larger ω_m is likely an artifact of using infinite cutoff for the integration over the dispersion. (b) Numerical verification of the scaling of $\Sigma^{(2)}(\omega_m) / \Sigma^{(E)}(\omega_m) \sim \lambda_E |\log \lambda_E|$ at small ω_m .

$\lambda_E > 1$, except for the smallest frequencies $\omega_m < \omega_D / \lambda_E$, where they remain small. Because $\lambda_E = g^2 / (\omega_D E_F)$, this result implies that one cannot extend the Eliashberg theory of electron-phonon interaction to $\omega_D = 0$. For a finite E_F , Eliashberg theory is valid only when ω_D exceeds a certain value. At smaller ω_D , a new description is required.

We summarize these results in Fig. 7.

III. QUANTUM CRITICAL METAL

We now analyze the validity of an Eliashberg-type description of a system of electrons interacting with soft collective bosons representing order parameter fluctuations. For definiteness, we focus on the system near a $= 0$ Ising-ferromagnetic or Ising-nematic instability (see e.g., Ref. 40 and references therein). The two cases differ by the form-factor for fermion-boson coupling (it has d-wave form in the Ising-nematic case). We verified that

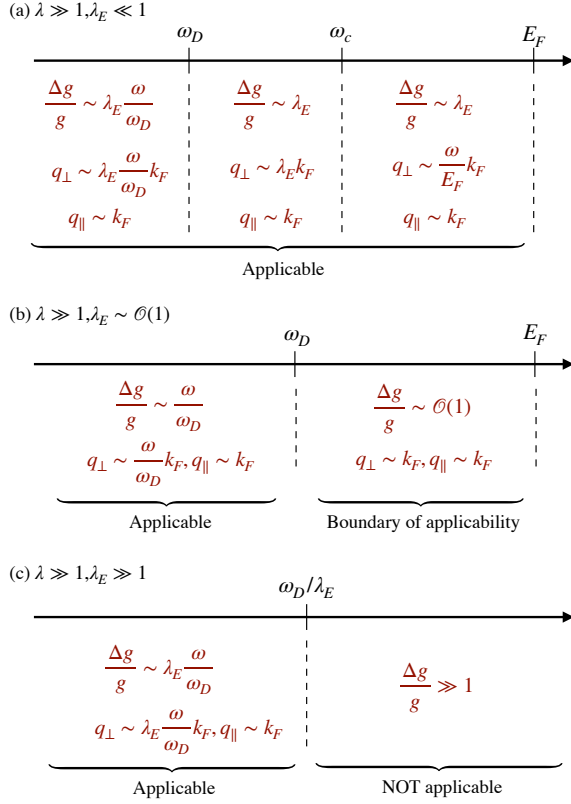


FIG. 7. Illustration of the applicability of Eliashberg theory for electron-phonon interactions at different energy scales: (a) $\lambda \gg 1$, $\lambda_E \ll 1$; (b) $\lambda \gg 1$, $\lambda_E \sim 1$; (c) $\lambda \gg 1$, $\lambda_E \gg 1$.

the form-factor does not play a critical role in our considerations and can be safely set to one (see also Ref. [60]). We do not consider $SU(2)$ ferromagnetic case as the ordering transition there is affected by non-analytic corrections to spin susceptibility [61].

The set of Eliashberg-type equations is the same as Eqs. (3) and (4). Like for the electron-phonon case, it can be either derived diagrammatically [13, 62] as self-consistent one-loop equations, or obtained as stationary solutions of the Luttinger-Ward functional [40]. The bare fermionic Green's function is the same as in Eq. (2). For the bare bosonic propagator we choose the conventional Ornstein-Zernike form

$$\chi_0(\mathbf{q}) = \frac{\chi_0}{\xi^{-2} + |\mathbf{q}|^2}. \quad (30)$$

We will define a related frequency scale $\omega_D^* = v_F \xi^{-1}$. The full generic $\chi_0(\mathbf{q})$ also contains a dynamical $(\Omega_n/v_F)^2$ term, however, this term only becomes relevant at frequencies above the upper limit of quantum-critical behavior (see below). For this reason, we neglect the Ω_n^2 term in χ_0 in our analysis.

We label the fermion-boson coupling as \tilde{g} and introduce the effective dimension-full interaction $g^* = \tilde{g}^2 \chi_0$. Like in the electron-phonon case, there are three energy scales in this model: the effective coupling g^* , the bosonic

energy ω_D^* , and the Fermi energy E_F . We again assume that E_F is the largest scale in the problem, i.e., set $E_F \gg \omega_D^*, g^*$. Like before, one can construct two dimensionless ratios out of these couplings. We choose them to be

$$\lambda^* = \frac{g^*}{4\pi\omega_D^*}, \quad \lambda_E^* = \frac{g^*}{E_F}. \quad (31)$$

By construction, $\lambda_E^* \ll 1$, but λ^* can be either small or large. We will see that the self-energy at the lowest ω_m is $\Sigma = \lambda^* \omega_m$, i.e., $\lambda^* \sim 1$ separates weak coupling ($\lambda^* < 1$) and strong coupling ($\lambda^* > 1$) regimes, respectively.

A. Solution of Eliashberg equations

The analysis of the Eliashberg equations for fermions near a $q = 0$ QCP has been done before in various contexts [18, 31–38, 40]. We list the existing results and present some new ones that will allow a direct comparison with the electron-phonon case.

We first discuss the calculation of the fermionic self-energy and bosonic polarization. Like for the electron-phonon case, we assume and then verify that the self-energy $\Sigma(k, \omega)$ can be approximated by a local $\Sigma(\omega_m)$. For such a self-energy, earlier calculations found that for $v_F |\mathbf{q}| \gg |\Omega_n|$, and both much smaller than E_F , the polarization can be written as a sum of static and dynamic terms $\Pi(\mathbf{q}, \Omega_n) = \Pi(\mathbf{q}, 0) + \delta\Pi(\mathbf{q}, \Omega_n)$, where $\delta\Pi(\mathbf{q}, \Omega_n)$ has the form of Landau damping:

$$\delta\Pi(\mathbf{q}, \Omega_n) = \frac{g^* k_F}{\chi_0 \pi v_F^2} \frac{|\Omega_n|}{|\mathbf{q}|}. \quad (32)$$

This is similar to the electron-phonon polarization, Eq. (8) at $q \ll k_F$. Like there, the prefactor for the Landau damping term does not depend on the fermionic self-energy, i.e., Eq. (32) has the same form for free fermions and for interacting fermions. We incorporate the static $\Pi(\mathbf{q}, 0)$ into $\chi_0(|\mathbf{q}|)$, like we did for the electron-phonon case. Substituting into $\chi(q)$, we then obtain

$$\chi(q) = \frac{\chi_0}{\xi^{-2} + |\mathbf{q}|^2 + \alpha \frac{|\Omega_n|}{|\mathbf{q}|}}, \quad (33)$$

where $\alpha = \frac{g^* k_F}{\pi v_F^2}$. Because relevant \mathbf{q} are much smaller than k_F , the dynamical Landau damping term in $\chi(q)$ becomes relevant when Ω_n exceeds the scale

$$\omega^* = \frac{(\omega_D^*)^3}{g^* E_F} \sim \frac{\omega_c^*}{(\lambda^*)^3}, \quad (34)$$

where $\omega_c^* \sim E_F (\lambda_E^*)^2$ (the exact definition with the prefactor is in Eq. (39) below). Because we consider λ_E^* to be small, we have $\omega_c^* \ll E_F$. We will see that at strong coupling, when $\lambda^* > 1$, ω^* is the upper edge of Fermi liquid behavior, while $\omega_c^* > \omega^*$ is the upper edge of quantum-critical non-Fermi liquid behavior. This identification

implies that the Landau damping term in Eq. (33) is essential outside of the Fermi liquid regime. This distinguishes this case from the electron-phonon one, where relevant $q \sim k_F$ and the Landau damping term is irrelevant for all frequencies, as long as λ_E is small.

$$\Sigma(\omega_m) = g^* \int \frac{d\Omega_n}{2\pi} \int \frac{d^2\mathbf{q}}{4\pi^2} \frac{1}{\xi^{-2} + |\mathbf{q}|^2 + \alpha \frac{|\Omega_n|}{|\mathbf{q}|}} \frac{1}{i\tilde{\Sigma}(\omega_m + \Omega_n) - \epsilon_{\mathbf{k}_F + \mathbf{q}}}, \quad (35)$$

where, as before, $\tilde{\Sigma}(\omega_m) = \omega_m + \Sigma(\omega_m)$. A straightforward analysis of the relevant scales in this equation shows that relevant $\Omega_n \sim \omega_m$, relevant $\epsilon_{\mathbf{k}_F + \mathbf{q}} \sim \tilde{\Sigma}(\omega_m)$, and relevant $|\mathbf{q}| \sim \xi^{-1}$ for $\omega_m < \omega^*$ and $|\mathbf{q}| \sim (\alpha|\omega_m|)^{1/3}$ for $\omega_m > \omega^*$. For the fermionic dispersion $\epsilon_{\mathbf{k}_F + \mathbf{q}}$ we will be using

$$\epsilon_{\mathbf{k}_F + \mathbf{q}} = v_F q_\perp + \frac{q_\parallel^2}{2m}, \quad (36)$$

where q_\perp and q_\parallel are momentum components perpendicular and parallel to the Fermi surface (along \mathbf{k}_F and perpendicular to \mathbf{k}_F , respectively). Comparing relevant scales, we immediately see that for $\lambda_E^* \ll 1$, relevant q_\parallel are much larger than q_\perp as long as $\omega_m < \omega_{\max} \equiv (g^* E_F)^{1/2} \sim \omega_c^*/(\lambda_E^*)^{3/2}$. For frequencies below ω_{\max} we then can factorize the momentum integration, i.e., re-express $d^2\mathbf{q}$ as $(1/v_F)d\epsilon_{\mathbf{k}_F + \mathbf{q}}dq_\parallel$ and integrate the fermionic propagator over $\epsilon_{\mathbf{k}_F + \mathbf{q}}$ and bosonic $\chi(|\mathbf{q}|, \Omega_n) \approx \chi(|q_\parallel|, \Omega_n)$ over q_\parallel . Using

$$\int \frac{d\epsilon}{i\tilde{\Sigma}(\omega_m + \Omega_n) - \epsilon} = -i\pi \text{sgn}(\omega_m + \Omega_n) \quad (37)$$

we find that the right-hand side of Eq. (3) does not depend on the self-energy, i.e., has the same form as for free fermions. Completing the integration, we obtain

$$\Sigma^{(E)}(\omega_m) = \lambda^* \omega_m F\left(\frac{\omega_m}{\omega^*}\right) \quad (38)$$

in terms of a function F , where $F(0) = 1$ and $F(x \gg 1) = ((4\pi)^{1/3}/\sqrt{3})/x^{1/3}$. One can obtain the full analytical formula for $F(x)$, but it is rather cumbersome and not particularly enlightening so we do not present it here.

At small frequencies, $\Sigma^{(E)}(\omega)$ has the Fermi liquid form, $\Sigma(\omega_m) = \lambda^* \omega_m$. For large λ^* , $\Sigma^{(E)}(\omega_m) \gg \omega_m$. In this regime, relevant $q_\parallel \sim \xi^{-1}$ while relevant $q_\perp \sim (\omega_m/\omega^*)(\omega_D^*/E_F)\xi^{-1}$ are far smaller. This is consistent with the “fast electrons/slow bosons” criterion. In this regime, we also have typical $q_\parallel^2/k_F \sim (\omega_D^*/E_F)\xi^{-1}$. This will be relevant to our analysis of vertex corrections below.

At larger $\omega_m > \omega^*$, the self-energy crosses over to a non-Fermi liquid, quantum-critical form $\Sigma^{(E)}(\omega_m) =$

We now substitute $\chi(q)$ from Eq. (33) into the formula for the self-energy, Eq. (3). As before, we assume and verify that $\Sigma(\mathbf{k}, \omega_m) \approx \Sigma(\mathbf{k}_F, \omega_m) = \Sigma(\omega_m)$. The self-consistent equation for $\Sigma(\omega_m)$ is

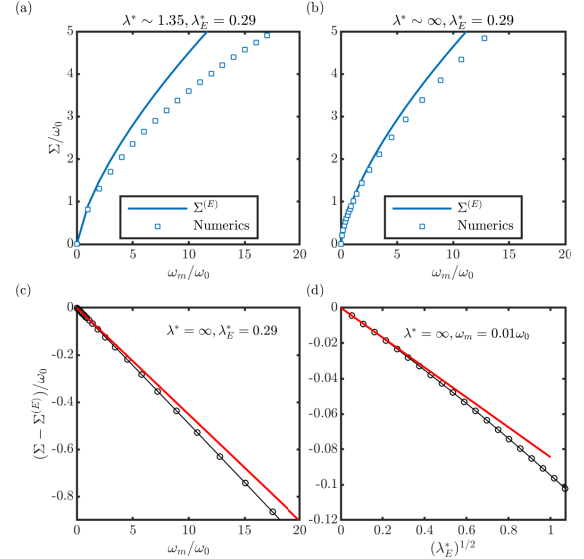


FIG. 8. (Color online) Numerical results for the self-energy in Eliashberg theory for $\lambda_E^* = 0.29$ at (a) intermediate coupling $\lambda^* = 1.35$ and (b) infinitely strong coupling $\lambda^* = \infty$ at a QCP. The solid line is the perturbative expression $\Sigma^{(E)}$, Eq. (38), obtained by factorizing the momentum integration, and squares are the numerical solutions of the full equation (35) for $\Sigma(\omega_m)$. (c) The difference between $\Sigma(\omega_m)$ and $\Sigma^{(E)}(\omega_m)$ at the QCP ($\lambda^* = \infty$). Red solid line is the analytic result, Eq. (41). (d) Verification of the scaling relation $\Sigma(\omega_m) - \Sigma^{(E)}(\omega_m) \propto (\lambda_E^*)^{1/2}$ (red line) at a particular ω_m .

$\omega_m^{2/3}(\omega_c^*)^{1/3}$, where

$$\omega_c^* = \frac{1}{16\pi^2\sqrt{27}} \frac{(g^*)^2}{E_F} = \frac{1}{16\pi^2\sqrt{27}} E_F (\lambda_E^*)^2. \quad (39)$$

In terms of ω_c^* ,

$$\omega^* = \frac{\sqrt{27}}{8} \frac{\omega_c^*}{(\lambda^*)^3}, \omega_{\max} = 16\pi^2\sqrt{27} \frac{\omega_c^*}{(\lambda_E^*)^{3/2}} \quad (40)$$

In the quantum-critical regime, relevant $q_\parallel \sim (\alpha\omega_m)^{1/3}$ and relevant $q_\perp \sim \omega_m^{2/3}\omega_0^{1/3}/v_F$. These can be re-expressed as $q_\parallel \sim k_F(\omega_m/\omega_c^*)^{1/3}\lambda_E^*$ and $q_\perp \sim k_F(\omega_m/\omega_c^*)^{2/3}(\lambda_E^*)^2$. We see that relevant q_\parallel are again

larger than q_\perp , i.e., the “fast electrons/slow bosons” criterion is well satisfied when λ_E^* is small. The conditions look similar to the Fermi liquid regime, but there is one important difference: relevant values of q_\parallel^2/k_F are now comparable to relevant q_\perp . We will discuss below how this affects vertex corrections.

At even larger $\omega_m > \omega_c^*$, the self-energy becomes smaller than ω_m , although it still preserves its $\omega_m^{2/3}$ form. In this last regime relevant $q_\parallel \sim (\alpha\omega_m)^{1/3}$, like before, but relevant $q_\perp \sim \omega_m/v_F$. These can be re-expressed as $q_\parallel \sim k_F(\lambda_E^*)^{1/2}(\omega_m/\omega_{\max})^{1/3}$ and $q_\perp \sim k_F(\lambda_E^*)^{1/2}(\omega_m/\omega_{\max})$. The “fast electrons/slow bosons” criterion is satisfied as long as $\omega_m < \omega_{\max}$.

At $\omega_m > \omega_{\max}$ the criterion of fast electrons and slow bosons is no longer valid and, as a result, one cannot factorize the momentum integration. In this regime, however, the Ω_m^2 term in the bare $\chi_0(q)$ cannot be neglected. We don’t analyze this high-frequency regime here.

B. Subleading terms in the self-energy

As for the electron-phonon case, there are three corrections to the perturbative one-loop self-energy. They come from (i) the momentum dependence of $\Sigma(\mathbf{k}, 0)$, (ii) from non-factorization of momentum integration and (iii) from keeping $\tilde{\Sigma}(\omega_m + \Omega_n)$ in the right-hand side of Eq. (35). The momentum dependent part of the self-energy is $\Sigma(\mathbf{k}, 0) \sim \epsilon_k(\lambda_E^*)^{1/2}$ [32]. It gives a small renormalization of the dispersion at $\lambda_E^* \ll 1$. The correction from non-factorization of momentum integration is of order of the ratio of relevant scales $(q_\perp/q_\parallel)^2$. In the quantum-critical regime this yields a relative correction to the self-energy of order $(\omega/\omega_c^*)^{2/3}(\lambda_E^*)^2$, which is at most $O((\lambda_E^*)^2)$. In the Fermi liquid regime the correction is even smaller. In the Fermi gas regime the correction increases and becomes of order one at $\omega \sim \omega_{\max}$, which, we remind, is the largest frequency up to which

the concept the “fast electron/slow boson” is applicable.

The correction $\delta\Sigma(\omega_m)$ from keeping $\tilde{\Sigma}(\omega_m + \Omega_n)$ is of order $\omega_m(\lambda_E^*)^{1/2}$ for all $\omega_m < \omega_{\max}$. We verified that it comes from internal $\Omega_n \sim \omega_{\max}$ and momenta $q_\perp \sim q_\parallel \sim k_F(\lambda_E^*)^{1/2}$. For such Ω_n , $\tilde{\Sigma}(\omega_m + \Omega_n) \approx \omega_m + \Omega_n$, i.e., the actual self-energy of an internal fermion is irrelevant. In explicit form we find for $\delta\Sigma(\omega_m)$

$$\delta\Sigma(\omega_m) = -\omega_m(\lambda_E^*)^{1/2}J \quad (41)$$

where

$$J = \frac{1}{2\pi^2} \int_0^\infty dx \int_0^\infty dy \frac{x^2 y}{(4x^2 + y^2)^{3/2} (x^3 + \frac{y}{4\pi})} \approx 0.084. \quad (42)$$

and the variables are $y = \Omega_n/\omega_{\max}$ and $x = q/(k_F(\lambda_E^*)^{1/2})$. We caution, however, that at $y = O(1)$, i.e., at $\Omega_n \sim \omega_{\max}$, the (Ω_n/v_F^2) term in $\chi(q)$, which we neglected, becomes comparable with the Landau damping, and this may change the value of J . Still, it remains a number of order one, and $\delta\Sigma(\omega_m)$ is much smaller than $\Sigma^{(E)}(\omega_m)$.

In Fig. 8, we show the numerical result of the self-energy for a small but finite λ_E^* . The solid line is $\Sigma^{(E)}(\omega_m)$ at $k = k_F$ and the dotted line is the full self-consistent numerical result $\Sigma(\omega_m) = \Sigma^{(E)}(\omega_m) + \delta\Sigma(\omega_m)$. We see that $\delta\Sigma(\omega_m)$ is linear in ω_m , and the prefactor scales as $(\lambda_E^*)^{1/2}$, as in (41). The numerical data matches Eq. (41) very well.

C. Vertex correction

Vertex corrections to Eliashberg theory have been discussed in detail in Ref. 32. We present several new results below.

The lowest order vertex correction is given by the diagram in Fig. 4 (a). In explicit form

$$\frac{\Delta g(k, q)}{g} = g^* \int \frac{d\Omega'_n d^2\mathbf{q}'}{(2\pi)^3} \frac{1}{i\tilde{\Sigma}(\omega_m + \Omega'_n) - \epsilon_{\mathbf{k}+\mathbf{q}'}} \frac{1}{i\tilde{\Sigma}(\omega_m + \Omega'_n + \Omega_n) - \epsilon_{\mathbf{k}+\mathbf{q}'+\mathbf{q}}} \frac{1}{|\mathbf{q}'|^2 + \alpha \frac{|\Omega'_n|}{|\mathbf{q}'|}}, \quad (43)$$

where we recall, $q = (\mathbf{q}, \Omega_n)$ is the external bosonic momentum and $k = (\mathbf{k}, \omega_m)$ is the external fermionic momentum.

Like for the electron-phonon case, vertex correction at $\mathbf{q} = 0$ has to obey the Ward identity, Eq. (19), associated with the conservation of the particle number. We show that this indeed holds.

In the Fermi liquid regime at $\omega_m < \omega^*$, an explicit calculation of $\Delta g(k, \Omega_n, \mathbf{q} = 0)/g$ yields,

$$\frac{\Delta g(k, \Omega_n, \mathbf{q} = 0)}{g} = \frac{\lambda}{1 + \lambda} \quad (44)$$

At large λ , the one-loop vertex correction is close to one. To get the full vertex $g_{\text{full}}(k, \Omega_n, \mathbf{q} = 0)$ one has to sum the series of ladder vertex correction diagrams, just as we did for the electron-phonon case. The summation yields the expected result

$$\frac{g_{\text{full}}(k, \Omega_n, \mathbf{q} = 0)}{g} = \frac{1}{1 - \frac{\lambda}{1+\lambda}} = 1 + \lambda, \quad (45)$$

which is equal to $(\tilde{\Sigma}(\omega_m + \Omega_n) - \tilde{\Sigma}(\omega_m))/\Omega_n$ in the FL frequency regime, and thus satisfies the Ward identity, Eq. (19).

In the quantum-critical regime, the one-loop vertex correction is again of $O(1)$, but has a more complicated structure. Introducing dimensionless variables $x = \omega_m/\Omega_n$ and $x' = -(\omega_m + \Omega_n')/\Omega_n$ and setting both ω_m and Ω_n to be positive, we find one-loop vertex correction in the form

$$\frac{\Delta g(x, \Omega_n)}{g} = \frac{2}{3} \int_0^1 \frac{dx'}{|x' - x|^{1/3}} \times \frac{1}{(1 - x')^{2/3} + (x')^{2/3} + \left(\frac{\Omega_n}{\omega_c^*}\right)^{1/3}}. \quad (46)$$

The ladder series of vertex corrections in this situation can be re-expressed as an integral equation for $g_{\text{full}}(x, \Omega_n)$:

$$g_{\text{full}}(x, \Omega_n) = 1 + \frac{2}{3} \int_0^1 \frac{dx'}{|x' - x|^{1/3}} \frac{g_{\text{full}}(x', \Omega_n)}{(1 - x')^{2/3} + (x')^{2/3} + \left(\frac{\Omega_n}{\omega_c^*}\right)^{1/3}} \quad (47)$$

This is an integral equation in the variable x , while Ω_n just acts as a parameter. One can easily verify that the solution of Eq. (47) is

$$g_{\text{full}}(x, \Omega_n) = \left(\frac{\omega_c^*}{\Omega_n}\right)^{1/3} \left((1 - x)^{2/3} + x^{2/3} + \left(\frac{\Omega_n}{\omega_c^*}\right)^{1/3} \right). \quad (48)$$

In the original variables Eq. (48) becomes

$$g_{\text{full}}(\omega_m, \Omega_n) = 1 + \frac{((\omega_m + \Omega_n)^{2/3} - (\omega_m)^{2/3}) \omega_0^{2/3}}{\Omega_n} = \frac{\tilde{\Sigma}(\omega_m + \Omega_n) - \tilde{\Sigma}(\omega_m)}{\Omega_n} \quad (49)$$

as in Eq. (19).

For finite \mathbf{q} , the result is more involved. The one-loop vertex correction for \mathbf{k} on the Fermi surface is a scaling function of five variables

$$\frac{\Delta g(k, q)}{g} = \Phi \left(\frac{\omega_m}{\Omega_n}, \frac{v_F q_{\perp}}{\tilde{\Sigma}(\omega_m)}, \frac{q_{\parallel}^2/2m}{\tilde{\Sigma}(\omega_m)}, \frac{\omega_m}{\tilde{\Sigma}(\omega_m)}, \frac{(\omega_D^*)^2}{E_F \tilde{\Sigma}(\omega_m)} \right) \quad (50)$$

In the Fermi liquid regime at $\omega_m < \omega^*$, typical $v_F q_{\perp} \sim \Sigma(\omega_m) = \lambda^* \omega_m$ and typical $(q_{\parallel}^2/2m)/\Sigma(\omega_m) \sim \omega^*/\omega_m$. We verified numerically that in this case Φ is of order $\omega_m/\omega^* \ll 1$, i.e., $\Delta g(k, q)/g \sim \omega_m/\omega^* \ll 1$.

In the quantum-critical regime the last two variables in Eq. (50) can be set to zero. The scaling function of the other three variables, $\Phi(x, y, z, 0, 0)$, is

$$\Phi(x, y, z, 0, 0) = \frac{3^{3/4}}{4\pi} \int_0^1 dr \int_0^\infty \frac{ds}{s^{3/2} + 3^{3/4} |r + x|} \times \left[\frac{1}{(1 - r)^{2/3} + r^{2/3} + i(y + z + 2\sqrt{|z|s})} + \frac{1}{(1 - r)^{2/3} + r^{2/3} + i(y + z - 2\sqrt{|z|s})} \right]. \quad (51)$$

The scaling function Φ in Eq. (50) is generally complex

and can be parametrized as $\Phi = |\Phi| e^{i\psi_\Phi}$. We plot $|\Phi|$ and ψ_Φ in Fig. 9 for several values of parameters. We see that in general Φ is of order one, but not particularly close to one, in distinction to the case $q = 0$. In this situation, we expect that higher-order vertex corrections remain of order one, but do not change substantially the value of $\Delta g/g$ compared to one-loop result.

Finally, in the Fermi gas regime $\omega_c^* < \omega_m < \omega_{\text{max}}$, we have for typical \mathbf{q} , and $\Omega_m \sim \omega_D^*$, $q_{\parallel}^2/(2m)/\tilde{\Sigma}(\omega_m) \ll 1$, $(\omega_D^*)^2/(E_F \tilde{\Sigma}(\omega_m)) \ll 1$, $v_F q_{\perp}/\tilde{\Sigma}(\omega_m) \sim 1$ and $\omega_m/\tilde{\Sigma}(\omega_m) \approx 1$. In this situation, we find $\Phi \sim \Sigma(\omega_m)/\omega_m \sim (\omega_c^*/\omega)^{1/3} \ll 1$, and hence $\Delta g(k, q)/g \ll 1$.

For qualitative analysis we do the same as for the electron-phonon case and estimate $\Delta g(k, q)/g$ by using typical values of internal momenta and frequency. A simple experimentation shows that by order of magnitude

$$\frac{\Delta g(k, q)}{g} \sim \frac{\Sigma(\omega_m)}{\tilde{\Sigma}(\omega_n) + v_F q_{\perp}^{\text{typ}} + \frac{(q_{\parallel}^{\text{typ}})^2}{2m}} \quad (52)$$

In the Fermi liquid and Fermi gas regimes the vertex correction is small because either $(q_{\parallel}^{\text{typ}})^2/(2m) \gg \Sigma(\omega_m)$ or $\tilde{\Sigma}(\omega_n) \gg \Sigma(\omega_n)$. In the quantum-critical, non-Fermi liquid regime all parameters in Eq. (52) are of the same order, and $\Delta g(k, q)/g$ is generally of order one.

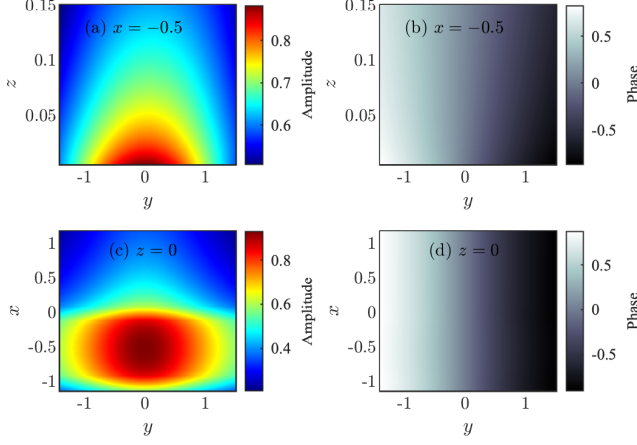


FIG. 9. Scaling function $\Phi(x, y, z, 0, 0) = |\Phi|e^{i\psi_\Phi}$, describing the one-loop vertex correction in the quantum-critical regime (cf. Eq. (50)). The functions $|\Phi|$ and ψ_Φ are plotted in y, z plane for $x = -0.5$ (a,b) and in (x, y) plane for $z = 0$ (c,d).

The quantitative measure of the strength of the vertex correction is the magnitude of the two-loop contribution to the self-energy with the vertex correction included, $\Sigma^{(2)}(\omega_m)$ compared to the one-loop $\Sigma^{(E)}(\omega_m)$. We show the corresponding diagram in Fig. 4 (b). In the Fermi liquid and Fermi gas regimes, we find that $\Sigma^{(2)}(\omega_m) \ll \Sigma^{(E)}(\omega_m)$, consistent with the smallness of $\Delta g/g$. In the quantum-critical regime, earlier order of magnitude studies [31, 32, 46, 47, 63] have found that $\Sigma^{(2)}(\omega_m)$ is of the same order as $\Sigma^{(E)}(\omega_m)$, unless one extends the theory to large N or assumes that the prefactor for q_{\parallel}^2 in $\epsilon_{\mathbf{k}_F+\mathbf{q}}$ is much larger than v_F/k_F .

We confirmed this results, but went further and computed $\Sigma^{(2)}(\omega_m)$ numerically with the prefactor. We found

$$\Sigma^{(2)}(\omega_m) \simeq 0.038 \Sigma^{(E)}(\omega_m). \quad (53)$$

We see that while by order of magnitude $\Sigma^{(2)}(\omega_m)$ is of order $\Sigma^{(E)}(\omega_m)$, it is far smaller numerically. It has been argued [46–48, 64, 65] that self-energies with higher-loop order vertex corrections included contain logarithmic singularities. We didn't compute these terms explicitly, but based on Eq. (53) we expect them to contain small prefactors and remain small down to very small frequencies, before logarithmic singularities become relevant. We recall in this regard that the ground state near an Ising-nematic/Ising-ferromagnetic transition is a superconductor, hence in practice the behavior at the lowest frequencies is relevant only if for some reason superconductivity does not develop.

For completeness, we also present results at weak coupling, when both λ^* and λ_E^* are small. There are two weak coupling regimes: $\lambda^* \ll (\lambda_E^*)^{1/2} \ll 1$ and $(\lambda_E^*)^{1/2} \ll \lambda^* \ll 1$. We focus on the second regime as it borders strong-coupling regime at $\lambda^* = O(1)$. In this regime, $\omega_c^* \ll \omega^*$, i.e., there is no range of non-Fermi

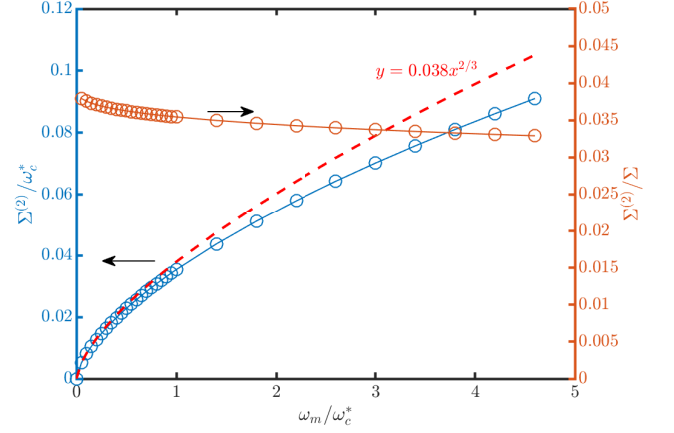


FIG. 10. Two-loop fermionic self energy with vertex correction included, $\Sigma^{(2)}(\omega_m)$ (see Fig. 4 (b)). Left axis: $\Sigma^{(2)}(\omega_m)$ in units of the characteristic frequency ω_c^* (the upper edge of non-Fermi liquid behavior). Right axis: the ratio of $\Sigma^{(2)}(\omega_m)$ and the full one-loop Eliashberg self-energy $\Sigma(\omega_m) = \Sigma^{(E)}(\omega_m) + \delta\Sigma(\omega_m)$ (see Section III B).

liquid behavior, although the self-energy still interpolates between $\lambda^*\omega_m$ at small frequencies and $(\omega_m)^{2/3}(\omega_c^*)^{1/3}$ at higher frequencies (the relation between all characteristic frequencies is $\omega_c^* < \omega^* < \omega_D < \omega_{max}$). A simple calculation shows that in this case vertex correction is small in λ^* and typical q_{\perp}^{typ} are parametrically larger than $q_{\parallel}^{\text{typ}}$ for all $\omega_m < \omega_{max}$ (see Fig. 11a for their values). Hence, Eliashberg theory is applicable and self-energy can be computed within perturbative one-loop approximation.

D. Summary of Section III

Like for the electron-phonon system, there are three energy scales in the problem: the bosonic energy ω_D^* , the coupling g^* and the Fermi energy E_F . This allows one to introduce two dimensionless ratios $\lambda^* = g^*/(4\pi\omega_D^*)$ and $\lambda_E^* = g^*/E_F$. The latter is a small parameter for the low-energy theory. The strong coupling regime occurs at $\lambda^* > 1, \lambda_E^* < 1$. In this regime, the system displays Fermi liquid behavior at $\omega < \omega^* \sim \omega_D^*\lambda_E^*/(\lambda^*)^2$, quantum-critical, non-Fermi liquid behavior with $\Sigma(\omega_m) \propto \omega_m^{2/3}$ at $\omega^* < \omega_m < \omega_c^*$, where $\omega_c^* \sim g^*\lambda_E^*$, and Fermi-gas behavior at larger frequencies.

The three key results for the system near an Ising-nematic/Ising-ferromagnetic QCP are the following.

First, Eliashberg theory is rigorously justified at strong coupling $\lambda^* \gg 1$ in the Fermi liquid and Fermi gas regimes, but not in the quantum-critical regime. In the latter, the leading vertex correction is $O(1)$ even when λ_E^* is small. It is nevertheless small numerically, as evidenced by numerical smallness of the two-loop self-energy with vertex correction included.

Second, the small parameter λ_E^* simplifies the cal-

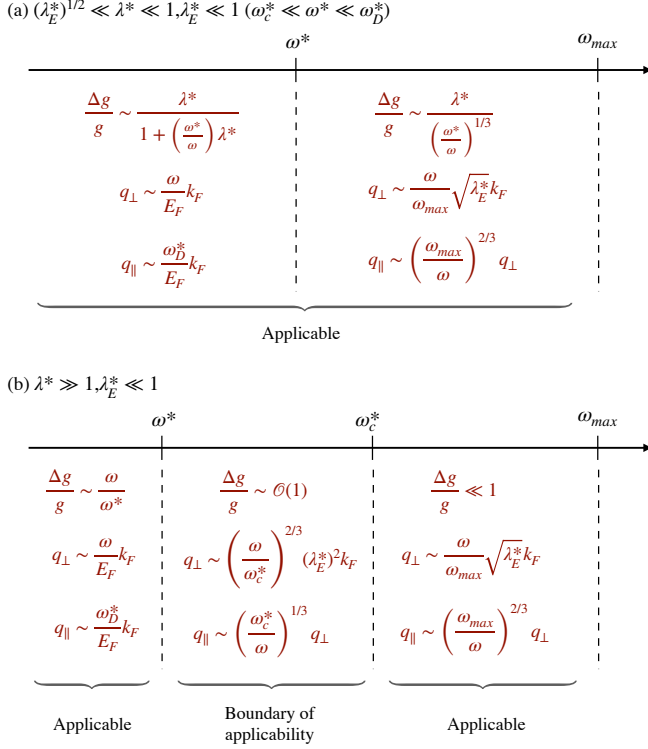


FIG. 11. Illustration of the applicability of Eliashberg theory near Ising-nematic/Ising-ferromagnetic critical point in a metal in (a) weak coupling regime at some distance from a critical point and (b) strong coupling regime near and at a critical point.

culations within Eliashberg theory: the self-energy $\Sigma^{(E)}(\mathbf{k}, \omega_m)$ is well approximated by the local $\Sigma^{(E)}(\omega_m)$, and the momentum integration in the computation of $\Sigma^{(E)}(\omega_m)$ can be factorized by invoking the Migdal “fast electron/slow boson” criterion. This factorization is rigorously justified at all ω_m up to $\omega_{max} \sim \omega_c^*/(\lambda_E^*)^{3/2} \gg \omega_c^*$. In this respect, λ_E^* plays the same role as λ_E for the electron-phonon case.

Third, and most important, the corrections to Eliashberg theory remain $\mathcal{O}(1)$ (and numerically small) even at a QCP, where $\omega_D^* = 0$ and $\lambda^* = \infty$. Furthermore, the “fast electron/slow boson” criterion also remains valid at a QCP as long as λ_E^* is small. In other words, Eliashberg theory near a nematic QCP, while not rigorously justified, is quite accurate numerically and to the same numerical accuracy can be extended right to the QCP.

We summarize the results for the Ising-nematic case in Fig. 11.

IV. CONCLUSIONS

In this paper we compared the validity of Eliashberg theory for electrons interacting with an Einstein phonon and with soft nematic fluctuations near an Ising-

nematic QCP and soft magnetic fluctuations near Ising-ferromagnetic QCP. Eliashberg theory is the set of coupled one-loop self-consistent equations for the fermionic self-energy and polarization operator, with vertex corrections neglected.

For electron-phonon interaction, Eliashberg theory has been justified by the argument that an Einstein phonon is a slow mode compared to a fermion (the effective boson velocity is smaller than Fermi velocity). In the Ising-nematic/Ising-ferromagnetic case this argument is not valid as soft fluctuations are collective modes of electrons and their velocity is of the same order as v_F .

We examined self-consistent Eliashberg theory and two-loop corrections to it for both cases in the strong coupling regime, where the system displays Fermi liquid behavior at the lowest energies, non-Fermi liquid behavior in a wide range of intermediate energies, and Fermi gas behavior at the largest energies.

For the electron-phonon case, Eliashberg theory is rigorously justified when the Eliashberg parameter λ_E is small. Namely, the two-loop self-energy with vertex correction is small in λ_E . Simultaneously, Eliashberg equations can be simplified using the same smallness of λ_E to uncoupled perturbative one-loop equations, which can be easily solved. In physical terms, the smallness of λ_E follows directly from the condition that an Einstein phonon is a slow mode compared to electrons.

For the Ising-nematic/Ising-ferromagnetic case, soft bosons are collective modes of the electrons, and in the non-Fermi liquid quantum-critical regime there is no parametric smallness of the two-loop self-energy with vertex correction compared to the one-loop self-energy in the Eliashberg theory. Nevertheless, we found that the two-loop self-energy is numerically much smaller than the one-loop one. Additionally, the low-energy theory contains a small parameter λ_E^* , which plays the role of λ_E in the sense that it again allows one to reduce the coupled self-consistent one-loop Eliashberg equations to uncoupled perturbative one-loop equations, which one can easily solve. The implication of these results is that Eliashberg theory for the Ising-nematic/Ising-ferromagnetic case is on rather solid grounds. It is very likely that this holds also for other cases when fermions interact with their soft collective excitations in the charge or spin channel.

There is one aspect in which the Eliashberg description of fermions coupled to soft collective bosons works even better than for fermions interacting with an Einstein phonon. Namely, for the collective boson case, the parameter λ_E^* is independent of the distance to a QCP, and two-loop self-energy remains numerically smaller than the one-loop one even at a QCP. As a result, Eliashberg theory can be extended right to the QCP. For the electron-phonon case, the Eliashberg parameter λ_E^* is inversely proportional to the dressed Debye frequency, and Eliashberg theory inevitably breaks down at some distance from the point where ω_D would vanish. Beyond this point, a completely new description in terms of po-

larons is needed [50, 52, 53, 66, 67].

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