

Geodesic extensions of mechanical systems with nonholonomic constraints

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Abstract

For a Lagrangian system with nonholonomic constraints, we construct extensions of the equations of motion to sets of second-order ordinary differential equations. In the case of a purely kinetic Lagrangian, we investigate the conditions under which the nonholonomic trajectories are geodesics of a Riemannian metric, while preserving the constrained Lagrangian. We interpret the algebraic and PDE conditions of this problem as infinitesimal versions of the relation between the nonholonomic exponential map and the Riemannian metric. We discuss the special case of a Chaplygin system with symmetries and we end the paper with a worked-out example.

Keywords: Lagrangian systems, nonholonomic constraints, second-order ordinary differential equations, Riemannian metrics, SODE extensions, exponential map, Chaplygin system.

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1 Introduction

In many applications involving mechanical systems, the motion is often limited by velocity-dependent constraints. If these are given by equations, and if they are non-integrable, they are called nonholonomic constraints. Some examples of such systems are those that involve wheels that roll without slipping, or a skate that is prevented to move in a direction perpendicular to the blade.

Let $L = T - V$ be the Lagrangian of such a mechanical system (given by the difference in kinetic and potential energy) on a configuration manifold Q and let the nonholonomic constraints (assumed to be linear, throughout the paper) be represented by a distribution \mathcal{D} on Q . The equations of motion for such systems (L, \mathcal{D}) follow from the so-called Lagrange-d'Alembert principle (see e.g. [7, 13]). It is well-known that this is not a purely variational principle (for the Lagrangian functional associated to L), since it only makes use of variations that are required to follow the constraints. The resulting equations, the so-called Lagrange-d'Alembert equations of (L, \mathcal{D}) , are therefore not simply the Euler-Lagrange equations of L . Instead, the nonholonomic equations of motion contain Lagrangian multipliers, and when eliminated, they form a set of first- and second-order ordinary differential equations (see their expressions in Section 2).

We will call the solutions of the Lagrange-d'Alembert equations the nonholonomic trajectories of (L, \mathcal{D}) . In many papers, it has been recognized that it is often useful if the nonholonomic trajectories can be interpreted, in one way or another, as part of the solutions of the Euler-Lagrange equations of some unrelated (and possibly non-mechanical) Lagrangian \hat{L} . This question is often referred to as the ‘Hamiltonization problem of nonholonomic mechanics’. It was probably first raised in the context of quantization of classical nonholonomic systems (see [1, 10, 20]), but there are also other important areas of possible applications, such as e.g. that of geometric integrators of nonholonomic systems (see e.g. [22, 26]). The subtlety of the Hamiltonization problem lies in the fact that, in contrast to the nonholonomic equations of (L, \mathcal{D}) , the Euler-Lagrange equations of \hat{L} are purely second-order ordinary differential equations. There have been many different approaches to this question. For example, in [9, 6] the main idea is to enlarge the mixed first- and second order system of (L, \mathcal{D}) to a (chosen) full system of SODEs and to find a Lagrangian \hat{L} for that system using techniques from the so-called inverse problem of Lagrangian mechanics. In other papers [4, 5, 11, 19] one rather uses the symmetries of the nonholonomic system to (roughly speaking) first reduce it to a SODE on a quotient manifold. Then, one tries to find a symplectic or Poisson structure on that quotient for which it is a Hamiltonian vector field.

The results of the current paper belong to the first approach. In short, we will investigate the conditions under which it is possible to have a Hamiltonization by means of the geodesic equations of a Riemannian metric. If that can be achieved, we will speak of a ‘geodesic extension’ of the nonholonomic dynamics (even though this question seems to be dubbed the kinetic Lagrangianization problem in the conclusions of the paper [3]).

In Section 2 we recall the most elementary concepts in this context, in an attempt to keep the paper self-contained. Here it is important to notice that we will use a specific technique throughout the paper. Indeed, to suppress the use of long expressions, and to gain ready geometric insight, it will be advantageous to express the intrinsic and coordinate-independent tools that we will use in terms of a so-called anholonomic frame of vector fields. In particular, we will often choose the frame in such a way that the vanishing of a part of its corresponding quasi-velocities represents the constraints and its distribution \mathcal{D} . In the last sections, it will be of further advantage to specify the frame in such a way that it also follows the orthogonal decomposition of the tangent spaces by a given Riemannian metric.

In Section 3 we will first define the notion of a SODE extension of a nonholonomic system as a set of (only) second-order differential equations whose solutions set contains the nonholonomic trajectories. The name comes from [3], but the idea can already be found in e.g. [8] as ‘Newton laws with the restriction property’ and, implicitly, in [9] where the SODE extensions are called ‘associated systems’. In the first part of the paper, we will show that such SODE extensions always exist. We will give two natural constructions of a SODE extension in Propositions 1 and 2. We will motivate our constructions by showing in Proposition 3 that they represent, in fact, a generalization of the well-known nonholonomic connection (see e.g. [24, 13]) to the level of arbitrary Lagrangians. We will say that a nonholonomic system is purely kinetic, if its potential V is constant and if its kinetic energy can be derived from a Riemannian metric g . Specifically for a purely kinetic nonholonomic systems, there is also a third construction of a SODE extension at the end of Section 6.

From Section 4 onwards, we restrict the attention to only purely kinetic nonholonomic systems. We will investigate the conditions under which the Riemannian metric g of the kinetic energy can be modified to a metric \hat{g} , such that the nonholonomic trajectories become part of its set of geodesics (and therefore also of the Euler-Lagrange equations that can be associated to the Lagrangian \hat{L} of the kinetic energy \hat{g}). The specific kind of geodesic extension we envisage here

will be such that the constrained Lagrangian $L_C = L|_{\mathcal{D}}$ remains preserved. This means that we will assume that the restrictions of the Riemannian metrics g and \hat{g} to the constraint distribution \mathcal{D} are the same. We will say, in that case, that \hat{g} is a \mathcal{D} -preserving modification of g . We are able to show in Lemma 2 that this has the advantage that the question narrows to the search for a suitable 1-form (with components θ_i along the distribution \mathcal{D}) that satisfies a set of algebraic and PDE conditions. If these have a solution, we show in Proposition 4 that we can always find a Riemannian metric \hat{g} that satisfies our purposes. In Section 5 we relate (in Propositions 5 and 6) the conditions of Lemma 2 to the kind of SODE extensions we had encountered in Section 3.

The exponential map of a Riemannian metric plays an important role in differential geometry and geometric mechanics, for example in results concerning Jacobi fields, conjugate points, variations of the energy and focal points. In this paper we will also address the search for a geodesic extension from the viewpoint of the so-called nonholonomic exponential map. This concept has been recently introduced in the paper [2], and further developed in [3]. There, among other results, it was used to show that the nonholonomic trajectories of a purely kinetic system, originating from a fixed point q , can be interpreted as the radial geodesics of a Riemannian metric on a submanifold. This Riemannian metric should not be confused with the (full) Riemannian metric \hat{g} we are looking for, since it (and the submanifold on which it is defined) is different for each different point q . Notwithstanding, like [3], we will be inspired here by the same underlying result from Riemannian geometry: the so-called Gauss Lemma (see e.g. [18]). This Lemma states that the exponential map of a Riemannian metric at q is a radial isometry. In Proposition 7 we show that the algebraic condition of Lemma 2 follows, as soon as the exponential map and the sought-for modified metric \hat{g} satisfy a certain Gauss-type condition. Next, in Proposition 8 we show that also the PDE condition can be interpreted as an infinitesimal condition. Along each nonholonomic trajectory, it can be seen to be equivalent with a conservation law that involves the parallel transport along a linear connection whose quadratic spray is, in fact, again a certain SODE extension. For these reasons, we will call the algebraic and PDE conditions infinitesimal Gauss conditions.

In Section 7 we investigate the presence of symmetry. Chaplygin nonholonomic systems are such that the Lagrangian and the constraints are both invariant under the action of a symmetry Lie group G . Moreover, it is assumed that $Q \rightarrow Q/G$ is a principal fibre bundle and that the constraint distribution \mathcal{D} is the horizontal distribution of a principal connection. We show in Proposition 10 that the conditions of Lemma 2 can be simplified, and that the geodesic extension problem can be stated as the search for linear first integrals that satisfy some further algebraic conditions.

We illustrate the results at the end of the paper with the example of a disk that rolls vertically without slipping, and we compare the metrics we find with those that had already appeared in the literature for this example [3, 21, 9].

2 Tangent bundle geometry, Lagrangian systems and nonholonomic systems

We will assume throughout that Q is an n -dimensional real differentiable manifold, with local coordinates (q^α) . We will assume that $\{X_\alpha\}$ represents an anholonomic frame of vector fields on Q . The adjective anholonomic means here that the Lie brackets of these vector fields do not necessarily vanish, and satisfy the relation $[X_\alpha, X_\beta] = R_{\alpha\beta}^\gamma X_\gamma$. We will call the components v^α

in the decomposition $v_q = v^\alpha X_\alpha(q)$ the quasi-velocities of a tangent vector $v_q \in T_q Q$. We may then use (q^α, v^α) as (non-standard) coordinates on the tangent manifold TQ , where most of the objects of interest live.

Any vector field X on Q can be lifted to vector fields on TQ , by means of the complete and vertical lift, respectively: If $X = X^\alpha \partial / \partial q^\alpha$, then in natural coordinates $(q^\alpha, \dot{q}^\alpha)$,

$$X^C = X^\alpha \frac{\partial}{\partial q^\alpha} + \dot{q}^\beta \frac{\partial X^\alpha}{\partial q^\beta} \frac{\partial}{\partial \dot{q}^\alpha} \quad \text{and} \quad X^V = X^\alpha \frac{\partial}{\partial \dot{q}^\alpha}.$$

One may easily verify that, if $\{X_\alpha\}$ is a basis for $\mathcal{X}(Q)$ (the set of vector fields on Q), then $\{X_\alpha^C, X_\alpha^V\}$ forms a basis for $\mathcal{X}(TQ)$. An important property we will frequently use is that, on the quasi-velocities v^α of the frame,

$$X_\alpha^V(v^\beta) = \delta_\alpha^\beta, \quad X_\alpha^C(v^\beta) = -R_{\alpha\gamma}^\beta v^\gamma,$$

see e.g. [15] (and the references therein) for more details.

We recall some aspects of the calculus of forms along the tangent bundle projection $\tau : TQ \rightarrow Q$ (see e.g. [25, 29] for a survey). A section X of the pullback bundle $\tau^* TQ = \{(u, v) \in TQ \times TQ \mid \tau(u) = \tau(v)\} \rightarrow TQ$ is said to be a vector field along τ . We denote the set of all these vector fields by $\mathcal{X}(\tau)$. We may interpret any vector field Y on Q as a “basic” vector field $Y \circ \tau$ along τ , but we will no longer make a notational distinction. For that reason, we may express $X \in \mathcal{X}(\tau)$ in the given frame $\{X_\alpha\}$ as $X = \xi^\alpha X_\alpha$, with $\xi^\alpha \in C^\infty(TQ)$. It is easy to see that vertical vector fields on TQ are in 1-1 correspondence with vector fields along τ . Indeed, each $X = \xi^\alpha X_\alpha \in \mathcal{X}(\tau)$ can be vertically lifted to $X^V \in \mathcal{X}(TQ)$, given by

$$X^V = \xi^\alpha X_\alpha^V.$$

In particular, there exist a canonical vector field $\mathbf{T} = v^\alpha X_\alpha \in \mathcal{X}(\tau)$. It corresponds to the section $v_q \mapsto (v_q, v_q)$ of the pullback bundle, and its vertical lift gives the Liouville vector field on TQ .

A vector field Γ on TQ is a second-order ordinary differential equations field (SODE, in short) if all its integral curves $\gamma : I \rightarrow TQ$ are lifted curves, that is of the type $\gamma = \dot{c}$. We call $c : I \rightarrow Q$ a base integral curve of Γ . When expressed in terms of the frame $\{X_\alpha^C, X_\alpha^V\}$, a SODE Γ is of the form

$$\Gamma = v^\alpha X_\alpha^C + F^\alpha X_\alpha^V,$$

for some functions $F^\alpha \in C^\infty(TQ)$. The quasi-velocity components $(v^\alpha(t))$ of the integral curves $\gamma(t)$ of Γ satisfy $\dot{v}^\alpha = F^\alpha$.

A SODE Γ can be used to introduce a horizontal lift $X^H \in \mathcal{X}(TQ)$ of $X \in \mathcal{X}(\tau)$. Its coordinate expression in the anholonomic frame is

$$X^H = \xi^\alpha \left(X_\alpha^C - \Gamma_\alpha^\beta X_\beta^V \right), \quad \Gamma_\alpha^\beta = -\frac{1}{2} \left(X_\alpha^V(F^\beta) + v^\gamma R_{\alpha\gamma}^\beta \right).$$

With this, any vector field Z on TQ can be decomposed into the sum of a horizontal and a vertical lift, $Z = X_1^H + X_2^V$ for $X_1, X_2 \in \mathcal{X}(\tau)$. For any vector field X along τ , we may define

$$[\Gamma, X^V] = -X^H + (\nabla X)^V \quad \text{and} \quad [\Gamma, X^H] = (\nabla X)^H + (\Phi(X))^V.$$

The $(1,1)$ -tensor field Φ along τ , the so-called Jacobi endomorphism, is of no concern for the current paper. Below, we will often use the dynamical covariant derivative ∇ of Γ . It acts as a derivation on $\mathcal{X}(\tau)$. For $f \in C^\infty(TQ)$ and $X \in \mathcal{X}(\tau)$:

$$\nabla(fX) = f\nabla X + \Gamma(f)X \quad \text{and} \quad \nabla X_\alpha = \Gamma_\alpha^\beta X_\beta.$$

The equations of motion for an (unconstrained) Lagrangian systems (with regular Lagrangian $L : TQ \rightarrow \mathbb{R}$) are given by the Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = 0.$$

The solutions of these equations are base integral curves of the Lagrangian vector field $\Gamma_L \in \mathcal{X}(TQ)$, which can be defined by as the unique SODE that satisfies

$$\Gamma_L(X^V(L)) - X^C(L) = 0, \quad \forall X \in \mathcal{X}(Q).$$

The main focus of the paper is on nonholonomic mechanical systems.

Definition 1. *A nonholonomic system on a manifold Q consists of a pair (L, \mathcal{D}) , where $L : TQ \rightarrow \mathbb{R}$ is the Lagrangian of the system and \mathcal{D} is the m -dimensional distribution on TQ that describes the nonholonomic constraints.*

In what follows, we will often use \mathcal{C} , when we interpret the distribution \mathcal{D} specifically as a submanifold of TQ . We will also use k for $n - m$.

The equations of motion follow from considering an extended principle of Hamilton (also called Lagrange-d'Alembert principle, see [7, 13]). The nonholonomic trajectory $c_{v_q}(t) = (q^\alpha(t))$ that starts at $v_q \in \mathcal{D}_q$ is a solution of the equations

$$\begin{cases} \mu_\alpha^i \dot{q}^\alpha = 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = \sum_{i=1}^k \Lambda_i \mu_\alpha^i. \end{cases}$$

The first set of equations represent the nonholonomic constraints. They ensure that the velocity $\dot{c}_{v_q}(t)$ remains in the distribution \mathcal{D} during the motion. In the second set of equations, the functions $\Lambda_i(t)$ are called the Lagrange multipliers and are part of the unknowns.

As before for Lagrangian systems, we will represent the nonholonomic trajectories as the base integral curves of a vector field $\Gamma_{(L, \mathcal{D})}$. Suppose that the vector fields $\{X_a\}$ span \mathcal{D} , and that we complete this set to a frame $\{X_\alpha\} = \{X_a, X_i\}$ for all vector fields on Q . This has the advantage that the equations for the nonholonomic constraints become very simple: if $(v^\alpha) = (v^a, v^i)$ are the quasi-velocities with respect to this frame, then $v_q \in \mathcal{D}_q$ if and only if its quasi-velocities satisfy $v^i = 0$. In this paper, the indices α run from 1 to n , a from 1 to m and i from 1 to k .

We will say that the Lagrangian is regular with respect to \mathcal{D} if the matrix

$$(X_a^V(X_b^V(L)))$$

is non-singular everywhere on TQ . This is actually a small modification from the terminology in [15, 16], where the matrix is only assumed to be non-singular on \mathcal{C} (and see also the regularity requirements in [17, 13]). If satisfied, there will exist a unique vector field $\Gamma_{(L, \mathcal{D})}$ of the type

$$\Gamma_{(L, \mathcal{D})} = v^a X_a^C + f^a X_a^V$$

that satisfies, on \mathcal{C} ,

$$\Gamma_{(L,\mathcal{D})}(X^V(L)) - X^C(L) = 0, \quad \forall X \in \text{Sec}(\mathcal{D}).$$

We will call this vector field the nonholonomic vector field. Given the current frame, it can be completely determined from the relation

$$\Gamma_{(L,\mathcal{D})}(X_a^V(L)) - X_a^C(L) = 0, \quad \text{on } \mathcal{C}.$$

This equation represents a version of the Lagrange-d'Alembert equations where the Lagrangian multipliers have already been eliminated. Once $\Gamma_{(L,\mathcal{D})}$ is determined, the functions $\lambda_i := \Gamma_{(L,\mathcal{D})}(X_i^V(L)) - X_i^C(L)$ on \mathcal{C} play the role of the Lagrangian multipliers, in the current frame.

Let L be a regular Lagrangian. The Hessian g of L , with respect to fibre coordinates, can be thought of as a $(0,2)$ -tensor field along the tangent bundle projection $\tau : TQ \rightarrow Q$ (see [15, 29] for details). To define it, it is enough to state its action along 'basic' vector fields in $\mathcal{X}(\tau)$ (i.e. vector fields on Q):

$$g(X, Y) = X^V(Y^V(L)), \quad X, Y \in \mathcal{X}(Q).$$

and extend it to $\mathcal{X}(\tau)$ by $C^\infty(TQ)$ -linearity. Since there is a 1-1 correspondence between τ -vertical vector fields on TQ and vector fields along τ , we may also think of it as acting on vertical vector fields.

For what follows, it is important to realize that this Hessian is familiar in the case of a purely kinetic Lagrangian, i.e. when $L(v) = \frac{1}{2}G(v, v)$ for a Riemannian metric G on Q . In this case, we may identify the above defined Hessian g of L (interpreted as a $(0,2)$ tensor field along τ) with the 'basic' Riemannian metric G (interpreted as a $(0,2)$ tensor field on Q). We will not make a notational distinction between G and g in that case, and simply write $L(v) = \frac{1}{2}g(v, v)$.

Let's assume now that L is again an arbitrary regular Lagrangian that is also regular with respect to \mathcal{D} . Let g be its Hessian. The pullback bundle $\tau^*\mathcal{D} \rightarrow TQ$ is a vector subbundle of τ^*TQ whose set of sections we will denote by $\mathcal{X}(\tau, \mathcal{D})$. Since L is regular with respect to \mathcal{D} , we can consider the set $\mathcal{X}^g(\tau, \mathcal{D})$ of vector fields along τ that are orthogonal to $\mathcal{X}(\tau, \mathcal{D})$, with respect to g . That is, $X \in \mathcal{X}^g(\tau, \mathcal{D})$ when

$$g(X, Y) = 0, \quad \forall Y \in \mathcal{X}(\tau, \mathcal{D}).$$

With this, we can decompose $\mathcal{X}(\tau)$ as

$$\mathcal{X}(\tau) = \mathcal{X}(\tau, \mathcal{D}) \oplus \mathcal{X}^g(\tau, \mathcal{D})$$

and define two complementary projection operators:

$$P : \mathcal{X}(\tau) \rightarrow \mathcal{X}(\tau, \mathcal{D}) \quad \text{and} \quad Q : \mathcal{X}(\tau) \rightarrow \mathcal{X}^g(\tau, \mathcal{D}).$$

Starting from a frame $\{X_a, X_i\}$ for $\mathcal{X}(Q)$, we can easily construct a frame for $\mathcal{X}(\tau)$ that is adapted to the decomposition. Indeed, if $g_{ab} = g(X_a, X_b)$, $g_{ai} = g(X_a, X_i)$ and $g_{ij} = g(X_i, X_j)$, then g_{ab} has an inverse (because of the assumed regularity with respect to \mathcal{D}). Then, the set $\{X_a, \tilde{X}_i\}$, with

$$\tilde{X}_i = X_i - g^{ab}g_{bi}X_a = X_i + K_i^a X_a$$

is such a frame. The vector fields $\tilde{X}_i \in \mathcal{X}(\tau)$ are, however, not basic vector fields.

Next, we introduce the set $\mathcal{X}^T(TQ)$ of vector fields on TQ that are tangent to the constraint manifold \mathcal{C} , i.e.

$$\mathcal{X}^T(TQ) = \{Z \in \mathcal{X}(TQ) \mid Z(v^i) = 0\}.$$

In view of the relations $X_\alpha^C(v^i) = -R_{\alpha\gamma}^i v^\gamma$ and $X_\alpha^V(v^i) = \delta_\alpha^i$, one easily verifies that the vector fields on TQ , given by

$$\overline{X_i^C} = X_i^C + R_{i\gamma}^j v^\gamma \tilde{X}_j^V, \quad \overline{X_a^C} = X_a^C + R_{a\gamma}^j v^\gamma \tilde{X}_j^V \quad \text{and} \quad X_a^V$$

form a basis for $\mathcal{X}^T(TQ)$. We may augment this basis with the vector fields $\{\tilde{X}_i^V\}$ that span $(\mathcal{X}^g(\tau, \mathcal{D}))^V$ to obtain a full basis of $\mathcal{X}(TQ)$. From this, we can conclude that we can decompose $\mathcal{X}(TQ)$ as

$$\mathcal{X}(TQ) = \mathcal{X}^T(TQ) \oplus (\mathcal{X}^g(\tau, \mathcal{D}))^V$$

and that we can define two complementary projectors p and q ,

$$p : \mathcal{X}(TQ) \rightarrow \mathcal{X}^T(TQ) \quad \text{and} \quad q : \mathcal{X}(TQ) \rightarrow (\mathcal{X}^g(\tau, \mathcal{D}))^V.$$

If $Z = Z^a X_a^C + Z^i X_i^C + \zeta^a X_a^V + \zeta^i X_i^V \in \mathcal{X}(TQ)$, then

$$p(Z) = Z^a \overline{X_a^C} + Z^i \overline{X_i^C} + (\zeta^a - \zeta^i K_i^a) X_a^V \quad \text{and} \quad q(Z) = (-Z^a R_{a\gamma}^j v^\gamma - Z^i R_{i\gamma}^j v^\gamma + \zeta^j) \tilde{X}_j^V.$$

3 Extensions of the nonholonomic vector field to a SODE

In what follows, we will assume throughout that the Lagrangian L is both regular and regular with respect to \mathcal{D} . Let $c_{v_q}(t)$ be the nonholonomic trajectory that starts at $v_q \in \mathcal{D}_q$. We formalize the notion of a SODE extension (implicit in [2, 3]) in the following definition.

Definition 2. A SODE Γ is a SODE extension of the nonholonomic vector field $\Gamma_{(L, \mathcal{D})}$, if each nonholonomic trajectory $c_{v_q}(t)$ is a base integral curve of Γ .

An equivalent way to define a SODE extension is that it should satisfy

$$\Gamma|_{\mathcal{D}} = \Gamma_{(L, \mathcal{D})}.$$

It's useful to see this property translated into terms of a frame $\{X_\alpha\} = \{X_a, X_i\}$ on Q whose first elements $\{X_a\}$ span \mathcal{D} , and whose corresponding quasi-velocities are $(v^\alpha) = (v^a, v^i)$. The SODE Γ and the vector field $\Gamma_{(L, \mathcal{D})}$ can then be represented, respectively, by

$$\Gamma = v^\alpha X_\alpha^C + F^\alpha X_\alpha^V \quad \text{and} \quad \Gamma_{(L, \mathcal{D})} = v^a X_a^C + f^a X_a^V.$$

Let $v^a(t)$ be such that $\dot{v}^a = f^a$. From the definition, Γ is a SODE extension if the curve $(v^a(t), v^i(t) = 0)$ is an integral curve of Γ , i.e. a solution of $\dot{v}^a = F^a$ and $\dot{v}^i = F^i$. This will be the case, if and only if,

$$F^a|_{\mathcal{C}} = f^a, \quad \text{and} \quad F^i|_{\mathcal{C}} = 0.$$

For the rest of the paper it is important to realize that the Lagrangian vector field Γ_L of L is *not* a SODE extension of $\Gamma_{(L, \mathcal{D})}$. The SODE $\Gamma_L = v^\alpha X_\alpha^C + F^\alpha X_\alpha^V$ is uniquely determined by the relations

$$\Gamma_L(X_a^V(L)) - X_a^V(L) = 0, \quad \Gamma_L(X_i^V(L)) - X_i^V(L) = 0.$$

From the first set of equations we get, in particular on \mathcal{C} ,

$$v^b X_b^C(X_a^V(L)) + F^b|_{\mathcal{C}} X_b^V(X_a^V(L)) + F^i|_{\mathcal{C}} X_i^V(X_a^V(L)) - X_a^C(L) = 0.$$

On the other hand, on \mathcal{C} , the nonholonomic vector field $\Gamma_{(L,\mathcal{D})} = v^a X_a^C + f^a X_a^V$ satisfies:

$$\Gamma_{(L,\mathcal{D})}(X_a^V(L)) - X_a^C(L) = 0,$$

which becomes when written in full

$$v^b X_b^C(X_a^V(L)) + f^b X_b^V(X_a^V(L)) - X_a^C(L) = 0.$$

In the assumed regularity with respect to \mathcal{D} , this last equation determines the coefficients f^b of the nonholonomic vector field. This means that the relation between the forces F^a , F^i and f^a is

$$f^a = F^a|_{\mathcal{C}} - K_i^a F^i|_{\mathcal{C}} \quad \text{on } \mathcal{C}.$$

This is not the required property for Γ_L to be a SODE extension of $\Gamma_{(L,\mathcal{D})}$, but it is a useful property for what follows.

We now show that a SODE extension of $\Gamma_{(L,\mathcal{D})}$ always exists. The construction we present below is, in essence, also contained in [16] (Proposition 2) and [17] (formulated as Proposition 2.1 in [2]).

Proposition 1. *The vector field $\Gamma^1 = p(\Gamma_L)$ is a SODE extension of the nonholonomic vector field $\Gamma_{(L,\mathcal{D})}$. It can, equivalently, be defined as the unique SODE Γ^1 which is tangent to the constraints, and whose difference from Γ_L is g -orthogonal.*

Proof. If we rewrite Γ_L in terms of the basis $\{\tilde{X}_i^V, \overline{X}_i^C, \overline{X}_a^C, X_a^V\}$ we get

$$\begin{aligned} \Gamma^1 &= p(\Gamma_L) = v^a \overline{X}_a^C + v^i \overline{X}_i^C + (F^a - F^i K_i^a) X_a^V \\ &= v^a (X_a^C + R_{a\gamma}^j v^\gamma \tilde{X}_j^V) + v^i (X_i^C + R_{i\gamma}^j v^\gamma \tilde{X}_j^V) + (F^a - F^i K_i^a) X_a^V \\ &= v^a X_a^C + v^i X_i^C + v^\mu R_{\mu\gamma}^j v^\gamma \tilde{X}_j^V + (F^a - F^i K_i^a) X_a^V \\ &= v^\alpha X_\alpha^C + (F^a - F^i K_i^a) X_a^V + 0 X_i^V. \end{aligned}$$

From the first term we may conclude that Γ^1 is indeed a SODE. When restricted to \mathcal{C} , we know that $F^a - F^i K_i^a$ coincides with f^a . Therefore, we can conclude that $p(\Gamma_L)$ is a SODE extension.

To prove the second statement, we set $\Gamma^1 = \Gamma_L + U^V = v^\alpha X_\alpha^C + F^a X_a^V + U^\alpha X_\alpha^V$. From the first condition, $\Gamma^1(v^i) = 0$, we obtain $U^i = -F^i$. The second condition can be expressed as $U \in (\mathcal{X}^g(\tau, \mathcal{D}))^V$, and thus $U^b = K_i^b U^i = -K_i^b F^i$. With this we see that, again, $\Gamma^1 = v^\alpha X_\alpha^C + (F^a - F^i K_i^a) X_a^V$. \square

In the next paragraph, we give a second construction of a SODE extension and show how it appears in the literature (in a special case).

Proposition 2. *Let ∇ be the dynamical covariant derivative of Γ_L and Q the projection on $\mathcal{X}^\perp(\tau, \mathcal{D})$, defined by g . The vector field*

$$\Gamma^2 = \Gamma_L - [\nabla(Q(\mathbf{T}))]^V$$

is a SODE extension of the nonholonomic vector field $\Gamma_{(L,\mathcal{D})}$.

Proof. For the canonical vector field \mathbf{T} along τ we can write $\mathbf{T} = v^a X_a + v^i X_i = \tilde{v}^a X_a + \tilde{v}^i \tilde{X}_i$ with $\tilde{v}^a = v^a + v^i K_i^a$ and $\tilde{v}^i = v^i$. Remark that if $v^i = 0$ (on \mathcal{C}) then $\tilde{v}^i = 0$ and $\tilde{v}^a = v^a$. From this, $Q(\mathbf{T}) = v^i \tilde{X}_i$ and

$$\nabla(Q(\mathbf{T})) = \nabla(v^i \tilde{X}_i) = \Gamma_L(v^i) \tilde{X}_i + v^i \nabla \tilde{X}_i = F^i \tilde{X}_i + v^i \nabla \tilde{X}_i.$$

The last term doesn't really concern us, since it vanishes on \mathcal{C} . Then,

$$\Gamma^2 = v^\alpha X_\alpha^C + F^a X_a^V + F^i X_i^V - F^i \tilde{X}_i^V - v^i (\nabla \tilde{X}_i)^V = v^\alpha X_\alpha^C + (F^a - F^i K_i^a) X_a^V - v^i (\nabla \tilde{X}_i)^V,$$

from which it is clear that $\Gamma^2|_{\mathcal{C}} = \Gamma_{L,\mathcal{D}}$. \square

The above construction is actually familiar, in the case of a purely kinetic Lagrangian.

Definition 3. A nonholonomic system (L, \mathcal{D}) is purely kinetic if its Lagrangian is of the type $L(v) = \frac{1}{2}g(v, v)$, for a Riemannian metric g on Q .

We have already remarked before that, in this case, the Hessian of L (interpreted as a (0,2) tensor field along τ) is directly related to the Riemannian metric g . Since g is a Riemannian metric, and therefore positive-definite, L is both regular and regular with respect to \mathcal{D} . We first derive an expression for Γ_L and $\Gamma_{(L,\mathcal{D})}$. Next, we introduce the nonholonomic connection ∇^{nh} , and we show that the SODE extension Γ^2 is in fact the quadratic spray of ∇^{nh} .

We may use the Riemannian metric g to define an orthogonal complement \mathcal{D}^g of \mathcal{D} in TQ . With a slight abuse of notation, we will write $P : \mathcal{X}(Q) \rightarrow \text{Sec}(\mathcal{D})$ and $Q : \mathcal{X}(Q) \rightarrow \text{Sec}(\mathcal{D}^g)$ for the two corresponding projections, since these operators can clearly be extended to their counterparts we had encountered in the previous section (with the same notation) on $\mathcal{X}(\tau)$.

Many papers make use of the so-called nonholonomic connection ∇^{nh} (see e.g. [24, 13]). It can be defined by the expression

$$\nabla_X^{nh} Y = P(\nabla_X^g Y) + \nabla_X^g(Q(Y)).$$

Herein, ∇^g is the Levi-Civita connection of g .

We will assume (until the end of the paper) that the frame $\{X_\alpha\} = \{X_a, X_i\}$ is chosen in such a way that $\{X_a\}$ represents a frame for \mathcal{D} and that $\{X_i\}$ is a frame for \mathcal{D}^g . This means that, from now on, $g_{ai} = g(X_a, X_i) = 0$. As before, we denote the quasi-velocities w.r.t. this frame by $(v^\alpha) = (v^a, v^i)$.

It is well-known that the Levi-Civita connection of g satisfies the Koszul-formula

$$2g(\nabla_X^g Y, Z) = Xg(Y, Z) + Y(g(X, Z)) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X).$$

If we write $\nabla_{X_a}^g X_b = \Gamma_{ab}^c X_c + \Gamma_{ab}^i X_i$, $[X_a, X_b] = R_{ab}^c X_c + R_{ab}^i X_i$, and so forth, we get by taking $X = X_a, Y = X_b, Z = X_c$ (among other choices)

$$2g_{cd}\Gamma_{ab}^d = X_a(g_{bc}) + X_b(g_{ac}) - X_c(g_{ab}) + g_{dc}R_{ab}^d + g_{db}R_{ca}^d + g_{da}R_{cb}^d.$$

After multiplying both sides by $v^a v^b$ and canceling a factor 2, we obtain

$$g_{cd}\Gamma_{ab}^d v^a v^b = X_a(g_{bc}) v^a v^b - \frac{1}{2}X_c(g_{ab}) v^a v^b - g_{db}R_{ac}^d v^a v^b.$$

Lemma 1. For a purely kinetic Lagrangian $L = \frac{1}{2}g_{\alpha\beta}v^\alpha v^\beta = \frac{1}{2}(g_{ij}v^i v^j + g_{ab}v^a v^b)$ the Lagrangian vector field is the geodesic spray Γ_g of the Levi-Civita connection. It is given by the expression

$$\Gamma_L = \Gamma_g = v^\alpha X_\alpha^C - \Gamma_{\beta\gamma}^\alpha v^\beta v^\gamma X_\alpha^V.$$

The nonholonomic vector field is given by

$$\Gamma_{(L,\mathcal{D})} = v^d X_d^C - \Gamma_{ab}^d v^a v^b X_d^V.$$

The spray of the nonholonomic connection ∇^{nh} is of the form

$$\Gamma_{\nabla^{nh}} = v^\gamma X_\gamma^C - \left(\Gamma_{ab}^c v^a v^b + 2\Gamma_{aj}^c v^a v^j + \Gamma_{ib}^c v^i v^b + 2\Gamma_{ij}^c v^i v^j \right) X_c^V - \left(\Gamma_{aj}^k v^a v^j + \Gamma_{ij}^k v^i v^j \right) X_k^V.$$

Proof. We only prove the second expression, since the expression for the first can be found by a similar reasoning.

Recall that on \mathcal{C} all $v^i = 0$ and that $X_d^C(v^b) = -R_{de}^b v^e$ and $X_d^V(v^b) = \delta_d^b$. If we set $\Gamma_{(L_g,\mathcal{D})} = v^d X_d^C + f^d X_d^V$, the coefficients f^d satisfy on \mathcal{C}

$$\begin{aligned} 0 &= \Gamma_{(L_g,\mathcal{D})}(X_a^V(L)) - X_a^C(L) \\ &= v^d X_d^C(g_{ab}v^b) + f^d X_d^V(g_{ab}v^b) - \frac{1}{2}X_a(g_{ij})v^i v^j - g_{ij}X_a^C(v^i)v^j - \frac{1}{2}X_a(g_{cd})v^c v^d - g_{cd}X_a^C(v^c)v^d \\ &= v^d X_d(g_{ab})v^b - v^d g_{ab}R_{de}^b v^e + f^d g_{ab}\delta_d^b - \frac{1}{2}X_a(g_{cd})v^c v^d + g_{cd}R_{ae}^c v^e v^d \\ &= g_{ab}f^b + \left(X_d(g_{ab})v^d v^b - g_{ab}R_{de}^b v^d v^e - \frac{1}{2}X_a(g_{cd})v^c v^d + g_{cd}R_{ae}^c v^e v^d \right) \\ &= g_{ab}f^b + g_{ab}\Gamma_{ef}^b v^e v^f. \end{aligned}$$

We conclude that, indeed, $f^b = -\Gamma_{ef}^b v^e v^f$.

If, in the frame $\{X_\alpha\}$, the connection coefficients of the nonholonomic connection are given by $\nabla_{X_\alpha}^{nh} X_\beta = \tilde{\Gamma}_{\alpha\beta}^\gamma X_\gamma$, the corresponding quadratic spray can be expressed as $\Gamma_{\nabla^{nh}} = v^\gamma X_\gamma^C - \tilde{\Gamma}_{\alpha\beta}^\gamma v^\alpha v^\beta X_\gamma^V$ (see e.g. [14]). Specifically for the nonholonomic connection, we may compute:

- $\nabla_{X_a}^{nh} X_b = P(\Gamma_{ab}^\gamma X_\gamma) = \Gamma_{ab}^c X_c + 0X_i,$
- $\nabla_{X_a}^{nh} X_j = P\left(\Gamma_{aj}^\gamma X_\gamma\right) + \nabla_{X_a}^g X_j = \Gamma_{aj}^c X_c + \Gamma_{aj}^k X_k + \Gamma_{aj}^c X_c = 2\Gamma_{aj}^c X_c + \Gamma_{aj}^k X_k,$
- $\nabla_{X_i}^{nh} X_j = P\left(\Gamma_{ij}^\gamma X_\gamma\right) + \nabla_{X_i}^g X_j = \Gamma_{ij}^c X_c + \Gamma_{ij}^k X_k + \Gamma_{ij}^c X_c = 2\Gamma_{ij}^c X_c + \Gamma_{ij}^k X_k,$

from which the expression in the statement then follows. \square

We now show that the second construction of a SODE extension is natural in the following way: it represents a generalization of the quadratic spray associated to the nonholonomic connection.

Proposition 3. For a purely kinetic Lagrangian $L(v) = \frac{1}{2}g(v, v)$, the spray $\Gamma_{\nabla^{nh}}$ of the nonholonomic connection is the SODE extension Γ^2 of $\Gamma_{(L,\mathcal{D})}$, i.e.

$$\Gamma_{\nabla^{nh}} = \Gamma_g - [\nabla(Q(\mathbf{T}))]^V.$$

Proof. When one restricts the expression of $\Gamma_{\nabla^{nh}}$ in the statement of Lemma 1 to $v^i = 0$, one easily verifies that $\Gamma_{\nabla^{nh}}$ is indeed a SODE extension of $\Gamma_{(L,\mathcal{D})}$. Besides, we can make the vector field Γ_g appear in the expression of $\Gamma_{\nabla^{nh}}$, as follows:

$$\Gamma_{\nabla^{nh}} = \Gamma_g - \Gamma_{\alpha j}^c v^\alpha v^j X_c^V + \Gamma_{\alpha a}^k v^\alpha v^a X_k^V.$$

In case of the geodesic spray Γ_g , the coefficients of the dynamical covariant derivative are given by $\Gamma_i^\beta = \Gamma_{i\alpha}^\beta v^\alpha$ and $\Gamma_a^\beta = \Gamma_{a\alpha}^\beta v^\alpha$. A more compact way of writing the spray of the nonholonomic connection is therefore

$$\Gamma_{\nabla^{nh}} = \Gamma_g - \Gamma_j^c v^j X_c^V + \Gamma_a^k v^a X_k^V.$$

For any quadratic spray (like Γ_g) is $\nabla \mathbf{T} = 0$ (see e.g. [25]). For the projections P and Q (now interpreted as acting on $\mathcal{X}(\tau)$), we have

$$\begin{aligned} \nabla(Q(\mathbf{T})) &= (\nabla Q)(\mathbf{T}) = \nabla(Q(\mathbf{T})) - Q(\nabla \mathbf{T}) = \nabla(v^i X_i) - Q(\Gamma^g(v^a) X_a + v^a \nabla X_a + \nabla(v^i X_i)) \\ &= P(\nabla v^i X_i) - v^a Q(\nabla X_a) = v^i P(\nabla X_i) - v^a Q(\nabla X_a) = v^i P(\Gamma_i^\beta X_\beta) - v^a Q(\Gamma_a^\beta X_\beta) \\ &= v^i \Gamma_i^b X_b - v^a \Gamma_a^k X_k. \end{aligned}$$

When we take the vertical lift of this expression, it represents the difference between $\Gamma_{\nabla^{nh}}$ and Γ_g , as required. \square

4 Geodesic extensions

In this section we consider only purely kinetic nonholonomic systems. The SODE extension $\Gamma_{\nabla^{nh}}$ (described in Proposition 3) is a quadratic spray, but it is not necessarily the geodesic spray of a (pseudo-)Riemannian metric \hat{g} .

Definition 4. A geodesic extension of a purely kinetic nonholonomic system (L, \mathcal{D}) is a Riemannian metric \hat{g} whose geodesic spray $\Gamma_{\hat{g}}$ is a SODE extension of $\Gamma_{(L,\mathcal{D})}$.

In that case, the nonholonomic trajectories could be interpreted as geodesics with initial velocities in \mathcal{D} . In the rest of the paper, we look at this problem in more detail.

Consider a distribution \mathcal{D} and a Riemannian metric g . We may use the metric g to decompose

$$TQ = \mathcal{D} \oplus \mathcal{D}^\perp.$$

In what follows we will, again, make use of a frame $\{X_a, X_i\}$ that respects this decomposition and of its corresponding quasi-velocities (v^a, v^i) .

If \hat{g} is a pseudo-Riemannian metric (i.e. non-degenerate, but not necessarily positive-definite), the integral curves of the geodesic spray $\Gamma_{\hat{g}}$ of \hat{g} satisfy the equations

$$\begin{cases} \dot{v}^a = -\hat{\Gamma}_{\alpha\beta}^a v^\alpha v^\beta = -\hat{\Gamma}_{bc}^a v^b v^c - \hat{\Gamma}_j^a v^j, \\ \dot{v}^i = -\hat{\Gamma}_{\alpha\beta}^i v^\alpha v^\beta = -\hat{\Gamma}_{ab}^i v^a v^b - \hat{\Gamma}_j^i v^j, \end{cases}$$

where $\hat{\Gamma}_{\beta\gamma}^\alpha$ are the connection coefficients of the Levi-Civita connection $\nabla^{\hat{g}}$ associated to the metric \hat{g} , in the frame $\{X_a, X_i\}$. For our purpose, the terms with a factor v^j are of no special interest, since they vanish after restriction to \mathcal{C} . Recall that these equations are equivalent with the Euler-Lagrange equations of the Lagrangian $\hat{L}(v) = \frac{1}{2}\hat{g}(v, v) = \frac{1}{2}(\hat{g}_{ab}v^a v^b + 2\hat{g}_{ai}v^a v^i + \hat{g}_{ij}v^i v^j)$.

When compared to the nonholonomic equations of motion of (L, \mathcal{D}) , the integral curves of any SODE extension of $\Gamma_{(L, \mathcal{D})}$ should satisfy on \mathcal{C} :

$$\begin{cases} \dot{v}^a = -\Gamma_{bc}^a v^b v^c, \\ \dot{v}^i = 0. \end{cases}$$

We conclude that the pseudo-Riemannian metric \hat{g} we are looking for must be such that its connection coefficients satisfy the conditions

$$(*) \quad \begin{cases} \hat{\Gamma}_{ab}^c v^a v^b = \Gamma_{ab}^c v^a v^b, \\ \hat{\Gamma}_{bc}^i v^b v^c = 0. \end{cases}$$

The factors $\hat{\Gamma}_{ab}^c$ are functions on Q and do not depend on the quasi-velocities v^a . We may also write the conditions $(*)$ in a version without the factors $v^b v^c$. However, we should keep in mind that (even though the Levi-Civita connection is torsionless) the Christoffel symbols are not symmetric in the current anholonomic frame. For example, the last condition of $(*)$ is equivalent with

$$2\hat{\Gamma}_{bc}^i + R_{bc}^i = 0,$$

and similar for the first. For compactness of our formulae, we prefer the first notation.

In what follows, we will make one further assumption:

Definition 5. *A symmetric $(0,2)$ tensor field \hat{g} on Q is a \mathcal{D} -preserving modification of a Riemannian metric g on Q if \hat{g} and g are related in such a way that*

$$\hat{g}|_{\mathcal{D} \times \mathcal{D}} = g|_{\mathcal{D} \times \mathcal{D}}.$$

In terms of the nonholonomic dynamics, this assumption has the interpretation that we are looking for an extension \hat{L} of the mechanical Lagrangian L , where we do not wish to alter the constraint Lagrangian $L_c = L|_{\mathcal{C}} = \hat{L}|_{\mathcal{C}}$. In the current frame $\{X_a, X_i\}$, this assumption means that g and \hat{g} are of the form

$$g = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ij} \end{pmatrix} \quad \text{and} \quad \hat{g} = \begin{pmatrix} g_{ab} & \hat{g}_{ai} \\ \hat{g}_{ai} & \hat{g}_{ij} \end{pmatrix}.$$

In this, for example, (\hat{g}_{ij}) is the matrix that corresponds to the restriction $\hat{g}|_{\mathcal{D}^g \times \mathcal{D}^g}$, i.e. $\hat{g}_{ij} = \hat{g}(X_i, X_j)$.

Recall that the principal restriction $g|_{\mathcal{D} \times \mathcal{D}}$ of a Riemannian metric remains positive-definite, because g is as a whole. Among other, this means that the matrix (g_{ab}) with $g_{ab} = g(X_a, X_b)$ has an inverse (g^{ab}) . Under the current assumptions, this means that there exists also an orthogonal decomposition by \hat{g} ,

$$TQ = \mathcal{D} \oplus \mathcal{D}^{\hat{g}},$$

where $\text{Sec}(\mathcal{D}^{\hat{g}}) = \text{span}\{\hat{X}_i = X_i - \hat{g}_{bi} g^{ab} X_a\}$. In the next proposition, we will not assume from the outset that \hat{g} is a Riemannian metric. The condition, in the Lemma below, that the restriction $\hat{g}|_{\mathcal{D}^{\hat{g}} \times \mathcal{D}^{\hat{g}}}$ is non-degenerate is equivalent with asking the matrix $(\hat{g}(\hat{X}_i, \hat{X}_j)) = (\hat{g}_{ij} - \hat{g}_{ai} g^{ab} \hat{g}_{jb})$ to be invertible.

With these extra assumptions, we will show that we can re-write the conditions $(*)$ as algebraic and PDE conditions in the unknown functions \hat{g}_{ai} , which represents the restriction of \hat{g} to $\mathcal{D} \times \mathcal{D}^g$. Besides, the Lemma below shows that the choice of \hat{g}_{ij} is (almost) completely free.

There is also a formulation of the conditions that solely makes use of unknown functions $\theta_i = \hat{g}_{ai}v^a$. In that case, also the Lagrangian multipliers $\lambda_i = \Gamma_{(L,\mathcal{D})}(X_i^V(L)) - X_i^C(L)$ (restricted to \mathcal{C}) make their appearance in the equations. These are both components of a semi-basic 1-form on TQ . This is a 1-form that vanishes when applied to vertical lifts and which, for this reason, can be fixed by its action on complete lifts. Relying on the decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^g$, we may consider the semi-basic 1-form defined by

$$\theta(X^C) = 0, \quad \forall X \in \text{Sec}(\mathcal{D}), \quad \theta(X^C) = \hat{g}(\mathbf{T}, X), \quad \forall X \in \text{Sec}(\mathcal{D}^g).$$

In the above, we have interpreted the Riemannian metric \hat{g} again as the Hessian of its corresponding Lagrangian, i.e. as a (basic) (0,2) tensor field along τ . We will be mainly interested in $\theta|_{\mathcal{C}}$, with components $\theta_i = \hat{g}_{ai}v^a$, in the dual frame of $\{X_a^C, X_i^C, X_a^V, X_i^V\}$.

Next, we consider the semi-basic 1-form, defined by

$$\lambda(X^C) = 0, \quad \forall X \in \text{Sec}(\mathcal{D}), \quad \lambda(X^C) = \Gamma_{(L,\mathcal{D})}(X^V(L)) - X^C(L), \quad \forall X \in \text{Sec}(\mathcal{D}^g),$$

and we consider its restriction to \mathcal{C} , $\lambda|_{\mathcal{C}}$. Again, its components are the restrictions of the multipliers $\lambda_i = \Gamma_{(L,\mathcal{D})}(X_i^V(L)) - X_i^C(L)$ to \mathcal{C} we had encountered before.

Lemma 2. *Suppose that g is a Riemannian metric and that \hat{g} is a \mathcal{D} -preserving pseudo-Riemannian modification of g whose restriction $\hat{g}|_{\mathcal{D}^g \times \mathcal{D}^g}$ is non-degenerate. Then, the equations (*) are equivalent with the equations*

$$\begin{aligned} (A) \quad & \hat{g}_{bk}R_{ac}^k v^a v^b = 0, \\ (B) \quad & \hat{g}_{di}\Gamma_{ab}^d v^a v^b = (g_{ki}\Gamma_{ab}^k + X_a(\hat{g}_{bi}) - \hat{g}_{bk}R_{ai}^k)v^a v^b, \end{aligned}$$

in the unknown restriction $\hat{g}|_{\mathcal{D} \times \mathcal{D}^g} = (\hat{g}_{ai})$. When written in terms of the 1-form $\theta = (\theta_i)$ the algebraic condition (A) and the PDE condition (B) become

$$\begin{aligned} (A) \quad & \theta_k R_{ac}^k v^a = 0, \\ (B) \quad & \Gamma_{(L,\mathcal{D})}(\theta_i) + \theta_k R_{ia}^k v^a + \lambda_i = 0. \end{aligned}$$

Proof. In the frame $\{X_a, X_i\}$, we have $g_{ai} = 0$. Koszul's formula (see Section 3) for the metric g leads to

$$\begin{cases} 2g_{cd}\Gamma_{ab}^d v^a v^b = (2X_a(g_{bc}) - X_c(g_{ab}) - 2g_{db}R_{ac}^d)v^a v^b, \\ 2g_{ki}\Gamma_{ab}^k v^a v^b = (-X_i(g_{ab}) - 2g_{db}R_{ai}^d)v^a v^b. \end{cases}$$

Likewise, when applied on the metric \hat{g} Koszul's formula gives

$$\begin{cases} 2(\hat{g}_{cd}\hat{\Gamma}_{ab}^d + \hat{g}_{ck}\hat{\Gamma}_{ab}^k)v^a v^b = (2X_a(\hat{g}_{bc}) - X_c(\hat{g}_{ab}) - 2\hat{g}_{db}R_{ac}^d - 2\hat{g}_{bk}R_{ac}^k)v^a v^b, \\ 2(\hat{g}_{di}\hat{\Gamma}_{ab}^d + \hat{g}_{ki}\hat{\Gamma}_{ab}^k)v^a v^b = (2X_a(\hat{g}_{bi}) - X_i(\hat{g}_{ab}) - 2\hat{g}_{db}R_{ai}^d - 2\hat{g}_{bk}R_{ai}^k)v^a v^b. \end{cases}$$

Since $\hat{g}_{ab} = g_{ab}$, we can recognize some terms of the first set in the second to obtain the following identities

$$\begin{cases} 2(g_{cd}\hat{\Gamma}_{ab}^d + \hat{g}_{ck}\hat{\Gamma}_{ab}^k)v^a v^b = (2g_{cd}\Gamma_{ab}^d - 2\hat{g}_{bk}R_{ac}^k)v^a v^b, \\ 2(\hat{g}_{di}\hat{\Gamma}_{ab}^d + \hat{g}_{ki}\hat{\Gamma}_{ab}^k)v^a v^b = (2X_a(\hat{g}_{bi}) + 2g_{ki}\Gamma_{ab}^k - 2\hat{g}_{bk}R_{ai}^k)v^a v^b. \end{cases}$$

When conditions (*) hold, the first identity above leads to condition (A) and the second to condition (B). We only need the extra assumptions in the statement to prove the converse.

If we suppose that the conditions (A) and (B) hold, we get from the identities that

$$\begin{cases} (g_{cd}\hat{\Gamma}_{ab}^d + \hat{g}_{kc}\hat{\Gamma}_{ab}^k)v^av^b = g_{cd}\Gamma_{ab}^dv^av^b, \\ (\hat{g}_{di}\hat{\Gamma}_{ab}^d + \hat{g}_{ki}\hat{\Gamma}_{ab}^k)v^av^b = \hat{g}_{di}\Gamma_{ab}^dv^av^b. \end{cases}$$

We may rewrite the first as $\hat{\Gamma}_{ab}^dv^av^b = (\Gamma_{ab}^d - g^{cd}\hat{g}_{ck}\hat{\Gamma}_{ab}^k)v^av^b$. When plugged in the second, this gives $(-\hat{g}_{di}g^{cd}\hat{g}_{ck} + \hat{g}_{ki})\hat{\Gamma}_{ab}^kv^av^b = 0$. Under the current assumptions, we get indeed $\hat{\Gamma}_{ab}^kv^av^b = 0$ and therefore also $\hat{\Gamma}_{ab}^dv^av^b = \Gamma_{ab}^dv^av^b$.

For the statements involving θ_i , it is clear that we may immediately rewrite (A) in the desired form. For (B), recall first that, on \mathcal{C} ,

$$\begin{aligned} \lambda_i &= \Gamma_{(L,\mathcal{D})}(X_i^V(L)) - X_i^C(L) = \Gamma_{(L,\mathcal{D})}(g_{ij}v^j) - \frac{1}{2}X_i(g_{ab})v^av^b - g_{ab}X_i^C(v^a)v^b \\ &= g_{ij}\Gamma_{(L,\mathcal{D})}(v^j) - \frac{1}{2}X_i(g_{ab})v^av^b - g_{ab}R_{ic}^av^b = g_{ki}\Gamma_{ab}^kv^av^b + g_{db}R_{ai}^dv^av^b + g_{ab}R_{ic}^av^c v^b \\ &= -g_{ki}\Gamma_L(v^k). \end{aligned}$$

In the before-last equality, we have again made use of Koszul's formula for the metric g . Condition (B) can then be expressed as

$$\begin{aligned} 0 &= -\hat{g}_{di}\Gamma_{ab}^dv^av^b + g_{ki}\Gamma_{ab}^kv^av^b + X_a(\hat{g}_{bi})v^av^b - \hat{g}_{bk}R_{ai}^kv^av^b \\ &= \hat{g}_{di}\Gamma_{(L,\mathcal{D})}(v^d) - g_{ki}\Gamma_L(v^k) + \Gamma_{(L,\mathcal{D})}(\hat{g}_{bi})v^b - \hat{g}_{bk}R_{ai}^kv^av^b \\ &= \Gamma_{(L,\mathcal{D})}(\hat{g}_{bi}v^b) - g_{ki}\Gamma_L(v^k) - \hat{g}_{bk}R_{ai}^kv^av^b \\ &= \Gamma_{(L,\mathcal{D})}(\hat{g}_{bi}v^b) + \lambda_i - \hat{g}_{bk}v^b R_{ai}^kv^a \\ &= \Gamma_{(L,\mathcal{D})}(\theta_i) + \lambda_i + \theta_k R_{ia}^kv^a. \end{aligned}$$

□

The next proposition shows that the problem of finding a geodesic extension by means of a Riemannian metric is solved, once we have found a solution for the algebraic condition (A) and the PDE equation (B).

Proposition 4. *Consider the equations (A) and (B) from the previous Lemma.*

1. *If \hat{g} is a Riemannian metric that is a \mathcal{D} -preserving modification of g and a geodesic extension of (L, \mathcal{D}) , then its restriction $\hat{g}|_{\mathcal{D} \times \mathcal{D}^g} = (\hat{g}_{ai})$ satisfies the algebraic equations (A) and the PDE equations (B).*
2. *If \hat{g} is $(0,2)$ -tensor field that is a \mathcal{D} -preserving modification of g and whose restriction $\hat{g}|_{\mathcal{D} \times \mathcal{D}^\perp} = (\hat{g}_{ai})$ satisfies the algebraic equations (A) and the PDE equations (B), then there exists a geodesic extension of (L, \mathcal{D}) by a Riemannian metric.*

Proof. Given what we have said in the proof of Lemma 2, the first statement is obvious. To prove the second, we only need to check that a restriction $\hat{g}|_{\mathcal{D} \times \mathcal{D}^g} = (\hat{g}_{ai})$ (solving (A) and (B)) can always be completed to a full Riemannian metric \hat{g} , by an appropriate choice of the freedom in $\hat{g}|_{\mathcal{D}^\perp \times \mathcal{D}^g} = (\hat{g}_{ij})$. In this proof, we will use the shorthand notation $\hat{g} = \begin{pmatrix} g_{ab} & \hat{g}_{ai} \\ \hat{g}_{ai} & \hat{g}_{ij} \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$. Here, A is a positive-definite $m \times m$ -matrix, B a $m \times k$ -matrix and D a $k \times k$ -matrix.

First, we show that we can choose D such that \hat{g} is a pseudo-Riemannian metric (i.e. non-degenerate). Let $C := B^T A^{-1} B$. Since

$$\det(\hat{g}) = \det(A) \det(D - B^T A^{-1} B) = \det(A) \det(D - C),$$

with $\det(A) \neq 0$, we need to find a symmetric and invertible D in such a way that $\det(D - C) \neq 0$. These requirements can be achieved by considering an $\alpha \neq 0$ which is not an eigenvalue of C and by choosing $D = \alpha I_k$ (where I_k is the identity matrix of order $k \times k$). Since $\det(D - B^T A^{-1} B) \neq 0$ is, in the current notations, equivalent with the condition that the restriction $\hat{g}|_{\mathcal{D}^{\hat{g}} \times \mathcal{D}^{\hat{g}}}$ is non-degenerate, all the conditions of Lemma 2 are now satisfied.

We now check whether we can specify α further, in such a way that \hat{g} is also positive-definite. This means that for every non-zero $v \in T_q Q$ we must have

$$\hat{g}_q(v, v) = g_{ab} v^a v^b + 2\hat{g}_{ai} v^a v^i + \alpha \delta_{ij} v^i v^j > 0.$$

Since g is positive-definite we already know that $g_{ab} v^a v^b > 0$. If we assume, for our convenience, that the frame $\{X_a\}$ of \mathcal{D} is an orthogonal eigenbasis for the symmetric g_{ab} , then we may write this as

$$g_{ab} v^a v^b = \tau_1 (v^1)^2 + \tau_2 (v^2)^2 + \dots + \tau_m (v^m)^2 > 0,$$

where τ_i are the (positive) eigenvalues of g_{ab} . Let $\tau_{\min} = \min\{\tau_1, \tau_2, \dots, \tau_m\} > 0$. Then

$$g_{ab} v^a v^b \geq \tau_{\min} ((v^1)^2 + \dots + (v^m)^2) = \tau_{\min} \delta_{ab} v^a v^b.$$

If we can show that there exists an α such that

$$\tau_{\min} \delta_{ab} v^a v^b + 2\hat{g}_{ai} v^a v^i + \alpha \delta_{ij} v^i v^j > 0$$

for every $(v^a, v^i) \neq (0, 0)$, we also get that

$$\hat{g}_q(v, v) = g_{ab} v^a v^b + 2\hat{g}_{ai} v^a v^i + \alpha \delta_{ij} v^i v^j > 0.$$

In conclusion, we can finish the proof if we show that there exist a value for α for which the matrix $\begin{pmatrix} \tau_{\min} I_m & B \\ B^T & \alpha I_k \end{pmatrix}$ is positive-definite, i.e. all its eigenvalues λ are strictly positive.

If λ is an eigenvalue of the above matrix, it will satisfy

$$\det \begin{pmatrix} (\tau_{\min} - \lambda) I_m & B \\ B^T & (\alpha - \lambda) I_k \end{pmatrix} = 0.$$

We distinguish between two cases. Suppose, first, that $\lambda = \tau_{\min}$, then it is clearly strictly positive. Next, if $\lambda \neq \tau_{\min}$, then it will satisfy the equation:

$$\det((\tau_{\min} - \lambda) I_m) \det((\alpha - \lambda) I_m - B^T \frac{1}{\tau_{\min} - \lambda} I_k B) = 0$$

and therefore also

$$(\tau_{\min} - \lambda)^{m-k} \det((\alpha - \lambda)(\tau_{\min} - \lambda) I_m - B^T B) = 0.$$

Since the first factor is not zero, we see that $\lambda \neq \tau_{\min}$ will be an eigenvalue of $\begin{pmatrix} \tau_{\min} I_m & B \\ B^T & \alpha I_k \end{pmatrix}$ if and only if $\mu = (\alpha - \lambda)(\tau_{\min} - \lambda)$ is an eigenvalue of $B^T B$. This last relation can also be written as

$$\lambda^2 - (\tau_{\min} + \alpha)\lambda - \mu + \tau_{\min}\alpha = 0.$$

Of all eigenvalues μ of $B^T B$, take $\mu_{\max} = \max\{\mu\}$. To end the proof, we will assume that $\alpha \neq 0$ is not an eigenvalue of C , and is such that $\alpha > -\tau_{\min}$ and $\alpha > \frac{\mu_{\max}}{\tau_{\min}}$. In that case, the two values λ that satisfy the above quadratic equation are indeed strictly positive, because their sum and product satisfy, respectively,

$$\tau_{\min} + \alpha > 0, \quad \text{and} \quad -\mu + \tau_{\min}\alpha \geq -\mu_{\max} + \tau_{\min}\alpha > 0.$$

□

In Section 8.1 we will derive and integrate the conditions (A) and (B) explicitly for the example of a vertically rolling disk. We will also provide Riemannian metrics \hat{g} which are geodesic extensions of the nonholonomic system.

The conditions (A) and (B), when written in terms of θ_i are familiar from a completely different context. If we set $\phi_i = -\theta_i$ (and forget that here θ_i is assumed to be linear in the v^a), the conditions

$$\phi_k R_{ac}^k v^a = 0 \quad \text{and} \quad \Gamma_{(L, \mathcal{D})}(\phi_i) + \phi_k R_{ia}^k v^a - \lambda_i = 0$$

also appear in [15] (Corollary 1). The correspondence with our conditions is only ostensibly, however, since there the conditions have a completely different interpretation and role in the theory: there, they are related to conditions for the nonholonomic problem to be consistent with a vakonomic problem. It would lead us too far to explain here deeper the relation between this consistency problem and the geodesic extension problem we are now dealing with.

5 Relation to SODE extensions

Let $(L = \frac{1}{2}g(v, v), \mathcal{D})$ be a purely kinetic nonholonomic system. If we take a basis $\{X_a, X_i\}$ that respects the orthogonal decomposition by g (i.e. with $\text{span}\{X_a\} = \mathcal{D}$ and $\text{span}\{X_i\} = \mathcal{D}^\perp$), we have already seen that the nonholonomic vector field is

$$\Gamma_{(L, \mathcal{D})} = v^a X_a^C + f^a X_a^V = v^a X_a^C - \Gamma_{bc}^a v^b v^c X_a^V.$$

Let \hat{g} be a Riemannian metric with $\hat{g}_{ab} = g_{ab}$. We have already mentioned that we may also use the metric \hat{g} to decompose TQ into $TQ = \mathcal{D} \oplus \mathcal{D}^{\hat{g}}$. A basis $\{\hat{X}_i\}$ that spans $\mathcal{D}^{\hat{g}}$ is given by the vector fields $\hat{X}_i = X_i - g^{ab} \hat{g}_{bi} X_a$. As in Section 3, this decomposition will lead to a decomposition of the type

$$\mathcal{X}(TQ) = \mathcal{X}^T(TQ) \oplus (\mathcal{X}^{\hat{g}}(\tau, \mathcal{D}))^V$$

where $(\mathcal{X}^{\hat{g}}(\tau, \mathcal{D}))^V = \text{span}\{\hat{X}_i^V\}$. Let's denote the projections that correspond to this decomposition by

$$\hat{p} : \mathcal{X}(TQ) \rightarrow \mathcal{X}^T(TQ) \quad \text{and} \quad \hat{q} : \mathcal{X}(TQ) \rightarrow (\mathcal{X}^{\hat{g}}(\tau, \mathcal{D}))^V.$$

The geodesic spray $\Gamma_{\hat{g}}$ (i.e. the Lagrangian vector field $\Gamma_{\hat{L}}$ of $\hat{L}(v) = \frac{1}{2}\hat{g}(v, v)$) is given in terms of the basis $\{X_a, X_i\}$ by

$$\Gamma_{\hat{g}} = v^\alpha X_\alpha^C + \hat{F}^a X_a^V + \hat{F}^i X_i^V = v^\alpha X_\alpha^C - \hat{\Gamma}_{\alpha\beta}^a v^\alpha v^\beta X_a^V - \hat{\Gamma}_{\alpha\beta}^i v^\alpha v^\beta X_i^V,$$

where $\hat{\Gamma}_{\alpha\beta}^\gamma$ are the Christoffel symbols of \hat{g} , as before.

Similarly as in Propositions 1 and 2, we may construct from $\Gamma_{\hat{g}}$ two new SODEs. The first is

$$\Gamma_{\hat{g}}^1 = \hat{p}(\Gamma_{\hat{g}}) = v^\alpha X_\alpha^C + (\hat{F}^a + g^{ab} \hat{g}_{bi} \hat{F}^i) X_a^V + 0 X_i^V.$$

The second is (according to Proposition 3) the quadratic spray of the nonholonomic connection (now the one associated to \hat{g} and \mathcal{D}). To derive its expression, we see that the orthogonal projection $\hat{Q} : \mathcal{X}(\tau) \rightarrow \mathcal{X}^{\hat{g}}(\tau, \mathcal{D})$ has the property

$$\hat{Q}(\mathbf{T}) = \hat{Q}(v^a X_a + v^i X_i) = \hat{Q}((v^a + v^i g^{ab} \hat{g}_{bi}) X_a + v^i \hat{X}_i) = v^i \hat{X}_i.$$

With that, we may define

$$\begin{aligned} \Gamma_{\hat{g}}^2 &= \Gamma_{\hat{L}} - (\hat{\nabla}(\hat{Q}(\mathbf{T})))^V \\ &= \Gamma_{\hat{g}} - (\Gamma_{\hat{g}}(v^i) \hat{X}_i + v^i \hat{\nabla}(\hat{X}_i))^V \\ &= v^\alpha X_\alpha^C + (\hat{F}^a + g^{ab} \hat{g}_{bi} \hat{F}^i) X_a^V - v^i (\hat{\nabla}(\hat{X}_i))^V. \end{aligned}$$

In the next proposition, we relate the condition (A) to SODE extensions of $\Gamma_{(L, \mathcal{D})}$ by one of these vector fields.

Proposition 5. *Let g and \hat{g} be two Riemannian metrics which are a \mathcal{D} -preserving modification of each other. The following statements are equivalent:*

- (i) *The condition (A) is satisfied for \hat{g} ,*
- (ii) *The vector field $\Gamma_{\hat{g}}^1$ is a SODE extension of $\Gamma_{(L, \mathcal{D})}$,*
- (iii) *The vector field $\Gamma_{\hat{g}}^2$ is a SODE extension of $\Gamma_{(L, \mathcal{D})}$,*
- (iv) *The nonholonomic vector field $\Gamma_{(L, \mathcal{D})}$ satisfies on \mathcal{C} ,*

$$\Gamma_{(L, \mathcal{D})}(X^V(\hat{L})) - X^C(\hat{L}) = 0, \quad \forall X \in \text{Sec}(\mathcal{D}).$$

Proof. We have already remarked in the proof of Lemma 2 that the Christoffel symbols $\hat{\Gamma}_{\alpha\beta}^\gamma$ of \hat{g} and $\Gamma_{\alpha\beta}^\gamma$ of g are always related as follows:

$$2(g_{cd} \hat{\Gamma}_{ab}^d + \hat{g}_{kc} \hat{\Gamma}_{ab}^k) v^a v^b = (2g_{cd} \Gamma_{ab}^d - \hat{g}_{bk} R_{ac}^k - \hat{g}_{ak} R_{bc}^k) v^a v^b.$$

The condition (A) is satisfied if, and only if, we have

$$2(g_{cd} \hat{\Gamma}_{ab}^d + \hat{g}_{kc} \hat{\Gamma}_{ab}^k) v^a v^b = (2g_{cd} \Gamma_{ab}^d) v^a v^b.$$

We may, equivalently, rewrite this as

$$g_{dc} \hat{F}^d|_{\mathcal{C}} + \hat{g}_{ck} \hat{F}^k|_{\mathcal{C}} = g_{dc} f^d.$$

Since the matrix g_{cd} is invertible we get

$$\hat{F}^d|_{\mathcal{C}} + g^{dc} \hat{g}_{ck} \hat{F}^k|_{\mathcal{C}} = f^d.$$

Since the steps are reversible, this condition is in fact equivalent with the condition in (i). We continue the proof by showing that also all of the other statements are equivalent to this.

Indeed, given that $v^i = 0$ on \mathcal{C} , it is easy to see from the expressions we had obtained before that

$$\Gamma_{\hat{g}}^1|_{\mathcal{C}} = v^a X_a^C + f^a X_a^V = \Gamma_{(L, \mathcal{D})}$$

if and only if $\hat{F}^d|_{\mathcal{C}} + g^{dc}\hat{g}_{ck}\hat{F}^k|_{\mathcal{C}} = f^d$, and similarly for $\Gamma_{\hat{g}}^2$.

The statement (iv) says that the coefficients f^a of the nonholonomic vector field $\Gamma_{(L,\mathcal{D})}$ satisfy, on \mathcal{C} ,

$$\begin{aligned} 0 &= \Gamma_{(L_g,\mathcal{D})}(X_a^V(\hat{L})) - X_a^C(\hat{L}) \\ &= v^b X_b^C X_a^C(\hat{L}) + f^b X_b^V X_a^V(\hat{L}) - X_a^C(\hat{L}) \\ &= v^b X_b^C X_a^C(\hat{L}) + f^b g_{ab} - X_a^C(\hat{L}). \end{aligned}$$

Now, recall that the Lagrangian vector field $\Gamma_{\hat{L}} = \Gamma_{\hat{g}}$ always satisfies

$$\Gamma_{\hat{L}}(X^V(\hat{L})) - X^C(\hat{L}) = 0, \quad \forall X \in \mathcal{X}(Q).$$

In particular, we have (after restriction to \mathcal{C})

$$\begin{aligned} 0 &= \Gamma_{\hat{L}}(X_a^V(\hat{L})) - X_a^C(\hat{L}) = v^b X_b^C X_a^V(\hat{L}) + \hat{F}^b|_{\mathcal{C}} X_b^V X_a^V(\hat{L}) + F^i|_{\mathcal{C}} X_i^V X_a^V(\hat{L}) - X_a^C(\hat{L}) \\ &= v^b X_b^C X_a^V(\hat{L}) + \hat{F}^b|_{\mathcal{C}} g_{ab} + F^i|_{\mathcal{C}} \hat{g}_{ai} - X_a^C(\hat{L}). \end{aligned}$$

If we subtract this from the previous, the statement in (iv) is indeed equivalent with $\hat{F}^b|_{\mathcal{C}} + F^i|_{\mathcal{C}} \hat{g}_{ai} g^{ab} = f^b$. \square

We have already shown in Proposition 4 that $\Gamma_{\hat{g}}$ is a SODE extension of $\Gamma_{(L_g,\mathcal{D})}$ if and only if (A) and (B) are satisfied. From the previous we know that, if only (A) is satisfied, it is not enough to guarantee that $\Gamma_{\hat{g}}$ is a SODE extension: Proposition 5 says that in that case only $\Gamma_{\hat{g}}^1$ or $\Gamma_{\hat{g}}^2$ is one. Therefore, we need to add the condition that also the restriction $\hat{\nabla}(\hat{Q}(\mathbf{T}))|_{\mathcal{C}}$ vanishes. The proof of the next proposition is now obvious.

Proposition 6. *Let g and \hat{g} be two Riemannian metrics which are a \mathcal{D} -preserving modification of each other. The following statements are equivalent:*

- (i) $\Gamma_{\hat{g}}$ is a SODE extension of $\Gamma_{(L,\mathcal{D})}$,
- (ii) The conditions (A) and (B) are satisfied,
- (ii) (A) is satisfied and $\hat{\nabla}(\hat{Q}(\mathbf{T}))|_{\mathcal{C}} = 0$,
- (iii) $\Gamma_{\hat{g}}|_{\mathcal{C}} = \Gamma_{\hat{g}}^1|_{\mathcal{C}} = \Gamma_{\hat{g}}^2|_{\mathcal{C}}$.

6 Infinitesimal Gauss-type conditions

Throughout this section, given $v_q \in T_q Q$, we will often make use of the canonical identification of $T_{v_q}(T_q Q)$ as the tangent space $T_q Q$:

$$(w_q)_{v_q}^V = w^\alpha X_\alpha^V(v_q) \longleftrightarrow w_q = w^\alpha X_\alpha(q).$$

Let $\phi_t^{nh} : \mathcal{D} \rightarrow \mathcal{D}$ be the flow of the nonholonomic vector field $\Gamma_{(L,\mathcal{D})}$. The nonholonomic exponential map is defined in [2] as

$$\exp_q^{nh}(v_q) = \tau(\phi_1^{nh}(v_q)),$$

where $\tau : TQ \rightarrow Q$ is the tangent bundle projection. Among other properties, it has been shown in [3] specifically for a purely kinetic nonholonomic system $(L(v) = \frac{1}{2}g(v, v), \mathcal{D})$ that for each fixed $q \in Q$, there exists a submanifold M_q of Q (with $q \in M_q$) such that the nonholonomic exponential map restricts to a diffeomorphism $\exp_q^{nh} : U_0 \subset \mathcal{D}_q \rightarrow M_q$, where U_0 is a starshaped open subset of \mathcal{D}_q around $0_q \in U_0$. Moreover, $\exp_q^{nh}(0_q) = q$ and the morphism $T_{0_q}(\exp_q^{nh}) : \mathcal{C}_q \rightarrow T_q Q$ (under the identification between \mathcal{C}_q and $T_{0_q} U_0$) is the canonical inclusion. Important for us, is that it has also been shown in [3] that, for every $v_q \in U_0$,

$$\exp_q^{nh}(tv_q) = c_{v_q}(t),$$

with $c_{v_q} : [0, 1] \rightarrow M_q$ the nonholonomic trajectory satisfying $c_{v_q}(0) = q$ and $\dot{c}_{v_q}(0) = v_q$.

In what follows we will use a description of the nonholonomic exponential in quasi-velocities. Given a frame $\{X_\alpha\} = \{X_a, X_i\}$, adapted to $TQ = \mathcal{D} \oplus \mathcal{D}^g$, as before, we set in local coordinates

$$\exp_q^{nh} : v_q = v^a X_a(q) \mapsto \tilde{q}^\alpha = \epsilon^\alpha(v).$$

We will also use $((E_\beta^\alpha)(v))$ for the matrix representation of the tangent map of \exp_q^{nh} in v_q in quasi-velocities: For $u_q \in \mathcal{D}_q$, we set

$$T_{v_q}(\exp_q^{nh})(u_q) = u^a \left(E_a^b(v) X_b(\tilde{q}) + E_a^i(v) X_i(\tilde{q}) \right) \in T_{\tilde{q}} Q.$$

Lemma 3. *At $0_q \in T_q Q$, we have $E_a^b(0) = \delta_a^b$, $E_a^i(0) = 0$,*

$$\frac{\partial E_a^d}{\partial v^b}(0) = \frac{1}{2} R_{ab}^d(q) - \frac{1}{2} (\Gamma_{ab}^d(q) + \Gamma_{ba}^d(q)) \quad \text{and} \quad \frac{\partial E_a^i}{\partial v^b}(0) = \frac{1}{2} R_{ab}^i(q).$$

Proof. The first two properties follow from the fact that, when $v_q = 0_q$, $T_{0_q}(\exp_q^{nh})$ can be regarded as the inclusion. We will compute the other expressions in two steps.

Let $v_q = v^a X_a(q) \in T_q Q$. Consider a nonholonomic trajectory $c_{v_q}(t)$ and its derivative $\dot{c}_{v_q}(t) = V^a(t) X_a(c_{v_q}(t)) \in \mathcal{D}_{c_{v_q}(t)}$. The functions $V^a(t)$ then satisfy $\dot{V}^a(t) = -\Gamma_{bc}^a(c_{v_q}(t)) V^b(t) V^c(t)$. Since in the purely kinetic case $c_{v_q}(t) = \exp(tv_q)$, we may also write this as

$$\dot{c}_{v_q}(t) = E_a^\alpha(tv) v^a X_\alpha(c_{v_q}(t)).$$

We get from this

$$E_a^d(tv_q) v^a = V^d(t) \quad \text{and} \quad E_a^i(tv_q) v^a = 0.$$

By applying $\frac{d}{dt}|_{t=0}$ on both sides of both equations we get

$$\frac{\partial E_a^d}{\partial v^\mu}(0) v^\mu v^a = \dot{V}^d(0) = -\Gamma_{ef}^d(q) v^e v^f$$

and after symmetrization

$$\frac{\partial E_a^d}{\partial v^c}(0) + \frac{\partial E_c^d}{\partial v^a}(0) = -(\Gamma_{ac}^d(q) + \Gamma_{ca}^d(q)).$$

Likewise, it follows from $\partial E_a^i / \partial v^\mu(0) v^\mu v^a = \partial E_a^i / \partial v^b(0) v^b v^a = 0$ that $\partial E_a^i / \partial v^b(0) + \partial E_b^i / \partial v^a(0) = 0$.

Now we compute the expression for the differences $\partial E_a^d/\partial v^c(0) - \partial E_c^d/\partial v^a(0)$ and $\partial E_a^i/\partial v^b(0) - \partial E_b^i/\partial v^a(0)$. We know that in natural coordinates and in quasi-velocities, we get (respectively)

$$T_{v_q}(\exp_q^{nh})(u_q) = u^a \frac{\partial \epsilon^\alpha}{\partial v^a}(v) \frac{\partial}{\partial q^\alpha} \big|_{\tilde{q}} = E_a^\beta(v) u^a X_\beta(\tilde{q}).$$

Since $\{\partial/\partial q^\alpha|_{\tilde{q}}\}$ is a basis for $T_{\tilde{q}}Q$ there exists an invertible matrix $(A_\beta^\alpha(\tilde{q}))$ of functions such that $X_\beta(\tilde{q}) = A_\beta^\alpha(\tilde{q})\partial/\partial q^\alpha|_{\tilde{q}}$. Therefore $\partial \epsilon^\alpha/\partial v^a(v) = E_a^\beta(v) A_\beta^\alpha(\epsilon(v))$, and in particular $\partial \epsilon^\alpha/\partial v^a(0) = E_a^\beta(0) A_\beta^\alpha(q) = A_a^\alpha(q)$. When we take a $\partial/\partial v^b$ -derivative on both sides of the first relation, we get

$$\frac{\partial^2 \epsilon^\alpha}{\partial v^a \partial v^b}(v) = \frac{\partial E_a^\beta}{\partial v^b}(v) A_\beta^\alpha(\epsilon(v)) + E_a^\beta(v) \frac{\partial A_\beta^\alpha}{\partial q^\mu}(\epsilon(v)) \frac{\partial \epsilon^\mu}{\partial v^b}(v),$$

and at $v = 0$:

$$\frac{\partial^2 \epsilon^\alpha}{\partial v^a \partial v^b}(0) = \frac{\partial E_a^\beta}{\partial v^b}(0) A_\beta^\alpha(q) + \delta_a^\beta \frac{\partial A_\beta^\alpha}{\partial q^\mu}(q) A_b^\mu(q).$$

If we change in this last equality a and b , and subtract we get the identity

$$0 = \left(\frac{\partial E_a^\beta}{\partial v^b}(0) - \frac{\partial E_b^\beta}{\partial v^a}(0) \right) A_\beta^\alpha(q) + \left(\frac{\partial A_a^\alpha}{\partial q^\mu}(q) A_b^\mu(q) - \frac{\partial A_b^\alpha}{\partial q^\mu}(q) A_a^\mu(q) \right).$$

The last term is, in fact, related to the Lie bracket $R_{ba}^\beta X_\beta = [X_b, X_a] = (A_b^\mu \partial A_a^\alpha / \partial q^\mu - A_a^\mu \partial A_b^\alpha / \partial q^\mu) \partial / \partial q^\alpha$. The difference now becomes

$$0 = \left[\frac{\partial E_a^\beta}{\partial v^b}(0) - \frac{\partial E_b^\beta}{\partial v^a}(0) - R_{ab}^\beta(q) \right] A_\beta^\alpha(q).$$

For $\beta = d$ and $\beta = i$, we get, respectively,

$$\frac{\partial E_a^d}{\partial v^b}(0) - \frac{\partial E_b^d}{\partial v^a}(0) = R_{ab}^d(q), \quad \frac{\partial E_a^i}{\partial v^b}(0) - \frac{\partial E_b^i}{\partial v^a}(0) = R_{ab}^i(q).$$

We conclude that

$$\begin{cases} \frac{\partial E_a^d}{\partial v^b}(0) - \frac{\partial E_b^d}{\partial v^a}(0) = R_{ab}^d(q), & \frac{\partial E_a^i}{\partial v^b}(0) - \frac{\partial E_b^i}{\partial v^a}(0) = R_{ab}^i(q), \\ \frac{\partial E_a^d}{\partial v^c}(0) + \frac{\partial E_c^d}{\partial v^a}(0) = -(\Gamma_{ac}^d(q) + \Gamma_{ca}^d(q)), & \frac{\partial E_a^i}{\partial v^b}(0) + \frac{\partial E_b^i}{\partial v^a}(0) = 0. \end{cases}$$

This leads to the expressions in the statement of the Lemma. \square

We can now relate condition (A) to the Gauss Lemma of Riemannian geometry. Let h be a Riemannian metric on a manifold Q with exponential map \exp_q^h . Suppose that $v_q \in T_q Q$ is such that $\exp_q^h(v_q)$ is defined. The Gauss Lemma says, in essence, that $\exp_q^h : T_q Q \rightarrow Q$ is a radial isometry (under appropriate identification). This means that $\mathcal{H}^0 = (\exp_q^h)^* h$ satisfies the ‘Gauss condition’:

$$\mathcal{H}_{v_q}^0(v_q, u_q) = \mathcal{H}_{0_q}^0(v_q, u_q),$$

for all $u_q \in T_v T_q Q \cong T_q Q$. The expression for \mathcal{H}^0 can equivalently be written as

$$h_{\exp_q^h(v_q)}(T_{v_q} \exp_q^h(v_q), T_{v_q} \exp_q^h(u_q)) = h_q(v_q, u_q).$$

We refer to e.g. [18] for the details.

We do not need the Gauss Lemma as such. In [3] it is shown that a certain Gauss-type condition is necessary and sufficient for the existence of a Riemannian metric g_q^{nh} on the submanifold M_q with the property that the radial kinetic nonholonomic trajectories departing from q are minimizing geodesics of g_q^{nh} . This is also not the property that we will be interested in here, but our result has a similar flavour. In the next proposition we give a sufficient condition under which the condition (A) (that appears in Lemma 2 and Proposition 4) always holds.

Proposition 7. *Let \exp_q^{nh} be the nonholonomic exponential map of a purely kinetic (L, \mathcal{D}) and let \hat{g} be a Riemannian metric on Q which is a \mathcal{D} -preserving modification of g . If $\mathcal{G}^0 = (\exp_q^{nh})^* \hat{g}$ satisfies the following Gauss-type condition*

$$\mathcal{G}_{v_q}^0(v_q, u_q) = \mathcal{G}_{0_q}^0(v_q, u_q), \quad \forall v_q, u_q \in \mathcal{D}_q,$$

or, equivalently,

$$\hat{g}_{\exp_q^{nh}(v_q)} \left(T_{v_q} \left(\exp_q^{nh} \right) (v_q), T_{v_q} \left(\exp_q^{nh} \right) (u_q) \right) = \hat{g}_q(v_q, u_q),$$

then condition (A) is satisfied for \hat{g} .

Proof. The equivalent condition is the first written in full, after having taken into account that the tangent map of the exponential map in zero is the inclusion (with the identification between $T_q Q$ and $T_{v_q}(T_q Q)$).

Given that $v_q, u_q \in \mathcal{D}_q$, we can write them as $v_q = v^a X_a(q)$ and $u_q = u^a X_a(q)$. With this the Gauss condition becomes

$$\hat{g}_{\alpha\beta}(\exp_q^{nh}(v_q)) E_a^\alpha(v) v^a E_b^\beta(v) = g_{ab}(q) v^a.$$

Now, we substitute v_q by tv_q and cancel a factor t on both sides,

$$\hat{g}_{\alpha\beta}(c_{v_q}(t)) E_a^\alpha(tv) v^a E_b^\beta(tv) = g_{ab} v^a,$$

where we have also used $\exp_q^{nh}(tv_q) = c_{v_q}(t)$. If we apply $\frac{d}{dt}|_{t=0}$ on both sides, we obtain

$$\frac{\partial \hat{g}_{\alpha\beta}}{\partial v^\mu}(q) (\dot{c}_{v_q})^\mu(0) E_a^\alpha(0) v^a E_b^\beta(0) + \hat{g}_{\alpha\beta}(q) \frac{\partial E_a^\alpha}{\partial v^\mu}(0) v^\mu v^a E_b^\beta(0) + \hat{g}_{\alpha\beta}(q) E_a^\alpha(0) v^a \frac{\partial E_b^\beta}{\partial v^\mu}(0) v^\mu = 0.$$

In the first term, we may rewrite $(\dot{c}_{v_q})^\mu(0) \partial/\partial q^\mu|_q$ as $v^f X_f(q) + v^i X_i(q) = v^f X_f(q)$, since the vector $v_q = \dot{c}_{v_q}(0) \in \mathcal{D}_q$ has components $v^i = 0$. If we also insert the expressions that we had found in Lemma 3, we get

$$\begin{aligned} 0 &= v^f X_f(\hat{g}_{ab}) v^a + \hat{g}_{db} \left(-(\Gamma_{af}^d + \Gamma_{fa}^d) \right) v^f v^a + \frac{1}{2} \hat{g}_{ad} \left(-(\Gamma_{bf}^d + \Gamma_{fb}^d) + R_{bf}^d \right) v^f v^a + \frac{1}{2} \hat{g}_{ai} R_{bf}^i v^f v^a \\ &= \left(X_f(\hat{g}_{ab}) - \frac{1}{2} \hat{g}_{db} (2\Gamma_{af}^d + 0) - \frac{1}{2} \hat{g}_{ad} (2\Gamma_{fb}^d) + \frac{1}{2} (\hat{g}_{ai} R_{bf}^i + \hat{g}_{fi} R_{ba}^i) \right) v^a v^f \\ &= \left(X_f(\hat{g}_{ab}) - \hat{g}_{db} \Gamma_{af}^d - \hat{g}_{ad} \Gamma_{fb}^d + \frac{1}{4} (\hat{g}_{ai} R_{bf}^i + \hat{g}_{fi} R_{ba}^i) \right) v^a v^f. \end{aligned}$$

Recall first that for the metric g , $\nabla_{X_f}^g g = 0$. In the current frame this gives

$$\left(X_f(g_{ab}) - \hat{g}_{db} \Gamma_{af}^d - g_{ad} \Gamma_{fb}^d \right) v^a v^f = 0.$$

Since we assume that $\hat{g}_{ab} = g_{ab}$, we may use this to see the first terms in the previous expression vanish. What remains is then indeed condition (A) from Lemma 2,

$$(\hat{g}_{ai}R_{bf}^i + \hat{g}_{fi}R_{ba}^i)v^av^f = 0,$$

or, equivalently, $\hat{g}_{ai}R_{bf}^iv^av^f = 0$. □

We end this paragraph with an equivalent characterization of the condition (B). It will be most convenient to come back to the version of (B), in the form where we regard it as a PDE in the unknowns $\theta_i = \hat{g}_{ai}v^a$ (i.e. the restriction of some semi-basic 1-form θ to \mathcal{C}). We now show that (B) can be interpreted as the infinitesimal version of a geometric property along $c_{v_q}(t)$. To do so, we introduce yet another SODE extension $\Gamma_{\bar{\nabla}}$ of $\Gamma_{(L,\mathcal{D})}$, which is the quadratic spray of a linear connection $\bar{\nabla}$ on Q , different from the nonholonomic connection ∇^{nh} . The construction of the connection $\bar{\nabla}$ is inspired by a result that can be found in [28].

We have already introduced the two projection operators $P : TQ \rightarrow \mathcal{D}$ and $Q : TQ \rightarrow \mathcal{D}^g$, related to g . Consider the Levi-Civita connection ∇^g of g . We may use it to define the operator ∇^P as

$$\nabla^P : \text{Sec}(\mathcal{D}) \times \text{Sec}(\mathcal{D}) \rightarrow \mathcal{X}(Q), \quad (X, Y) \mapsto \nabla_X^P Y = P(\nabla_X^g Y).$$

Proposition 2.1 in [28] states that we may extend this to a covariant derivative

$$\bar{\nabla}^P : \mathcal{X}(Q) \times \text{Sec}(\mathcal{D}) \rightarrow \mathcal{X}(Q), \quad (X, Y) \mapsto \nabla_{P(X)}^P Y + P([Q(X), Y]).$$

We can do the same for the projection operator Q . Next, from Theorem 2.2 in [28], we obtain that, for $X, Y \in \mathcal{X}(Q)$, the operator

$$\bar{\nabla} : \mathcal{X}(Q) \times \mathcal{X}(Q) \rightarrow \mathcal{X}(Q), \quad (X, Y) \mapsto \bar{\nabla}_X^P(P(Y)) + \bar{\nabla}_X^Q(Q(Y))$$

is a linear connection on Q . In the current frame, the coefficients of this linear connection are

$$\bar{\nabla}_{X_a} X_b = \Gamma_{ab}^c X_c, \quad \bar{\nabla}_{X_a} X_i = R_{ai}^k X_k, \quad \bar{\nabla}_{X_i} X_a = R_{ia}^c X_c, \quad \bar{\nabla}_{X_i} X_j = \Gamma_{ij}^k X_k,$$

and its corresponding spray is

$$\Gamma_{\bar{\nabla}} = v^\alpha X_\alpha^C + (\Gamma_{ab}^c v^a v^b + R_{ia}^c v^i v^a) X_c^V + (R_{ai}^k v^a v^i + \Gamma_{ij}^k v^i v^j) X_k^V.$$

By restricting it to $v^i = 0$ we can easily see that it is, indeed, a SODE extension of $\Gamma_{(L,\mathcal{D})}$.

Let $c_{v_q}(t)$ be a nonholonomic trajectory, with $\dot{c}_{v_q}(t) = V^a(t)X_a(c_{v_q}(t)) \in \mathcal{D}_{c_{v_q}(t)}$. Since $\Gamma_{\bar{\nabla}}$ is a SODE extension of $\Gamma_{(L,\mathcal{D})}$, we may interpret $c_{v_q}(t)$ as one of its base integral curves. We may then use the linear connection $\bar{\nabla}$ to parallel transport a vector $w_q \in T_q Q$ along $c_{v_q}(t)$. In this way, we obtain a vector field $W(t)$ along $c_{v_q}(t)$ whose components along $\{X_a, X_i\}$ are solutions of the initial value problem

$$\dot{W}^\alpha + \bar{\Gamma}_{\gamma\beta}^\alpha V^\gamma W^\beta = 0, \quad W^\alpha(0) = w^\alpha.$$

For our specific case, we get

$$\begin{cases} \dot{W}^c = -\Gamma_{ab}^c V^a W^b, \\ \dot{W}^i = -R_{aj}^i V^a W^j. \end{cases}$$

The above differential equations are separated. If we take the initial value $w_q \in \mathcal{D}_q^g$, we have $w^a = 0$ and therefore is the solution of the first simply $W^a(t) = 0$. This means that $W(t) \in \mathcal{D}_{c_{v_q}(t)}^g$.

For a 1-form $h(t)$ along the curve $c_{v_q}(t)$, its covariant derivative $\bar{\nabla}_{\dot{c}_{v_q}(t)} h(t)$ is again a 1-form along $c_{v_q}(t)$. It can be defined below by means of its action on vector fields $X(t)$ along $c_{v_q}(t)$:

$$\left(\bar{\nabla}_{\dot{c}_{v_q}(t)} h(t)\right)(X(t)) = \frac{d}{dt} \left((h(t))(X(t)) \right) - (h(t))(\bar{\nabla}_{\dot{c}_{v_q}(t)} X(t)).$$

Consider $\{X^i, X^a\}$ the basis of 1-forms, dual to the basis $\{X_i, X_a\}$ of vector fields. Then $\text{span}\{X^i\}$ is the annihilator of \mathcal{D} , say \mathcal{D}^o . Since $\dot{c}_{v_q}(t)$ has only components in $\mathcal{D}_{c_{v_q}(t)}$, we can compute that, along $c_{v_q}(t)$, the covariant derivative of $h(t) = h_i(t)X^i(c_{v_q}(t)) + h_b(t)X^b(c_{v_q}(t))$ is given by

$$\bar{\nabla}_{\dot{c}_{v_q}(t)} h = \left(\dot{h}_c - \Gamma_{ac}^b V^a h_b\right) X^c + \left(\dot{h}_i - R_{ai}^k V^a h_k\right) X^i.$$

Now, consider the 1-form $\lambda(t) = \lambda(\dot{c}_{v_q}(t))$ along $c_{v_q}(t)$, i.e. the restriction of the semi-basic 1-form λ that represented the Lagrangian multipliers (in Section 4). For this, we can define a unique 1-form $h(t)$ along $c_{v_q}(t)$ that satisfies

$$\bar{\nabla}_{\dot{c}_{v_q}} h(t) = \lambda(t), \quad h(0) = 0_q \in T_q^* Q.$$

This $h(t)$ is the unique solution of the initial value problem

$$\begin{cases} \dot{h}_c(t) = \Gamma_{ac}^b(c_{v_q}(t)) V^a(t) h_b(t), \\ \dot{h}_i(t) = R_{ai}^j(c_{v_q}(t)) V^a(t) h_j(t) + \lambda_i(t), \end{cases} \quad \begin{cases} h_c(0) = 0, \\ h_i(0) = 0. \end{cases}$$

From the first relation, we may conclude that $h_b(t) = 0$.

We have now gathered all the tools to define an equivalent characterization of the PDE condition (B) from Proposition 4.

Proposition 8. *Consider $q \in Q, v_q \in \mathcal{D}_q$ and the 1-form $h(t)$, as defined above. A \mathcal{D} -preserving modification \hat{g} of g satisfies (B) if and only if*

$$\hat{g}_{c_{v_q}(t)}(\dot{c}_{v_q}(t), W(t)) + (h(t))(W(t)) = \hat{g}_q(v_q, w_q), \quad \forall w_q \in \mathcal{D}_q^g,$$

where $W(t)$ is the parallel transport of $w_q \in \mathcal{D}_q^g$ with respect to $\bar{\nabla}$.

Apart from the term in $h(t)$ this condition is again reminiscent to a Gauss-type condition. If we multiply the relation in the proposition on both sides with an extra factor t , we can express it equivalently in terms of the exponential map $\exp_q^{nh}(tv_q) = c_{v_q}(t)$, as follows:

$$\hat{g}_{\exp_q^{nh}(tv_q)}(T_{tv_q}(\exp_q^{nh}(tv_q)), W(t)) + t(h(t))(W(t)) = \hat{g}_q(tv_q, w_q).$$

Proof. From Lemma 2 we know that the expression (B) can be written as

$$\Gamma_{(L, \mathcal{D})}(\theta_i) + \lambda_i + \theta_k R_{ib}^k v^b = 0,$$

or, after expanding and multiplying both sides with w^i :

$$X_f(\hat{g}_{ai})v^f v^a w^i + \hat{g}_{ai} f^a w^i + \lambda_i w^i + \hat{g}_{ka} v^a R_{ib}^k v^b w^i = 0.$$

This expression is valid for arbitrary choices of q, v^a and w^k . If we replace them, specifically, by $c_{v_q}(t)$, $V^a(t)$ and $W^k(t)$, respectively, we get, for all t :

$$X_f(\hat{g}_{ai})V^f(t)V^a(t)W^i(t) + \hat{g}_{ai}\dot{V}^a(t)W^i(t) + \lambda_i(t)W^i(t) + \hat{g}_{ak}V^a(t)R_{ib}^kV^b(t)W^i(t) = 0$$

or,

$$\frac{d}{dt}(\hat{g}_{ai}V^a(t))W^i(t) + \left(\dot{h}_i(t) - R_{ai}^j(c_{v_q}(t))V^a(t)h_j(t)\right)W^i(t) + \hat{g}_{ak}V^a(t)\dot{W}^k(t) = 0.$$

We may simplify this to

$$\frac{d}{dt}(\hat{g}_{ai}V^a(t)W^i(t)) + \left(\dot{h}_i(t)W^i(t) + \dot{W}^j(t)h_j(t)\right) = 0,$$

from which

$$\frac{d}{dt}(\hat{g}_{ai}V^a(t)W^i(t) + h_i(t)W^i(t)) = 0.$$

This means that

$$\hat{g}_{ai}V^a(t)W^i(t) + h_i(t)W^i(t) = \hat{g}_{ai}v^aw^i + h_i(0)w^i.$$

Since $h_i(0) = 0$, we get the result.

The other direction follows if we start from the expression above, take a derivative by t and set $t = 0$, afterward. \square

From Propositions 4, 7 and 8 we may conclude:

Proposition 9. *Let \exp_q^{nh} be the nonholonomic exponential map of a purely kinetic (L, \mathcal{D}) and let \hat{g} be a Riemannian metric which is a \mathcal{D} -preserving modification of g . If, for all $v_q \in \mathcal{D}_q$, they satisfy both*

$$\hat{g}_{\exp_q^{nh}(v_q)}\left(T_{v_q}\left(\exp_q^{nh}\right)(v_q), T_{v_q}\left(\exp_q^{nh}\right)(w_q)\right) = \hat{g}_q(v_q, w_q), \quad \forall u_q \in \mathcal{D}_q$$

and

$$\hat{g}_{\exp_q^{nh}(tv_q)}\left(T_{tv_q}(\exp_q^{nh})(tv_q), W(t)\right) + t\left(h(t)\right)\left(W(t)\right) = \hat{g}_q(tv_q, w), \quad \forall w_q \in \mathcal{D}_q^g,$$

then \hat{g} is a geodesic extension of (L, \mathcal{D}) .

Because of the similarity with the statement of the Gauss Lemma, we call the conditions (A) and (B) the infinitesimal version of the above Gauss-type conditions.

7 Chaplygin systems

Chaplygin systems (also called generalized Chaplygin systems or nonholonomic systems of principle type, see e.g. [7, 12, 23]) are essentially nonholonomic systems with extra symmetry properties. The goal of this section is to use that symmetry to simplify the conditions (A) and (B).

Suppose that we have a purely kinetic Chaplygin system. This means that both the Riemannian metric g (and its Lagrangian L) and the constraint manifold \mathcal{C} are invariant under the action

$G \times Q \rightarrow Q$ of a Lie group G on the configuration manifold Q (when lifted to TQ). In addition, we assume that $\pi : Q \rightarrow Q/G$ is a principal fibre bundle and that the nonholonomic distribution \mathcal{D} is the horizontal distribution of a principal connection ω on π . Let $\{E_i\}$ be a basis for the Lie algebra of G and $\{(E_i)_Q\}$ its corresponding infinitesimal generators. If (x^a) are coordinates on Q/G , the horizontal lifts X_a of the vector fields $\partial/\partial x^a$ on Q/G (by means of ω) are invariant vector fields on Q that span \mathcal{D} . The set $\{X_a, (E_i)_Q\}$ is then a basis for $\mathcal{X}(Q)$. If we construct the basis in this way, we get

$$[X_a, (E_i)_Q] = 0, \quad [X_a, X_b] = B_{ab}^j (E_j)_Q, \quad [(E_i)_Q, (E_j)_Q] = -C_{ij}^k (E_k)_Q.$$

Here, B_{ab}^j can be interpreted as the curvature of the connection ω and C_{ij}^k as the structure constants of the Lie algebra.

Let, for now, $G_{ab} = g(X_a, X_b)$, $G_{ai} = g(X_a, (E_i)_Q)$ and $G_{ij} = g((E_i)_Q, (E_j)_Q)$. Similarly as in Section 2, we can create a basis for \mathcal{D}^g , by defining the vector fields \tilde{X}_i as

$$\tilde{X}_i = (E_i)_Q - G^{ab} G_{ia} X_b = (E_i)_Q + K_i^b X_b.$$

In the orthogonal basis $\{X_a, \tilde{X}_i\}$, we have $g_{ab} = g(X_a, X_b) = G_{ab}$, $g_{aj} = g(X_a, \tilde{X}_j) = 0$ and $g_{ij} = g(\tilde{X}_i, \tilde{X}_j)$. We first compute the bracket coefficients R_{ia}^k and R_{bc}^k in terms of the current bracket coefficients B_{ab}^k and C_{ij}^k .

On the one hand, we have $[X_a, \tilde{X}_i] = [X_a, (E_i)_Q + K_i^b X_b] = X_a(K_i^b) X_b + K_i^b B_{ab}^j (E_j)_Q$, while on the other hand is $[X_a, \tilde{X}_i] = R_{ai}^b X_b + R_{ai}^j \tilde{X}_j = (R_{ai}^b + K_j^b R_{ai}^j) X_b + R_{ai}^j (E_j)_Q$. Therefore, $R_{ai}^j = K_i^b B_{ab}^j$. Likewise, from $[X_a, X_b] = B_{ab}^j (E_j)_Q$ and $[X_a, X_b] = R_{ab}^c X_c + R_{ab}^j \tilde{X}_j = (R_{ab}^c - K_j^c R_{ab}^j) X_c + R_{ab}^j (E_j)_Q$, we can conclude that $R_{ab}^j = B_{ab}^j$.

The Lagrangian is still $L = \frac{1}{2} g_{ab} v^a v^b + \frac{1}{2} g_{ij} v^i v^j$. We now give an expression for the multipliers λ_i in the Chaplygin case. From the invariance of the Lagrangian it follows that $(E_i)_Q^C(L) = 0$ (see e.g. [27]). Moreover, on \mathcal{C} , $X_b^C(L) = \Gamma_{(L, \mathcal{D})}(X_b^V(L))$. With that, we have, on \mathcal{C} ,

$$\begin{aligned} \lambda_i &= \Gamma_{(L, \mathcal{D})}(\tilde{X}_i^V L) - \tilde{X}_i^C(L) = \Gamma_{(L, \mathcal{D})}(g_{ij} v^j) - K_i^b X_b^C(L) - \dot{K}_i^b X_b^V(L) \\ &= -K_i^b \Gamma_{(L, \mathcal{D})}(X_b^V(L)) - \dot{K}_i^b X_b^V(L) = -\Gamma_{(L, \mathcal{D})}(K_i^b X_b^V(L)) = \Gamma_{(L, \mathcal{D})}(G_{ai} v^a). \end{aligned}$$

We may now re-express the conditions (A) and (B). The (A) condition, $\theta_k R_{ac}^k v^a = 0$, becomes here $\theta_k B_{ac}^k v^a = 0$. For the (B) condition, we obtain after plugging in (A) and the expression for λ_i ,

$$\begin{aligned} 0 &= \Gamma_{(L, \mathcal{D})}(\theta_i) + \lambda_i + \theta_k R_{ia}^k v^a = \Gamma_{(L, \mathcal{D})}(\hat{g}_{ai} v^a) + \Gamma_{(L, \mathcal{D})}(G_{ai} v^a) - \theta_k K_i^b B_{ab}^k v^a \\ &= \Gamma_{(L, \mathcal{D})}((\hat{g}_{ai} + G_{ai}) v^a). \end{aligned}$$

As a consequence, $\mu_i = (\hat{g}_{ai} + G_{ai}) v^a$ are first integrals of the nonholonomic vector field $\Gamma_{(L, \mathcal{D})}$. Recall that (G_{ai}) in the Proposition below stands for the restriction $g|_{\mathcal{D} \times V\pi}$, where $V\pi$ is the vertical distribution of the principal fibre bundle $\pi : Q \rightarrow Q/G$.

Proposition 10. *Let (L, \mathcal{D}) be a purely kinetic Chaplygin nonholonomic system. If $\Gamma_{(L, \mathcal{D})}$ has linear first integrals $\mu_i = \mu_{ai} v^a$ that satisfy*

$$\mu_k B_{ac}^k v^a = G_{bk} B_{ac}^k v^b v^a,$$

then there exists a geodesic extension of (L, \mathcal{D}) by a Riemannian metric \hat{g} .

Proof. The proof follows from Proposition 4, since the conditions ensure that a (0,2) tensor field \hat{g} with $\hat{g}(X_a, X_b) = g_{ab}$ and $\hat{g}(X_a, \tilde{X}_i) = \mu_{ai} - G_{ai}$ can be extended to a Riemannian metric. \square

8 An example: the vertically rolling disk

The purpose of this section is to determine a geodesic extension \hat{g} , on a concrete example. We have chosen the example of the disk (rolling vertically without slipping), not only because it is well-known and simple, but also because it is involved enough to enlighten some of the constructions and results we have obtained in the previous sections. Moreover, for this example, already, some quadratic Lagrangians \hat{L} are known in the literature [9, 3] and at the end of the section we will compare them with the ones we find through our methods.

We will use the following coordinates, see the figures in [21] or in [26]. The couple (x, y) stands for the position of the center of the mass C , φ is the orientation angle of the disk (measured between the tangent of the contact point and the positive x -axis) and θ is the angle that a point fixed on the disk makes with respect to the vertical. In this example the configuration manifold is $Q = \mathbb{R}^2 \times S^1 \times S^1$. Because we assume the disk to remain vertical during its motion, the potential is constant and the Lagrangian is of purely kinetic type. If we choose all relevant constants to be unit, the Lagrangian function is given by

$$L = \frac{1}{2}g(v, v) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2 + \dot{\theta}^2)$$

and the linear nonholonomic constraints are

$$\begin{cases} \dot{x} = \cos(\varphi)\dot{\theta}, \\ \dot{y} = \sin(\varphi)\dot{\theta}. \end{cases}$$

This means that the distribution \mathcal{D} can be spanned by

$$\mathcal{D} = \text{span}\{X_a\} = \text{span}\left\{X_\varphi = \frac{\partial}{\partial\varphi}, X_\theta = \frac{\partial}{\partial\theta} + \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y}\right\}.$$

The constrained Lagrangian is

$$L_c = L|_C = \frac{1}{2}(\dot{\varphi}^2 + 2\dot{\theta}^2).$$

With this input one may write down the Lagrange-d'Alembert equations. It is well-known (see e.g. [7, 13, 9]) that the center of mass will make either a circular movement, or a motion along a straight line.

8.1 The vertically rolling disk as a purely kinetic nonholonomic system

Remark first that the metric g is such that $g(\partial/\partial q^i, \partial/\partial q^j) = \delta_{ij}$, with $(q^i) = (x, y, \theta, \varphi)$. In this subsection we choose the following basis for the orthogonal complement \mathcal{D}^g :

$$\mathcal{D}^g = \text{span}\{X_i\} = \text{span}\left\{X_x = \frac{\partial}{\partial x} - \cos(\varphi)\frac{\partial}{\partial\theta}, X_y = \frac{\partial}{\partial y} - \sin(\varphi)\frac{\partial}{\partial\theta}\right\}.$$

It is important to realize that none of our results depend on the specific choices we have made for the frames. We will make this clear in the next subsection.

When compared to the expressions in the previous sections, the index a will run over φ and θ , while the index i is here either x or y . If we compute the metric coefficients with respect to this basis, we get $g_{ai} = 0$, meaning here that $g_{\theta x} = g_{\theta y} = g_{\varphi x} = g_{\varphi y} = 0$. Moreover

$$(g_{ab}) = \begin{pmatrix} g_{\varphi\varphi} & g_{\varphi\theta} \\ g_{\varphi\theta} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad (g_{ij}) = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 + \cos^2(\varphi) & \cos(\varphi)\sin(\varphi) \\ \cos(\varphi)\sin(\varphi) & 1 + \sin^2(\varphi) \end{pmatrix}.$$

With this, the Lagrangian in quasi-velocities $(v_x, v_y, v_\theta, v_\varphi)$ becomes

$$L = \frac{1}{2} (2v_\theta^2 + v_\varphi^2) + \frac{1}{2} ((1 + \cos^2(\varphi)) v_x^2 + 2 \cos(\varphi) \sin(\varphi) v_x v_y + (1 + \sin^2(\varphi)) v_y^2).$$

We may read off the coefficients $R_{\alpha\beta}^\gamma$ from the Lie brackets $[X_\alpha, X_\beta] = R_{\alpha\beta}^c X_c + R_{\alpha\beta}^k X_k$. Here they are

$$\begin{aligned} [X_\varphi, X_\theta] &= -\sin(\varphi) X_x + \cos(\varphi) X_y, & [X_x, X_y] &= 0, & [X_x, X_\theta] &= 0, & [X_y, X_\theta] &= 0, \\ [X_\varphi, X_x] &= \frac{1}{2} \sin(\varphi) X_\theta - \frac{1}{2} \sin(\varphi) \cos(\varphi) X_x - \frac{1}{2} \sin^2(\varphi) X_y, \\ [X_\varphi, X_y] &= -\frac{1}{2} \cos(\varphi) X_\theta + \frac{1}{2} \cos^2(\varphi) X_x + \frac{1}{2} \cos(\varphi) \sin(\varphi) X_y. \end{aligned}$$

Since all g_{ab} are constant and $R_{ab}^a = 0$, we get from the Koszul formula,

$$2g_{cd}\Gamma_{ab}^d = X_a(g_{bc}) + X_b(g_{ac}) - X_c(g_{ab}) + g_{dc}R_{bc}^d - g_{db}R_{ac}^d - g_{ad}R_{bc}^d,$$

that all $\Gamma_{ab}^d = 0$.

We are now ready to find a solution for the equations (A) and (B) of Lemma 2. For (A), we get the following two algebraic conditions that should be satisfied by $\hat{g}_{\varphi x}, \hat{g}_{\varphi y}, \hat{g}_{\theta x}$ and $\hat{g}_{\theta y}$:

$$\begin{cases} \hat{g}_{\varphi x} \sin(\varphi) - \hat{g}_{\varphi y} \cos(\varphi) = 0, \\ -\hat{g}_{\theta x} \sin(\varphi) + \hat{g}_{\theta y} \cos(\varphi) = 0. \end{cases}$$

Condition (B), written in a version without $v^a v^b$ (and with $\Gamma_{bc}^d = 0$) is here:

$$0 = X_a(\hat{g}_{bi}) + X_b(\hat{g}_{ai}) - \hat{g}_{bk}R_{ai}^k - \hat{g}_{ak}R_{bi}^k.$$

If we first take $a = b = \varphi$, this gives $X_\varphi(\hat{g}_{\varphi i}) - \hat{g}_{\varphi k}R_{\varphi i}^k = 0$, or:

$$X_\varphi(\hat{g}_{\varphi x}) - \hat{g}_{\varphi x}R_{\varphi x}^x - \hat{g}_{\varphi y}R_{\varphi x}^y = 0 \quad \text{and} \quad X_\varphi(\hat{g}_{\varphi y}) - \hat{g}_{\varphi x}R_{\varphi y}^x - \hat{g}_{\varphi y}R_{\varphi y}^y = 0.$$

After substituting the values of $R_{\varphi x}^y$, etc, and after substituting the relations we had derived from the algebraic condition (A), we may write this as

$$\frac{\partial \hat{g}_{\varphi x}}{\partial \varphi} + \frac{1}{2} \hat{g}_{\varphi x} \tan(\varphi) = 0 \quad \text{and} \quad \frac{\partial \hat{g}_{\varphi y}}{\partial \varphi} - \frac{1}{2} \hat{g}_{\varphi y} \cot(\varphi) = 0.$$

The solutions of these equations are $\hat{g}_{\varphi x} = C\sqrt{\cos(\varphi)}$ and $\hat{g}_{\varphi y} = D\sqrt{\sin(\varphi)}$ (with C, D possibly functions of (x, y, θ)), but the only ones satisfying again the algebraic conditions are $\hat{g}_{\varphi x} = \hat{g}_{\varphi y} = 0$.

When we put $a = \theta$ and $b = \varphi$, we get:

$$0 = X_\varphi(\hat{g}_{\theta i}) - g_{\theta\theta}R_{\varphi i}^\theta - \hat{g}_{\theta k}R_{\varphi i}^k,$$

which becomes here:

$$\frac{\partial \hat{g}_{\theta x}}{\partial \varphi} - \sin(\varphi) + \frac{1}{2} \hat{g}_{\theta x} \sin(\varphi) \cos(\varphi) + \frac{1}{2} \hat{g}_{\theta y} \sin^2(\varphi) = 0$$

and

$$\frac{\partial \hat{g}_{\theta y}}{\partial \varphi} + \cos(\varphi) - \frac{1}{2} \hat{g}_{\theta x} \cos^2(\varphi) - \frac{1}{2} \hat{g}_{\theta y} \cos(\varphi) \sin(\varphi) = 0,$$

from which (with E, F functions of (x, y, θ))

$$\hat{g}_{\theta x} = E \sqrt{\cos(\varphi)} - 2 \cos(\varphi) \quad \text{and} \quad \hat{g}_{\theta y} = F \sqrt{\sin(\varphi)} - 2 \sin(\varphi).$$

Again, only for $E = 0$ and $F = 0$ the algebraic conditions are satisfied. Finally, we get for $a = b = \theta$ that $X_\theta(\hat{g}_{\theta i}) = 0$ (since $R_{\theta i}^\alpha = 0$). This is automatically satisfied since the $\hat{g}_{\theta i}$ we had found only depend on φ .

We conclude that the trajectories of the vertically rolling disk with initial velocities in \mathcal{D} are geodesics of all metrics \hat{g} with metric coefficients in the frame $\{X_a, X_i\}$ given by

$$\hat{g}_{ab} = g_{ab}, \quad \hat{g}_{\theta x} = -2 \cos(\varphi), \quad \hat{g}_{\varphi x} = 0, \quad \hat{g}_{\theta y} = -2 \sin(\varphi), \quad \hat{g}_{\varphi y} = 0,$$

and where the coefficients \hat{g}_{ik} can be anything as long as \hat{g} is positive-definite. We may look at the constructive proof of Proposition 4 for one possible (but not exclusive) way to realize this: we only need to find a constant α such that $\hat{g} = \begin{pmatrix} A & B \\ B^T & \alpha I \end{pmatrix}$ is positive-definite. The matrices A and B are here given by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 \cos(\varphi) & -2 \sin(\varphi) \\ 0 & 0 \end{pmatrix}.$$

The constant $\alpha \neq 0$ is not allowed to be an eigenvalue of $C = B^T A^{-1} B$, and should be larger than $-\tau_{\min}$ and $\frac{\mu_{\max}}{\tau_{\min}}$, where $\tau_{\min} = \min\{\text{eigenvalues of } A\}$ and $\mu_{\max} = \max\{\text{eigenvalues of } B^T B\}$. Here, the eigenvalues of A are 1 and 2, the eigenvalues of $B^T B$ are 0 and 4 and the eigenvalues of C are 0 and 2. We see that any $\alpha > 4$ fulfills all requirements. We can take for example $\alpha = 42$ and then

$$\hat{L} = \frac{1}{2} [v_\varphi^2 + 2v_\theta^2 - 4 \cos(\varphi) v_\theta v_x - 4 \sin(\varphi) v_\theta v_y + 42v_x^2 + 42v_y^2].$$

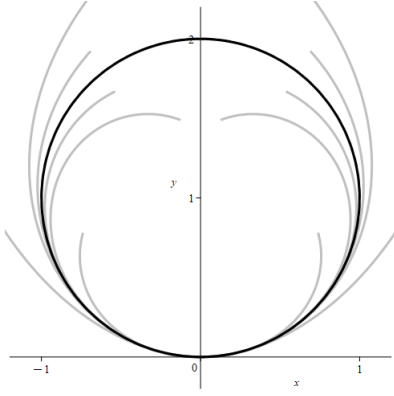
The relation between the quasi-velocities $\{v_x, v_y, v_\theta, v_\varphi\}$ and the natural fibre bundle coordinates $\{\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}\}$ is

$$\dot{x} = v_x + v_\theta \cos(\varphi), \quad \dot{y} = v_y + v_\theta \sin(\varphi), \quad \dot{\varphi} = v_\varphi, \quad \dot{\theta} = v_\theta - v_x \cos(\varphi) - v_y \sin(\varphi),$$

or conversely

$$\begin{cases} v_\varphi = \dot{\varphi}, \\ v_\theta = \frac{1}{2}(\dot{\theta} + \cos(\varphi)\dot{x} + \sin(\varphi)\dot{y}), \\ v_x = \frac{1}{2}(-\cos(\varphi)\dot{\theta} + (\sin^2(\varphi) + 1)\dot{x} - \cos(\varphi)\sin(\varphi)\dot{y}), \\ v_y = \frac{1}{2}(-\sin(\varphi)\dot{\theta} - \cos(\varphi)\sin(\varphi)\dot{x} + (\cos^2(\varphi) + 1)\dot{y}). \end{cases}$$

After substituting this, one obtains an expression for \hat{L} in natural coordinates.



On the left, we have plotted some geodesics for \hat{g} with $\alpha = 42$ (their projections for the center of mass (x, y)). The black circle through the origin is the nonholonomic trajectory through the origin with initial velocity

$$(\dot{x}_0 = 1, \dot{y}_0 = 0, \dot{\theta}_0 = 1, \dot{\varphi}_0 = 1) \in \mathcal{D}_{(0,0,0,0)}.$$

The nearby grey geodesics are perturbations in either \dot{x}_0 , \dot{y}_0 and $\dot{\theta}_0$, with $\epsilon = \pm 0.01$.

We can now compare our results with the Lagrangians that have been derived in the literature, as a solution of the Hamiltonization problem. A second attempt to obtain a Riemannian metric \hat{g} from the same solution (g_{ab}, \hat{g}_{ai}) is to choose \hat{g}_{ij} in such a way that it is a multiple of the original g_{ij} , i.e. $\hat{g}_{ij} = \beta g_{ij}$. In [9] (see also [21]) we can find the following regular quadratic Lagrangian (among other non-quadratic Lagrangians):

$$\hat{L}_{[9]} = \frac{1}{2}(\dot{\theta}^2 + \dot{\varphi}^2) - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \dot{\theta}(\dot{x} \cos(\varphi) + \dot{y} \sin(\varphi)).$$

When written in our current quasivelocities, we find for the corresponding metric

$$(\hat{g}_{ab})_{[9]} = (\hat{g}_{ab}), \quad (\hat{g}_{ai})_{[9]} = (\hat{g}_{ai}), \quad (\hat{g}_{ij})_{[9]} = - \begin{pmatrix} 1 + \cos^2(\varphi) & \cos(\varphi) \sin(\varphi) \\ \cos(\varphi) \sin(\varphi) & 1 + \sin^2(\varphi) \end{pmatrix},$$

meaning that it corresponds with $\beta = -1$. This metric has therefore the same constrained Lagrangian L_c as L and as our Lagrangian \hat{L} . However, the choice $\beta = -1$ leads only to a pseudo-Riemannian metric.

The following Lagrangian is mentioned in [3]:

$$\hat{L}_{[3]} = \frac{1}{2}(2\dot{\theta}^2 + \dot{\varphi}^2 + \dot{x}^2 + \dot{y}^2) - \dot{\theta}(\dot{x} \cos(\varphi) + \dot{y} \sin(\varphi)).$$

It comes from a Riemannian metric, but it does not preserve the constrained Lagrangian (i.e. it is not a \mathcal{D} -preserving modification of g), since

$$(g_{ab})_{[3]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, there are also some results that can be related to the geodesic extension problem in [8]. However, the authors of [8] do not make use of a \mathcal{D} -preserving modification, and they mention that they are unable to find a geodesic extension by their methods, specifically, for the vertically rolling disk.

8.2 The vertically rolling disk as a Chaplygin system

It is well-known ([7, 13]) that the configuration space $Q = S^1 \times S^1 \times \mathbb{R}^2$ of the vertically rolling disk is the total space of a principal fibre bundle with structure group $G = \mathbb{R}^2$. Both the Lagrangian and the constraints are invariant under this symmetry group, and the constraints

can be thought of as the horizontal distribution of a principal connection. This means that the vertically rolling disk can be regarded as a Chaplygin system.

For the vertical distribution of $\pi : Q \rightarrow Q/G$ we can take $V\pi = \text{span}\{(E_x)_Q = \frac{\partial}{\partial x}, (E_y)_Q = \frac{\partial}{\partial y}\}$, and for the horizontal distribution we may set $\mathcal{D} = \text{span}\{X_\varphi = \frac{\partial}{\partial \varphi}, X_\theta = \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}\}$, as before. We may compute that

$$G_{x\varphi} = 0, \quad G_{x\theta} = \cos(\varphi), \quad G_{y\varphi} = 0, \quad G_{y\theta} = \sin(\varphi),$$

and

$$[X_\varphi, X_\theta] = -\sin(\varphi)\frac{\partial}{\partial x} + \cos(\varphi)\frac{\partial}{\partial y}.$$

It is important to remark, however, that the result in Proposition 10 makes use of a frame of \mathcal{D}^g that is different from the frame $\{X_x, X_y\}$ we had encountered earlier. The vector fields $\tilde{X}_i = (E_i)_Q - G^{bc}G_{ci}X_b = (E_i)_Q + K_i^b X_b$ are here

$$\tilde{X}_x = \frac{\partial}{\partial x} - \frac{1}{2}\cos(\varphi)\left(\frac{\partial}{\partial \theta} + \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y}\right)$$

and

$$\tilde{X}_y = \frac{\partial}{\partial y} - \frac{1}{2}\sin(\varphi)\left(\frac{\partial}{\partial \theta} + \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y}\right),$$

and the relation between $\{\tilde{X}_i\}$ and the frame $\{X_i\}$ of the previous subsection is

$$\begin{cases} \tilde{X}_x = (1 - \frac{1}{2}\cos^2(\varphi))X_x - \frac{1}{2}\cos(\varphi)\sin(\varphi)X_y, \\ \tilde{X}_y = -\frac{1}{2}\sin(\varphi)\cos(\varphi)X_x + (1 - \frac{1}{2}\sin^2(\varphi))X_y. \end{cases}$$

According to Proposition 10 we should look for linear first integrals μ_x and μ_y of $\Gamma_{(L,\mathcal{D})}$ that satisfy

$$(\mu_x - G_{x\theta}v_\theta - G_{x\varphi}v_\varphi)B_{\varphi\theta}^x v_\varphi + (\mu_y - G_{y\theta}v_\theta - G_{y\varphi}v_\varphi)B_{\varphi\theta}^y v_\varphi = 0,$$

which is here

$$-\sin(\varphi)\mu_x + \cos(\varphi)\mu_y = 0.$$

If we take a $\Gamma_{(L,\mathcal{D})}$ -derivative we also get $\cos(\varphi)\mu_x + \sin(\varphi)\mu_y = 0$ (because $\Gamma_{(L,\mathcal{D})}(\sin(\varphi)) = \cos(\varphi)\dot{\varphi}$). We can, therefore conclude that the first integrals should be $\mu_x = \mu_y = 0$. With that, we may get the coefficients of the metric \hat{g} with respect to the frame $\{X_a, \tilde{X}_i\}$ (see the proof of Proposition 10),

$$\hat{g}(X_a, X_b) = g_{ab}, \quad \widetilde{\hat{g}_{ai}} = \hat{g}(X_a, \tilde{X}_i) = \mu_{ai} - G_{ai},$$

which are here

$$\begin{cases} \widetilde{\hat{g}_{x\varphi}} = 0 & \text{and} & \widetilde{\hat{g}_{x\theta}} = -\cos(\varphi) \\ \widetilde{\hat{g}_{y\varphi}} = 0 & \text{and} & \widetilde{\hat{g}_{y\theta}} = -\sin(\varphi) \end{cases}.$$

We see that this method is way shorter and more efficient since we had already solved the involved differential equations.

We end this section by showing that the new coefficients we had found are compatible with the old ones. Indeed, given the relation between the two bases $\{X_i\}$ and $\{\tilde{X}_i\}$ one easily verifies that, for example,

$$\widetilde{\hat{g}_{\theta y}} = -\frac{1}{2}\sin(\varphi)\cos(\varphi)\hat{g}_{\theta x} + \left(1 - \frac{1}{2}\sin^2(\varphi)\right)\hat{g}_{\theta y},$$

as it should.

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