# Generalized Converses of Operator Jensen's Inequalities with Applications to Hypercomplex Function Approximations and Bounds Algebra 

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April 19, 2024


#### Abstract

Mond and Pecaric proposed a powerful method, namd as MP method, to deal with operator inequalities. However, this method requires a real-valued function to be convex or concave, and the normalized positive linear map between Hilbert spaces. The objective of this study is to extend the MP method by allowing non-convex or non-concave real-valued functions and nonlinear mapping between Hilbert spaces. The Stone-Weierstrass theorem and Kantorovich function are fundamental components employed in generalizing the MP method inequality in this context. Several examples are presented to demonstrate the inequalities obtained from the conventional MP method by requiring convex function with a normalized positive linear map. Various new inequalities regarding hypercomplex functions, i.e., operator-valued functions with operators as arguments, are derived based on the proposed MP method. These inequalities are applied to approximate hypercomplex functions using ratio criteria and difference criteria. Another application of these new inequalities is to establish bounds for hypercomplex functions algebra, i.e., an abelian monoid for the addition or multiplication of hypercomplex functions, and to derive tail bounds for random tensors ensembles addition or multiplication systematically.


Index terms- Operator inequality, Jensen's inequality, hypercomplex function, Stone-Weierstrass theorem, Kantorovich function.

## 1 Introduction

Operator Jensen's inequality extends the concept of conventional Jensen's inequality from real value argument function to operator argument function and is generally used in optimization, especially in convex optimization involving operator variables, e.g., matrices, tensors. The inequality states that for an operator convex function $f$, we have

$$
\begin{equation*}
f(\Phi(\boldsymbol{A})) \leq \Phi(f(\boldsymbol{A})), \tag{1}
\end{equation*}
$$

where $\boldsymbol{A}$ is a self-adjoint operator, $\Phi$ is a normalized positive linear map and $\leq$ is by Loewner ordering sense [1,2]. Operator Jensen's inequality is a powerful device in diverse areas, along with machine learning, signal processing, and control principle. In gadget learning, for example, it is used inside the analysis of convex optimization problems related to matrices, consisting of matrix factorization and matrix completion. In signal processing, it finds applications in the design and analysis of convex optimization algorithms for jobs like blind source separation and beamforming. In the field of control science, it is carried out in the

[^0]evaluation and synthesis of convex control structures regarding matrix variables. Operator Jensen's inequality gives a way to generalize the idea of convex features to operator variables and presents insights into the function convexity related to matrices or tensors, that are frequently encountered in many mathematical and engineering tasks [3-15].

Mond and Pečarić demonstrated multiple extensions of Kantorovich-type operator inequalities concerning normalized positive linear maps. They highlighted that determining upper bounds for the difference and ratio in Jensen's inequality can be simplified to solving a single-variable maximization or minimization problem by leveraging the concavity of a real-valued function $f$. Building on this approach, they established complementary inequalities to the Hölder-McCarthy inequality and Kantorovich-type inequalities, provided estimates for the difference and ratio of operator means, and explored various converses of Jensen's inequality applicable to normalized positive linear maps. In cases where $f$ exhibits concavity, they derived the dual problem. This method, known as the Mond-Pečarić (MP) method, has proven to be highly fruitful in the realm of operator inequalities, offering valuable insights and results [1,2].

The purpose of this work is to extend MP method by allowing $f$ in Eq. (1) to be non-convex or nonconcave and the mapping $\Phi$ in Eq. (1) to be a nonlinear mapping. The Stone-Weierstrass theorem and Kantorovich function serve as the primary ingredients employed in generalizing MP method inequality in this context. Several examples are shown here to have those inequalities obtained from the conventional MP method by setting $f$ to be a convex function with $\Phi$ as a normalized positive linear map. Various new inequalities about hypercomplex functions, i.e., operator-valued functions with operators as arguments, are derived based on the proposed MP method. These new inequalities are applied to approximate hypercomplex functions by ratio criteria and by difference criteria. The other application of these new inequalities is to establish hypercomplex functions lower/upper bounds algebra, i.e., these bounds form a abelian monoid under the addition or the multiplication of hypercomplex functions. Besides, we also can derive random tensors tail bounds for the addition or the multiplication of random tensors ensembles by an uniform way.

The remainder of this paper is organized as follows. In Section 2 , fundamental inequalities about hypercomplex functions are established. In Section 3, generalized converses of operator Jensen's inequalities for ratio kinds with nonlinear $\Phi$ are established. On the other hand, generalized converses of operator Jensen's inequalities for different kinds with nonlinear $\Phi$ are established in Section 4. The first application about applying newly derived inequalities to hypercomplex functions approximation is studied in Section 5 . The second application about applying newly derived inequalities to investigate lower and upper bounds algebra is presented in Section 6

Nomenclature: Inequalities $\geq,>, \leq$, and $<$, when applied to operators, follow the Loewner ordering. The symbol $\Lambda(\boldsymbol{A})$ denotes the spectrum of the operator $\boldsymbol{A}$, i.e., the set of eigenvalues of $\boldsymbol{A}$. If $\Lambda(\boldsymbol{A})$ consists of real numbers, $\min (\Lambda(\boldsymbol{A}))$ and $\max (\Lambda(\boldsymbol{A}))$ represent the minimum and maximum values within $\Lambda(\boldsymbol{A})$, respectively. For given $M>m>0$ and any $r \in \mathbb{R}$ where $r \neq 1$, the Kantorovich function with respect to $m, M$, and $r$ is defined as follows:

$$
\begin{equation*}
\mathcal{K}(m, M, r)=\frac{\left(m M^{r}-M m^{r}\right)}{(r-1)(M-m)}\left[\frac{(r-1)\left(M^{r}-m^{r}\right)}{r\left(m M^{r}-M m^{r}\right)}\right]^{r} . \tag{2}
\end{equation*}
$$

## 2 Fundamental Inequalities for Hypercomplex Functions with Nonlinear Map

### 2.1 Upper Bound and Lower Bound of Function $f$

Let's start with the Stone-Weierstrass Theorem, which asserts that any continuous real-valued function $f(x)$ defined on the closed interval $[m, M]$, where $m, M \in \mathbb{R}$ and $M>m$, can be uniformly approximated by a polynomial $p(x)$. This approximation ensures that the absolute difference between $f(x)$ and $p(x)$ is less
than any given positive value $\epsilon$ across the entire interval $[m, M]$. Mathematically, this difference is bounded by the supremum norm, denoted as $\|f-p\|_{\infty}$, which remains less than $\epsilon$ [16].

Given a continuous real-valued function $f(x)$ and a positive error bound $\epsilon$, we can employ the Lagrange polynomial interpolation method, grounded in the Stone-Weierstrass Theorem, to ascertain both an upper polynomial $p_{\mathcal{U}}(x) \geq f(x)$ and a lower polynomial $p_{\mathcal{L}}(x) \leq f(x)$ over the interval $[m, M]$. These polynomials are guaranteed to satisfy the following inequalities:

$$
\begin{align*}
& 0 \leq p_{u}(x)-f(x) \leq \epsilon, \\
& 0 \leq f(x)-p_{\mathcal{L}}(x) \leq \epsilon, \tag{3}
\end{align*}
$$

Additionally, in this paper, we assume that $f(\boldsymbol{A})$ is a self-adjoint operator if $\boldsymbol{A}$ is a self-adjoint operator.

### 2.2 Polynomial Map $\Phi$

Consider two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K} . \mathbb{B}(\mathfrak{H})$ and $\mathbb{B}(\mathfrak{K})$ represent the semi-algebras comprising all bounded linear operators on these respective spaces. In Choi-Davis-Jensen's Inequality, the mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow$ $\mathbb{B}(\mathfrak{K})$ is defined as a normalized positive linear map. Such a map is precisely defined by Definition 1 below [1,2].

Definition 1 A map $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is considered a normalized positive linear map if it satisfies the following conditions:

- Linearity: $\Phi(a \boldsymbol{X}+b \boldsymbol{Y})=a \Phi(\boldsymbol{X})+b \Phi(\boldsymbol{Y})$ for any $a, b \in \mathbb{C}$ and any $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{B}(\mathfrak{H})$.
- Positivity: If $\boldsymbol{X} \geq \boldsymbol{Y}$, then $\Phi(\boldsymbol{X}) \geq \Phi(\boldsymbol{Y})$.
- Normalization: $\Phi\left(\boldsymbol{I}_{\mathfrak{H}}\right)=\boldsymbol{I}_{\mathfrak{F}}$, where $\boldsymbol{I}_{\mathfrak{H}}$ and $\boldsymbol{I}_{\mathfrak{K}}$ are the identity operators of the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively.

In this study, we will explore a broader class of $\Phi$ by defining $\Phi$ as follows:

$$
\begin{align*}
\Phi(\boldsymbol{X}) & =\boldsymbol{V}^{*}\left(\sum_{i=0}^{I} a_{i} \boldsymbol{X}^{i}\right) \boldsymbol{V} \\
& =\boldsymbol{V}^{*}\left(\sum_{i_{+} \in S_{I_{+}}} a_{i_{+}} \boldsymbol{X}^{i_{+}}+\sum_{i_{-} \in S_{I_{-}}} a_{i_{-}} \boldsymbol{X}^{i_{-}}\right) \boldsymbol{V} \tag{4}
\end{align*}
$$

where, $\boldsymbol{V}$ stands as an isometry within $\mathfrak{H}$, adhering to $\boldsymbol{V}^{*} \boldsymbol{V}=\boldsymbol{I}_{\mathfrak{H}}$. In $a_{i}, a_{i_{+}}$signifies the nonnegative coefficients, while $a_{i_{-}}$denotes the negative coefficients. The set of indices corresponding to positive coefficients is denoted as $S_{I_{+}}$, and those for negative coefficients form $S_{I_{-}}$. It's noteworthy that no constraints regarding linearity, positivity, or normalization are imposed on $\Phi$ as outlined in Eq. (4). With this premise, the conventional notion of a normalized positive linear map, delineated in Definition 11, emerges as a special case. This is accomplished by configuring the polynomial $\sum_{i=0}^{I} a_{i} \boldsymbol{X}^{i}$ as the identity map, wherein all coefficients $a_{i}$ except $a_{1}$ are zero.

Contents from Section 2.1] and Section 2.2 are based on [4], however, we present them here again for self-contained presentation purpose.

### 2.3 Fundamental Inequalities

In this section, general converses of operator Jensen's inequalities for any polynomial map $\Phi$ will be obtained. Let us recall Lemma 2 from [4].

Lemma 1 Given a self-adjoint operator $\boldsymbol{A}$ with $\Lambda(\boldsymbol{A})$, such that

$$
\begin{align*}
& 0 \leq p_{\mathcal{U}}(x)-f(x) \leq \epsilon, \\
& 0 \leq f(x)-p_{\mathcal{L}}(x) \leq \epsilon, \tag{5}
\end{align*}
$$

for $x \in[\min (\Lambda(\boldsymbol{A})), \max (\Lambda(\boldsymbol{A}))]$ with polynomials $p_{\mathcal{L}}(x)$ and $p_{u}(x)$ expressed by

$$
\begin{equation*}
p_{\mathcal{L}}(x)=\sum_{k=0}^{n_{\mathcal{L}}} \alpha_{k} x^{k}, \quad p_{\mathcal{U}}(x)=\sum_{j=0}^{n_{\mathcal{U}}} \beta_{j} x^{j} . \tag{6}
\end{equation*}
$$

Under the definition of $\Phi$ provided by Eq. (4), we have

$$
\begin{align*}
\Phi(f(\boldsymbol{A})) \leq & \boldsymbol{V}^{*}\left\{\sum_{i_{+} \in S_{I_{+}}} a_{i_{+}} \mathcal{K}\left(\min \left(\Lambda\left(p_{u}(\boldsymbol{A})\right)\right), \max \left(\Lambda\left(p_{u}(\boldsymbol{A})\right)\right), i_{+}\right) p_{\mathcal{u}}^{i_{+}}(\boldsymbol{A})\right. \\
& \left.+\sum_{i_{-} \in S_{I_{-}}} a_{i_{-}} \mathcal{K}^{-1}\left(\min \left(\Lambda\left(p_{\mathcal{L}}(\boldsymbol{A})\right)\right), \max \left(\Lambda\left(p_{\mathcal{L}}(\boldsymbol{A})\right)\right), i_{-}\right) p_{\mathcal{L}}^{i_{-}}(\boldsymbol{A})\right\} \boldsymbol{V} \\
& \stackrel{\text { def }}{=} \boldsymbol{V}^{*} \text { Poly }_{f, u}(\boldsymbol{A}) \boldsymbol{V}, \tag{7}
\end{align*}
$$

where $\Lambda\left(p_{u}(\boldsymbol{A})\right.$ and $\Lambda\left(p_{\mathcal{L}}(\boldsymbol{A})\right.$ are spectrms of operators $p_{u}\left(\boldsymbol{A}\right.$ and $p_{\mathcal{L}}(\boldsymbol{A}$, respectively. On the other hand, we also have

$$
\begin{align*}
\Phi(f(\boldsymbol{A})) \geq & \boldsymbol{V}^{*}\left\{\sum_{i_{+} \in S_{I_{+}}} a_{i_{+}} \mathcal{K}^{-1}\left(\min \left(\Lambda\left(p_{\mathcal{L}}(\boldsymbol{A})\right)\right), \max \left(\Lambda\left(p_{\mathcal{L}}(\boldsymbol{A})\right)\right), i_{+}\right) p_{\mathcal{L}}^{i_{+}}(\boldsymbol{A})\right. \\
& \left.+\sum_{i_{-} \in S_{I_{-}}} a_{i_{-}} \mathcal{K}\left(\min \left(\Lambda\left(p_{\mathcal{u}}(\boldsymbol{A})\right)\right), \max \left(\Lambda\left(p_{u}(\boldsymbol{A})\right)\right), i_{-}\right) p_{u}^{i_{u}}(\boldsymbol{A})\right\} \boldsymbol{V} \\
& \stackrel{\text { def }}{=} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}(\boldsymbol{A}) \boldsymbol{V} . \tag{8}
\end{align*}
$$

Given $\Lambda(\boldsymbol{A}) \in[m, M]$, the value range for the spectrum $\Lambda\left(\operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A})\right)$ is represented by $\widetilde{\operatorname{Poly}}_{f, \mathcal{L}}(m, M)$. Similarly, the value range for the spectrum $\Lambda\left(\operatorname{Poly}_{f, u}(\boldsymbol{A})\right)$ is represented by $\widetilde{\operatorname{Poly}}_{f, u}(m, M)$.

The main theorem of this work is presented below. Theorem 1 will give operator inequalties, lower and uppber bounds, of functional with respect to $\Phi(f(\boldsymbol{A}))$.

Theorem 1 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6). The function $g$ is also a real
valued continous function defined on the range $\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}_{f, \mathcal{L}}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Pol}}_{f, u}\left(m_{j}, M_{j}\right)\right)$. We also have a real valued function $F(u, v)$ with operator monotone on the first variable $u$ defined on $U \times V$ such that $f\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right) \subset U$, and $g\left(\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }_{f, u}}\left(m_{j}, M_{j}\right)\right)\right) \subset V$.

Then, we have the following upper bound:

$$
\begin{equation*}
F\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right), g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \leq \max _{x \in \bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}_{f, u}\left(m_{j}, M_{j}\right)}} F(x, g(x)) \mathbf{1}_{\mathfrak{K}} . \tag{9}
\end{equation*}
$$

Similarly, we also have the following lower bound:

$$
\begin{equation*}
F\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right), g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \geq \min _{x \in \bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)} F(x, g(x)) \mathbf{1}_{\mathfrak{K}} . \tag{10}
\end{equation*}
$$

Proof: We begin with the proof for the upper bound provided by Eq. (9). From Lemma 1, we have

$$
\begin{equation*}
\Phi(f(\boldsymbol{A})) \leq \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}(\boldsymbol{A}) \boldsymbol{V} \tag{11}
\end{equation*}
$$

By replacing $\boldsymbol{A}$ with $\boldsymbol{A}_{j}$ in Eq. (11) and applying $w_{j}$ with respect to each $\boldsymbol{A}_{j}$, we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V} \tag{12}
\end{equation*}
$$

According to the function $F(u, v)$ condition, we have

$$
\begin{align*}
& F\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right), g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \\
& \quad \leq F\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}, g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \\
& \quad \leq \max _{x \in \bigcup_{j=1}^{k} w_{j}{\underset{\mathrm{Poly}}{f, u}}\left(m_{j}, M_{j}\right)} F(x, g(x)) \mathbf{1}_{\mathfrak{K}}, \tag{13}
\end{align*}
$$

where the last inequality comes from that the spectrum $\Lambda\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)$ is in the range of $\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, u}\left(m_{j}, M_{j}\right)$. The desired inequality provided by Eq. (9) is established.

Now, we will prove the lower bound provided by Eq. 10. From Lemma 1, we have

$$
\begin{equation*}
\Phi(f(\boldsymbol{A})) \geq \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A}) \boldsymbol{V} \tag{14}
\end{equation*}
$$

By replacing $\boldsymbol{A}$ with $\boldsymbol{A}_{j}$ in Eq. (11) and applying $w_{j}$ with respect to each $\boldsymbol{A}_{j}$, we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V} \tag{15}
\end{equation*}
$$

From the function $F(u, v)$ condition, we also have

$$
\begin{align*}
& F\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right), g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \\
& \geq F\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}, g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \tag{16}
\end{align*}
$$

where the last inequality comes from that the spectrum $\Lambda\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)$ is in the range of $\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)$. The desired inequality provided by Eq. (10) is established.

By applying the function $F(u, v)$ with the following format:

$$
\begin{equation*}
F(u, v)=u-\alpha v, \tag{17}
\end{equation*}
$$

where $\alpha \in \mathbb{R}$, we can have the following Theorem 2. Theorem 2 will provide generalized converses of operator Jensen's inequalities with ratio and difference kinds in coming sections.

Theorem 2 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6). The function $g$ is also a real valued continous function defined on the range $\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }_{f, \mathcal{L}}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{P o l y_{f, u}}\left(m_{j}, M_{j}\right)\right)$. We also have a real valued function $F(u, v)$ defined as Eq. (17) with support domain on $U \times V$ such that $f\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right) \subset U$, and $g\left(\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, u}\left(m_{j}, M_{j}\right)\right)\right) \subset V$.

Then, we have the following upper bound:

Similarly, we also have the following lower bound:

Proof: By setting $F(u, v)=u-\alpha v$ in Eq. (9) in Theorem 11, we have the desired inequality provided by Eq. (18). Similarly, By setting $F(u, v)=u-\alpha v$ in Eq. (10) in Theorem 1, we have the desired inequality provided by Eq. 19].

Corollary 1 below is provided to give upper and lower bounds for special types of the function $g$ by applying Theorem 2 .

Corollary 1 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6).
(I) If $g(x)=x^{q}$, where $q \in \mathbb{R}$ and $\left(\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }}_{f, u}\left(m_{j}, M_{j}\right)\right)\right) \geq 0$, we have the upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \alpha\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q}+{\underset{x \in \bigcup_{j=1}^{k} w_{j}}{ } \max _{\operatorname{Pol}_{f, u}\left(m_{j}, M_{j}\right)}}\left(x-\alpha x^{q}\right) \mathbf{1}_{\mathfrak{K}}, \tag{20}
\end{equation*}
$$

and we have the lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{align*}
& \sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \alpha\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q}+{\underset{x \in \bigcup_{j=1}^{k} w_{j}}{\min _{\operatorname{Poly}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}}\left(x-\alpha x^{q}\right) \mathbf{1}_{\mathfrak{K}} \text {. }}_{\text {(II) If } g(x)=\log (x) \text { and }\left(\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, u}\left(m_{j}, M_{j}\right)\right)\right)>0 \text {, we have }}=\text {, } \tag{21}
\end{align*}
$$ the upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

and we have the lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \alpha \log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+{\underset{x \in \bigcup_{j=1}^{k} w_{j}}{\min _{\mathcal{P o l}}^{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}}(x-\alpha \log (x)) \mathbf{1}_{\mathfrak{N}} . \tag{23}
\end{equation*}
$$

(III) If $g(x)=\exp (x)$, we have the upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \alpha \exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+{\underset{x \in \bigcup_{j=1}^{k} w_{j}}{\max _{\mathcal{P o l y}_{f, u}\left(m_{j}, M_{j}\right)}}(x-\alpha \exp (x)) \mathbf{1}_{\mathfrak{K}}, ~}_{\text {, }} \tag{24}
\end{equation*}
$$

and we have the lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

In the following Example 1 , we will assume that the function $f$ is bounded by linear functions and derive related inequalties given by Theorem 1, Theorem 2, and Corollary 1 .

Example 1 In this example, we assume that the function $f$ satsifies the following:

$$
\begin{align*}
& 0 \leq \overbrace{\left(\beta_{0}+\beta_{1} x\right)}^{p_{u}(x)}-f(x) \leq \epsilon, \\
& 0 \leq f(x)-\overbrace{\left(\alpha_{0}+\alpha_{1} x\right)}^{p_{\mathcal{L}}(x)} \leq \epsilon, \tag{26}
\end{align*}
$$

where $x \in[m, M]$ with positive $\alpha_{0}, \alpha_{1}, \beta_{0}$ and $\beta_{1}$. We also assume that $\Lambda\left(\boldsymbol{A}_{j}\right) \in[m, M]$ and $f\left(\boldsymbol{A}_{j}\right) \geq \mathbf{0}$ for all $j \in 1,2, \cdots, k$. Moreover, the mapping $\Phi$ defined by Eq. (4) has all coefficients $a_{i} \geq 0$. Then, from Lemma [] we have

$$
\begin{equation*}
\Phi(f(\boldsymbol{A})) \leq \boldsymbol{V}^{*}\left\{\sum_{i=0}^{I} a_{i} \mathcal{K}\left(\beta_{0}+\beta_{1} m, \beta_{0}+\beta_{1} M, i\right)\left(\beta_{0}+\beta_{1} \boldsymbol{A}\right)^{i}\right\} \boldsymbol{V} \tag{27}
\end{equation*}
$$

Besides, we also have

$$
\begin{equation*}
\Phi(f(\boldsymbol{A})) \geq \boldsymbol{V}^{*}\left\{\sum_{i=0}^{I} a_{i} \mathcal{K}^{-1}\left(\alpha_{0}+\alpha_{1} m, \alpha_{0}+\alpha_{1} M, i\right)\left(\alpha_{0}+\alpha_{1} \boldsymbol{A}\right)^{i}\right\} \boldsymbol{V} \tag{28}
\end{equation*}
$$

From Eq. (27), we can express Poly $_{f, u}(\boldsymbol{A})$ as:

$$
\begin{equation*}
\operatorname{Poly}_{f, u}(\boldsymbol{A})=\sum_{i=0}^{I} c_{i} \boldsymbol{A}^{i} \tag{29}
\end{equation*}
$$

where coeffcients $c_{i}$ are given by

$$
\begin{equation*}
c_{i}=\sum_{j=0}^{I-i}\binom{i+j}{i} a_{i+j} \mathcal{K}\left(\beta_{0}+\beta_{1} m, \beta_{0}+\beta_{1} M, i+j\right) \beta_{1}^{i} \beta_{0}^{j} . \tag{30}
\end{equation*}
$$

Similarly, from Eq. (28), we can express Poly $_{f, \mathcal{L}}(\boldsymbol{A})$ as:

$$
\begin{equation*}
\operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A})=\sum_{i=0}^{I} d_{i} \boldsymbol{A}^{i}, \tag{31}
\end{equation*}
$$

where coeffcients $c_{i}$ are given by

$$
\begin{equation*}
d_{i}=\sum_{j=0}^{I-i}\binom{i+j}{i} a_{i+j} \mathcal{K}^{-1}\left(\alpha_{0}+\alpha_{1} m, \alpha_{0}+\alpha_{1} M, i+j\right) \alpha_{1}^{i} \alpha_{0}^{j} . \tag{32}
\end{equation*}
$$

Suppose conditions provided by Theorem 1 are valid, from polynomials given by Eq. 29) and Eq. (31), we have

$$
\begin{equation*}
F\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right), g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \leq \underset{x \in \underset{\text { Poly }_{f, u}(m, M)}{ }}{\max } F(x, g(x)) \mathbf{1}_{\mathfrak{\mathfrak { R }}} . \tag{33}
\end{equation*}
$$

Similarly, we also have the following lower bound:

$$
\begin{equation*}
F\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right), g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right) \geq \min _{x \in \widetilde{\text { Poly }_{f, \mathcal{L}}(m, M)}} F(x, g(x)) \mathbf{1}_{\mathfrak{K}} . \tag{34}
\end{equation*}
$$

If $F(u, v)=u-\alpha v$, we can apply Theorem 2 to the function $f$ and $\Phi$ provided in this example to obtain the following:

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \alpha g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+\underset{x \in \overparen{\operatorname{Pol}_{f, u}(m, M)}}{ }(x-\alpha g(x)) \mathbf{1}_{\mathfrak{K}} . \tag{35}
\end{equation*}
$$

Similarly, we also have the following lower bound:

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \alpha g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+\underset{x \in \underset{\operatorname{Pol}_{f, \mathcal{L}}(m, M)}{ }}{\min }(x-\alpha g(x)) \mathbf{1}_{\mathfrak{K}} . \tag{36}
\end{equation*}
$$

Finally, the application of Corollary $\mathbb{1}$ to the function $f$ and $\Phi$ provided in this example will get:
(I) If $g(x)=x^{q}$, where $q \in \mathbb{R}$ and $\left({\left.\widetilde{\text { Poly }_{f, \mathcal{L}}}(m, M) \cup \widetilde{\operatorname{Poly}}_{f, u}(m, M)\right) \geq 0 \text {, we have the upper bound }}\right.$ for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \alpha\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q}+\underset{x \in \operatorname{Poly}_{f, u}(m, M)}{ }\left(x-\alpha x^{q}\right) \mathbf{1}_{\mathfrak{K}}, \tag{37}
\end{equation*}
$$

and we have the lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \alpha\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q}+\underset{x \in \underset{\text { Poly }_{f, \mathcal{L}}(m, M)}{ }}{\min }\left(x-\alpha x^{q}\right) \mathbf{1}_{\mathfrak{K}} . \tag{38}
\end{equation*}
$$

(II) If $g(x)=\log (x)$ and $\left(\widetilde{\text { Poly }}_{f, \mathcal{L}}(m, M) \bigcup \widetilde{\operatorname{Poly}}_{f, u}(m, M)\right)>0$, we have the upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right):$

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \alpha \log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+\underset{x \in \underset{\operatorname{Pol}_{f, u}(m, M)}{ }}{\max }\left(x-\alpha \log (x) \mathbf{1}_{\mathfrak{K}},\right. \tag{39}
\end{equation*}
$$

and we have the lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :
(III) If $g(x)=\exp (x)$, we have the upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \alpha \exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+\underset{x \in{\underset{\text { Poly }}{f, u}}(m, M)}{\max }(x-\alpha \exp (x)) \mathbf{1}_{\mathfrak{K}} \tag{41}
\end{equation*}
$$

and we have the lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \alpha \exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)+\min _{x \in \widehat{\operatorname{Poly}_{f, \mathcal{L}}(m, M)}}(x-\alpha \exp (x)) \mathbf{1}_{\mathfrak{K}} . \tag{42}
\end{equation*}
$$

## 3 Generalized Converses of Operator Jensen's Inequalities: Ratio Kind

In this section, we will derive the lower and upper bounds for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ in terms of ratio criteria related to the function $g$.

Theorem 3 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6). The function $g$ is also a real valued continous function defined on the range $\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, u}\left(m_{j}, M_{j}\right)\right)$ and $g(x) \neq 0$ for $x \in\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Pol}}_{f, u}\left(m_{j}, M_{j}\right)\right)$.
(I) If we also assume that $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*}\right.$ Poly $\left._{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)>\boldsymbol{O}$ and $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)>\boldsymbol{O}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq\left[\max _{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\text { Poly }}}^{f, u}\left(m_{j}, M_{j}\right)<t g^{-1}(x)\right] g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \tag{43}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\text { Poly }}}^{f, \mathcal{L}}\left(m_{j}, M_{j}\right)<1 g^{-1}(x)\right] g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \tag{44}
\end{equation*}
$$

(II) If we also assume that $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*}\right.$ Poly $\left._{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)<\boldsymbol{O}$ and $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*}\right.$ Poly $\left._{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)<\boldsymbol{O}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j} \text { Poly }_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)} x g^{-1}(x)\right] g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) ; \tag{45}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq\left[\max _{\substack{x \in \bigcup_{j=1}^{k} w_{j} \\ w_{j} o l y \\ f, u \\\left(m_{j}, M_{j}\right)}} x g^{-1}(x)\right] g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) . \tag{46}
\end{equation*}
$$

Proof: For part (I), we will apply $F(u, v)$ as

$$
\begin{equation*}
F(u, v)=v^{-1 / 2} u v^{-1 / 2} \tag{47}
\end{equation*}
$$

to Eq. (97) in Theorem 1, then, we will obtain

$$
\begin{align*}
\left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right)^{-1 / 2}\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)\right) & \left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right)^{-1 / 2} \\
& \leq\left[\underset{\left.x \in \underset{j=1}{k} \max _{j} \underset{\operatorname{Poly}_{f, u}\left(m_{j}, M_{j}\right)}{ } x g^{-1}(x)\right] \mathbf{1}_{\mathfrak{K}} \cdot(4}{ }\right. \tag{48}
\end{align*}
$$

By multiplying $\left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right)^{1 / 2}$ at both sides of Eq. 48), we obtain the desired inequality provided by Eq. (43). By applying $F(u, v)$ with Eq. (47) again to Eq. (10) in Theorem11 then, we will obtain

$$
\begin{align*}
\left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right)^{-1 / 2}\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)\right) & \left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right)^{-1 / 2} \\
& \geq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)} x g^{-1}(x)\right] \mathbf{1}_{\mathfrak{K}} \cdot\left(\mathcal{L}^{2}\right. \tag{49}
\end{align*}
$$

By multiplying $\left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)\right)^{-1 / 2}$ at both sides of Eq. (49), we obtain the desired inequality provided by Eq. (44).

The proof of Part (II) is immediate obtained by setting $g(x)$ as $-g(x)$ in Eq. (48) and Eq. (49).
Next Corollary 2 is obtained by applying Theorem 3 to special types of function $g$.
Corollary 2 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6).
(I) If we also assume that $\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q}>\boldsymbol{O}$ and $\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q}>\boldsymbol{O}$ for $q \in \mathbb{R}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq\left[{\underset{x \in \bigcup_{j=1}^{k}}{\text { wax }_{j} \text { Poly }_{f, u}\left(m_{j}, M_{j}\right)}}^{\max ^{1-q}}\right]\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q} \tag{50}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j}{\underset{\text { Poly }}{f, \mathcal{L}}}\left(m_{j}, M_{j}\right)} x^{1-q}\right]\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q} \tag{51}
\end{equation*}
$$

(II) If we also assume that $\log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)>\boldsymbol{O}$ and $\log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)>$ O for $q \in \mathbb{R}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :
and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\text { Poly }_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}} \frac{x}{\log x}\right] \log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \tag{53}
\end{equation*}
$$

(II') If we also assume that $\log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)<\boldsymbol{O}$ and $\log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*}\right.$ Poly $\left._{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)<$ $\boldsymbol{O}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :
and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq\left[\max _{x \in \bigcup_{j=1}^{k} \underset{w_{j}}{ } \widetilde{\text { Poly }_{f, u}\left(m_{j}, M_{j}\right)}} \frac{x}{\log x}\right] \log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \tag{55}
\end{equation*}
$$

(III) Since we have $\exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*}\right.$ Poly $\left._{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)>\boldsymbol{O}$ and $\exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*}\right.$ Poly $\left._{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)>\boldsymbol{O}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq\left[\max _{x \in \bigcup_{j=1}^{k} w_{j} \text { Poly }_{f, u}\left(m_{j}, M_{j}\right)} \frac{x}{\exp x}\right] \exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \tag{56}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

Proof: Part (I) of this Corollary is proved by applying Theorem 3 Part (I) with the function $g$ as $g(x)=x^{q}$. Part (II) of this Corollary is proved by applying Theorem 3 Part (I) with the function $g$ as $g(x)=\log (x)$, where $\log (x)>0$. Part (II') of this Corollary is proved by applying Theorem 3 Part (II) with the function $g$ as $g(x)=\log (x)$, where $\log (x)<0$. Finally, Part (III) of this Corollary is proved by applying Theorem 3 Part (I) with the function $g$ as $g(x)=\exp (x)$.

## 4 Generalized Converses of Operator Jensen's Inequalities: Difference Kind

In this section, we will derive the lower and upper bounds for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ in terms of difference criteria related to the function $g$.

Theorem 4 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6). The function $g$ is also a real valued continous function defined on the range $\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}_{f, \mathcal{L}}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{P o l y}_{f, u}\left(m_{j}, M_{j}\right)\right)$. We also have a real valued function $F(u, v)$ defined as Eq. (17) with support domain on $U \times V$ such that $f\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right) \subset U$, and $g\left(\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\operatorname{Poly}}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)\right) \cup\left(\bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }}_{f, u}\left(m_{j}, M_{j}\right)\right)\right) \subset V$.

Then, we have the following upper bound:

Similarly, we also have the following lower bound:

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \geq \min _{x \in \bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}}(x-g(x)) \mathbf{1}_{\mathfrak{K}} . \tag{59}
\end{equation*}
$$

Proof: The upper bound of this theorem is proved by setting $\alpha=1$ in Eq. (18) from Theorem 2 and rearrangement of the term $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)$ to obtain Eq. (58).

Similarly, the lower bound of this theorem is proved by setting $\alpha=1$ in Eq. (19) from Theorem 2 and rearrangement of the term $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)$ to obtain Eq. (59).

Next Corollary 3 is obtained by applying Theorem 4 to special types of function $g$.
Corollary 3 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfes thsoe conditions provided by Eq. (5) and Eq. (6).
(I) If we also assume that $g(x)=x^{q}$, where $q \in \mathbb{R}$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right):$

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q} \leq \max _{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\operatorname{Poly}}_{f, u}\left(m_{j}, M_{j}\right)}\left(x-x^{q}\right) \mathbf{1}_{\mathfrak{K}} \tag{60}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)^{q} \geq \min _{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\operatorname{Pol}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}}\left(x-x^{q}\right) \mathbf{1}_{\mathfrak{K}} \tag{61}
\end{equation*}
$$

(II) If we also assume that $g(x)=\log (x)$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-\log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \operatorname{Poly}_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \leq{\underset{x \in \bigcup_{j=1}^{k} w_{j} \widehat{\operatorname{Poly}_{f, u}\left(m_{j}, M_{j}\right)}}{ }(x-\log (x)) \mathbf{1}_{\mathfrak{R}} ; ~}_{\max } \tag{62}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-\log \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \geq \min _{x \in \bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}}(x-\log (x)) \mathbf{1}_{\mathfrak{K}} . \tag{63}
\end{equation*}
$$

(III) If we also assume that $g(x)=\exp (x)$, then, we have the following upper bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-\exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, u}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \leq \max _{\substack{x \in \bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }_{f, u}\left(m_{j}, M_{j}\right)}}}(x-\exp (x)) \mathbf{1}_{\mathfrak{K}} ; \tag{64}
\end{equation*}
$$

and, the following lower bound for $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ :

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-\exp \left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} \text { Poly }_{f, \mathcal{L}}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \geq \min _{x \in \bigcup_{j=1}^{k} w_{j} \widetilde{\text { Poly }_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)}}(x-\exp (x)) \mathbf{1}_{\mathfrak{K}} . \tag{65}
\end{equation*}
$$

Proof: For Part (I), we will use $g(x)=x^{q}$ in Eq. (58) in Theorem 4 to obtain Eq. (60). We will also use $g(x)=x^{q}$ in Eq. (59) in Theorem4 4 to obtain Eq. (61).

For Part (II), we will use $g(x)=\log (x)$ in Eq. (58) in Theorem4 to obtain Eq. (62). We will also use $g(x)=\log (x)$ in Eq. (59) in Theorem 4 to obtain Eq. (63).

For Part (III), we will use $g(x)=\exp (x)$ in Eq. (58) in Theorem44to obtain Eq. (64). We will also use $g(x)=\exp (x)$ in Eq. (59) in Theorem 4 to obtain Eq. (65).

## 5 Hypercomplex Function Approximation

In this section, we will consider hypercomplex function approximation problem in terms of the ratio error discussed in Section 5.1 and the difference error in Section 5.2.

### 5.1 Ratio Type Approximation

Let $\boldsymbol{A}$ be a self-adjoint operator with $\Lambda(\boldsymbol{A}) \in[m, M]$ for real scalars $m<M$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow$ $\mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). Let $f$ be any real valued continuous functions defined on the range $[m, M]$, represented by $f \in \mathcal{C}([m, M])$. The ratio type approximation problem is to find the function $g$ and the polynomial function $p_{1}$ to satisfy the following:

$$
\begin{equation*}
\frac{\Phi(f(\boldsymbol{A}))}{g\left(\boldsymbol{V}^{*} p_{1}(\boldsymbol{A}) \boldsymbol{V}\right)} \leq \alpha_{1} \mathbf{1}_{\mathfrak{K}}, \tag{66}
\end{equation*}
$$

where $\alpha_{1}$ is some specified positive real number. Similarly, we also can find the polynomial function $p_{2}$ to satisfy the following:

$$
\begin{equation*}
\frac{\Phi(f(\boldsymbol{A}))}{g\left(\boldsymbol{V}^{*} p_{2}(\boldsymbol{A}) \boldsymbol{V}\right)} \geq \alpha_{2} \mathbf{1}_{\mathfrak{K}}, \tag{67}
\end{equation*}
$$

where $\alpha_{2}$ is some specified positive real number.
From Theorem3, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, u}(\boldsymbol{A}) \boldsymbol{V}\right)>\boldsymbol{O}$, Eq. (66) can be established by setting

$$
\begin{equation*}
\alpha_{1} \geq\left[\underset{\left.x \in \underset{\operatorname{Poly}_{f, u}(m, M)}{\max } x g^{-1}(x)\right] . . . ~ . ~}{\text {. }}\right. \tag{68}
\end{equation*}
$$

Similarly, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A}) \boldsymbol{V}\right)>\boldsymbol{O}$, Eq. (67) can be established by setting

$$
\begin{equation*}
\alpha_{2} \leq\left[\min _{x \in \widetilde{\operatorname{Poly}_{f, \mathcal{L}}(m, M)}} x g^{-1}(x)\right] . \tag{69}
\end{equation*}
$$

On the other hand, from Theorem 3, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, u}(\boldsymbol{A}) \boldsymbol{V}\right)<\boldsymbol{O}$, Eq. 66) can be established by setting

$$
\begin{equation*}
\alpha_{1} \geq\left[{\left.\underset{x \in \operatorname{Poly}_{f, \mathcal{L}}(m, M)}{ } x g^{-1}(x)\right] . . . . ~}_{\min }\right. \tag{70}
\end{equation*}
$$

Similarly, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A}) \boldsymbol{V}\right)<\boldsymbol{O}$, Eq. 67) can be established by setting

$$
\begin{equation*}
\alpha_{2} \leq\left[\max _{x \in \operatorname{Poly}_{f, u}(m, M)} x g^{-1}(x)\right] \tag{71}
\end{equation*}
$$

For $k$ terms, let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continuous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfies the conditions provided by Eq. (5) and Eq. (6). The ratio type approximation problem is to find the function $g$ and the polynomial functions $p_{1,1}(x), \cdots, p_{1, k}(x)$ to satisfy the following:

$$
\begin{equation*}
\frac{\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)}{g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} p_{1, j}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)} \leq \alpha_{1} \mathbf{1}_{\mathfrak{K}} \tag{72}
\end{equation*}
$$

where $\alpha_{1}$ is some specfied positive real number. Similarly, we also can find the polynomial function $p_{2,1}(x), \cdots, p_{2, k}(x)$ to satisfy the following:

$$
\begin{equation*}
\frac{\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)}{g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} p_{2, j}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right)} \geq \alpha_{2} \mathbf{1}_{\mathfrak{K}}, \tag{73}
\end{equation*}
$$

where $\alpha_{2}$ is some specfied positive real number.
From Theorem 3, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, u}(\boldsymbol{A}) \boldsymbol{V}\right)>\boldsymbol{O}$, Eq. (72) can be established by setting

$$
\begin{equation*}
\alpha_{1} \geq\left[\max _{x \in \bigcup_{j=1}^{k} w_{j} \operatorname{Poly}_{f, u}\left(m_{j}, M_{j}\right)} x g^{-1}(x)\right] . \tag{74}
\end{equation*}
$$

Similarly, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A}) \boldsymbol{V}\right)>\boldsymbol{O}$, Eq. (73) can be established by setting

$$
\begin{equation*}
\alpha_{2} \leq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j} \operatorname{Poly}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)} x g^{-1}(x)\right] \tag{75}
\end{equation*}
$$

On the other hand, from Theorem 3 , if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, u}(\boldsymbol{A}) \boldsymbol{V}\right)<\boldsymbol{O}$, Eq. (72) can be established by setting

$$
\begin{equation*}
\alpha_{1} \geq\left[\min _{x \in \bigcup_{j=1}^{k} w_{j} \operatorname{Poly}_{f, \mathcal{L}}\left(m_{j}, M_{j}\right)} x g^{-1}(x)\right] . \tag{76}
\end{equation*}
$$

Similarly, if $g\left(\boldsymbol{V}^{*} \operatorname{Poly}_{f, \mathcal{L}}(\boldsymbol{A}) \boldsymbol{V}\right)<\boldsymbol{O}$, Eq. (73) can be established by setting

$$
\begin{equation*}
\alpha_{2} \leq\left[\max _{x \in \bigcup_{j=1}^{k} w_{j} \operatorname{Poly}_{f, u}\left(m_{j}, M_{j}\right)} x g^{-1}(x)\right] \tag{77}
\end{equation*}
$$

Remark 1 Given requirements of Eq. (66), Eq. (67), Eq. (72), and Eq. (73), we conjecture the existence of function $g$ and polynomials $p_{1}(x), p_{2}(x), p_{1,1}(x), \cdots, p_{1, k}(x)$, and $p_{2,1}(x), \cdots, p_{2, k}(x)$ for any given $\alpha_{1}$ or $\alpha_{2}$. If existence, how to find these functions?

The following Example 2 is provided to evaluate the upper ratio bound given by Eq. (72) and evaluate the lower ratio bound given by Eq. (73).

Example 2 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in[m, M]$ for real scalars $m<M$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined as $\Phi(\boldsymbol{X})=\boldsymbol{V}^{*}(\boldsymbol{X}) \boldsymbol{X}$. The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be a convex and
differentiable function such that $a x+b^{\prime} \leq f(x) \leq a x+b$ for $x \in[m, M]$, where

$$
\begin{align*}
a & =\frac{f(M)-f(m)}{M-m} \\
b & =\frac{M f(m)-m f(M)}{M-m} \\
b^{\prime} & =f\left(x_{0}\right)-\frac{f(M)-f(m)}{M-m} x_{0} \tag{78}
\end{align*}
$$

where $f^{\prime}\left(x_{0}\right)=\frac{f(M)-f(m)}{M-m}$.
If we require $g(x)>0$ for all $x \in[m, M]$, from Theorem 3 we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \max _{m<x<M} \frac{a x+b}{g(x)} g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{79}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \min _{m<x<M} \frac{a x+b^{\prime}}{g(x)} g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{80}
\end{equation*}
$$

Let us consider several special cases of the function $g(x)$. If $g(x)=x^{q}$, where $q \in \mathbb{R}$, and $m>0$, we have

$$
\begin{align*}
& \alpha_{1} \geq \max _{m<x<M} \frac{a x+b}{x^{q}} \\
& \alpha_{2} \leq \min _{m<x<M} \frac{a x+b^{\prime}}{x^{q}} . \tag{81}
\end{align*}
$$

If $g(x)=\log (x)$ and $m>1$, we have

$$
\begin{align*}
& \alpha_{1} \geq \max _{m<x<M} \frac{a x+b}{\log (x)}, \\
& \alpha_{2} \leq \min _{m<x<M} \frac{a x+b^{\prime}}{\log (x)} . \tag{82}
\end{align*}
$$

If $g(x)=\exp (x)$, we have

$$
\begin{align*}
& \alpha_{1} \geq \max _{m<x<M} \frac{a x+b}{\exp (x)}, \\
& \alpha_{2} \leq \min _{m<x<M} \frac{a x+b^{\prime}}{\exp (x)} . \tag{83}
\end{align*}
$$

If we require $g(x)<0$ for all $x \in[m, M]$, from Theorem 3 we also have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \min _{m<x<M} \frac{a x+b}{g(x)} \tag{84}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \max _{m<x<M} \frac{a x+b^{\prime}}{g(x)} \tag{85}
\end{equation*}
$$

Let us consider several special cases of the function $g(x)$. If $g(x)=-x^{q}$, where $q \in \mathbb{R}$, and $m>0$, we have

$$
\begin{align*}
& \alpha_{1} \geq \min _{m<x<M}-\frac{a x+b}{x^{q}} \\
& \alpha_{2} \leq \max _{m<x<M}-\frac{a x+b^{\prime}}{x^{q}} \tag{86}
\end{align*}
$$

If $g(x)=\log (x)$ and $0<m<M<1$, we have

$$
\begin{align*}
\alpha_{1} & \geq \min _{m<x<M} \frac{a x+b}{\log (x)} \\
\alpha_{2} & \leq \max _{m<x<M} \frac{a x+b^{\prime}}{\log (x)} \tag{87}
\end{align*}
$$

### 5.2 Difference Type Approximation

Let $\boldsymbol{A}$ be a self-adjoint operator with $\Lambda(\boldsymbol{A}) \in[m, M]$ for real scalars $m<M$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow$ $\mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). Let $f$ be any real valued continuous functions defined on the range $[m, M]$, represented by $f \in \mathcal{C}([m, M])$. The difference type approximation problem is to find the function $g$ and the polynomial function $p_{1}(x)$ to satisfy the following:

$$
\begin{equation*}
\Phi(f(\boldsymbol{A}))-g\left(\boldsymbol{V}^{*} p_{1}(\boldsymbol{A}) \boldsymbol{V}\right) \leq \beta_{1} \mathbf{1}_{\mathfrak{K}} \tag{88}
\end{equation*}
$$

where $\beta_{1}$ is some specified real number. Similarly, we also can find the polynomial function $p_{2}(x)$ to satisfy the following:

$$
\begin{equation*}
\Phi(f(\boldsymbol{A}))-g\left(\boldsymbol{V}^{*} p_{2}(\boldsymbol{A}) \boldsymbol{V}\right) \geq \beta_{2} \mathbf{1}_{\mathfrak{K}} \tag{89}
\end{equation*}
$$

where $\beta_{2}$ is some specified real number.
From Theorem 4. Eq. 88) can be established by setting

$$
\begin{equation*}
\beta_{1} \geq\left[\max _{x \in \operatorname{Poly}_{f, u}(m, M)} x-g(x)\right] \tag{90}
\end{equation*}
$$

Similarly, Eq. (89) can be established by setting

$$
\begin{equation*}
\beta_{2} \leq\left[\min _{x \in \operatorname{Poly}_{f, \mathcal{L}}(m, M)} x-g(x)\right] \tag{91}
\end{equation*}
$$

For $k$ terms, let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in\left[m_{j}, M_{j}\right]$ for real scalars $m_{j}<M_{j}$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined by Eq. (4). The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be any real valued continuous functions defined on the range $\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]$, represented by $f \in \mathcal{C}\left(\bigcup_{j=1}^{k}\left[m_{j}, M_{j}\right]\right)$. Besides, we assume that the function $f$ satisfies the conditions provided by Eq. (5) and Eq. (6). The difference type
approximation problem is to find the function $g$ and the polynomial functions $p_{1,1}(x), \cdots, p_{1, k}(x)$ to satisfy the following:

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} p_{1, j}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \leq \beta_{1} \mathbf{1}_{\mathfrak{K}} \tag{92}
\end{equation*}
$$

where $\beta_{1}$ is some specfied real number. Similarly, we also can find the polynomial function $p_{2,1}(x), \cdots, p_{2, k}(x)$ to satisfy the following:

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)-g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{V}^{*} p_{2, j}\left(\boldsymbol{A}_{j}\right) \boldsymbol{V}\right) \geq \beta_{2} \mathbf{1}_{\mathfrak{K}} \tag{93}
\end{equation*}
$$

where $\beta_{2}$ is some specfied real number.
The following Example 3 is provided to evaluate the upper difference bound given by Eq. (92) and evaluate the lower difference bound given by Eq. 93 .

Example 3 Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in[m, M]$ for real scalars $m<M$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined as $\Phi(\boldsymbol{X})=\boldsymbol{V}^{*}(\boldsymbol{X}) \boldsymbol{X}$. The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f$ be a convex and differentiable function such that $a x+b^{\prime} \leq f(x) \leq a x+b$ for $x \in[m, M]$, where

$$
\begin{align*}
a & =\frac{f(M)-f(m)}{M-m} \\
b & =\frac{M f(m)-m f(M)}{M-m} \\
b^{\prime} & =f\left(x_{0}\right)-\frac{f(M)-f(m)}{M-m} x_{0} \tag{94}
\end{align*}
$$

where $f^{\prime}\left(x_{0}\right)=\frac{f(M)-f(m)}{M-m}$.
From Theorem 4 we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \max _{m<x<M}(a x+b-g(x)) \tag{95}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \min _{m<x<M}\left(a x+b^{\prime}-g(x)\right) \tag{96}
\end{equation*}
$$

Let us consider several special cases of the function $g(x)$. If $g(x)=x^{q}$, where $q \in \mathbb{R}$ and $m>0$, we have

$$
\begin{align*}
& \beta_{1} \geq \max _{m<x<M}\left(a x+b-x^{q}\right) \\
& \beta_{2} \leq \min _{m<x<M}\left(a x+b^{\prime}-x^{q}\right) \tag{97}
\end{align*}
$$

If $g(x)=\log (x)$ and $m>0$, we have

$$
\begin{align*}
& \beta_{1} \geq \max _{m<x<M}(a x+b-\log (x)) \\
& \beta_{2} \leq \min _{m<x<M}\left(a x+b^{\prime}-\log (x)\right) \tag{98}
\end{align*}
$$

If $g(x)=\exp (x)$, we have

$$
\begin{align*}
& \beta_{1} \geq \max _{m<x<M}(a x+b-\exp (x)), \\
& \beta_{2} \leq \min _{m<x<M}\left(a x+b^{\prime}-\exp (x)\right) . \tag{99}
\end{align*}
$$

Remark 2 Given requirements of Eq. (88), Eq. (89), Eq. (92), and Eq. (93), we conjecture the existence of function $g$ and polynomials $p_{1}(x), p_{2}(x), p_{1,1}(x), \cdots, p_{1, k}(x)$, and $p_{2,1}(x), \cdots, p_{2, k}(x)$ for any given $\beta_{1}$ or $\beta_{2}$. If existence, how to find these functions?

## 6 Bounds Algebra

In this section, we will apply results from Section 3 to build bounds algebra for addition and multiplication of hypercomplex functions $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$, where functions $f$ and $h$ share some common properties.

### 6.1 Hypercomplex Function Bounds Algebra

Let $\boldsymbol{A}_{j}$ be self-adjoint operator with $\Lambda\left(\boldsymbol{A}_{j}\right) \in[m, M]$ for real scalars $m<M$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow$ $\mathbb{B}(\mathfrak{K})$ is defined as $\Phi(\boldsymbol{X})=\boldsymbol{V}^{*}(\boldsymbol{X}) \boldsymbol{X}$. The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f, h$ be two convex and differentiable functions defined in $[m, M]$ such that

$$
\begin{align*}
a x+b^{\prime} & \leq f(x) \\
c x+d^{\prime} & \leq h(x) \tag{100}
\end{align*}
$$

where $x \in[m, M]$. If we require $g(x)>0$ for all $x \in[m, M]$, from Theorem 3, we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \leq \underbrace{\max _{m<x<M} \frac{a x+b}{g(x)}}_{:=\alpha_{f, u}} g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{101}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right) \geq \underbrace{\min _{m<x<M} \frac{a x+b^{\prime}}{g(x)}}_{:=\alpha f, \mathcal{L}} g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{102}
\end{equation*}
$$

Similarly, from Theorem 3, we also have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right) \leq \underbrace{\max _{m<x<M} \frac{c x+d}{g(x)}}_{:=\alpha_{h, u}} g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{103}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right) \geq \underbrace{\min _{m<x<M} \frac{c x+d^{\prime}}{g(x)}}_{:=\alpha_{h, \mathcal{L}}} g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{104}
\end{equation*}
$$

Consider the addition between $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$, from Eq. (101) and Eq. 103), then, we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)+\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right) \leq\left(\alpha_{f, u}+\alpha_{h, u}\right) g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{105}
\end{equation*}
$$

and, from Eq. (102) and Eq. (104), we also have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)+\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right) \geq\left(\alpha_{f, \mathcal{L}}+\alpha_{h, \mathcal{L}}\right) g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right) \tag{106}
\end{equation*}
$$

If any two functions $f, h$ satisfying Eq. (100), from Eq. (112) and Eq. (106), we can bound the addition between $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$ by coefficients of $g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)$ with the term $g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)$, which is independent of the functions $f$ and $h$. Therefore, the bounding coefficients of $g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)$ form an algebraic system of interval numbers, which is an abelian monoid with respect to the opration addition accoding to Theorem 2.14 [17].

Consider the multiplication between $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$, from Eq. (101) and Eq. (103) with assumptions of positive $\alpha_{f, \mathcal{U}}, \alpha_{h, u}, \alpha_{f, \mathcal{L}}$ and $\alpha_{h, \mathcal{L}}$ and $g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)>\boldsymbol{O}$, then, we have

$$
\begin{equation*}
\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)\right) \times\left(\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)\right) \leq\left(\alpha_{f, u} \times \alpha_{h, u}\right)\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\right)^{2} \tag{107}
\end{equation*}
$$

and, from Eq. (102) and Eq. (104), we also have

$$
\begin{equation*}
\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)\right) \times\left(\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)\right) \geq\left(\alpha_{f, \mathcal{L}} \times \alpha_{h, \mathcal{L}}\right)\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\right)^{2} \tag{108}
\end{equation*}
$$

Analogly, if any two functions $f, h$ satisfying Eq. (100), from Eq. (107) and Eq. (108), we can bound the multiplication between $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$ by coefficients of $\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)^{2}\right.$ with the term $\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\right)^{2}$, which is independent of the functions $f$ and $h$. Therefore, the bounding coefficients of $\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\right)^{2}$ form an algebraic system of interval numbers, which is an abelian monoid with respect to the opration multiplication accoding to Theorem 2.14 [17].

Remark 3 The norms of the addition (or multiplication) between $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$ can also be bounded by coefficients of $g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\left(\operatorname{or}\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\right)^{2}\right)$ with abelian monoid algebraic structure and norms of $g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\left(\operatorname{or}\left(g\left(\sum_{j=1}^{k} w_{j} \Phi\left(\boldsymbol{A}_{j}\right)\right)\right)^{2}\right)$.

### 6.2 Random Tensor Tail Bounds Algebra

The purpose of this section is to show a method to obtain tail bound for the addition (or multiplication) between two random tensors $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$ via Theorem 3 .

Let $\boldsymbol{A}_{j}$ be random Hermitian tensor with $\Lambda\left(\boldsymbol{A}_{j}\right) \in[m, M]$ for real scalars $m<M$. The mapping $\Phi: \mathbb{B}(\mathfrak{H}) \rightarrow \mathbb{B}(\mathfrak{K})$ is defined as $\Phi(\boldsymbol{X})=\boldsymbol{X}$. The index $j$ is in the range of $1,2, \cdots, k$, and we have a probability vector $\boldsymbol{w}=\left[w_{1}, w_{2}, \cdots, w_{k}\right]$ with the dimension $k$, i.e., $\sum_{j=1}^{k} w_{j}=1$. Let $f, h$ be two convex and differentiable functions defined in $[m, M]$ such that

$$
\begin{align*}
& f(x) \leq a x+b, \\
& h(x) \leq c x+d, \tag{109}
\end{align*}
$$

where $x \in[m, M]$. If we require $g(x)>0$ for all $x \in[m, M]$ and any positive number $\theta$, from Theorem 3 , we have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} f\left(\boldsymbol{A}_{j}\right) \leq \underbrace{\max _{m<x<M} \frac{a x+b}{g(x)}}_{:=\alpha_{f, u}} g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{A}_{j}\right) . \tag{110}
\end{equation*}
$$

Similarly, from Theorem 3, we also have

$$
\begin{equation*}
\sum_{j=1}^{k} w_{j} h\left(\boldsymbol{A}_{j}\right) \leq \underbrace{\min _{m \times x<M} \frac{c x+d}{g(x)}}_{:=\alpha_{h, u}} g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{A}_{j}\right) \tag{111}
\end{equation*}
$$

Consider tail bounds for the addition between $\sum_{j=1}^{k} w_{j} f\left(\boldsymbol{A}_{j}\right)$ and $\sum_{j=1}^{k} w_{j} h\left(\boldsymbol{A}_{j}\right)$, from Eq. (110) and Eq. 111, then, we have

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left(\sum_{j=1}^{k} w_{j} f\left(\boldsymbol{A}_{j}\right)\right)+\left(\sum_{j=1}^{k} w_{j} h\left(\boldsymbol{A}_{j}\right)\right)\right\|_{\ell} \geq \theta\right) \\
& \quad \leq \operatorname{Pr}\left(\left\|\left(\alpha_{f, u}+\alpha_{h, u}\right) g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{A}_{j}\right)\right\|_{\ell} \geq \theta\right) \tag{112}
\end{align*}
$$

where $\|\cdot\|_{(\ell)}$ is Ky Fan $\ell$-norm. R.H.S. of Eq. (112), where the random tensors summation part is independent of functions $f$ and $h$ can be upper bounded by those theorems in Section IV in [18].

Consider the multiplication between $\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)$ and $\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)$, from Eq. 110) and Eq. (111) with assumptions of positive $\alpha_{f, u}$ and $\alpha_{h, u}$ and $g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{A}_{j}\right)>\boldsymbol{O}$, then, we have

$$
\begin{gather*}
\operatorname{Pr}\left(\left\|\left(\sum_{j=1}^{k} w_{j} \Phi\left(f\left(\boldsymbol{A}_{j}\right)\right)\right) \times\left(\sum_{j=1}^{k} w_{j} \Phi\left(h\left(\boldsymbol{A}_{j}\right)\right)\right)\right\|_{\ell} \geq \theta\right) \\
\leq \operatorname{Pr}\left(\|\left(\alpha_{\left.\left.f, u \times \alpha_{h, u}\right)\left(g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{A}_{j}\right)\right)^{2} \|_{\ell} \geq \theta\right)}=\operatorname{Pr}\left(\left\|g\left(\sum_{j=1}^{k} w_{j} \boldsymbol{A}_{j}\right)\right\|_{\ell} \geq \sqrt{\frac{\theta}{\left(\alpha_{f, u} \times \alpha_{h, u}\right)}}\right),\right.\right.
\end{gather*}
$$

where the last term of Eq. (113) can be upper bounded by those theorems in Section IV in [18].

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