Inflationary dynamics in modified gravity models

R. R. Cuzinatto[1] and L. G. Medeiros[2]

[1]Instituto de Ciência e Tecnologia, Universidade Federal de Alfenas, Rodovia José Aurélio Vilela, 11999, CEP 37715-400, Poços de Caldas, MG, Brazil

[2]Escola de Ciência e Tecnologia, Universidade Federal do Rio Grande do Norte, Campus Universitário, s/n–Lagoa Nova, CEP 59078-970, Natal, RN, Brazil

Abstract

Higher-order theories of gravity are a branch of modified gravity wherein the geometrodynamics of the four-dimensional Riemannian manifold is determined by field equations involving derivatives of the metric tensor of order higher than two. This paper considers a general action built with the Einstein-Hilbert term plus additional curvature-based invariants, viz. the Starobinsky R^2-type term, a term scaling with R^3, and a correction of the type $R\Box R$. The focus is on the background inflationary regime accommodated by these three models. For that, the higher-order field equations are built and specified for the FLRW line element. The dynanical analysis in the phase space is carried in each case. This analysis shows that the Starobinsky-plus-R^3 model keeps the good features exhibited by the pure Starobinsky inflationary model, although the set of initial conditions for the inflaton field χ leading to a graceful exit scenario is more contrived; the coupling constant α_0 of the R^3 invariant is also constrained by the dynamical analysis. The Starobinsky-plus-$R\Box R$ model turns out being a double-field inflation model; it consistently enables an almost-exponential primordial acceleration followed by a radiation dominated universe if its coupling β_0 takes values in the interval $0 \leq \beta_0 \leq 3/4$. The models introducing higher-order correction to Starobinsky inflation are interesting due to the possibility of a running spectral index n_s, something that is allowed by current CMB observations.

1 Introduction

General relativity (GR) currently stands as the canonical theory describing the gravitational interaction. Since its proposition early in the XX century, GR was able to explain and predict a plethora of phenomena in the realms of physics, astrophysics and cosmology. Among them are the examples of gravitational redshift [1], gravitational lensing [2], prediction of existence of black holes [3] and gravitational waves [4], and the description of the universe's large scale evolution [5]. Even so, there are indications that GR is not a definitive theory of gravity. The hints are structural in nature—e.g. the existence of singularities within GR—or particularly related to high-energy regimes: GR can not be trivially quantized [6, 7] and it does not provide a completely consistent description of the primeval universe (around the energy scales related to inflation) [8, 9]. Therefore, it is only natural to propose modification to GR in an attempt to overcome these challenges.

From a purely theoretical point of view, GR is built by considering that gravity is described by a metric-compatible four-dimensional Riemannian manifold, which is endowed with a single rank-2 tensorial field—the metric tensor $g_{\mu\nu}$—, which is invariant under diffeomorphisms, and which exhibits second-order equations of motion (cf. the Lovelock theorem) [10]. Modifications to GR are implemented by relaxing anyone of the aforementioned hypotheses. For instance, Horndeski theories [11] stem from violating the hypothesis that the metric is the only fundamental field: an extra degree of freedom is also assumed. A different pathway is to admit a Riemann-Cartan-type of spacetime substrate, a manifold equipped with an affine connection bearing a non-null antisymmetric sector; in this case, torsion is included as a gravitational entity and the Einstein-Cartan theories are born [12]. Another possibility is to eliminate curvature while keeping a non-null torsion; this is a feature of Weitzenböck manifold and the teleparallel equivalent of general relativity [13, 14, 15, 16, 17, 18].[1] On the other hand, if the fields equations for the metric tensor are allowed to include derivatives of order greater than two—while simultaneously maintaining all the other hypotheses—then the higher-order gravity theories are obtained [19, 20].

Higher-order theories of gravity feature additional terms to the Einstein-Hilbert (EH) action engendering higher-order derivatives in the field equations. Such extra terms in the action may be seen as correction terms, classified according to their typical mass/energy scale. Following this classification, zero-order terms are those counted in units of square mass; they correspond to the curvature scalar R and the cosmological constant Λ in the EH action. First-order corrections to EH action involve term of mass to the fourth power; these are built with the invariants

$$R^2 \qquad \text{and} \qquad R_{\mu\nu}R^{\mu\nu}. \tag{1}$$

It is worth mentioning that the other two possible first-order invariants, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ and $\Box R$, do not contribute to the field equations.[2] Second-order terms are corrections to EH action having units of mass to the sixth power; they made up with the following invariants [21]

$$\begin{gathered} R\Box R,\ R_{\mu\nu}\Box R^{\mu\nu}, \\ R^3,\ RR_{\mu\nu}R^{\mu\nu},\ R_{\mu\nu}R^{\nu}{}_{\alpha}R^{\alpha\mu}, \\ RR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta},\ R_{\mu\alpha}R_{\nu\beta}R^{\mu\nu\alpha\beta},\ \text{and}\ R_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\kappa\rho}R^{\kappa\rho\mu\nu}. \end{gathered} \tag{2}$$

Among the various applications of higher-order theories of gravity [22, 23, 24, 25], one class of particular interest is that of inflationary cosmology [26, 27, 28, 29].

In the end of 1979, Alexei A. Starobinsky proposed that quantum gravitational effects, presumably significant in the primordial universe, produce a quasi-de Sitter cosmic dynamics, i.e. an almost-exponential inflationary regime [9, 30]. In fact, A. A. Starobinsky showed that the inclusion of the term R^2 in the EH action is able to generate an early accelerated expansion ending in a radiation-dominated decelerated universe. Starobinsky model is an enormous success: nowadays, it is one of the most promising candidates for realizing the inflationary dynamics. The main reason for this accomplishment is its being a single-parameter model fitting perfectly the most recent observations of the cosmic microwave background radiation (CMB) [31, 32]. Moreover, the theoretical motivation for Starobinsky model is quite robust. In effect, Starobinsky inflation occurs in energy scales of about 10^{15} GeV; in such period the action containing the term R^2 may be considered as part of a higher-order theory expected in the context of quantization of gravity [33].

The main goal of this contribution is to review the basic aspects of the cosmic dynamics predicted by Starobinsky inflation and to study its extension to models containing second-order derivative corrections involving the curvature scalar. Section 2 presents a general action integral encompassing the regular EH term, plus Starobinsky R^2-contributions, and the novel higher-order corrections; the field equations for this modified gravity are also derived therein. Section 3 summarizes the conditions for inflation in a homogeneous and isotropic background; the field equations are also specified in FLRW spacetime. Subsections 3.1, 3.2, and 3.3 analyse the inflationary dynamics (in the phase space) in three separate models, viz. the original Starobinsky proposal, the model supplementing Starobinsky term with a R^3 contribution, and a higher-order model adding a correction of the type $R\Box R$ to the traditional R^2-term. Section 4 brings our final comments.

[1] Regarding the teleparallel framework for gravity, we also point the reader to the contribution by P. J. Pompeia for this book and the references cited in that paper.

[2] The term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ may be written as a linear combination of R^2, $R_{\mu\nu}R^{\mu\nu}$, and the Gauss-Bonnet topological invariant. The term $\Box R$ is explicitly a surface term.

2 Fundamentals of the proposed modified gravity models

The most general action presenting up to second order correction to the EH action involving the curvature scalar reads:

$$S=\frac{M_{\mathrm{Pl}}^{2}}{2}\int d^{4}x\sqrt{-g}\left[R+\frac{1}{2\kappa_{0}}R^{2}+\frac{\alpha_{0}}{3\kappa_{0}^{2}}R^{3}-\frac{\beta_{0}}{2\kappa_{0}^{2}}R\Box R\right]. \tag{3}$$

Herein κ_0 has units of square mass while α_0 and β_0 are dimensionless parameters. Starobinsky R^2 term introduce the first-order correction to Einstein-Hilbert R term. The last two terms of (3) account for all the possible second-order corrections built with the curvature scalar. Parameter κ_0 sets the energy scale for inflation; α_0 and β_0 regulate the deviations from Starobinsky model.

It is convenient to perform a conformal metric transformation and to introduce dimensionless fields as follows:

$$\bar{g}_{\mu\nu}=e^{\chi}g_{\mu\nu},\qquad \lambda=\frac{R}{\kappa_{0}}\quad\text{and}\quad e^{\chi}=1+\lambda+\alpha_{0}\lambda^{2}-\frac{\beta_{0}}{\kappa_{0}}\Box\lambda. \tag{4}$$

The above allows one to cast (3) in the Einstein frame [29]:

$$\bar{S}=\frac{M_{\mathrm{Pl}}^{2}}{2}\int d^{4}x\sqrt{-\bar{g}}\left[\bar{R}-3\left(\frac{1}{2}\bar{\nabla}_{\rho}\chi\bar{\nabla}^{\rho}\chi-\frac{\beta_{0}}{6}e^{-\chi}\bar{\nabla}_{\rho}\lambda\bar{\nabla}^{\rho}\lambda+V\left(\chi,\lambda\right)\right)\right], \tag{5}$$

where

$$V\left(\chi,\lambda\right)=\frac{\kappa_{0}}{3}e^{-2\chi}\lambda\left(e^{\chi}-1-\frac{1}{2}\lambda-\frac{\alpha_{0}}{3}\lambda^{2}\right), \tag{6}$$

stands for the multi-field potential of our model. The latter is a gravity model described in terms of the metric tensor $\bar{g}_{\mu\nu}$ along with two scalar fields, viz. χ and λ.

The field equations follow from setting to zero the variations of the action (5) with respect to the fields $\bar{g}_{\mu\nu}$, χ and λ. Executing this procedure for the metric tensor yields:

$$\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}=\frac{1}{M_{\mathrm{Pl}}^{2}}\bar{T}_{\mu\nu}^{(\mathrm{eff})}, \tag{7}$$

with the effective energy momentum tensor given by

$$\frac{1}{M_{\mathrm{Pl}}^{2}}\bar{T}_{\mu\nu}^{(\mathrm{eff})}=\frac{3}{2}\left(\bar{\nabla}_{\mu}\chi\bar{\nabla}_{\nu}\chi-\frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}^{\rho}\chi\bar{\nabla}_{\rho}\chi\right)-\frac{\beta_{0}e^{-\chi}}{2}\left(\bar{\nabla}_{\mu}\lambda\bar{\nabla}_{\nu}\lambda-\frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}^{\rho}\lambda\bar{\nabla}_{\rho}\lambda\right)-\frac{3}{2}\bar{g}_{\mu\nu}V\left(\chi,\lambda\right). \tag{8}$$

The field equations for the scalar fields are:

$$\bar{\Box}\chi-\frac{\beta_{0}}{6}e^{-\chi}\bar{\nabla}_{\rho}\lambda\bar{\nabla}^{\rho}\lambda-V_{\chi}=0, \tag{9}$$

$$\beta_{0}e^{-\chi}\left(\bar{\nabla}^{\rho}\chi\bar{\nabla}_{\rho}\lambda-\bar{\Box}\lambda\right)-3V_{\lambda}=0. \tag{10}$$

The shorthand notations $V_{\chi}=\frac{\partial V}{\partial\chi}$ and $V_{\lambda}=\frac{\partial V}{\partial\lambda}$ were used.

3 Inflation on the FLRW background

Generically, inflation may be regarded as an early period of near-exponential accelerated expansion taking place at some point roughly in between 10 MeV and 10^{16} GeV. The motivations for this early vertiginous expansion range from the need to explain the observed flat universe, to the attempt to justify the high degree of homogeneity and isotropy displayed by the CMB, and, more importantly, to predict the causally connected density fluctuations that are correlated to the large-scale structure in the present-day universe [8, 34, 35].

Inflationary cosmology addresses basically three points:

1. Initial conditions leading to the quasi-exponential expansion;
2. The details of the early accelerated regime and its connections with observations;
3. The ending of the accelerated expansion and reheating.

The first point is addressed in two ways. Approach number one is more thorough; it admits a broad range of possible initial conditions in a non-homogeneous and anisotropic spacetime. The second approach to point number 1 is a simplified approach assuming generic initial conditions while the spacetime is restricted to being described by FLRW line element at the background level [36]. Notice that the flatness problem and the problem of generating the primordial fluctuations can be treated via both the above approaches; however, the problem of explaining homogeneity and isotropy can only be addressed by the first, more complete approach. Regardless the approach, a robust inflationary model should be able to produce an accelerated expansion from fairly general initial conditions.

Point number 2 is the most relevant one since it directly connects inflation to observations. In fact, initial fluctuations are generated during the inflationary dynamics; these density perturbations are the very seeds of the universe's large-scale structure. The necessary condition for achieving an inflationary regime accommodating causally connected perturbation is an accelerated expansion:[3]

$$\text{Inflation} \Longleftrightarrow \ddot{a} > 0 \Longleftrightarrow \frac{d}{dt}\left(aH\right)^{-1} < 0.$$

The scale $\left(aH\right)^{-1}$ is known as Hubble horizon or Hubble radius; it delimits the region wherein two points are momentarily causally connected. The Hubble radius decreases during inflation allowing quantum fluctuations to exit the the horizon. These initially correlated perturbations are then frozen, and later produce the necessary conditions for structure formation [37] (after horizon crossing at the end of the accelerated period).

Point number 3 addresses the end of the inflationary regime. The particles the eventually populated the primeval universe were diluted to such a degree during the almost-exponential expansion that any hint of a thermalized universe disappears after inflation. Hence, a viable inflationary model should be able to repopulate the universe after its ending, then producing a hot Big Band phase dominated by radiation (ultra-relativistic particles). The period bridging inflation to a radiation-dominated era is called reheating [38, 39].

The three points above can be (partially) studied in a Friedmann-Lemaître-Robertson-Walker (FLRW) background. The homogeneous and isotropic FLRW flat spacetime is described by the line element

$$ds^2 = -dt^2 + a^2\left(t\right)\left(dx^2 + dy^2 + dz^2\right), \tag{11}$$

where $a\left(t\right)$ is the scale factor; natural units are assumed: $c = 1$. Specifying the field equations (7), (9), and (10) on the spacetime (11), leads to:

$$h^2 = \frac{1}{2}\left(\frac{1}{2}\chi_t^2 - \frac{\beta_0}{6}e^{-\chi}\lambda_t^2 + \bar{V}\left(\chi,\lambda\right)\right), \tag{12}$$

$$h_t = -\frac{3}{4}\chi_t^2 + \frac{1}{4}\beta_0 e^{-\chi}\lambda_t^2, \tag{13}$$

and

$$\chi_{tt} + 3h\chi_t - \frac{\beta_0}{6}e^{-\chi}{\lambda_t}^2 + \bar{V}_\chi = 0, \tag{14}$$

$$\beta_0 e^{-\chi}\left[\lambda_{tt} - \left(\chi_t - 3h\right)\lambda_t\right] - 3\bar{V}_\lambda = 0. \tag{15}$$

[3]The Hubble function is defined as usual: $H = \dot{a}/a$, where an overdot denotes differentiation with reespect to the cosmic time t.

For convenience, the above equations were written in terms of the dimensionless Hubble function h and the dimensionless potential $\bar{V}$:

$$h \equiv \frac{1}{\sqrt{\kappa_0}}\frac{\dot{a}}{a} \quad \text{and} \quad \bar{V}\left(\chi,\lambda\right) \equiv \frac{1}{\kappa_0}V\left(\chi,\lambda\right). \tag{16}$$

Moreover, use is made of the dimensionless time derivative

$$A_t \equiv \frac{1}{\sqrt{\kappa_0}}\dot{A}. \tag{17}$$

The following three subsection deal with particular solutions to Eqs. (12), (13), (14), and (15).

3.1 Starobinsky model

Starobinsky inflation adds the first-order correction to EH action via the term proportional R^2. In this case, S is simplified by taking $\alpha_0 = \beta_0 = 0$. Consequently, the field equation for λ—Eq. (15)—becomes a constraint equation given by:

$$\bar{V}_\lambda = 0 \Rightarrow \lambda = e^\chi - 1. \tag{18}$$

Inserting (18) into Eqs. (12), (13) and (14), leads to:

$$h^2 = \frac{1}{2}\left(\frac{1}{2}\chi_t^2 + \bar{V}^{\mathrm{St}}\right), \tag{19}$$

$$h_t = -\frac{3}{4}\chi_t^2, \tag{20}$$

and

$$\chi_{tt} + 3h\chi_t + \bar{V}_\chi^{\mathrm{St}} = 0. \tag{21}$$

The χ-related potential $\bar{V}^{\mathrm{St}}\left(\chi\right)$ for the Starobinsky model (label $^{\mathrm{St}}$) reads

$$\bar{V}^{\mathrm{St}}\left(\chi\right) = \frac{1}{6}\left(1 - e^{-\chi}\right)^2. \tag{22}$$

Notice that Starobinsky inflation is achieved by the dynamics of the scalar field χ alone. This dynamics is obtained from Eqs. (19) and (21). By taking χ as the variable describing the evolution of the system, one rewrites Eq. (21) in the form:

$$\frac{d\chi_t}{d\chi} = \frac{-3\chi_t\sqrt{\frac{1}{4}\chi_t^2 + \frac{1}{2}\bar{V}^{\mathrm{St}}} - \bar{V}_\chi^{\mathrm{St}}}{\chi_t}. \tag{23}$$

The above equation is an autonomous first-order ordinary differential equation; its structure is studied by means of the direction fields related to (χ, χ_t). Fig. 1 shows the phase space for system of Eq. (23).

There are two noticeable features in 1: an approximately horizontal attractor line in the vicinity of $\chi_t \approx 0$ and an accumulation point at the origin.

The attractor line realizes an (almost-)exponential expansion regime since $\chi_t^2 \ll \bar{V}^{\mathrm{St}}$ ($\chi_t \ll 1$ and $\bar{V}^{\mathrm{St}} \sim 1/6$) along this trajectory. In fact, by using these conditions in Eqs. (19) and (20), we build a slow-roll parameter ϵ satisfying

$$\epsilon = -\frac{h_t}{h^2} = \frac{3\chi_t^2}{\left(\chi_t^2 + 2\bar{V}^{St}\right)} \ll 1. \tag{24}$$

The condition $\epsilon \ll 1$ yields the inflationary period because

$$h_t \ll h^2 \Rightarrow h \approx \text{constant} \Rightarrow a\left(t\right) \propto \exp\left(\sqrt{\kappa_0}ht\right). \tag{25}$$

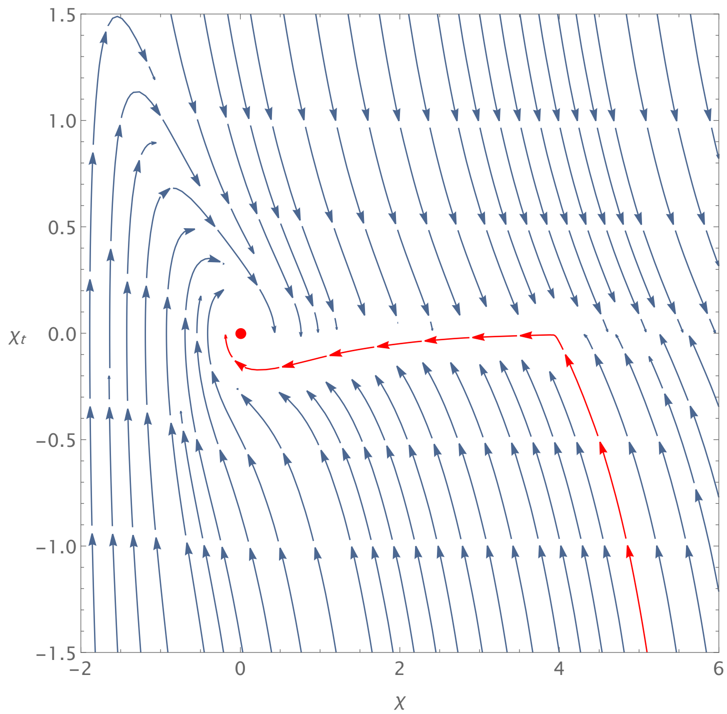

Figure 1: Phase space (χ, χ_t) for the inflaton field in Starobinsky model. The red dot corresponds to the accumulation point at $(0, 0)$; the red oriented line highlights a possible trajectory in the phase space.

Moreover, Fig.1 makes it transparent that a broad range of initial conditions ($\chi^i > 2$ and χ_t^i arbitrary) set the system towards the attractor line. Starobinsky model is therefore capable of producing an inflationary regime starting from a very general set of initial conditions.[4]

The accumulation point at $(\chi, \chi_t) = (0, 0)$ is the point of inflation's end. The dynamics of χ in the vicinity of this point is oscillatory. This means that χ transfers energy to the matter fields it is coupled with while it oscillates coherently about the origin. The process just described is known as pre-heating; it is the initial phase of the reheating, when a large number of matter particles is produced. Since pre-heating is essentially a non-thermal process, a subsequent thermalization stage is demanded to lead the universe to a radiation-dominated era where all kinds of matter particles are in thermal equilibrium [40, 41].

In spite of being a preliminar analysis, the above study based on Fig. 1 shows that Starobinsky model successfully addresses the three basic points of interest listed at the beginning of Section 3. In the next two subsection, it will be checked if that continues to be the case for the models including the R^3- and $R\Box R$-type corrections to Starobinsky inflation.

3.2 Starobinsky-plus-R^3 model

A term of the type R^3 can be added to Starobinsky action ($\propto R + R^2$) thus generating the Starobinsky-plus-R^3 model. The inflationary dynamics accommodated by this modified gravity model respects Eqs. (19), (20), and (21) provided that $\bar{V}^{\rm St}$ is generalized to the potential [28]

$$\bar{V}^{\alpha_0}(\chi) = \frac{e^{-2\chi}}{72\alpha_0^2}\left(1 - \sqrt{1 - 4\alpha_0 (1 - e^{\chi})}\right)\left(-1 + 8\alpha_0 (1 - e^{\chi}) + \sqrt{1 - 4\alpha_0 (1 - e^{\chi})}\right). \tag{26}$$

The potential is real-valued regardless of the value taken by χ under the constraint: $0 \leq 4\alpha_0 \leq 1$.

[4] Starobinsky inflation is an example of the chaotic inflationary scenario [35].

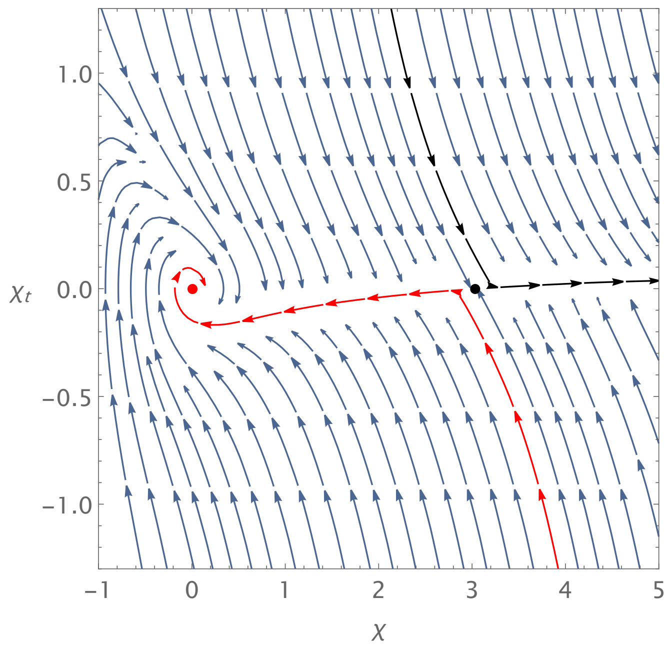

Figure 2: Phase-space representation (χ, χ_t) for Starobinsky-plus-R^3 model with parameter $\alpha_0 = 10^{-2}$. The red dot and the black dot in the plot mark the critical points $(0, 0)$ and $(\chi_c, 0)$ where $\chi_c = 3.06$. The red line and the black line show two opposite trajectories with respect to the critical point $(\chi_c, 0)$. Source: Ref. [28].

The phase-space analysis for the Starobinsky-plus-R^3 model is performed along the lines of what was done in Section 3.1, by employing Eq. (23) with the substitution $\bar{V}^{\mathrm{St}} \to \bar{V}^{\alpha_0}$. This leads to Fig. 2.

The main difference between the Figs. 2 and 1 is the appearance of a new critical point

$$P_c = (\chi_c, 0) = \left(\ln\left(4 + \sqrt{3\alpha_0^{-1}}\right), 0\right). \tag{27}$$

This critical point is a saddle point that splits the phase space into two distinct regions in regard to the direction field lines. The sector of Fig. 2 to the left of the vertical red attractor line yields an inflationary regime ending in the stable accumulation point $(\chi, \chi_t) = (0, 0)$. If the inflaton field χ starts from $\left(\chi^i, \chi_t^i\right)$ in this region, inflation occurs in the usual way: the accelerated expansion subsequently gives off into a decelerated phase with χ oscillating about the origin (potential minimum). On the other hand, the trajectories to the right from the vertical black line yield a inflationary dynamics that never ends. In fact, Ref. [28] details how the field χ grows indefinitely in this sector; its dynamics leading to the transition of an initial almost-exponential expansion to the asymptotically accelerated phase of the power-law type $a(t) \sim t^{12}$.

The precise location of the point P_c is sensitive to the value of parameter α_0 accompanying the term R^3 in the action (3). The smaller the value of α_0, the greater the value of χ_c. Reassuringly enough, in the (Starobinsky) limit $\alpha_0 \to 0$, it is $\chi_c \to \infty$, and Fig. 2 degenerates into 1, as it should. The most notable difference between the Starobinsky-plus-R^3 setup and the standard Starobinsky inflation is the fact that the initial conditions leading to a physically meaningful inflation cannot be chosen arbitrarily.[5] In effect, the greater the value of α_0 the smaller the set of initial conditions $\left(\chi^i, \chi_t^i\right)$ capable of producing an inflationary regime that evolves to a radiation epoch. In this sense, a introduction of the R^3 correction

[5] By "physically meaningful inflation" it is meant a primordial accelerated expansion that ends in a radiation-dominated Hot Big-Bang phase.

to the Starobinsky model requires some sort of fine tuning in the initial conditions of the inflaton field [28].

3.3 Starobinsky-plus-$R\Box R$ model

This section deals with the changes to inflation resulting from the inclusion of a $R\Box R$-type correction to Starobinsky model.

This scenario is called the Starobinsky-plus-$R\Box R$ model; its main distinctive feature with respect to the previous cases (Subsections 3.1 and 3.2) is the presence of two scalar fields χ and λ that are both responsible for the background dynamics, i.e. wherein a multi-field inflation will be realized. The related phase-space analysis is performed rewriting the second-order equations (14) and (15) as a system of four first-order equations. Accordingly, by defining

$$\chi_t = \psi \quad \text{and} \quad \lambda_t = \phi \tag{28}$$

it results:

$$\chi_t = \psi, \tag{29}$$

$$\psi_t = -3h\psi + \frac{\beta_0}{6}e^{-\chi}\phi^2 - \bar{V}^{\beta_0}_{\chi}, \tag{30}$$

$$\lambda_t = \phi, \tag{31}$$

$$\beta_0\phi_t = \beta_0\left(\psi - 3h\right)\phi + 3e^{\chi}\bar{V}^{\beta_0}_{\lambda}, \tag{32}$$

with

$$h = \sqrt{\frac{1}{2}\left(\frac{1}{2}\psi^2 - \frac{\beta_0}{6}e^{-\chi}\phi^2 + \bar{V}^{\beta_0}\right)} \tag{33}$$

and

$$\bar{V}^{\beta_0}\left(\chi,\lambda\right) = \lim_{\alpha_0\to 0}\bar{V}\left(\chi,\lambda\right) = \frac{1}{3}e^{-2\chi}\lambda\left(e^{\chi} - 1 - \frac{1}{2}\lambda\right). \tag{34}$$

Notice that: (i) the phase space is four dimensional in the higher-order Starobinsky model—it is built with χ, χ_t, λ, and λ_t; (ii) the dimensionless Hubble function h depends explicitly on the parameter β_0—the coupling of the $R\Box R$-term in the action (3); and, predictably (iii) the dimensionless potential $\bar{V}^{\beta_0}$ is a double-field quantity.

The autonomous system formed by Eqs. (29), (30), (31), and (32) admits a single critical point at the origin:

$$P_0 = \left(\chi_0, \lambda_0, \psi_0, \phi_0\right) = \left(0,0,0,0\right). \tag{35}$$

Eqs. (29)—(32) can be linearized about point P_0. Thereby, it follows that Lyapunov exponents r_0 related to the stability of the critical point satisfy the algebraic equation

$$\beta_0 r_0^4 + r_0^2 + \frac{1}{3} = 0. \tag{36}$$

The solution of Eq. (36),

$$r_0 = \pm\sqrt{\frac{-1 \pm \sqrt{1 - \frac{4\beta_0}{3}}}{2\beta_0}}, \tag{37}$$

and the analysis of the direction fields in the phase-space, lead to the conclusion that P_0 is a stable fixed point only within the interval

$$0 \leq \beta_0 \leq \frac{3}{4}. \tag{38}$$

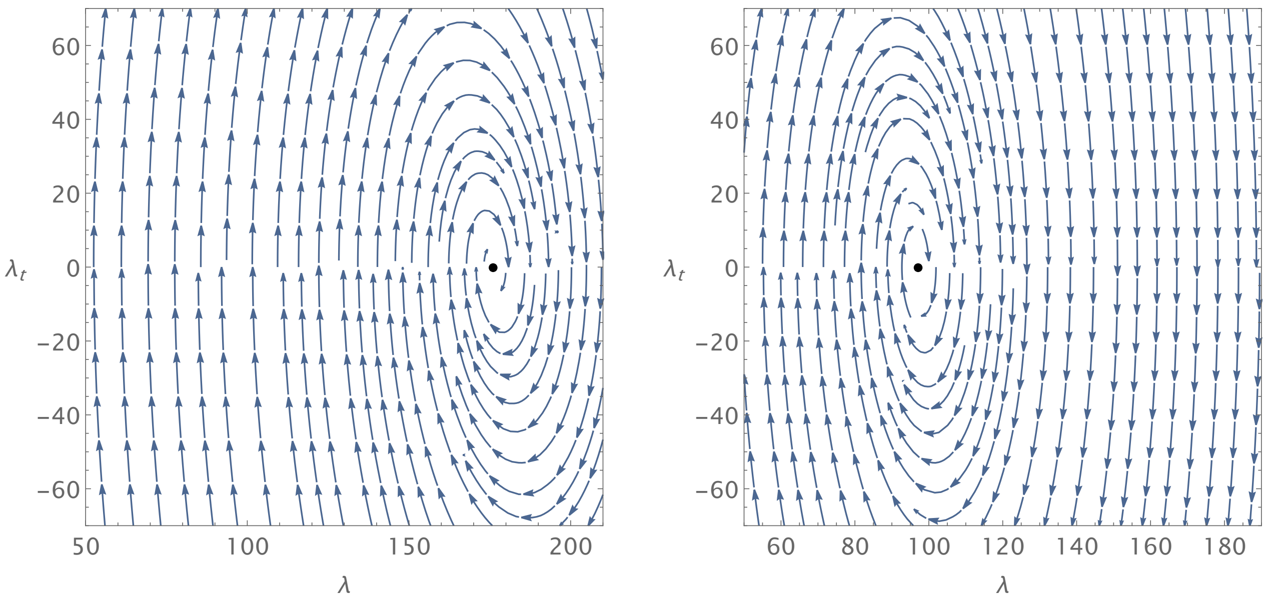

Figure 3: Two-dimensional (λ, λ_t)-slices of the four-dimensional $(\chi, \chi_t, \lambda, \lambda_t)$-phase space corresponding to $\beta_0 = 10^{-3}$ and $\chi_t = 0$ with $\chi = 5.18$ (left panel) and $\chi = 4.58$ (right panel). The black dots mark the position of the accumulation points: $(\lambda, \lambda_t) \simeq (177, 0)$ in the left panel; $(\lambda, \lambda_t) \simeq (98, 0)$ in the right panel.

In fact, any value of β_0 outside the above interval leads to $\mathrm{Re}\,[r_0] > 0$ for at least one of the four possible r_0 in Eq. (37). To put it another way, the equilibrium point P_0 is unstable whenever condition (38) is violated. Moreover, it is worth mentioning that the stability of P_0 is a necessary condition for a graceful exit from inflation into a radiation-dominated universe. This result was first published in Ref. [27] and later reanalyzed by [29].

Details about the dynamics of the double-field higher-order Starobinsky inflation are obtained from the numerical analysis of the four-dimensional phase space $(\chi, \chi_t, \lambda, \lambda_t)$. For this end, Eqs. (14) and (15) are cast into the form:

$$\frac{d\chi_t}{d\chi} = \frac{-3h\chi_t + \frac{\beta_0}{6}e^{-\chi}\lambda_t^2 - \bar{V}_\chi^{\beta_0}}{\chi_t}, \tag{39}$$

$$\frac{d\lambda_t}{d\lambda} = (\chi_t - 3h) + \frac{3e^\chi}{\beta_0\lambda_t}\bar{V}_\lambda^{\beta_0}, \tag{40}$$

with h given by Eq. (33).

By using Eqs. (39) and (40), two-dimensional slices of the phase space can be performed, e.g. plots of (χ, χ_t) and (λ, λ_t) are built for fixed values of β_0 and of the remaining dynamical variables. Specifically, the direction fields in the (χ, χ_t) slice are obtained by choosing adequate values for β_0, λ, and λ_t; Fig. 1 is representative of the (χ, χ_t) plane thus constructed: it is verified the existence of an attractor line close to $\chi_t \simeq 0$ in the Starobinsky-plus-$R\Box R$ model. The attractor line in the (χ, χ_t)-plane is very robust in the sense that it exists for arbitrary values of β_0, λ, and λ_t that are consistent with a physical inflation—i.e. β_0 within the interval in (38) and a real-valued h given by Eq. (33). On the other hand, the direction fields for the two-dimensional slice (λ, λ_t) are built by fixing the values assumed by β_0, χ, and χ_t; Fig. 3 illustrates two such examples of (λ, λ_t)-plane slices.

A joint analysis of Figs. 1 and 3 indicates that the field χ approaches the attractor line $\chi_t \simeq 0$ simultaneously as the field λ tends to the accumulation point where $\lambda_t \to 0$ and $\lambda \simeq e^\chi$. This attractor trajectory $(\chi, \lambda, \chi_t, \lambda_t) \simeq (\chi, e^\chi, 0, 0)$ in the four-dimensional phase space corresponds to the configuration realizing the inflationary regime. This fact is verified from the first-order approximation slow-roll parameter

$$\epsilon \simeq \frac{4e^{-2\chi}}{(3-\beta_0 e^{\chi})}. \tag{41}$$

Internal consistency with the first-order approximations requires $\beta_0 e^{\chi}$ smaller than (but not to close to) 3. Accordingly, Eq. (41) shows that the inflationary regime ($\epsilon \ll 1$) takes place whenever $\chi \gtrsim 2$. Further details on the show-roll regime are available in Ref. [29].

We summarize the analysis of this subsection by stating the three basic conditions that must be satisfied for achieving a physical inflationary regime within the Starobinsky-plus-$R\Box R$ model: (1) Parameter β_0 should pertain to the interval of values specified in (38); (2) The initial condition for the field χ must comply with $\chi^i \gtrsim 2$; and (3) The dimensionless Hubble function should be well defined, i.e. $h(t) \in \mathbb{R}$, for all trajectories taken by the fields χ and λ.

4 Final remarks

This article recalls some of the motivations to consider extensions to general relativity for describing the gravitational interaction. A particular branch of modified gravity proposals is chosen as the focus, namely that of higher-order gravity. The latter admits a four-dimensional Riemaniann manifold endowed with a rank-2 metric tensor $g_{\mu\nu}$ which field equations include derivatives of order higher than two (Section 1). For this reason, Einstein-Hilbert action (wherein the Lagrangian density is $\mathcal{L}^{\text{EH}} \propto R \sim \partial^2 g$) is generalized into Starobinsky model ($\mathcal{L}^{\text{St}} \propto R + R^2$), and further into the higher-order Starobinsky model ($\mathcal{L}^{\text{HOSt}} \propto R + R^2 + \alpha_0 R^3 + \beta_0 R\Box R$)—cf. Section 2. The main scope of the paper was to specify the field equations for $g_{\mu\nu}$ and the extra scalar degree(s) of freedom χ (and λ) for the homogeneous and isotropic FLRW background of non-perturbative cosmology before studying the early-universe inflationary regime allowed within those modified gravity models (Section 3).

Three specific examples were scrutinized in Subsections 3.1 through 3.3. Starobinsky model was taken as the paradigm of successful realization of inflation. Its dynamics was studied carefully in the phase space because of its transparency and for setting the stage for the more complicated models that followed. Starobinsky's inflaton field dynamics follows an attractor line towards an accumulation point for arbitrary general initial conditions (Fig. 1). It engenders a quasi-exponential expansion that exits gracefully to a radiation-dominated universe.

The same possibility—that of an inflation ending in a Hot Big-Bang universe—is realized within the Starobinsky-plus-R^3 model (Subsection 3.2), albeit for a more restrict set of initial conditions. In fact, besides the accumulation point at the origin of the phase space (χ, χ_t) for the single inflaton field of this model, there is an unstable equilibrium point depending on $\chi_c = \ln\left(4 + \sqrt{3\alpha_0^{-1}}\right)$; the associated trajectories split the phase space into two region, one of those leading to eternal inflation (see Fig. 2). In order to tone down this possibility, the parameter α_0 could be constraint to assume small values.

Parameter β_0 typical of the Starobinsky-plus-$R\Box R$ model is also constrained based on similar arguments. However, this case is more evolved partially due to the fact that there are two scalar degrees of freedom (χ and λ) playing the role of the inflaton. The related double-field inflation is achieved by requiring β_0 to take on values within the interval $0 \leq \beta_0 \leq 3/4$. This requirement is based both on the phase-space analysis and on the demand for stability of the critical point at the origin of the four-dimensional space $(\chi, \lambda, \chi_t, \lambda_t)$. Subsection 3.3 contains the details of how to slice the phase space into two-dimensional sectors (such as those in Fig. 3) leading to these conclusions and to the additional requirement that it should be $\chi^i \gtrsim 2$ for an initial condition leading to a physical inflationary regime.

The constraints on the parameters α_0 and β_0 deduced here stem from a simplistic reasoning based on the backgroung evolution of the field equations. These constraints can be refined by the perturbative treatment of the modified gravity models. This technically sofisticated task is undertaken elsewhere—see e.g. [29, 28, 27]. In fact, the CMB data offers a contour region in a plot of the tensor-to-scalar ratio r in as a function of the scalar tilt n_s [32]. Starobinsky model is highly favored because its prediction for $r = r(n_s)$ for a number of e-folds in the interval $50 \lesssim N \lesssim 60$ respects $r \lesssim 0.01$. Starobinsky-plus-R^3 model [28] and Starobinsky-plus-$R\Box R$ model [29, 27] are also consistent with CMB observations; additionally, they allow for a larger variability of n_s values thus accommodating a greater flexibility for data constraining. This might be consistent with the possibility of a non-null running of n_s in the power spectrum parameterization [31].

Acknowledgements

RRC and LGM are grateful to Ruben Aldrovandi for his supervision, his friendship, the inspiration, and the time they have shared at IFT-UNESP. The authors thank CNPq-Brazil for partial financial support—Grants: 309984/2020-3 (RRC) and 307901/2022-0 (LGM).

References

[1] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley & Sons, 1972.

[2] M. Bartelmann, Gravitational lensing, Classical and Quantum Gravity 27, 233001 (2010) [arXiv:1010.3829 [astro-ph.CO]]. [DOI: https://doi.org/10.1088/0264-9381/27/23/233001]

[3] R. M. Wald, General Relativity, Chicago University Press, 1984.

[4] M. Maggiore, Gravitational Waves: Volume 1: Theory and Experiments, Oxford University Press, 2008.

[5] T. R. Padmanabahan, Structure Formation in the Universe, Cambridge University Press, 1993.

[6] R. Utiyama and B. S. DeWitt, Renormalization of a classical gravitational field interacting with quantized matter fields, Journal of Mathematical Physics 03, 608 (1962). [DOI: https://doi.org/10.1063/1.1724264]

[7] R. P. Woodard, How far are we from the quantum theory of gravity?, Reports on Progress in Physics 72, 126002 (2009) [arXiv:0907.4238 [gr-qc]]. [DOI: https://doi.org/10.1088/0034-4885/72/12/126002]

[8] A. H. Guth, Inflationary universe: A possible solution to the horizon and flatness problems, Physical Review D 23, 347 (1981). [DOI: https://doi.org/10.1103/PhysRevD.23.347]

[9] A. A. Starobinsky, A new type of isotropic cosmological models without singularity, Physics Letters B 91, 99 (1980). [DOI: https://doi.org/10.1016/0370-2693(80)90670-X]

[10] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Modified gravity and cosmology, Physics Reports 513, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]]. [DOI: https://doi.org/10.1016/j.physrep.2012.01.001]

[11] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, International Journal of Theoretical Physics 10, 363 (1974). [DOI: https://doi.org/10.1007/BF01807638]

[12] A. Trautman, Einstein-Cartan Theory, Encyclopedia of Mathematical Physics, edited by J.-P. Francoise, G.L. Naber and S.T. Tsou, Oxford: Elsevier, vol. 2, pages 189–195, 2006 [arXiv:gr-qc/0606062].

[13] R. Aldrovandi and J. G. Pereira, Teleparallel Gravity: An Introduction, Springer, 2013.

[14] J. G. Pereira and Y. N. Obukhov, Gauge structure of teleparallel gravity, Universe 5, 139 (2019) [arXiv:1906.06287 [gr-qc]]. [DOI: https://doi.org/10.3390/universe5060139]

[15] M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Boehmer, and A. A. Coley, Teleparallel theories of gravity: illuminating a fully invariant approach, Classical and Quantum Gravity 36, 183001 (2019) [arXiv:1810.12932 [gr-qc]]. [DOI: https://doi.org/10.1088/1361-6382/ab2e1f]

[16] R. Aldrovandi, J. G. Pereira, and K. H. Vu, Selected topics in teleparallel gravity, Brazilian Journal of Physics 34, 1374 (2004) [arXiv:gr-qc/0312008]. [DOI: https://doi.org/10.1590/S0103-97332004000700014]

[17] R. Aldrovandi, P. B. Barros, and J. G. Pereira, Gravitation as anholonomy, General Relativity and Gravitation 35, 991 (2003) [arXiv:gr-qc/0301077]. [DOI: https://doi.org/10.1023/A%3A1024060732690]

[18] V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira, Gravitational energy-momentum density in teleparallel gravity, Physical Review Letters 84, 4533 (2000) [arXiv:gr-qc/0003100]. [DOI: https://doi.org/10.1103/PhysRevLett.84.4533]

[19] R. R. Cuzinatto, C. A. M. de Melo, L. G. Medeiros, and P. J. Pompeia, Gauge formulation for higher order gravity, European Physical Journal C 53, 99 (2008) [arXiv:gr-qc/0611116]. [DOI: https://doi.org/10.1140/epjc/s10052-007-0441-1]

[20] R. R. Cuzinatto, C. A. M. de Melo, L. G. Medeiros, and P. J. Pompeia, Scalar-multi-tensorial equivalence for higher order $f\left(R, \nabla_\mu R, \nabla_{\mu_1}\nabla_{\mu_2} R, \ldots, \nabla_{\mu_1}\ldots\nabla_{\mu_n} R\right)$ theories of gravity, Physical Review D 93, 124034 (2016) [arXiv:1603.01563 [gr-qc]]. [DOI: https://doi.org/10.1103/PhysRevD.93.124034]

[21] Y. Décanini and A. Folacci, Irreducible forms for the metric variations of the action terms of sixth-order gravity and approximated stress-energy tensor, Classical and Quantum Gravity 24, 4777 (2007) [arXiv:0706.0691 [gr-qc]]. [DOI: https://doi.org/10.1088/0264-9381/24/18/014]

[22] R. R. Cuzinatto, C. A. M. de Melo, L. G. Medeiros, and P. J. Pompeia, Observational constraints to a phenomenological $f(R, \nabla R)$-model, General Relativity and Gravitation 47, 29 (2015) [arXiv:1311.7312 [gr-qc]]. [DOI: https://doi.org/10.1007/s10714-015-1862-z]

[23] G. Rodrigues-da-Silva and L. G. Medeiros, Spherically symmetric solutions in higher-derivative theories of gravity, Physical Review D 101, 124061 (2020) [arXiv:2004.04878 [gr-qc]]. [DOI: https://doi.org/10.1103/PhysRevD.101.124061]

[24] S. G. Vilhena, L. G. Medeiros, and R. R. Cuzinatto, Gravitational waves in higher-order R^2 gravity, Physical Review D 104, 084061 (2021) [arXiv:2108.06874 [gr-qc]]. [DOI: https://doi.org/10.1103/PhysRevD.104.084061]

[25] M. F. S. Alves, L. F. M. A. M. Reis, and L. G. Medeiros, Gravitational waves from inspiraling black holes in quadratic gravity, Physical Review D 107, 044017 (2023) [arXiv:2206.13672 [gr-qc]]. [DOI: https://doi.org/10.1103/PhysRevD.107.044017]

[26] R. R. Cuzinatto, C. A. M. de Melo, L. G. Medeiros, P. J. Pompeia, $f\left(R, \nabla_\mu R, \nabla_{\mu_1}\nabla_{\mu_2} R, \ldots, \nabla_{\mu_1}\ldots\nabla_{\mu_n} R\right)$ theories of gravity in Einstein frame: A higher order modified Starobinsky inflation model in the Palatini approach, Physical Review D 99, 084053 (2019) [arXiv:1806.08850 [gr-qc]]. [DOI: https://doi.org/10.1103/PhysRevD.99.084053]

[27] R. R. Cuzinatto, L. G. Medeiros, and P. J. Pompeia, Higher-order modified Starobinsky inflation, Journal of Cosmology and Astroparticle Physics 02, 055 (2019) [arXiv:1810.08911 [gr-qc]]. [DOI: https://doi.org/10.1088/1475-7516/2019/02/055]

[28] G. Rodrigues-da Silva, J. Bezerra-Sobrinho, and L.G. Medeiros, Higher-order extension of Starobinsky inflation: initial conditions, slow-roll regime, and reheating phase, Physical Review D 105, 063504 (2022) [arXiv:2110.15502 [astro-ph.CO]]. [DOI: https://doi.org/10.1103/PhysRevD.105.063504]

[29] G. Rodrigues-da-Silva and L. G. Medeiros, Second-order corrections to Starobinsky inflation, European Physical Journal C 83, 1032 (2023) [arXiv:2207.02103 [astro-ph.CO]]. [DOI: https://doi.org/10.1140/epjc/s10052-023-12149-8]

[30] A. A. Starobinsky, Spectrum of relict gravitational radiation and the early state of the universe, Journal of Experimental and Theoretical Physics Letters, 30, 682 (1979).

[31] Planck Collaboration: Y. Akrami et al., Planck 2018 results. X. Constraints on inflation, Astronomy & Astrophysics 641, A10 (2020) [arXiv:1807.06211 [astro-ph.CO]]. [DOI: https://doi.org/10.1051/0004-6361/201833887]

[32] BICEP/Keck Collaboration: P. A. R. Ade et al., BICEP/Keck XIII: Improved constraints on primordial gravitational waves using Planck, WMAP, and BICEP/Keck Observations through the 2018 Observing Season, Physical Review Letters 127, 151301 (2021) [arXiv:2110.00483 [astro-ph.CO]]. [DOI: https://doi.org/10.1103/PhysRevLett.127.151301]

[33] K. S. Stelle, Renormalization of higher-derivative quantum gravity, Phys. Rev. D 16, 953 (1977). [DOI: https://doi.org/10.1103/PhysRevD.16.953]

[34] V. Mukhanov, Physical Foundations of Cosmology, Cambridge University Press, 2005.

[35] A. D. Linde, Chaotic inflation, Physics Letters B 129, 177 (1983) [DOI: https://doi.org/10.1016/0370-2693(83)90837-7]

[36] R. Brandenberger, Initial conditions for inflation – a short review, International Journal of Modern Physics D 26, 1740002 [arXiv:1601.01918 [hep-th]]. [DOI: https://doi.org/10.1142/S0218271817400028]

[37] D. Baumann, Cosmology: Part III Mathematical Tripos, Lecture Notes (2021). Available at: https://cmb.wintherscoming.no/pdfs/baumann.pdf.

[38] M. A. Amin, M. P. Hertzberg, D. I. Kaiser, and J. Karouby, Nonperturbative dynamics of reheating after inflation: a review, International Journal of Modern Physics D 24, 1530003 (2015) [arXiv:1410.3808 [hep-ph]]. [DOI: https://doi.org/10.1142/S0218271815300037]

[39] K. D. Lozanov, Lectures on Reheating after Inflation, Lecture Notes (2019), arXiv:1907.04402 [astro-ph.CO]. [DOI: https://doi.org/10.48550/arXiv.1907.04402]

[40] L. Kofman, A. D. Linde, and A. A. Starobinsky, Towards the theory of reheating after inflation, Physical Review D 56, 3258 (1997) [arXiv:hep-ph/9704452]. [DOI: https://doi.org/10.1103/PhysRevD.56.3258]

[41] B. A. Bassett, S. Tsujikawa, and D. Wands, Inflation dynamics and reheating, Reviews of Modern Physics 78, 537 (2006) [arXiv:astro-ph/0507632]. [DOI: https://doi.org/10.1103/RevModPhys.78.537]