Odd-Derivative Couplings in Heterotic Theory

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Abstract

In this paper, our focus is on exploring the gauge invariant basis for bosonic couplings within the framework of heterotic string theories, specifically examining 3-, 5-, and 7-derivative terms. We thoroughly analyze the invariance of these couplings under T-duality transformations and make a notable observation: the T-duality constraint enforces the vanishing of these couplings. We speculate that this result likely holds true for all higher odd-derivative couplings as well. This is unlike the result in type I superstring theory, where, for example, the couplings of 5 Yang-Mills field strength are non-zero.

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1 Introduction

The classical effective action of string theory exhibits a global $O(d, d, \mathbb{R})$ symmetry at all orders of derivatives when the effective action is dimensionally reduced on a torus $T^{(d)}$ [1, 2, 3]. The non-geometric subgroup of this symmetry allows us to construct the covariant and gauge-invariant effective action of string theory at any order of derivatives. The construction involves first determining the independent basis at each order of derivatives with arbitrary coupling constants and then imposing T-duality symmetry to determine the coupling constants in each basis. For circular reduction, this method has been employed in previous works, such as [4, 5, 6], to derive the Neveu-Schwarz-Neveu-Schwarz (NS-NS) couplings up to eight-derivative orders. More recently, a truncated T-duality transformation has been shown to be applicable in finding the Yang-Mills (YM) couplings, as well as NS-NS couplings, in the heterotic theory [7].

When constructing the covariant and gauge invariant bases, it is important to consider all contractions of the YM and NS-NS field strengths at each order of derivatives. Subsequently, one should remove from the list of couplings those that are related by field redefinitions, integration by parts, and various Bianchi identities [8].

To enforce the Bianchi identities for the covariant NS-NS couplings, one can utilize the diffeomorphism symmetry to select a specific gauge where the Levi-Civita connection is zero, while its derivatives are non-zero. This gauge selection greatly simplifies the identification of independent gravity couplings within the given basis [9]. Similarly, in order to impose the Bianchi identities for the YM gauge-invariant couplings, the YM gauge symmetry can be employed to choose a specific gauge where the YM connection is zero, while its derivatives are non-zero. This gauge choice aids in determining the independent YM couplings within the basis, effectively eliminating the commutator term in the YM field strength [7].

To validate the elimination of commutator terms, even for the derivatives of the YM field strength, we employ the following reasoning. When the YM field is present, the derivative of the field strength involves both the Levi-Civita and YM connections. This derivative is given by $\tilde{\nabla}F = \nabla F + \frac{1}{\sqrt{\alpha'}}[A, F]$, where ∇ represents the ordinary covariant derivative involving only the Levi-Civita connection. We assume that the YM field is dimensionless. Therefore, in the chosen gauge where $A = 0$, the first derivative of the YM field strength simplifies to the ordinary covariant derivative.

The second derivative of the YM field strength can be expressed as the sum of symmetric and antisymmetric derivatives:

$$\tilde{\nabla}\tilde{\nabla}F = [\tilde{\nabla}, \tilde{\nabla}]F + \{\tilde{\nabla}, \tilde{\nabla}\}F\,. \tag{1}$$

The symmetric part involves $\{\nabla, \nabla\}F$, along with other terms involving the YM gauge field A without derivatives, which become zero in the chosen gauge. On the other hand, the antisymmetric part satisfies the following Bianchi identity:

$$[\tilde{\nabla}, \tilde{\nabla}]F = [\nabla, \nabla]F + \frac{1}{\sqrt{\alpha'}}[F, F]\,. \tag{2}$$

This indicates that either the couplings on the left-hand side are independent, or the couplings on the right-hand side are independent. The second term on the right-hand side has fewer derivatives than the other terms in this equation. On the other hand, in our procedure for finding the basis, we assume a derivative expansion for the bases and first construct the independent bases at the 2-derivative level. We then proceed to construct the bases at 3-derivatives, 4-derivatives, and so on. Consequently, the terms FF on the right-hand side are already chosen as independent couplings in the basis at the lower derivative order. Therefore, these terms can only modify the coefficient of the independent terms that have already been chosen as independent couplings. As a result, the FF terms can be removed from the antisymmetric part, and the second derivatives reduce to just the ordinary covariant derivatives. This pattern continues for all higher derivatives of the YM field strength.

Hence, in finding the basis at any order of derivatives, we utilize the following YM field strength and B-field strength:

$$\begin{aligned} F_{\mu\nu}{}^{ij} &= \partial_\mu A_\nu{}^{ij} - \partial_\nu A_\mu{}^{ij}, && (3)\\ H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} - \frac{3}{2}A_{[\mu}{}^{ij}F_{\nu\rho]ij}, && (4) \end{aligned}$$

and their ordinary covariant derivatives. Here, the YM gauge field is defined as $A_\mu{}^{ij} = A_\mu{}^I(\lambda^I)^{ij}$, where the antisymmetric matrices $(\lambda^I)^{ij}$ represent the adjoint representation of the gauge group $SO(32)$ or $E_8 \times E_8$ with the normalization $(\lambda^I)^{ij}(\lambda^J)_{ij} = \delta^{IJ}$.

Once the basis at the n-derivative order is found, it is important to also include the Lorentz Chern-Simons three-form $\Omega_{\mu\nu\rho}$ in the B-field strength. This three-form arises as a result of the Green-Schwarz mechanism [10]. Its inclusion leads to couplings at derivative orders higher than n with the same coefficients as those in the basis with n derivatives. Moreover, to establish a basis suitable for unfixed YM gauges, one should replace the ordinary covariant derivatives with covariant derivatives that incorporate both the Levi-Civita and YM connections. Additionally, the commutator term in the YM field strength should be replaced as follows:

$$F_{\mu\nu}{}^{ij} = \partial_\mu A_\nu{}^{ij} - \partial_\nu A_\mu{}^{ij} + \frac{1}{\sqrt{\alpha'}}[A_\mu{}^{ik}, A_{\nu k}{}^j]\,. \tag{5}$$

This replacement should also be applied to the covariant derivatives of the YM field strength.

Determining the coupling constants in the bases involves applying the non-geometric T-duality constraint to the reduction of the effective action. If the parent couplings are in unfixed YM gauges, the reduced couplings become covariant and invariant under YM gauge transformations in the base space. We can then leverage these symmetries to choose gauges in which the Levi-Civita connection and YM connections in the base space vanish, while their derivatives do not. This allows us to apply the logic described earlier in the base space, effectively eliminating all commutators in the YM field strength and its derivatives in the bases, resulting in ordinary covariant derivatives. On the other hand, if the parent couplings are in the chosen gauge where $A = 0$, the reduced couplings do not contain commutator terms, and the derivatives in the base space are already in the form of ordinary covariant derivatives. In our approach, we follow the latter case where the parent couplings are in the chosen gauge without commutator terms, and the derivatives in the base space are treated as ordinary covariant derivatives.

In the context of circular reduction, the non-geometric aspects of T-duality transformations at the leading order of derivatives are described by the Buscher transformations [11, 12]. These transformations, along with their higher derivative generalizations, play a crucial role in determining the coupling constants. Within the reduction scheme [13, 14], the Buscher transformations exhibit linearity for the NS-NS fields but introduce nonlinearity for the YM field in the base space. Specifically, the scalar component of the YM gauge field appears nonlinearly in the corresponding Buscher transformation [14]. Furthermore, this scalar field manifests in a nonlinear manner during the circular reduction of the effective actions.

A previous study [7] introduced a truncation approach for generalized Buscher transformations and the reduced action. This truncation involves removing terms that include derivatives of the scalar field, as well as terms with second and higher orders of the scalar field. It was proposed that these truncated Buscher transformations and reduced action contain sufficient information to determine the coupling constants at each derivative order. The validity of this proposal was confirmed by evaluating the four-derivative couplings in the heterotic theory, which were found to be consistent with the corresponding couplings obtained using the S-matrix method, as reported in the literature [16, 17]. Similar truncation schemes for the NS-NS fields were also observed in [4], where the removal of the Levi-Civita connection and all its derivatives in the base space still retains enough information in the T-duality constraint to determine all NS-NS couplings in a basis. In this paper, we adopt this method to investigate the couplings involving odd derivatives in the heterotic theory.

The paper is structured as follows: In Section 2, we explore bases that incorporate 3-, 5-, and 7-derivatives, encompassing both NS-NS and YM field strengths, along with their covariant derivatives. These bases include arbitrary coupling constants. In Section 3, we employ truncated T-duality techniques to explicitly determine the coupling constants in each basis. We find that T-duality fixes all the coupling constants in these bases to be zero. The result for the 3-derivative coupling is consistent with the one obtained from the S-matrix method. While higher odd-derivative couplings are extremely difficult to explicitly calculate, we speculate that all couplings involving odd-derivatives, which include field strengths and their derivatives, are zero. In other words, the couplings associated with odd derivatives are not YM gauge invariant and should arise from the commutator terms present in the even-derivative couplings of field strengths. Section 4 provides a concise discussion of our findings and their implications. Throughout our calculations, we utilize the "xAct" package [15] for computational purposes.

2 Minimal Bases

In this section, we focus on identifying the bases that involve 3-, 5-, and 7-derivative orders, incorporating both the NS-NS and YM field strengths. It becomes evident that the couplings in these bases must necessarily include the YM field strength. Without the presence of the YM field strength, it would not be possible to construct couplings with odd-derivatives solely from the NS-NS field strengths.

At the 3-derivative order, we find a single coupling given by:

$$\mathbf{L}^{(1/2)} \;=\; a_1\sqrt{\alpha'}F_{\alpha}{}^{\gamma}{}_{i}{}^{j}F^{\alpha\beta ik}F_{\beta\gamma kj} \;=\; a_1\sqrt{\alpha'}\,\mathrm{Tr}(F_{\alpha}{}^{\gamma}F_{\beta\gamma}F^{\alpha\beta})\,. \tag{6}$$

Here, a_1 represents the coupling constant, which will be determined in the subsequent section through T-duality.

At the 5-derivative order, we encounter a total of 51 couplings. However, by eliminating redundancies arising from field redefinitions, integration by parts, and employing the Bianchi identity, as demonstrated in previous work [7], we discover that there are 23 independent couplings remaining. These independent couplings in a particular scheme are given by:

$$\begin{aligned}\mathbf{L}^{(3/2)} = \alpha'\sqrt{\alpha'}\Big[& b_1F_{\alpha}{}^{\epsilon im}F^{\alpha\beta jk}F_{\beta}{}^{\gamma}{}_{i}{}^{l}F_{\gamma\delta km}F_{\epsilon}{}^{\delta}{}_{jl} + b_2F_{\alpha}{}^{\epsilon}{}_{il}F^{\alpha\beta il}F_{\beta}{}^{\varepsilon jk}F_{\epsilon}{}^{\gamma}{}_{j}{}^{m}F_{\varepsilon\gamma km}\\
&+b_3F_{\alpha\beta il}F^{\alpha\beta il}F_{\epsilon}{}^{\gamma}{}_{j}{}^{m}F^{\epsilon\varepsilon jk}F_{\varepsilon\gamma km} + b_4F_{\alpha}{}^{\epsilon}{}_{i}{}^{m}F^{\alpha\beta ik}F_{\beta}{}^{\nu jl}F_{\epsilon}{}^{\gamma}{}_{jl}F_{\nu\gamma km}\\
&+b_5F_{\alpha}{}^{\epsilon}{}_{i}{}^{l}F^{\alpha\beta ij}F_{\beta}{}^{\nu}{}_{j}{}^{k}F_{\epsilon}{}^{\gamma}{}_{k}{}^{m}F_{\nu\gamma lm} + b_6F_{\alpha}{}^{\epsilon}{}_{i}{}^{j}F^{\alpha\beta il}F_{\beta}{}^{\nu}{}_{j}{}^{k}F_{\epsilon}{}^{\gamma}{}_{k}{}^{m}F_{\nu\gamma lm}\\
&+b_7F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F_{\beta}{}^{\nu}{}_{j}{}^{l}F_{\epsilon}{}^{\gamma}{}_{k}{}^{m}F_{\nu\gamma lm} + b_8F_{\alpha\beta i}{}^{j}F^{\alpha\beta ik}F_{\varepsilon}{}^{\gamma}{}_{j}{}^{m}F^{\varepsilon\nu}{}_{k}{}^{l}F_{\nu\gamma lm}\\
&+b_9F_{\alpha\beta}{}^{lk}F^{\alpha\beta ji}F_{\gamma\delta km}F_{\nu}{}^{\delta}{}_{i}{}^{m}F^{\nu\gamma}{}_{jl} + b_{10}F_{\alpha\beta}{}^{ik}F^{\alpha\beta jl}F_{\gamma\delta km}F_{\varepsilon\varepsilon}{}^{\delta}{}_{i}{}^{m}F^{\nu\gamma}{}_{jl}\\
&+b_{11}F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F^{\nu\gamma}{}_{jk}H_{\beta\nu}{}^{\delta}H_{\epsilon\gamma\delta} + b_{12}F^{\alpha\beta ij}F^{\gamma\delta}{}_{jk}F^{\epsilon\varepsilon}{}_{i}{}^{k}H_{\alpha\beta\epsilon}H_{\varepsilon\gamma\delta}\\
&+b_{13}F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F^{\nu\gamma}{}_{jk}H_{\beta\epsilon}{}^{\delta}H_{\nu\gamma\delta} + b_{14}F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F_{\beta}{}^{\nu}{}_{jk}H_{\epsilon}{}^{\gamma\delta}H_{\nu\gamma\delta}\\
&+b_{15}F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F_{\beta\epsilon jk}H_{\nu\gamma\delta}H^{\nu\gamma\delta} + b_{16}F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F^{\nu\gamma}{}_{jk}R_{\beta\epsilon\nu\gamma}\\
&+b_{17}F_{\alpha}{}^{\beta ji}F_{\epsilon}{}^{\gamma}{}_{ik}F^{\epsilon\varepsilon}{}_{j}{}^{k}H_{\beta\varepsilon\gamma}\nabla^{\alpha}\Phi + b_{18}F_{\alpha}{}^{\beta ij}F_{\beta}{}^{\epsilon}{}_{i}{}^{k}F^{\nu\gamma}{}_{jk}H_{\epsilon\nu\gamma}\nabla^{\alpha}\Phi\\
&+b_{19}F_{\beta}{}^{\varepsilon}{}_{i}{}^{k}F^{\beta\gamma ij}F_{\gamma\varepsilon jk}\nabla_{\alpha}\Phi\nabla^{\alpha}\Phi + b_{20}F^{\alpha\beta ij}\nabla_{\alpha}F^{\epsilon\varepsilon}{}_{ik}\nabla_{\beta}F_{\epsilon\varepsilon j}{}^{k}\\
&+b_{21}F_{\alpha}{}^{\gamma ij}F_{\beta}{}^{\varepsilon}{}_{i}{}^{k}F_{\gamma\varepsilon jk}\nabla^{\alpha}\Phi\nabla^{\beta}\Phi+b_{22}F_{\alpha}{}^{\gamma ij}F_{\beta}{}^{\varepsilon}{}_{i}{}^{k}F_{\gamma\varepsilon jk}\nabla^{\beta}\nabla^{\alpha}\Phi+b_{23}F_{\alpha}{}^{\epsilon}{}_{i}{}^{k}F^{\alpha\beta ij}F^{\nu\gamma}{}_{jk}\nabla_{\epsilon}H_{\beta\nu\gamma}\Big],\end{aligned} \tag{7}$$

where $b_1, b_2, \ldots, b_{23}$ represent the corresponding coupling constants. It includes single-trace terms, namely $\mathrm{Tr}(FFF)$ and $\mathrm{Tr}(FFFFF)$, as well as two-trace term $\mathrm{Tr}(FF)\mathrm{Tr}(FFF)$.

At the 7-derivative order, a comprehensive analysis reveals the existence of a total of 19,310 couplings. However, through the elimination of redundancies resulting from field redefinitions, integration by parts, and the application of the Bianchi identity, it is possible to identify 1,168 independent couplings. These independent couplings can be expressed as follows:

$$\begin{aligned}\mathbf{L}^{(5/2)} \;=\; \alpha'^2\sqrt{\alpha'}\Big[&c_1F_{\alpha}{}^{\gamma kl}F^{\alpha\beta ij}F_{\beta}{}^{\delta}{}_{k}{}^{m}F_{\gamma}{}^{\epsilon}{}_{m}{}^{o}F_{\delta}{}^{\varepsilon}{}_{o}{}^{n}F_{\epsilon}{}^{\nu}{}_{ln}F_{\varepsilon\nu ij}+c_2F_{\alpha}{}^{\gamma}{}_{i}{}^{k}F^{\alpha\beta ij}F_{\beta}{}^{\delta lm}F_{\gamma}{}^{\epsilon on}F_{\delta}{}^{\varepsilon}{}_{on}F_{\epsilon}{}^{\nu}{}_{lm}F_{\varepsilon\nu jk}\\
&+\cdots+c_{1167}F^{\alpha\beta ij}F^{\gamma\delta}{}_{i}{}^{k}H_{\alpha\epsilon\nu}\nabla_{\varepsilon}H_{\beta\gamma\delta}\nabla^{\nu}F^{\epsilon\varepsilon}{}_{jk}+c_{1168}F_{\alpha}{}^{\gamma}{}_{i}{}^{k}F^{\alpha\beta ij}F_{\beta}{}^{\delta}{}_{jk}\nabla_{\varepsilon}H_{\delta\epsilon\nu}\nabla^{\nu}H_{\gamma}{}^{\epsilon\varepsilon}\Big].\end{aligned} \tag{8}$$

The expression above represents a subset of the independent couplings, with the ellipsis symbolizing an additional 1,164 terms that are not explicitly listed. These terms involve various combinations of F, ∇F, H, ∇F, $\nabla\Phi$, $\nabla\nabla\Phi$, and Riemann curvature. It includes single-trace terms, namely $\mathrm{Tr}(F^3)$, $\mathrm{Tr}(F^5)$, and $\mathrm{Tr}(F^7)$, as well as two-trace terms, namely $\mathrm{Tr}(F^2)\mathrm{Tr}(F^5)$ and $\mathrm{Tr}(F^3)\mathrm{Tr}(F^4)$. Additionally, there is a three-trace term $\mathrm{Tr}(F^2)\mathrm{Tr}(F^2)\mathrm{Tr}(F^3)$. The parameters $c_1, c_2, \ldots, c_{1168}$ correspond to the respective coupling constants associated with these independent couplings. The determination of these coupling constants will be addressed in the subsequent section.

3 T-duality Constraint on the Minimal Bases

The observation that the dimensional reduction of the classical effective action of string theory on a torus $T^{(d)}$ must be invariant under the $O(d, d, \mathbb{R})$ transformations [1, 2, 3] indicates that the circular reduction of the couplings in the effective Lagrangian should be invariant under the discrete group $O(1, 1, \mathbb{Z})$ or $\mathbb{Z}_2$-group, which consists only of non-geometrical transformations. Therefore, in order to impose this T-duality on the classical effective Lagrangian $\mathbf{L}_{\rm eff}$, we need to reduce the theory on a circle to obtain the $(D-1)$-dimensional effective Lagrangian $L_{\rm eff}(\psi)$, where ψ collectively represents the base space fields. Then, we transform this Lagrangian under the $\mathbb{Z}_2$-transformations to produce $L_{\rm eff}(\psi')$, where ψ' is the T-duality transformation of ψ. The T-duality constraint on the effective Lagrangian is given by:

$$L_{\rm eff}(\psi) - L_{\rm eff}(\psi') = 0\,. \tag{9}$$

If we apply the T-duality once more to the above equation, we find $L_{\rm eff}[(\psi')'] = L_{\rm eff}(\psi)$. This indicates that the T-duality transformation that satisfies the above relation must satisfy the $\mathbb{Z}_2$-group, i.e., $(\psi')' = \psi$.

To utilize the aforementioned constraint on the effective Lagrangian, one must first determine the basis in which only the redundancies resulting from field redefinitions and Bianchi identities are eliminated. This basis comprises terms that are total derivative terms. Subsequently, the aforementioned constraint can be employed to determine the coupling constants in the bases as well as the T-duality transformations [18]. However, to work with bases that have fewer couplings, it is necessary to remove the couplings in the bases that are total derivative terms, as demonstrated in finding the minimal bases in (7) and (8). In this case, T-duality should be imposed on the effective action $\mathbf{S}_{\rm eff}$ which includes the minimal bases. Since some total derivative terms are eliminated from the Lagrangian in the constraint (9), the remaining terms do not satisfy the relation (9) unless certain total derivative terms are included in the base space to compensate for the absence of total derivative terms in the effective action. The T-duality constraint in this scenario is given by:

$$S_{\rm eff}(\psi) - S_{\rm eff}(\psi') = \int d^{D-1}x\sqrt{-\bar{g}}\nabla_a\Big[e^{-2\bar{\phi}}J^a(\psi)\Big], \tag{10}$$

where $S_{\rm eff}(\psi)$ represents the circular reduction of the effective action, and $S_{\rm eff}(\psi')$ denotes its transformation under the T-duality transformation. In the above equation, J^a is an arbitrary covariant vector composed of the $(D-1)$-dimensional base space fields. In this case, if we apply T-duality once more to (10), we find $S_{\rm eff}[(\psi')'] = S_{\rm eff}(\psi)$ up to certain total derivative terms. Therefore, once again, the T-duality transformation that satisfies the above relation must adhere to the $\mathbb{Z}_2$-group, i.e., $(\psi')' = \psi$.

The constraint (10) can be employed to determine the coupling constants in the effective action $\mathbf{S}_{\rm eff}$ as well as the T-duality transformations. One approach is to impose the symmetry $(\psi')' = \psi$ on the T-duality transformation to reduce the parameters in the most general T-duality transformations. These reduced parameters can then be used in the above constraint to fix the coupling constants in the effective action, as well as the remaining parameters in the

T-duality transformations [4, 5, 6]. Alternatively, one can use the most general T-duality transformations in the above constraint and solve the equation to determine the coupling constants of the effective action and the corresponding T-duality transformation. It should be noted that the resulting T-duality transformation must automatically satisfy the symmetry $(\psi')' = \psi$. In this paper, we adopt the latter approach.

In the context of the heterotic theory, it has been proposed that the truncated reduced action, which should be invariant under truncated T-duality transformations, can fix all coupling constants in the effective action $\mathbf{S}_{\rm eff}$ and in the truncated T-duality transformations [7]. The proposed constraint is given by:

$$S^L_{\rm eff}(\psi) - S^L_{\rm eff}(\psi^L) = \int d^9x\sqrt{-\bar{g}}\nabla_a\Big[e^{-2\bar{\phi}}J^a(\psi)\Big]\,. \tag{11}$$

In the above equation, ψ^L represents the truncated T-duality transformation, which includes only the zeroth and first order terms of the scalar component of the YM gauge field in the base space. $S^L_{\rm eff}(\psi)$ also corresponds to the truncated reduction of the effective action. In the second term of the equation, only the zeroth and first order terms of the scalar should be retained.

The truncated T-duality transformation ψ^L can be expanded in powers of α' as follows:

$$\psi^L = \psi^L_0 + \sum_{n=1}^{\infty}\frac{\alpha'^n}{n!}\psi^L_n + \sum_{n=0}^{\infty}\frac{\alpha'^{n+\frac{1}{2}}}{n!}\psi^L_{n+\frac{1}{2}}. \tag{12}$$

Here, ψ^L_0 represents the truncated T-duality transformations at zeroth derivative, also known as the truncated Buscher rule, and ψ^L_k represents their deformations at order α'^k. By employing an α'-expansion for the 9-dimensional effective action $S^L_{\rm eff}$ and for the vector J^a, and using a Taylor expansion for the T-duality transformation of the 9-dimensional effective action at order α'^k around the truncated Buscher rule ψ^L_0, which is given by

$$S^{L(k)}(\psi^L) = \sum_{m=0,\frac{1}{2},1,\frac{3}{2},\cdots}\alpha'^m S^{L(k,m)}(\psi^L_0), \tag{13}$$

where $S^{L(k,0)} = S^{L(k)}$, the $\mathbb{Z}_2$-constraint in (11) can be written as:

$$\sum_{k=0,\frac{1}{2},1,\frac{3}{2},\cdots}\alpha'^k\left[S^{L(k)}(\psi) - \sum_{m=0,\frac{1}{2},1,\frac{3}{2},\cdots}\alpha'^m S^{L(k,m)}(\psi^L_0) - \int d^9x\,\partial_a\Big[e^{-2\bar{\phi}}J^a_{(k)}(\psi)\Big]\right] = 0, \tag{14}$$

where $J^a_{(k)}$ is an arbitrary covariant vector composed of the 9-dimensional base space fields at order $\alpha'^{k+1/2}$. In the above equation, we assume the base space is flat for simplicity, as the $\mathbb{Z}_2$-constraint for the coupling constants is the same for curved and flat base spaces, as observed in [4]. To determine the appropriate constraints on the effective actions, each term at every order of α' must be set to zero.

The T-duality constraint (14) at order α'^0 is given by

$$S^{L(0)}(\psi) - S^{L(0)}(\psi^L_0) - \int d^9x\,\partial_a\Big[e^{-2\bar{\phi}}J^a_{(0)}(\psi)\Big] = 0, \tag{15}$$

where $S^{L(0)}(\psi)$ is the circular reduction of the effective action at the two-derivative order, which includes the zeroth and the first order of the YM scalar. The effective action at the two-derivative order is [19, 16]

$$\mathbf{S}^{(0)} = -\frac{2}{\kappa^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\Big[R-\frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}+4\nabla_\alpha\Phi\nabla^\alpha\Phi-\frac{1}{4}F_{\mu\nu ij}F^{\mu\nu ij}\Big]. \quad (16)$$

To perform the dimensional reduction of this action on a circle, we utilize the reduction scheme for the NS-NS and YM fields as proposed in [13, 14, 7]:

$$G_{\mu\nu}=\begin{pmatrix}\bar{g}_{ab}+e^{\varphi}g_ag_b & e^{\varphi}g_a\\ e^{\varphi}g_b & e^{\varphi}\end{pmatrix},\;\; A_\mu{}^{ij}=\begin{pmatrix}\bar{A}_a{}^{ij}+e^{\varphi/2}\alpha^{ij}g_a\\ e^{\varphi/2}\alpha^{ij}\end{pmatrix},\; \Phi=\bar{\phi}+\varphi/4\,, \quad (17)$$

$$B_{\mu\nu}=\begin{pmatrix}\bar{b}_{ab}+\frac{1}{2}(b_ag_b-b_bg_a)+\frac{1}{2}e^{\varphi/2}\alpha_{ij}(g_a\bar{A}_b{}^{ij}-g_b\bar{A}_a{}^{ij}) & b_a-\frac{1}{2}e^{\varphi/2}\alpha_{ij}\bar{A}_a{}^{ij}\\ -b_b+\frac{1}{2}e^{\varphi/2}\alpha_{ij}\bar{A}_a{}^{ij} & 0\end{pmatrix},$$

where $\bar{g}_{ab}$, $\bar{b}_{ab}$, $\bar{\phi}$, and $\bar{A}_a^{ij}$ represent the metric, B-field, dilaton, and YM gauge field in the base space, respectively. Additionally, g_a and b_b denote two vectors, while φ and α^{ij} represent scalars within this space. By employing the above reduction scheme, one can derive the following reductions for different components of the H-field [13, 14]:

$$\begin{aligned}H_{aby} &= W_{ab}-\frac{1}{2}e^{\varphi}\alpha^2V_{ab}-e^{\varphi/2}\alpha^{ij}\bar{F}_{abij}\,,\\ H_{abc} &= \bar{H}_{abc}+3g_{[a}W_{bc]}-\frac{3}{2}e^{\varphi}\alpha^2g_{[a}V_{bc]}-3e^{\varphi/2}\alpha_{ij}g_{[a}\bar{F}_{bc]}{}^{ij}\,,\end{aligned} \quad (18)$$

where $\alpha^2=\alpha^{ij}\alpha_{ij}$, $W_{ab}=\partial_ab_b-\partial_bb_a$, $V_{ab}=\partial_ag_b-\partial_bg_a$, $\bar{F}_{ab}{}^{ij}=\partial_a\bar{A}_a{}^{ij}-\partial_b\bar{A}_a{}^{ij}$, and $\bar{H}$ represents the torsion in the base space:

$$\bar{H}_{abc} = 3\partial_{[a}\bar{b}_{bc]}-\frac{3}{2}g_{[a}W_{bc]}-\frac{3}{2}b_{[a}V_{bc]}-\frac{3}{2}\bar{A}_{[a}{}^{ij}\bar{F}_{bc]}{}^{ij}\,. \quad (19)$$

Our notation for antisymmetry is such that, for example, $3g_{[a}W_{bc]}=g_aW_{bc}-g_bW_{ac}-g_cW_{ba}$. The torsion obeys the following Bianchi identity:

$$d\bar{H}+\frac{3}{2}V\wedge W+\frac{3}{4}\bar{F}^{ij}\wedge\bar{F}_{ij}=0\,. \quad (20)$$

The truncated reduction of the action (16) up to certain total derivative terms, considering a flat base space, can be found in [7]

$$\begin{aligned}S^{L(0)}(\psi)=-\frac{2}{\kappa'^2}\int d^9x\,e^{-2\bar{\phi}}\Big[&-\frac{1}{12}\bar{H}^2+4\partial_a\bar{\phi}\partial^a\bar{\phi}-\frac{1}{4}\bar{F}_{abij}\bar{F}^{abij}\\ &-\frac{1}{4}\partial_a\varphi\partial^a\varphi-\frac{1}{4}e^{\varphi}V^2-\frac{1}{4}e^{-\varphi}W^2-\frac{1}{2}e^{\varphi/2}\bar{F}_{abij}V^{ab}\alpha^{ij}+\frac{1}{2}e^{-\varphi/2}\bar{F}_{abij}W^{ab}\alpha^{ij}\Big]\,,\end{aligned} \quad (21)$$

where κ' is related to the 9-dimensional Newton's constant. It has been observed in [7] that the above action satisfies the constraint (15) for the following truncated Buscher transformations ψ_0^L:

$$g_a \to b_a, b_a \to g_a, \varphi \to -\varphi, \alpha^{ij} \to -\alpha^{ij}, \bar{A}_a{}^{ij} \to \bar{A}_a{}^{ij}, \bar{\eta}_{ab} \to \bar{\eta}_{ab}, \bar{b}_{ab} \to \bar{b}_{ab}, \bar{\phi} \to \bar{\phi}\,, \tag{22}$$

where $\bar{\eta}_{ab}$ is the flat metric of the base space.

The constraint in (14) at order $\sqrt{\alpha'}$ becomes

$$-S^{L(0,1/2)}(\psi_0^L) - \int d^9x\, \partial_a \left[e^{-2\bar{\phi}} J^a_{(1/2)}(\psi)\right] = S^{L(1/2)}(\psi_0^L) - S^{L(1/2)}(\psi). \tag{23}$$

Using the reductions in (17), one finds the truncated reduction of the couplings in (6):

$$S^{L(1/2)}(\psi) = -\frac{2}{\kappa'^2}\int d^9x\, e^{-2\bar{\phi}}\left[a_1 \bar{F}_a{}^c{}_i{}^k \bar{F}^{abij}\bar{F}_{bcjk} + 3a_1 e^{\varphi/2}\bar{F}_a{}^c{}_i{}^k \bar{F}_{bcjk}V^{ab}\alpha^{ij}\right]. \tag{24}$$

Its transformation under the truncated Buscher transformations (22) is:

$$S^{L(1/2)}(\psi_0^L) = -\frac{2}{\kappa'^2}\int d^9x\, e^{-2\bar{\phi}}\left[a_1 \bar{F}_a{}^c{}_i{}^k \bar{F}^{abij}\bar{F}_{bcjk} + 3a_1 e^{-\varphi/2}\bar{F}_a{}^c{}_i{}^k \bar{F}_{bcjk}W^{ab}\alpha^{ij}\right]. \tag{25}$$

Since these terms are single-trace and include only field strengths $\bar{F}, V, W$ without their derivatives, it is impossible to produce such terms by total derivative or by corrections to the leading-order T-duality transformation (22). Hence, the terms on the left-hand side of (23) are zero, and the terms on the right-hand side must also be zero, which fixes the parameter to be zero. Therefore, T-duality dictates that

$$\psi^L_{\frac{1}{2}} = 0, \quad a_1 = 0. \tag{26}$$

The result for the effective action is consistent with the S-matrix method [16].

Since $S^{L(1/2)} = \psi^L_{\frac{1}{2}} = 0$, the constraint in (14) at order $\alpha'\sqrt{\alpha'}$ becomes:

$$-S^{L(0,3/2)}(\psi_0^L) - \int d^9x\, \partial_a \left[e^{-2\bar{\phi}} J^a_{(3/2)}(\psi)\right] = S^{L(3/2)}(\psi_0^L) - S^{L(3/2)}(\psi). \tag{27}$$

Using the reductions in (17), it is straightforward to find the truncated reduction of the couplings in (7) to generate $S^{L(3/2)}(\psi)$. Then, by applying the truncated Buscher transformations in (22), one can calculate its corresponding $S^{L(3/2)}(\psi_0^L)$. It is important to note that the right-hand side is odd under the truncated Buscher transformation. Hence, $S^{L(0,3/2)}(\psi_0^L)$ must also be odd, up to certain total derivative terms.

To determine the first term on the left-hand side of the above equation, we require the inclusion of 3-derivative corrections to the truncated Buscher rules (22). In other words, we need to consider the expanded form of the Buscher transformations that includes terms involving three derivatives.

$$\begin{aligned}
&\varphi^L=-\varphi+\alpha'\sqrt{\alpha'}\Delta\varphi^{(3/2)}(\psi)+\cdots\,,\quad g^L_a=b_a+\alpha'\sqrt{\alpha'}e^{\varphi/2}\Delta g^{(3/2)}_a(\psi)+\cdots\,,\\
&b^L_a=g_a+\alpha'\sqrt{\alpha'}e^{-\varphi/2}\Delta b^{(3/2)}_a(\psi)+\cdots\,,\quad \bar{g}^L_{ab}=\eta_{ab}+\alpha'\sqrt{\alpha'}\Delta\bar{g}^{(3/2)}_{ab}(\psi)+\cdots\,,\\
&\bar{H}^L_{abc}=\bar{H}_{abc}+\alpha'\sqrt{\alpha'}\Delta\bar{H}^{(3/2)}_{abc}(\psi)+\cdots\,,\quad \bar{\phi}^L=\bar{\phi}+\alpha'\sqrt{\alpha'}\Delta\bar{\phi}^{(3/2)}(\psi)+\cdots\,,\\
&(\bar{A}^L_a)^{ij}=\bar{A}_a{}^{ij}+\alpha'\sqrt{\alpha'}\Delta\bar{A}^{(3/2)ij}_a(\psi)+\cdots\,,\quad (\alpha^L)^{ij}=-\alpha^{ij}+\alpha'\sqrt{\alpha'}\Delta\alpha^{(3/2)ij}(\psi)+\cdots\,.
\end{aligned}\tag{28}$$

Note that the corrections also contain non-zero terms at the 2-derivative order, denoted as ψ^L_1, which have been derived in [7]. However, since we have $\psi^L_{\frac{1}{2}}=0$, these corrections do not play a role in studying the T-duality of the 5-derivative couplings.

The T-duality transformed base space fields ψ^L must satisfy the Bianchi identity (20). Therefore, the correction $\Delta\bar{H}^{(3/2)}_{abc}$ is related to the corrections $\Delta b^{(3/2)}_a$, $\Delta\bar{g}^{(3/2)}_{ab}$, $\Delta\bar{A}^{(3/2)ij}_a$, and an arbitrary two-form $\Delta B^{(3/2)}_{ab}$, all of which are at the 3-derivative order. This relation is given by [7]:

$$\Delta\bar{H}^{(3/2)}_{abc}=3\partial_{[a}\Delta B^{(3/2)}_{bc]}-3e^{\varphi/2}V_{[ab}\Delta g^{(3/2)}_{c]}-3e^{-\varphi/2}W_{[ab}\Delta b^{(3/2)}_{c]}-3\bar{F}_{[ab}{}^{ij}\Delta\bar{A}^{(3/2)}_{c]ij}\,.\tag{29}$$

Here, $\Delta B^{(3/2)}_{ab}$ represents an arbitrary two-form at the 3-derivative order. It is important to note that the truncated T-duality transformations (28) must satisfy the $\mathbb{Z}_2$-group. Consequently, the deformations at the order of $\alpha'\sqrt{\alpha'}$ need to adhere to the following constraint:

$$\begin{aligned}
-\Delta\varphi^{(3/2)}(\psi)+\Delta\varphi^{(3/2)}(\psi^L_0)=0\quad&;\quad \Delta b^{(3/2)}_a(\psi)+\Delta g^{(3/2)}_a(\psi^L_0)=0\,,\\
\Delta g^{(3/2)}_a(\psi)+\Delta b^{(3/2)}_a(\psi^L_0)=0\quad&;\quad \Delta\bar{g}^{(3/2)}_{ab}(\psi)+\Delta\bar{g}^{(3/2)}_{ab}(\psi^L_0)=0\,,\\
\Delta\bar{\phi}^{(3/2)}(\psi)+\Delta\bar{\phi}^{(3/2)}(\psi^L_0)=0\quad&;\quad \Delta B^{(3/2)}_{ab}(\psi)+\Delta B^{(3/2)}_{ab}(\psi^L_0)=0\,,\\
\Delta\bar{A}^{(3/2)ij}_a(\psi)+\Delta\bar{A}^{(3/2)ij}_a(\psi^L_0)=0\quad&;\quad -\Delta\alpha^{(3/2)ij}(\psi)+\Delta\alpha^{(3/2)ij}(\psi^L_0)=0\,.
\end{aligned}\tag{30}$$

The corrections need to be constructed by considering all possible contractions of $\partial\varphi$, $\partial\bar{\varphi}$, $e^{\varphi/2}V$, $e^{-\varphi/2}W$, $\bar{H}$, $\bar{F}_{ab}{}^{ij}$, and their derivatives at the order of $\alpha'\sqrt{\alpha'}$, with arbitrary coefficients. These corrections should include terms at both zeroth and first order in the scalar α^{ij}. One approach is to impose the above constraint on these corrections, which would reduce the number of parameters involved. These modified corrections can then be used in the constraint (27). Alternatively, one can directly insert the most general form of the corrections into the constraint (27). In this case, the constraint (27) would determine the coupling constants in the effective action and also fix the parameters in the T-duality transformations, such that they satisfy the relations mentioned above.

By employing the T-duality transformations (28) and the relation (29), one can derive the Taylor expansion of the T-duality transformation of the truncated reduced action at the leading order (21), centered around the truncated Buscher transformations ψ^L_0. The resulting expression is as follows:

$$S^{L(0,3/2)}(\psi^L_0)=-\frac{2}{\kappa'^2}\int d^9xe^{-2\bar{\phi}}\Big[\Big(\frac{1}{4}\partial^a\varphi\partial^b\varphi-2\partial^a\partial^b\bar{\phi}+\frac{1}{4}\bar{H}^{acd}\bar{H}^b{}_{cd}+\frac{1}{2}(e^{\varphi}V^{ac}V^b{}_c+e^{-\varphi}W^{ac}W^b{}_c)$$

$$
\begin{aligned}
&+\bar{F}^{acij}\bar{F}^{b}{}_{cij}+(e^{\varphi/2}V^{ac}-e^{-\varphi/2}W^{ac})\bar{F}^{b}{}_{cij}\alpha^{ij}\Big)\Delta\bar{g}^{(3/2)}_{ab}+\Big(2\partial_c\partial^c\bar{\phi}-2\partial_c\bar{\phi}\partial^c\bar{\phi}-\frac{1}{24}\bar{H}^2\\
&-\frac{1}{8}\partial_c\varphi\partial^c\varphi-\frac{1}{8}(e^{\varphi}V^2+e^{-\varphi}W^2)+\frac{1}{2}\bar{F}_{abij}\bar{F}^{abij}+(e^{\varphi/2}V^{ab}-e^{-\varphi/2}W^{ab})\bar{F}_{abij}\alpha^{ij}\Big)\\
&\times(\eta^{ab}\Delta\bar{g}^{(3/2)}_{ab}-4\Delta\bar{\phi}^{(3/2)})-\Big(\frac{1}{2}\partial_a\partial^a\varphi-\partial_a\bar{\phi}\partial^a\varphi+\frac{1}{4}(e^{\varphi/2}V^{ab}+e^{-\varphi/2}W^{ab})\bar{F}_{abij}\alpha^{ij}\\
&-\frac{1}{4}(e^{\varphi}V^2-e^{-\varphi}W^2)\Big)\Delta\varphi^{(3/2)}+\Big(2e^{-\varphi/2}\partial_b\bar{\phi}W^{ab}-e^{-\varphi/2}\partial_bW^{ab}+e^{-\varphi/2}\partial_b\varphi W^{ab}\\
&+\frac{1}{2}e^{\varphi/2}\bar{H}^{abc}V_{bc}-2(\partial_b\bar{\phi}+\frac{1}{4}\partial_b\varphi)\bar{F}^{abij}\alpha_{ij}+\partial_b\bar{F}^{abij}\alpha_{ij}\Big)\Delta g^{(3/2)}_{a}+\Big(2e^{\varphi/2}\partial_b\bar{\phi}V^{ab}\\
&-e^{\varphi/2}\partial_bV^{ab}-e^{\varphi/2}\partial_b\varphi V^{ab}+\frac{1}{2}e^{-\varphi/2}\bar{H}^{abc}W_{bc}+2(\partial_b\bar{\phi}-\frac{1}{4}\partial_b\varphi)\bar{F}^{abij}\alpha_{ij}\\
&-\partial_b\bar{F}^{abij}\alpha_{ij}\Big)\Delta b^{(3/2)}_{a}+\Big(\frac{1}{2}\partial_a\bar{H}^{abc}-\bar{H}^{abc}\partial_a\bar{\phi}\Big)\Delta B^{(3/2)}_{bc}+\Big(\frac{1}{2}\bar{H}^{abc}\bar{F}_{bcij}-\partial_b\bar{F}^{ab}{}_{ij}\\
&+2\partial_b\bar{\phi}\bar{F}^{ab}{}_{ij}+\alpha_{ij}[2(\partial_b\bar{\phi}-\frac{1}{4}\partial_b\varphi)e^{\varphi/2}V^{ab}-2(\partial_b\bar{\phi}+\frac{1}{4}\partial_b\varphi)e^{-\varphi/2}W^{ab}-e^{\varphi/2}\partial_bV^{ab}\\
&+e^{-\varphi/2}\partial_bW^{ab}]\Big)\Delta\bar{A}^{(3/2)ij}_{a}+\frac{1}{2}\bar{F}_{abij}(e^{\varphi/2}V^{ab}-e^{-\varphi/2}W^{ab})\Delta\alpha^{(3/2)ij}\Big]\,, \qquad (31)
\end{aligned}
$$

where we have also removed certain total derivative terms. It is important to emphasize that the terms presented above exhibit an odd behavior under the truncated Buscher transformations. Assuming that the correction $\Delta\alpha^{(3/2)ij}$ includes terms at the zeroth order of α^{ij}, it becomes necessary to consider the terms in the reduced action that involve the second order of α^{ij}. Consequently, this consideration introduces terms in the equation above that contain $\alpha_{ij}\Delta\alpha^{(3/2)ij}$. However, when multiplied by the appropriate terms resulting from the corresponding term in the reduced action, this contribution fails to maintain the desired odd behavior under the Buscher transformations (see the explicit form of this term in [7]). Therefore, it can be concluded that $\Delta\alpha^{(3/2)ij}$ cannot contain terms at the zeroth order of α^{ij}.

The vector $J^a_{(3/2)}$ in the total derivative term of the T-duality constraint (27) should be constructed by considering contractions of $\partial\varphi$, $\partial\bar{\varphi}$, $e^{\varphi/2}V$, $e^{-\varphi/2}W$, $\bar{H}$, and $\bar{F}_{ab}{}^{ij}$ at the 4-derivative order, with arbitrary coefficients. These constructions should also include terms at both the zeroth and first order in the scalar α^{ij}.

In order to solve the equation (27), it is crucial to impose the Bianchi identities associated with the field strengths $\bar{H}$, $\bar{F}$, V, and W. We impose the $\bar{H}$-Bianchi identity in its gauge-invariant form, while for the other Bianchi identities, we impose them in a non-gauge-invariant form. To enforce the $\bar{H}$-Bianchi identity, we introduce the gauge-invariant tensor $\mathbf{H}_{abcd}$, which is totally antisymmetric and of order α'. We then consider all possible scalar contractions involving $\partial\varphi$, $\partial\bar{\varphi}$, $e^{\varphi/2}V$, $e^{-\varphi/2}W$, $\bar{H}$, $\mathbf{H}$, $\bar{F}^{ij}$, and their derivatives up to the 5-derivative order. These contractions should include terms at both zeroth and first order in the scalar α^{ij}, and the coefficients of these gauge-invariant terms are arbitrary. Next, we incorporate the following gauge-invariant relation for the newly defined tensor:

$$
\mathbf{H}_{abcd}\;=\partial_{[a}\bar{H}_{bcd]}+\tfrac{3}{2}V_{[ab}W_{cd]}+\tfrac{3}{4}\bar{F}_{[ab}{}^{ij}\bar{F}_{cd]ij}\,. \qquad (32)
$$

We denote the resulting gauge-invariant scalar as $\mathbf{BI}(\psi)$. This scalar is then added to (27), thereby imposing the $\bar{H}$-Bianchi identity in a gauge-invariant form. To impose the Bianchi identities corresponding to $\bar{F}$, V, and W, whenever there are terms with derivatives of these field strengths, we replace them with their corresponding potentials. As a result, equation (27) can be expressed in terms of independent but non-gauge-invariant couplings. The coefficients of these independent terms must be set to zero, resulting in a system of linear algebraic equations involving all the parameters.

Upon solving the system of algebraic equations, we discover two distinct sets of solutions for the parameters involved. The first set represents relationships exclusively among the parameters in the deformations of the truncated Buscher rules, the parameters corresponding to the $\bar{H}$-Bianchi identity, and the parameters in the total derivative terms. These solutions satisfy the homogeneous equation corresponding to (27), with the right-hand side being zero. However, this particular solution is not of interest to us.

The other solution, which is the particular solution of the non-homogeneous equation (27) and the one of interest, provides an expression for these parameters in terms of the coupling constants $b_1, b_2, \ldots$. This solution also establishes the relationships between the coupling constants. In our analysis, we will consider this particular solution that sets all parameters in the effective action and their corresponding corrections to the Buscher rules to zero, i.e.,

$$\psi^L_{\frac{3}{2}} = 0\,, \qquad b_1 = b_2 = \cdots = b_{23} = 0\,. \tag{33}$$

In fact, there are 12 couplings with coefficients $b_1, b_5, b_6, b_7, b_8, b_9, b_{11}, b_{12}, b_{13}, b_{17}, b_{18}, b_{20}$ that appear solely in the algebraic equation, without any contribution from total derivatives, corrections to the Buscher transformations, or the Bianchi identities. As a result, they are constrained to be zero. The other coefficients appear in the algebraic equation in combination with the other parameters of the total derivative terms, corrections to the Buscher transformations, and the Bianchi identities. However, the algebraic equation fixes them to zero as well. Therefore, the effective action at the 5-derivative order vanishes identically.

Given that $S^{L(1/2)} = S^{L(3/2)} = \psi^L_{\frac{1}{2}} = \psi^L_{\frac{3}{2}} = 0$, the constraint in (14) at the order $\alpha'^2\sqrt{\alpha'}$ takes the form:

$$-S^{L(0,5/2)}(\psi^L_0) - \int d^9x\,\partial_a\Big[e^{-2\bar{\phi}}J^a_{(5/2)}(\psi)\Big] = S^{L(5/2)}(\psi^L_0) - S^{L(5/2)}(\psi). \tag{34}$$

By employing the circular reductions in (17), one can compute the truncated reduction of the couplings in (8) to generate $S^{L(5/2)}(\psi)$. Subsequently, by utilizing the truncated Buscher transformations in (22), one can calculate the corresponding $S^{L(5/2)}(\psi^L_0)$.

To determine the first term on the left-hand side of the equation (34), we consider the 5-derivative corrections to the truncated Buscher rules (22), given by:

$$\begin{aligned}
&\varphi^L = -\varphi + \alpha'\sqrt{\alpha'}\Delta\varphi^{(5/2)}(\psi) + \cdots\,, \quad g^L_a = b_a + \alpha'\sqrt{\alpha'}e^{\varphi/2}\Delta g^{(5/2)}_a(\psi) + \cdots\,,\\
&b^L_a = g_a + \alpha'\sqrt{\alpha'}e^{-\varphi/2}\Delta b^{(5/2)}_a(\psi) + \cdots\,, \quad \bar{g}^L_{ab} = \eta_{ab} + \alpha'\sqrt{\alpha'}\Delta\bar{g}^{(5/2)}_{ab}(\psi) + \cdots\,,\\
&\bar{H}^L_{abc} = \bar{H}_{abc} + \alpha'\sqrt{\alpha'}\Delta\bar{H}^{(5/2)}_{abc}(\psi) + \cdots\,, \quad \bar{\phi}^L = \bar{\phi} + \alpha'\sqrt{\alpha'}\Delta\bar{\phi}^{(5/2)}(\psi) + \cdots\,,\\
&(\bar{A}^L_a)^{ij} = \bar{A}_a{}^{ij} + \alpha'\sqrt{\alpha'}\Delta\bar{A}^{(5/2)ij}_a(\psi) + \cdots\,, \quad (\alpha^L)^{ij} = -\alpha^{ij} + \alpha'\sqrt{\alpha'}\Delta\alpha^{(5/2)ij}(\psi) + \cdots\,.
\end{aligned} \tag{35}$$

It is worth noting that these corrections also contain terms at the 2-derivative and 4-derivative orders, i.e., ψ_1^L and ψ_2^L. However, since $\psi_{\frac{1}{2}}^L = \psi_{\frac{3}{2}}^L = 0$, these corrections do not play a role in studying the T-duality of the 7-derivative couplings. Thus, the first term on the left-hand side of the constraint (34) can be obtained by replacing the 3-derivative correction of the Buscher rules in (31) with the 5-derivative corrections. The subsequent steps follow a similar procedure. The result we obtain is as follows:

$$\psi_{\frac{5}{2}}^L = 0\,, \qquad c_1 = c_2 = \cdots = c_{1168} = 0. \tag{36}$$

Hence, the effective action at the 7-derivative order is zero.

Given that $S^{L(1/2)} = S^{L(3/2)} = S^{L(5/2)} = \psi_{\frac{1}{2}}^L = \psi_{\frac{3}{2}}^L = \psi_{\frac{5}{2}}^L = 0$, the constraint in (14) at the order $\alpha'^3\sqrt{\alpha'}$ can be expressed as follows:

$$-S^{L(0,7/2)}(\psi_0^L) - \int d^9x\, \partial_a\Big[e^{-2\bar{\phi}} J^a_{(7/2)}(\psi)\Big] = S^{L(7/2)}(\psi_0^L) - S^{L(7/2)}(\psi). \tag{37}$$

However, performing explicit calculations at this order and beyond is extremely challenging for computers due to the extensive length of the calculations involved. While there are no fundamental obstacles in performing these calculations, the main difficulty lies in the extensive computational effort required. We expect that the same results we have obtained for the 3-, 5-, and 7-derivative cases hold true for all higher odd-derivative couplings, namely:

$$\psi^L_{n+\frac{1}{2}} = 0\,, \qquad S^{(n+\frac{1}{2})} = 0\,;\;\; n = 0, 1, 2, \cdots. \tag{38}$$

Hence, the truncated T-duality transformation (12) can only have even-derivative contributions, i.e.,

$$\psi^L = \psi_0^L + \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \psi_n^L. \tag{39}$$

Furthermore, the effective action also consists solely of even-derivative gauge invariant couplings.

However, it is important to note that this result does not imply the absence of contact terms in the sphere-level S-matrix element of the heterotic theory that involve an odd number of momenta. In fact, these contact terms should be accounted for by the non-zero commutator terms present in the even-derivative couplings of field strengths. For example, consider the three-point function of three YM vertex operators, which includes one momentum [16]. This particular configuration is indeed reproduced by the commutator term present in the leading-order action (16). Thus, while the odd-derivative couplings vanish, the contact terms involving odd numbers of momenta are still captured by the commutator terms within the even-derivative couplings.

4 Discussion

The classical effective action of string theory, when subjected to circular reduction, exhibits the $O(1,1,\mathbb{R})$ symmetry, as discussed in [1, 2, 3]. The determination of coupling constants within the effective action can be achieved by applying the non-geometric subgroup $O(1,1,\mathbb{Z})$ or the associated generalized Buscher transformations to the gauge invariant independent couplings at any order of derivatives. In the context of heterotic string theory, the circular reduction of the effective action and the generalized Buscher transformations demonstrate nonlinearity with respect to the scalar component of the YM gauge field. A recent paper has introduced a truncated approach, where the reduced action and generalized Buscher transformations treat the scalar field as a constant and retain only the zeroth and first orders of the scalar field. By employing this truncated version of the generalized Buscher transformations, the coupling constants within the effective action can still be determined. In this study, we have applied this constraint to the gauge invariant independent couplings with an odd number of derivatives. Specifically, we have provided explicit demonstrations that for 3-, 5-, and 7-derivative couplings, the odd-derivative terms vanish under this imposed constraint. We further speculate that this result holds true for all higher odd-derivative couplings as well. This is in contrast to the couplings in type I superstring theory, where non-zero couplings of, for example, 5 YM field strength are present [20].

In a recent paper [21], it was also observed that T-duality fixes the coupling constants in the NS-NS bases that have an odd number of B-fields to be zero, both in bosonic and superstring theories. This suggests that T-duality has the ability to determine the couplings in a basis up to an overall factor, as long as that factor is non-zero in the context of string theory. In other words, if a coupling is zero in string theory, T-duality will yield that result. However, if the overall factor of the couplings in a basis is non-zero in any of the string theories and their values depend on the specific type of string theory, then T-duality cannot fix it, as T-duality is a symmetry of all string theories.

It has been observed in [7] that T-duality imposes constraints on the single-trace couplings of the YM field strength at the n-derivative order, with the exception of $n = 2$, resulting in their vanishing. In the present paper, we have made a similar finding: the couplings involving single-trace terms with an odd number of YM field strengths, multiplied by other NS-NS terms or $\mathrm{Tr}(FF)$, are likewise forced to be zero. For instance, we have determined that the coupling $\mathrm{Tr}(FF)\mathrm{Tr}(FFF)$ is zero. This observation suggests that similar constraints may apply to even-derivative gauge invariant couplings as well. If this hypothesis holds true, one would expect the YM field strength and its derivatives to appear in the even-derivative couplings in the form of different orders of $\mathrm{Tr}(FF)$. Performing calculations for the 6-derivative and 8-derivative orders would be of great interest in order to verify this speculation.

We have observed how specifically fixing the YM gauge symmetry facilitates the identification of independent covariant couplings at various orders of derivatives. In both bosonic and superstring theory, the nonabelian YM field arises on stacks of D_p-branes. One might attempt to employ a similar method to identify the independent covariant YM couplings for D_p-branes. However, in this case, it is not clear how to impose world-volume diffeomorphism symmetry for the nonabelian case. For the abelian case, the independent world-volume covariant couplings can be constructed using the pull-back metric. Subsequently, the coupling constants can be determined by imposing T-duality [22, 23]. In the nonabelian case, it is known that the four YM field strength couplings coincide with the corresponding couplings in the nonabelian DBI action when a symmetric trace is imposed [24]. However, it is also known that the couplings involving higher YM field strengths are not provided by the nonabelian symmetrized trace DBI action [25]. Investigating the world-volume diffeomorphism symmetry for the nonabelian case would be intriguing, as it could help identify independent covariant and YM gauge invariant world-volume couplings. Subsequently, by imposing T-duality on these couplings, the coupling constants can be determined following a similar approach to what has been done in the context of the abelian case [22, 23].

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