A δ-FIRST WHITEHEAD LEMMA FOR JORDAN ALGEBRAS

AREZOO ZOHRABI AND PASHA ZUSMANOVICH

ABSTRACT. We compute δ-derivations of simple Jordan algebras with values in irreducible bimodules. They turn out to be either ordinary derivations ($\delta = 1$), or scalar multiples of the identity map ($\delta = \frac{1}{2}$). This can be considered as a generalization of the "First Whitehead Lemma" for Jordan algebras which claims that all such ordinary derivations are inner. The proof amounts to simple calculations in matrix algebras, or, in the case of Jordan algebras of a symmetric bilinear form, to more elaborated calculations in Clifford algebras.

INTRODUCTION

Let A be a (generally, nonassociative) algebra A, and M an A-bimodule, with the bimodule action denoted by $\bullet$. Recall that a *δ-derivation* of A with values in M is a linear map $D : A \to M$ such that

$$D(xy) = \delta D(x) \bullet y + \delta x \bullet D(y) \tag{1}$$

for any $x, y \in A$. Obviously, the ordinary derivations are 1-derivations, and in the case where $M = A$, the regular bimodule, elements of the centroid are the special cases of $\frac{1}{2}$-derivations. Despite being seemingly a straightforward generalization of derivations, δ-derivations appear in different situations and are proved to be a useful an interesting invariant. For example, in Lie algebras context, they are related, for various values of δ, to commutative 2-cocycles, arise in description of (ordinary) derivations of certain current Lie algebras, and can be used to construct non-semigroup gradings; see [Z] for further references.

There are quite a lot of investigations of δ-derivations of various algebras with values in itself, i.e., in the regular module – see, for example, in addition to the already mentioned [Z], also [K] and [ZZ2], and references therein – but very little was done concerning δ-derivations with values in more general modules.

In [ZZ1] we computed δ-derivations of simple finite-dimensional Lie algebras of characteristic zero with coefficients in finite-dimensional modules. The motivation of doing this was twofold: first, to establish a "δ-analog" of the classical First Whitehead Lemma, that is, to prove that δ-derivations of a finite-dimensional simple Lie algebra of characteristic zero with values in a finite-dimensional module are just inner derivations, with the exception of peculiar cases related to $\mathfrak{sl}(2)$. Second, to provide an alternative route to computation, done in [ZZ2], of δ-derivations of algebras of skew-Hermitian matrices over octonions, an interesting series of anticommutative nonassociative algebras.

The purpose of this note is to establish a "Jordan analog" of this result: we prove that any δ-derivation of a finite-dimensional simple Jordan algebra with values in a finite-dimensional unital irreducible bimodule is, in a sense, trivial, i.e., it is either an ordinary, and hence inner, derivation ($\delta = 1$), or is a scalar multiple of the identity map on the underlying Jordan algebra, in which case the bimodule is the regular bimodule ($\delta = \frac{1}{2}$). (Note that, unlike in the Lie algebras case, there are no exceptional cases related to Jordan algebras of small dimension). This can be considered as a generalization of the "First Whitehead Lemma" for Jordan algebras, that is, the classical result that (ordinary) derivations of a finite-dimensional simple Jordan algebra with values in an irreducible bimodule are inner.

As a corollary, we also present an alternative proof of the result, established in [ZZ2], that δ-derivations of the algebra of Hermitian $n \times n$ matrices over octonions are trivial (what, in its turn, helped to determine symmetric invariant bilinear forms on these algebras). These octonionic matrix algebras generalize the 3×3 case of the 27-dimensional exceptional simple Jordan algebra, and the 4×4 case appears in modern physical theories. (Note, however, that for $n \geq 4$ these algebras are neither Jordan, nor belong to any known variety of nonassociative algebras studied in the literature). The proof uses the fact that the $n \times n$ octonionic matrix algebra contains the simple Jordan subalgebra $M_n^+(K)$ of symmetric matrices, and restricting a δ-derivation to that particular subalgebra gives a δ-derivation of $M_n^+(K)$ with values in the whole octonionic matrix algebra.

NOTATION AND CONVENTIONS

Unless stated otherwise, the ground field K assumed to be arbitrary of characteristic different from 2, a usual assumption in the (classical) Jordan structure theory. As we deal simultaneously with Jordan algebras

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and associative algebras (as their associative envelopes), the multiplication in the formers (or, more generally, in an arbitrary commutative nonassociative algebra, like in Lemma 2 below) will be (traditionally) denoted by $\circ$, while multiplication in the latters will be denoted by juxtaposition. Action of a Jordan algebra on its bimodule is denoted by $\bullet$. The action of an algebra on itself by multiplications is called the *regular* representation.

For an associative algebra A, A^{op} denotes the "opposite" algebra, i.e., the same vector space A subject to multiplication $x \cdot y = yx$, and $A^{(+)}$ denotes its "plus" Jordan algebra, i.e., the same vector space A subject to multiplication $x \circ y = \frac{1}{2}(xy + yx)$. Assuming an algebra A has an involution (i.e., an antiautomorphism of order 2) $a \mapsto a^{\mathbb{J}}$, define $S^+(A,\mathbb{J}) = \{a \in A \mid a^{\mathbb{J}} = a\}$ and $S^-(A,\mathbb{J}) = \{a \in A \mid a^{\mathbb{J}} = -a\}$, the vector spaces of $\mathbb{J}$-symmetric and $\mathbb{J}$-skew-symmetric elements, respectively. If A is associative, then $S^+(A,\mathbb{J})$ is a Jordan subalgebra of $A^{(+)}$. $M_n(A)$ denotes the matrix algebra over the algebra A.

1. Recapitulation on Clifford algebras

Here we recall some facts related to Clifford algebras which will be used below, when dealing with Jordan algebras of a symmetric bilinear form.

Let V be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form f, and $C(V,f)$, or just $C(V)$ if there is no ambiguity what f is, a corresponding Clifford algebra.

If W is a subspace of V, then, denoting by abuse of notation the restriction of f to W by the same letter f, we have that $C(W,f)$ is a subalgebra of $C(V,f)$.

Fix an orthogonal, with respect to f, basis $\{u_1,\dots,u_n\}$ of V. Then the basis of $C(V)$ can be chosen to consist of elements

$$u_{i_1}u_{i_2}\dots u_{i_k}, \text{ where } 1 \le i_1 < i_2 < \dots < i_k \le n,\ 0 \le k \le n \tag{2}$$

(as usual in such situations, we tacitly assume that the above product for $k = 0$, i.e., with the zero number of factors, is equal to 1).

Let $C(V)^{(k)}$ be the $\binom{n}{k}$-dimensional subspace of $C(V)$ spanned by elements of the form (2) for a fixed k. (Thus, $C(V)^{(0)}$ is just the one-dimensional space $K1$, and $C(V)^{(1)} = V$). Then $C(V)$ is decomposed as the vector space direct sum $\bigoplus_{k=0}^n C(V)^{(k)}$.

Lemma 1. *In the Clifford algebra $C(V,f)$ the following equality holds for any $x, y_1, \dots, y_k \in V$:*

$$y_1 \cdots y_k x = (-1)^k x y_1 \cdots y_k + 2 \sum_{i=1}^k (-1)^{k+i} f(x,y_i) y_1 \cdots y_{i-1} \widehat{y_i} y_{i+1} \cdots y_k \tag{3}$$

(as usual, $\widehat{\ }$ means that the corresponding element is omitted in the product).

Proof. This is implicit in the corresponding Clifford-algebraic calculations in [J1] and [J2], and is easily proved by induction. □

For $k = 1$, the equality (3) reduces to $xy_1 + y_1x = 2f(x,y_1)$, the defining relation in the Clifford algebra, so (3) can be viewed as a generalization of this defining relation for an arbitrary number of elements.

Corollary 1.

$$u_i(u_1 \cdots u_k)u_i = \begin{cases} (-1)^{k+i} f(u_i,u_i) u_1 \cdots u_k, & \text{if } i \in \{1,\dots,k\} \\ (-1)^k \quad f(u_i,u_i) u_1 \cdots u_k, & \text{if } i \notin \{1,\dots,k\}. \end{cases}$$

Proof. Multiplying both sides of (3) on x from the left, and using the fact that $x^2 = f(x,x)1$, we get

$$x y_1 \cdots y_k x = (-1)^k f(x,x) y_1 \cdots y_k + 2 \sum_{i=1}^k (-1)^{k+i} f(x,y_i) x y_1 \cdots y_{i-1} \widehat{y_i} y_{i+1} \cdots y_k.$$

Substituting in the last equality the appropriate u_i's instead of x and y_i's, and using the fact that u_i and u_j anticommute for $i \ne j$, we get the required equalities. □

2. Recapitulation on simple Jordan algebras and their irreducible bimodules

Here we briefly review the necessary facts about Jordan algebras and their modules. The main sources are [J1] and [J2], Chapters II, V, and VII.

If J is a simple Jordan algebra, the associative universal envelope $U(J)$ (called universal associative algebra for the unital representations in [J1], and universal unital multiplication envelope in [J2]) is semisimple, and unital irreducible J-bimodules are in a bijective correspondence with simple components of $U(J)$ (note, however, that since not every Jordan algebra is special, J is not necessarily embedded into $U(J)^{(+)}$).

Let us recall the isomorphism types of finite-dimensional simple Jordan algebras and their associative universal envelopes.

(i) The ground field K. The associative universal envelope coincides with the K, and any unital irreducible K-bimodule is isomorphic to K itself.

(ii) The algebra $J(V,f) = K1 \oplus V$ of nondegenerate symmetric bilinear form f defined on a vector space V. The multiplication between elements $x,y \in V$ is defined by $x \circ y = f(x,y)1$. Define the vector space V^{ev} as follows:
(a) If $\dim V$ is even, set $V^{ev} = V$;
(b) If $\dim V$ is odd, set $V^{ev} = V \oplus Ku$, and extend f to V^{ev} by setting $f(u,u) = 1$ and $f(u,V) = f(V,u) = 0$.

The associative universal envelope is isomorphic to the so-called meson algebra, which can be realized as the unital subalgebra of the algebra of linear endomorphisms of the Clifford algebra $C(V^{ev},f)$ generated by Jordan multiplications on the elements of $J(V,f)$, i.e., by the maps $x \mapsto \frac{1}{2}(ax+xa)$, where $a \in J(V,f)$. The Jordan algebra $J(V,f)$ is embedded into the algebra $C(V^{ev},f)^{(+)}$, and the action of $J(V,f)$ on $C(V^{ev},f)$ is the restriction to $J(V,f)$ of the regular representation of $C(V^{ev},f)^{(+)}$. Below we describe the decomposition of $C(V^{ev},f)$ into the sum of subspaces invariant with respect to the action of $J(V,f)$. The unital irreducible $J(V,f)$-bimodules consist of action of $J(V,f)$ on these subspaces. The precise description, in terms of a fixed orthogonal basis of V, depends on the parity (more exactly, on the residue modulo 4) of n.

Case (a): $n = 2m$. The irreducible $J(V,f)$-invariant subspaces are:

$$C(V)^{(0)} \oplus C(V)^{(1)}, \quad C(V)^{(2)} \oplus C(V)^{(3)}, \quad \dots, \quad C(V)^{(2m-2)} \oplus C(V)^{(2m-1)}, \quad C(V)^{(2m)}.$$

Case (b): $n = 2m-1$. Extend the chosen orthogonal basis of V to an orthogonal basis of V^{ev} by adding u to it.

Case (b1): m is even. The irreducible $J(V,f)$-invariant subspaces are:

$$\begin{gathered} C(V)^{(0)} \oplus C(V)^{(1)}, \quad C(V)^{(2)} \oplus C(V)^{(3)}, \quad \dots, \quad C(V)^{(m-2)} \oplus C(V)^{(m-1)}, \\ C(V)^{(0)}u, \quad C(V)^{(1)}u \oplus C(V)^{(2)}u, \quad \dots, \quad C(V)^{(m-3)}u \oplus C(V)^{(m-2)}u. \end{gathered}$$

In addition to that, the subspace $C(V)^{(m-1)}u \oplus C(V)^{(m)}u$ decomposes as the direct sum of two irreducible $J(V,f)$-invariant subspaces of the same dimension $\frac{1}{2}\binom{2m}{m}$.

Case (b2): m is odd. The irreducible $J(V,f)$-invariant subspaces are:

$$\begin{gathered} C(V)^{(0)} \oplus C(V)^{(1)}, \quad C(V)^{(2)} \oplus C(V)^{(3)}, \quad \dots, \quad C(V)^{(m-3)} \oplus C(V)^{(m-2)}, \\ C(V)^{(0)}u, \quad C(V)^{(1)}u \oplus C(V)^{(2)}u, \quad \dots, \quad C(V)^{(m-2)}u \oplus C(V)^{(m-1)}u. \end{gathered}$$

In addition to that, the subspace $C(V)^{(m-1)} \oplus C(V)^{(m)}$ decomposes as the direct sum of two irreducible $J(V,f)$-invariant subspaces of the same dimension $\frac{1}{2}\binom{2m}{m}$.

(iii) The algebra $A^{(+)}$, where A is a central simple associative algebra. The associative universal envelope is $A \oplus A^{\mathbb{J}}$, where $\mathbb{J}$ is an involution interchanging A and $A^{\mathbb{J}}$. The unital irreducible $A^{(+)}$-bimodules are as follows. First, A (the regular bimodule), with the action $a \bullet b = \frac{1}{2}(ab+ba)$ (here and below, we assume a belongs to the Jordan algebra in question, and b to the corresponding bimodule); then, assuming A has an involution $\mathbb{K}$, there are additionally the following four bimodules:
(a) $S^+(A,\mathbb{K})$, $a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}})$;
(b) $S^+(A,\mathbb{K})$, $a \bullet b = \frac{1}{2}(a^{\mathbb{K}}b + ba)$;

(c) $S^-(A,\mathbb{K})$, $a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}})$;
(d) $S^-(A,\mathbb{K})$, $a \bullet b = \frac{1}{2}(a^{\mathbb{K}}b + ba)$.

(iv) $S^+(A,\mathbb{J})$, where A is a central simple associative algebra with an involution $\mathbb{J}$ of the first kind. The associative universal envelope is A, and the unital irreducible $S^+(A,\mathbb{J})$-bimodules are $S^+(A,\mathbb{J})$ (the regular bimodule) and $S^-(A,\mathbb{J})$, both with the action $a \bullet b = \frac{1}{2}(ab + ba)$.

(v) $S^+(A,\mathbb{J})$, where A is a simple associative algebra whose center $Z(A)$ is a quadratic extension of the base field K, $\mathbb{J}$ is an involution of the second kind on A, and A over $Z(A)$ has an involution $\mathbb{K}$ commuting with $\mathbb{J}$. The associative universal envelope is A, and the unital irreducible $S^+(A,\mathbb{J})$-bimodules are:
(a) $S^+(A,\mathbb{J})$, $a \bullet b = \frac{1}{2}(ab + ba)$ (the regular bimodule);
(b) $S^+(A,\mathbb{K})$, $a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}})$;
(c) $S^-(A,\mathbb{K})$, $a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}})$.

(vi) The 27-dimensional algebra of 3×3 Hermitian matrices over octonions. The associative universal envelope is isomorphic to the full 27×27 matrix algebra $M_{27}(K)$, and the only unital irreducible bimodule is the regular one.

The algebras specified in (i)–(v) are special, while the last one, in (vi), is exceptional.

To summarize the special cases: any unital irreducible bimodule of a special simple Jordan algebra J is a submodule of the restriction to J either of the regular representation of $U(J)^{(+)}$, or of the representation of $U(J)^{(+)}$ in itself with one of the following actions:

$$a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}}) \tag{4}$$

or

$$a \bullet b = \frac{1}{2}(a^{\mathbb{K}}b + ba), \tag{5}$$

where $a,b \in U(J)^{(+)}$, and $\mathbb{K}$ is an involution on $U(J)$. Note, however, that for an arbitrary associative algebra A with an involution $\mathbb{K}$, the action of type (5) coincides with the action of type (4) for the algebra A^{op} (with the same involution $\mathbb{K}$).

3. δ-DERIVATIONS, SIMPLE CASE

Lemma 2. *Let D be a nonzero δ-derivation of a commutative algebra A with unit with values in a unital symmetric A-bimodule M. Then either $\delta = 1$ (i.e., D is a derivation), or $\delta = \frac{1}{2}$ and $D(x) = x \bullet m$ for a certain element $m \in M$ such that*

$$(x \circ y) \bullet m = \frac{1}{2}\big(x \bullet (y \bullet m) + y \bullet (x \bullet m)\big)$$

for any $x,y \in A$.

Proof. An elementary proof consists of a repetitive substitution of 1 in the equation (1), and is an almost verbatim repetition of the proof of Lemma 9 from [ZZ1] which treats the case $M = A$ (which, in its turn, is just a slight reformulation of Theorem 2.1 from [K]). □

Lemma 3. *Let J be a special Jordan algebra with unit, embedded into an algebra $A^{(+)}$ for an associative algebra A. Let M be a submodule of the restriction to J of the regular $A^{(+)}$-bimodule, and $D : J \to M$ a nonzero δ-derivation. Then either $\delta = 1$, or $\delta = \frac{1}{2}$ and $D(x) = \frac{1}{2}(xm + mx)$ for a certain element $m \in M$ such that*

$$xym + yxm + mxy + myx - 2xmy - 2ymx = 0 \tag{6}$$

for any $x,y \in J$.

Lemma 4. *Let J be a special Jordan algebra with unit, embedded into an algebra $A^{(+)}$ for an associative algebra A with involution $\mathbb{K}$. Let M be a submodule of the restriction to J of the $A^{(+)}$-bimodule with the action $a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}})$, and $D : J \to M$ a nonzero δ-derivation. Then either $\delta = 1$, or $\delta = \frac{1}{2}$ and $D(x) = \frac{1}{2}(xm + mx^{\mathbb{K}})$ for a certain element $m \in M$ such that*

$$xym + yxm + mx^{\mathbb{K}}y^{\mathbb{K}} + my^{\mathbb{K}}x^{\mathbb{K}} - 2xmy^{\mathbb{K}} - 2ymx^{\mathbb{K}} = 0 \tag{7}$$

for any $x,y \in J$.

Proof. The proof of Lemma 3, respectively Lemma 4, amounts to application of Lemma 2 to the situation where $x \circ y = \frac{1}{2}(xy+yx)$, and $x \bullet m = \frac{1}{2}(xm+mx)$, respectively $x \bullet m = \frac{1}{2}(xm+mx^{\mathbb{K}})$. □

In what follows it will be useful the rewrite the condition (6) in equivalent forms. Substituting $y = x$ in (6), we get the equality

$$x^2m + mx^2 - 2xmx = 0, \tag{8}$$

valid for any $x \in J$ (actually, this is equivalent to (6) via linearization). The last equality, in its turn, can be rewritten as

$$[[m,x],x] = 0, \tag{9}$$

where $[a,b] = ab - ba$ is the commutator of two elements.

Similarly, (7) can be written in an equivalent form by setting $y = x$:

$$x^2m + m(x^2)^{\mathbb{K}} - 2xmx^{\mathbb{K}} = 0, \tag{10}$$

valid for any $x \in J$.

Theorem 1. *Let D be a nonzero δ-derivation of a finite-dimensional simple Jordan algebra with values in a finite-dimensional unital irreducible bimodule M. Then either $\delta = 1$ and D is an inner derivation, or $\delta = \frac{1}{2}$, M is the regular bimodule, and D is a scalar multiple of the identity map.*

Proof. If $\delta = 1$ then D is an ordinary derivation and hence is inner, so suppose $\delta \neq 1$. We proceed case-by-case according to §2. Note that since our problem – computation of δ-derivations – does not change under an extension of the ground field (see the remark after Theorem 8 in [ZZ2, §3]), we may assume that the ground field K is algebraically closed whenever appropriate.

(i) Trivial.

(ii) As $x^2 = f(x,x)1$ for any $x \in V$, the equality (8) reduces to

$$xmx = f(x,x)m \tag{11}$$

for any $x \in V$.

We have that either $m \in C(V)^{(k)} \oplus C(V)^{(k+1)}$, or, in the case of odd-dimensional V, $m \in C(V)^{(k)}u \oplus C(V)^{(k+1)}u$ for certain k. Decompose m as a linear combination of the basic elements of the form (2), and let $u_{i_1} \cdots u_{i_k}$, or $u_{i_1} \cdots u_{i_k}u$ in the case of odd-dimensional V, be one of those basic elements entering this sum with a nonzero coefficient. Assuming $k \geq 2$, Corollary 1 and equality (11) coupled together, yield

$$f(u_{i_1},u_{i_1})u_{i_1} \cdots u_{i_k} = u_{i_1}(u_{i_1} \cdots u_{i_k})u_{i_1} = (-1)^{k+1} f(u_{i_1},u_{i_1})u_{i_1} \cdots u_{i_k}$$

and

$$f(u_{i_2},u_{i_2})u_{i_1} \cdots u_{i_k} = u_{i_2}(u_{i_1} \cdots u_{i_k})u_{i_2} = (-1)^{k+2} f(u_{i_2},u_{i_2})u_{i_1} \cdots u_{i_k},$$

whence

$$u_{i_1} \cdots u_{i_k} = (-1)^{k+1}u_{i_1} \cdots u_{i_k} = (-1)^{k+2}u_{i_1} \cdots u_{i_k} = 0,$$

a contradiction. The same reasoning holds when applied to the element of the form $u_{i_1} \cdots u_{i_k}u$.

Hence, in both cases, $k \leq 1$, i.e., m belongs either to the regular bimodule $K \oplus V$, or to the module Ku, or $m \in Vu$. These remaining cases can be processed in the similar way as the generic case above, just the calculations are even simpler. Namely, in the case of the regular module we can either refer to [K, Lemma 2.3], or, assuming u_i is a nonzero component of m, consider the expression $u_ju_iu_j$ with $j \neq i$, and apply Corollary 1 again to get a contradiction. A similar reasoning works in the case $m = \lambda u$ for some $\lambda \in K$, and in the remaining case $m \in Vu$, assuming u_iu is a nonzero component of m, we get a contradiction in a similar fashion by considering expression $u_i(u_iu)u_i$. Therefore, the only remaining possibility is $m \in K1$.

(iii) For the case of the regular $A^{(+)}$-bimodule we can either refer to [K, Theorem 2.5], or proceed as follows. Assume that the ground field K is algebraically closed. Then A is isomorphic to the full matrix algebra over K, and substituting in (9) the matrix units E_{ii} instead of x, we get that m should be a diagonal matrix, with diagonal elements, say, $\lambda_1, \dots, \lambda_n$. Then, for example, substituting in (9) again the matrix having 1's in the first row and the first column, and zeros elsewhere, we get at the left-hand side the matrix whose first row, beginning from the 2nd element, consists of elements $\lambda_1 - \lambda_i$. This proves that all λ_i's are equal, and m is a multiple of the identity matrix.

Very similar reasonings will do in the case of bimodule $S^+(A,\mathbb{K})$ or $S^-(A,\mathbb{K})$ (i.e., the space of symmetric or skew-symmetric matrices in the case of algebraically closed ground field) with the action $a \bullet b = \frac{1}{2}(ab + ba^{\mathbb{K}})$, utilizing the equality (10). In these cases it is enough to substitute there the matrix units E_{ij} instead of x, to get $m = 0$.

The remaining two cases can be reduced to the already considered cases by passing to the algebra A^{op} with the opposite multiplication (which does not change the associated Jordan algebra $A^{(+)}$), as noted at the end of §2.

(iv) Over an algebraically closed field the Jordan algebra $S^+(A,\mathbb{J})$ is isomorphic to the algebra of symmetric matrices; we are again in the realm of Lemma 3, and exactly the same reasonings as in case (iii) will do. Indeed, when deriving from (9) the necessary conclusion about m by substituting various matrices instead of x, we used only symmetric matrices. Thus the same conclusion holds in this case, i.e., m is a scalar multiple of the identity matrix. This is only possible in the case when the bimodule coincides with $S^+(A,\mathbb{J})$, i.e., is the regular bimodule.

(v) Being extended to an algebraic closure of the ground field, these algebras and bimodules are isomorphic to those of the case (iii).

(vi) As in this case the question is reduced entirely to δ-derivations of simple exceptional Jordan algebra with values in itself, we can refer to [K, Theorem 2.5], or, to a more general case of Hermitian matrices over octonions of arbitrary size covered in [ZZ2, Theorem 8]. (Note that below, in §5, we outline a proof, different from those given in [ZZ2], of the triviality of δ-derivations of these algebras using Theorem 1. This does not lead to circular arguments, as in §5 we are using the part of Theorem 1 dealing with special matrix Jordan algebras).

To summarize: in all the cases we have proved that in the context of Lemma 2, M is the regular bimodule and m is a scalar multiple of 1, and hence D is a scalar multiple of the identity map on the underlying Jordan algebra. ☐

4. δ-DERIVATIONS, SEMISIMPLE CASE

As an immediate corollary of the just proved theorem, we get a description of δ-derivations of a finite-dimensional semisimple Jordan algebra J with values in an arbitrary finite-dimensional unital bimodule M. For that, we need first the following two simple lemmas, valid for an arbitrary nonassociative algebra A. By $\mathrm{Der}_\delta(A,M)$ we denote the vector space of all δ-derivations of an algebra A with values in an A-bimodule M.

Lemma 5. *Let M is decomposed as the direct sum of A-submodules: $M = \bigoplus_i M_i$. Then*

$$\mathrm{Der}_\delta(A,M) \simeq \bigoplus_i \mathrm{Der}_\delta(A,M_i). \tag{12}$$

Proof. The elementary proof of the Lie algebra case given in [ZZ1, Lemma 1] is valid for arbitrary algebras, and repeats the proof of the similar statement for ordinary derivations. ☐

Lemma 6. *Let A is decomposed as the direct sum of ideals: $A = \bigoplus_{i=1}^n A_i$, and $\delta \neq 0$. Then*

$$\mathrm{Der}_\delta(A,M) \simeq$$
$$\{(D_1,\dots,D_n) \in \bigoplus_{i=1}^n \mathrm{Der}_\delta(A_i,M) \mid D_i(x)\bullet y + x \bullet D_j(y) = 0 \textit{ for any } x \in L_i, y \in L_j,\ i,j = 1,\dots,n\}.$$

Proof. Again, a verbatim repetition of the proof of the Lie algebra case in [ZZ1, Lemma 2]. ☐

Now, assume first that J is simple (and M arbitrary). Since any finite-dimensional representation of J is completely reducible, M is decomposed as the direct sum of irreducible submodules: $M = \bigoplus_i M_i$. If $\mathrm{Der}_\delta(J,M)$ does not vanish, then, according to Theorem 1, either $\delta = 1$ and then $\mathrm{Der}_1(J,M)$ consists of inner derivations, or $\delta = \frac{1}{2}$ and the only nonzero summands in (12) are those for which M_i is isomorphic to the regular J-bimodule, in which cases $\mathrm{Der}_{\frac{1}{2}}(J,M_i) \simeq \mathrm{Der}_{\frac{1}{2}}(J,J)$ is the one-dimensional vector space linearly spanned by id_J.

If J is semisimple, then J is decomposed into the direct sum of simple ideals, and Lemma 6 reduces the situation to simple cases.

To give a precise statement describing δ-derivations of a semisimple Jordan algebra with values in a finite-dimensional module, by using combination of Theorem 1, Lemma 5, and Lemma 6, similar to Main Theorem in [ZZ1] in the Lie algebra case, would be somewhat cumbersome. The main reason for this is the absence of the tensor product construction for bimodules over Jordan algebras. However, the scheme described above allows to settle the question effectively in each concrete case.

5. δ-DERIVATIONS, OCTONIONIC MATRIX ALGEBRAS

Theorem 1 (or, more exactly, reasoning in the preceding section based on it), allows to give a somewhat streamlined proof of the result established in [ZZ2]: triviality of δ-derivations of the algebra $S^+(M_n(O),\mathbb{J})$ of Hermitian $n \times n$ matrices over octonions. Here O is the algebra of octonions, $\mathbb{J}$ is the composition of the matrix transposition and the standard involution on O, and the multiplication in $S^+(M_n(O),\mathbb{J})$ is defined according to the "Jordan" rule:

$$A \circ B = \frac{1}{2}(AB + BA). \tag{13}$$

The streamlined proof we present here is based on the following general simple observation which can be of independent interest as a tool to derive triviality of δ-derivations of algebras from those of their subalgebras. Let us call a (not necessarily Jordan) algebra A *δ-challenged* if it satisfies the conclusion of Theorem 1, i.e., if for any finite-dimensional irreducible bimodule over A, $\mathrm{Der}_\delta(A,M) \neq 0$ implies that either $\delta = 1$, or $\delta = \frac{1}{2}$, M is the regular bimodule, and $\mathrm{Der}_{\frac{1}{2}}(A,M) = \mathrm{Der}_{\frac{1}{2}}(A,A)$ is linearly spanned by the identity map on A.

Lemma 7. *Let A be a commutative algebra with unit, S a simple unital δ-challenged subalgebra of A such that A is decomposed into the direct sum of irreducible components as an S-bimodule, and among those components the only one isomorphic to the regular S-bimodule is S itself. If D is a nonzero δ-derivation of A with values in the regular bimodule, then either $\delta = 1$, or $\delta = \frac{1}{2}$ and D is a scalar multiple of the identity map.*

Proof. Let $\delta \neq 1$. Restriction of D to S is a δ-derivation of S with values in A, and Lemma 5 together with the assumption that S is δ-challenged, implies that $\delta = \frac{1}{2}$ and $D(x) = \lambda x$ for some $\lambda \in K$ and any $x \in S$; in particular, $D(1) = \lambda 1$. But according to Lemma 2, $D(x) = xD(1)$ for any $x \in A$, whence $D(x) = \lambda x$ for any $x \in A$. $\square$

Theorem 2 ([ZZ2], Theorem 8)**.** *Let D be a nonzero δ-derivation of the algebra $S^+(M_n(O),\mathbb{J})$ with values in the regular bimodule. Then either $\delta = 1$, or $\delta = \frac{1}{2}$ and D is a scalar multiple of the identity map.*

In the proof below we will use the following shorthand notation: $M_n^+(K) = S^+(M_n(K),^\top)$, the Jordan algebra of $n \times n$ symmetric matrices; $M_n^-(K) = S^-(M_n(K),^\top)$, the vector space of $n \times n$ skew-symmetric matrices; and $O^- = S^-(O,^-)$, the 7-dimensional vector space of octonions skew-symmetric with respect to the standard conjugation $^-$ in O (of course, the latter two "minus" vector spaces are a Lie algebra and a Malcev algebra respectively, but, in our entirely commutative "Jordan" setting here, we do not need these nice and important facts).

Proof. As noted in [ZZ2], the algebra $S^+(M_n(O),\mathbb{J})$ can be represented as the vector space direct sum $M_n^+(K) \oplus (M_n^-(K) \otimes O^-)$, where $M_n^+(K)$ is a (Jordan) subalgebra, the multiplication between $M_n^+(K)$ and $M_n^-(K) \otimes O^-$ is performed by the action of $M_n^+(K)$ on the first tensor factor $M_n^-(K)$ via the formula (13), leaving the second tensor factor O^- intact. ($M_n^-(K) \otimes O^-$ is not a subalgebra, but the exact nice formula for multiplication of elements in this subspace will not concern us here; the interested reader can consult [ZZ2] for details). Therefore, as an $M_n^+(K)$-bimodule, the whole algebra $S^+(M_n(O),\mathbb{J})$ is decomposed into the direct sum of 8 irreducible components: $M_n^+(K)$, the regular bimodule, and 7 copies of the bimodule $M_n^-(K)$ (parametrized by O^-). By Theorem 1, the Jordan algebra $M_n^+(K)$ is δ-challenged, and thus Lemma 7 is applicable. $\square$

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UNIVERSITY OF OSTRAVA, CZECH REPUBLIC
Email address: `arezoo.zohrabi@osu.cz`

Email address: `pasha.zusmanovich@osu.cz`