Preprint typeset in JHEP style

Solitonic ground state in supersymmetric theory in background

Muneto Nitta[a,b] **and Shin Sasaki**[c]

[a] *Department of Physics & Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Kanagawa 223-8521, Japan*

[b] *International Institute for Sustainability with Knotted Chiral Meta Matter(SKCM^2), Hiroshima University, 1-3-2 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8511, Japan*

[c] *Department of Physics, Kitasato University, Sagamihara 252-0373, Japan*

E-mail: nitta(at)phys-h.keio.ac.jp, shin-s(at)kitasato-u.ac.jp

Abstract: A solitonic ground state called a chiral soliton lattice (CSL) is realized in a supersymmetric theory with background magnetic field and finite chemical potential. To this end, we construct, in the superfield formalism, a supersymmetric chiral sine-Gordon model as a neutral pion sector of a supersymmetric two-flavor chiral Lagrangian with a Wess-Zumino-Witten term. The CSL ground state appears in the presence of either a strong magnetic field and/or large chemical potential, or a background fermionic condensate in the form of a fermion bilinear consisting of the gaugino and a superpartner of a baryon gauge field.

Contents

1 Introduction

Determination of the ground states or vacua and phase structures is quite important to understand any physical system. For such a purpose, it is becoming important to consider a possibility of spatially inhomogeneous ground states. Several examples can be found in condensed matter physics such as superconductors [1–4] and polyacetylene [5–7], and in quantum field theories such as the Gross–Neveu and Nambu–Jona-Lasino models [8–10] and quantum chromodynamics (QCD) [11–15]. For such a kind of ground states, the order parameter is characterized by a spatially varying function and several translational symmetries are spontaneously broken there. However, without any external field, spatial modulations are unstable due to the so-called Landau-Peierls instability [16, 17].

On the other hand, solitonic ground states are present in the presence of spin-orbit coupling in condensed matter systems or background fields in quantum field theories such as a magnetic field or rotation. The examples of the former contain chiral magnets and ultracold atomic gases with synthetic gauge fields. The ground states of the chiral magnets contain two kinds of inhomogeneous ground states made of topological solitons due to the presence of the so-called Dzyaloshinskii-Moriya (DM) interaction [18, 19]. One is a chiral soliton lattice (CSL), also called a spiral phase [20–24], where the energy of a single soliton or domain wall is negative and thus one dimensionally inhomogeneous states composed of solitons or domain walls have a lower energy than uniform states. The other is the 2D Skyrmion lattice (crystal) phase where the energy of a single Skyrmion is negative [25–31]. Ultracold atomic gases with spin-orbit couplings or synthetic gauge fields [32, 33] give the solitonic ground states of 2D Skyrmions [34] and 3D Skyrmions [35].

As the example of the latter, QCD under extreme conditions like high baryon density, pronounced magnetic fields, and rapid rotation has been extensively studied. Recently, the QCD phase diagram including such conditions gathers significant attention due to the relevance in the interior of neutron stars and heavy-ion collisions [36]. In particular, QCD in strong magnetic fields has received quite intense attention. At low energy, QCD can be described in terms of pions degrees of freedom by the chiral Lagrangian or chiral perturbation theory (ChPT) accompanied with the Wess-Zumiono-Witten (WZW) term [37]. In the presence of a background magnetic field B at finite chemical potential μ_{B}, the WZW term in QCD with two flavors contains an anomalous coupling of the neutral pion π_0 to the magnetic field via the chiral anomaly [38, 39] through the Goldstone-Wilczek current [40, 41]. Then, when the background gauge field/and or chemical potential are large enough

$$B\mu_{\mathrm{B}} \geq \frac{16\pi m_\pi}{f_\pi^2 e}, \tag{1.1}$$

with the pion's mass m_π and decay constant f_π and the gauge coupling constant e, the ground state is inhomogeneous in the form of a CSL consisting of a stack of solitons [39, 42, 43], analogous to chiral magnets. To show this, neglecting charged pions degrees of freedom, the neutral pion sector of the chiral Lagrangian reduces to the so-called chiral sine-Gordon model, in common with chiral magnets [20–22]. It was also discussed that thermal fluctuations enhance the stability of CSL [44–47]. Similar CSLs of the η (or η') meson also appear under rapid rotation instead of strong magnetic field [48–53]. Further investigations have been done into the quantum nucleation of CSLs [54, 55], quasicrystals [56], the domain-wall Skyrmion phase [57, 58], the interplay with Skyrmion crystals at zero magnetic field [59–61], and an Abrikosov's vortex lattice and baryon crystals [62, 63]. Among various studies of CSL, one of the most important directions is the stability of CSL beyond perturbations; CSL states in QCD-like theory such as SU(2) QCD and vector-like gauge theories were studied in Refs. [64, 65] for nonperturbative studies in lattice gauge theories without the sign problem. On the other hand, the motivation of this paper is to investigate a CSL in supersymmetric theories which should be also relevant for studies of its nonperturbative aspects.

Supersymmetry (SUSY) is a symmetry between bosons and fermions and was quite extensively studied for long time because of a lot of reasons [66]: it was expected to solve the so-called naturalness problem in the phenomenological side [67, 68], while in the theoretical side it is a useful or necessary tool to control nonperturbative effects in quantum field theory and string theory [69, 70]. Bogomol'nyi-Prasad-Sommerfiled (BPS) topological solitons and instantons preserve some fraction of SUSY and are nonperturbatively stable [71–75], thereby playing a crucial role for study of nonperturbative aspects. However, in reality SUSY is not present at low energy in nature, and thus SUSY breaking is one of the most important aspects in phenomenology [76]. Among various SUSY breaking mechanisms, a unconventional SUSY breaking mechanism due to a spatial or temporal modulation was proposed [77, 78] by extending modulated vacua in relativistic bosonic field theories [79, 80]. To this end, SUSY higher derivative terms free from ghost and auxiliary field problem [81–89] were essential. By contrast, in the configuration proposed in this paper, SUSY is broken by a solitonic ground state (as well as background fields).

In this paper, we discuss, in the superfield formalism, a manifestly supersymmetric extension of SU(2) chiral Lagrangian with the WZW term in magnetic field at finite density. Instead of the full Lagrangian, we succeed to construct the neutral pion sector which is a SUSY chiral sine-Gordon model. We then construct a BPS domain wall (chiral soliton) and show that in a certain parameter region of strong magnetic field and/or large chemical potential, the tension of the domain wall becomes negative. In such a case, the ground state is CSL, an alternate array of BPS and anti-BPS chiral solitons, where SUSY is broken. While this is the same with conventional QCD, we also find that a CSL also occurs in a constant fermion condensation background.

Some comments on previous works are in order. First, a SUSY sine-Gordon model was discussed in the literature [90–94], but SUSY *chiral* sine-Gordon model was not. Second, the WZW term has the same form with the previously known supersymmetric WZW term [95, 96].[1] Third, a BPS domain wall in SUSY chiral Lagrangian was studied without the WZW term [103].

This paper is organized as follows. In Sec. 2, we summarize ChPT in QCD in the magnetic field at finite density and the CSL ground state in it. In Sec. 3, we construct manifestly supersymmetric chiral Lagrangian and WZW term in the superfield formalism. In Sec. 4, we construct BPS solitons (domain walls), and discuss solitonic ground states (CSL) consisting of an array of BPS and anti-BPS solitons in the presence of an external gauge field background or a fermion background. Section 5 is devoted to a summary and discussion. In Appendix A, we show the incompatibility of the Fayet-Iliopoulos term.

[1] See Refs. [97–99] for further studies of SUSY WZW terms, see also Refs. [100–102].

2 Chiral soliton lattice in strong magnetic field: a review

In this section, we give a brief review on the two-flavor chiral Lagrangian with the WZW term at finite density and background gauge field, and the CSL ground state.

2.1 Chiral Lagrangian

We concentrate on the phase in which the chiral symmetry is spontaneously broken. The low-energy dynamics in this phase is described by the two-flavor ChPT, which is an effective field theory for pions. The pion fields $\phi_a\,(a = 1, 2, 3)$ are represented by a 2×2 unitary matrix $\Sigma = \mathrm{e}^{\mathrm{i}\tau_a \frac{\phi^a}{f_\pi}}$ which undergoes the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ chiral symmetry;

$$\Sigma \to L\Sigma R^\dagger. \tag{2.1}$$

where both L and R are 2×2 unitary matrices. Here $\tau_a\,(a = 1, 2, 3)$ are the Pauli matrices normalized as $\mathrm{tr}[\tau_a\tau_b] = 2\delta_{ab}$.

The $\mathrm{U}(1)_\mathrm{EM}$ electromagnetic gauge transformation is given by

$$\Sigma \to \mathrm{e}^{\mathrm{i}f_\pi^{-1}\lambda\frac{\tau_3}{2}}\Sigma\mathrm{e}^{-\mathrm{i}f_\pi^{-1}\lambda\frac{\tau_3}{2}} \quad \text{and} \quad A_m \to A_m - \frac{1}{e}\partial_m\lambda, \tag{2.2}$$

where λ, e are the gauge parameter and the coupling constant. The associated covariant derivative D_m is defined by

$$D_m\Sigma = \partial_m\Sigma + \mathrm{i}eA_m[Q, \Sigma]\,, \tag{2.3}$$

$$Q = \frac{1}{6}\mathbf{1}_2 + \frac{1}{2}\tau_3\,, \tag{2.4}$$

where the matrix Q represents the electric charge carried by quarks. The effective Lagrangian at $\mathcal{O}(p^2)$ is given by

$$\mathcal{L} = \mathcal{L}_\mathrm{ChPT} + \mathcal{L}_\mathrm{WZW} \tag{2.5}$$

where the first term is given by

$$\mathcal{L}_\mathrm{ChPT} = -\frac{f_\pi^2}{4}\,\mathrm{tr}\left(D_m\Sigma D^m\Sigma^\dagger\right) - \frac{f_\pi^2 m_\pi^2}{4}\,\mathrm{tr}(2\mathbf{1}_2 - \Sigma - \Sigma^\dagger). \tag{2.6}$$

The second term in (2.5) gives the coupling of the pions to the external $\mathrm{U}(1)_\mathrm{B}$ baryon gauge field $A_m^\mathrm{B} = (\mu_\mathrm{B}, 0, 0, 0)$ through the WZW term [39]. This is given in Refs. [38, 39] as

$$\mathcal{L}_\mathrm{WZW} = -\left(A_m^\mathrm{B} + \frac{e}{2}A_m\right) j_\mathrm{GW}^m. \tag{2.7}$$

Here, j_GW^m is the Goldstone-Wilczek current [40, 41] defined by

$$j_\mathrm{GW}^m = -\frac{\epsilon^{mnpq}}{24\pi^2}\mathrm{tr}\left(L_nL_pL_q - 3\mathrm{i}e\partial_n\left[A_pQ(L_q + R_q)\right]\right), \tag{2.8}$$

where $L_m = \Sigma\partial_m\Sigma^\dagger$, $R_m = \partial_m\Sigma^\dagger\Sigma$ are the left-invariant and the right-invariant Maurer-Cartan 1-forms respectively.

It is convenient to rewrite the Goldstone-Wilczek current (2.8) in the form where the $\mathrm{U}(1)_{\mathrm{em}}$ gauge symmetry is manifest [39];

$$j^m_{\mathrm{GW}} = -\frac{1}{24\pi^2}\varepsilon^{mnpq}\mathrm{tr}\left[(\Sigma D_n\Sigma^\dagger)(\Sigma D_p\Sigma^\dagger)(\Sigma D_q\Sigma^\dagger) - \frac{3ei}{2}F_{np}Q(\Sigma D_q\Sigma^\dagger + D_q\Sigma^\dagger\Sigma)\right]. \quad (2.9)$$

In the low energies, only the neutral pion $\phi^3 = \phi$ contributes to physics. We therefore set $\Sigma = e^{i\tau_3\frac{\phi}{f_\pi}}$ and the covariant derivative D_m is replaced by ∂_m

$$D_m\Sigma = \partial_m\Sigma + ieA_m\left[\frac{\tau_3}{2}, \Sigma\right] = \partial_m\Sigma. \quad (2.10)$$

Then the ChPT Lagrangian for the neutral pion is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{ChPT}} = &-\frac{f_\pi^2}{4}\mathrm{tr}\Big(\partial_m\Sigma^\dagger\partial^m\Sigma\Big) + \frac{f_\pi^2 m_\pi^2}{2}\Big(\mathrm{tr}\Sigma + \mathrm{h.c.}\Big) - \frac{f_\pi^2 m_\pi^2}{2}\mathrm{tr}\mathbf{1}_2 \\ &+ \frac{1}{24\pi^2}\varepsilon^{mnpq}\left(A^B_m + \frac{e}{2}A_m\right)\mathrm{tr}\left[(\Sigma\partial_n\Sigma^\dagger)(\Sigma\partial_p\Sigma^\dagger)(\Sigma\partial_q\Sigma^\dagger) - \frac{3i}{2}F_{np}\tau_3\Sigma\partial_q\Sigma^\dagger\right].\end{aligned} \quad (2.11)$$

2.2 Chiral soliton lattice

It is noteworthy to see that the model for the neutral pion (2.11) is given by the sine-Gordon model;

$$\mathcal{L} = -\frac{1}{2}(\partial_m\phi)^2 - f_\pi^2 m_\pi^2\left\{1 - \cos\left(\frac{\phi}{f_\pi}\right)\right\} + \frac{e\mu_{\mathrm{B}}}{4\pi}\vec{B}\cdot\vec{\nabla}\phi, \quad (2.12)$$

where we have assumed the external $\mathrm{U}(1)_{\mathrm{em}}$ constant magnetic field $\vec{B}$ and that the pion field is time independent $\partial_0\Sigma = 0$. This model is known as the chiral sine-Gordon model studied extensively in the context of chiral magnets [20–24].

The QCD background in the absence of the external background field is given by $\phi/f_\pi = 2\pi n$ with $n \in \mathbb{Z}$. where the energy (density) vanishes. The last term in (2.12) does not contribute to the equation motion:

$$\nabla^2\phi - f_\pi m_\pi^2 \sin\frac{\phi}{f_\pi} = 0. \quad (2.13)$$

The simplest solution to this equation is known as a single sine-Gordon (anti-)kink interpolating two adjacent vacua:

$$\phi(z) = 4f_\pi \arctan\left[e^{\pm m_\pi(z-c)}\right], \quad (2.14)$$

where c is the position of the center or the translational modulus. The tension (energy per unit area) is given by

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} dz \left[\frac{1}{2}(\partial_z \phi)^2 + f_\pi^2 m_\pi^2 \left(1 - \cos\left(\frac{\phi}{f_\pi}\right)\right) - \frac{e f_\pi \mu_{\mathrm{B}} B_z}{4\pi^2} \partial_z \phi \right] \\
&= 8 m_\pi f_\pi^2 \mp \frac{e \mu_{\mathrm{B}} B_z}{2\pi}.
\end{aligned} \tag{2.15}
$$

It is obvious that the finite baryon density and the external magnetic field give a negative (positive) contribution to the energy for a sine-Gordon (anti-)kink. This imbalance between a kink and an anti-kink is the origin of the term "*chiral*" of the chiral sine-Gordon model. At the boundary of Eq. (1.1), the tension of a single sine-Gordon kink in Eq. (2.15) with the upper sign becomes zero, $E = 0$, and the ground state is a single kink degenerated with the QCD vacuum. For larger background satisfying inequality of Eq. (1.1), one finds that the tension of the sine-Gordon kink is negative, $E < 0$, and the kink in Eq. (2.14) with the upper sign is energetically more stable against the ordinary QCD vacuum $\phi/f_\pi = 2n\pi$. However, one cannot create infinite numbers of kinks since two adjacent sine-Gordon kinks repel each other, increasing the energy. Consequently, the ground state in the presence of the large background fields is a stack of sine-Gordon kinks, that is a CSL [39, 42, 43].

3 Supersymmetric chiral perturbation theory

In this section, we construct a four-dimensional $\mathcal{N} = 1$ supersymmetric ChPT in the external background gauge fields. The basic elements are the supersymmetric generalization of the pion field Σ and the $\mathrm{U}(1)_{\mathrm{em}}$ and the baryon gauge fields (A_m, A_m^B). In the following, we follow the Wess-Bagger conventions [104] of superfields.

First, the target space of SUSY nonlinear sigma models must be Kähler, and thus pion fields SU(2) must be complexified:

$$
\mathrm{SU}(2)^{\mathbb{C}} \simeq \mathrm{SL}(2, \mathbb{C}) \simeq T^* \mathrm{SU}(2). \tag{3.1}
$$

We then introduce the $\mathrm{SU}(2)^{\mathbb{C}}$-valued chiral superfield $\boldsymbol{\Sigma}$ whose lowest component is given by Σ;

$$
\boldsymbol{\Sigma}(y, \theta) = \Sigma + \sqrt{2}\psi_\Sigma \theta + F_\Sigma \theta^2 \in \mathrm{SU}(2)^{\mathbb{C}} \tag{3.2}
$$

where ψ_Σ, F_Σ are the Weyl fermion and the auxiliary field and (y, θ) are the chiral coordinates. The real parts of Σ denote Nambu-Goldstone (NG) bosons (pions) associated with the chiral symmetry breaking while the imaginary parts of Σ are so-called quasi-NG bosons which are not associated with a symmetry breaking but are required from SUSY [105–112].

On the other hand, ψ_Σ are called quasi-NG fermions. The kinetic and the mass terms of $\boldsymbol{\Sigma}$ are given by[2]

$$\mathcal{L}_{\rm kin} = \frac{2}{\beta^2}\int d^4\theta\,{\rm tr}\big[\boldsymbol{\Sigma}^\dagger\boldsymbol{\Sigma}\big] + \left(\frac{2m}{\beta^2}\int d^2\theta\,{\rm tr}\boldsymbol{\Sigma} + {\rm h.c.}\right). \tag{3.3}$$

Here the trace is taken over the 2×2, $\mathrm{SU}(2)^{\mathbb{C}}$-valued matrices and m, β are real parameters of dimensions $+1$ and -1, respectively. Later, we will see that the correspondences of these parameters with those in the last section are $1/\beta = f_\pi$ and $m = m_\pi$.

The WZW part of the coupling to the external field A_m^B in the Goldstone-Wilczek current is constructed with the help of the SUSY WZW term elucidated in Refs. [95, 96];

$$\mathcal{L}_{\rm WZW} = \int d^4\theta\, V^B {\rm tr}\Big[(\boldsymbol{\Sigma}(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\boldsymbol{\Sigma}^\dagger)(\boldsymbol{\Sigma}\bar{D}^{\dot{\alpha}}\boldsymbol{\Sigma}^\dagger)(D^\alpha\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\dagger)\Big] + {\rm h.c.} \tag{3.4}$$

Here V^B is the U(1) vector multiplet for the external baryon number gauge field A_m^B. This is given, in the Wess-Zumino gauge, by

$$V^B(x,\theta,\bar\theta) = -(\theta\sigma^m\bar\theta)A_m^B + i\theta^2\bar\theta\bar\lambda^B - i\bar\theta^2\theta\lambda^B + \frac{1}{2}\theta^2\bar\theta^2 D^B, \tag{3.5}$$

where λ^B, D^B are the Weyl fermion and the auxiliary field, respectively. The last part in the Goldstone-Wilczek current may be given by

$$\mathcal{L}_{\rm GW} = \frac{i}{2}\int d^4\theta\, V^B W_\alpha\,{\rm tr}\Big[\tau_3\boldsymbol{\Sigma}^\dagger D^\alpha\boldsymbol{\Sigma}\Big] + {\rm h.c.} \tag{3.6}$$

where $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$ is the field strength for the $\mathrm{U}(1)_{\rm em}$ vector multiplet V. Although Eqs. (3.4) and (3.6) contain derivatives of superfields, which sometimes give rise to a potential auxiliary field problem [97–99], they cause no troubles. We will see this in due course.

In order to see that the above interactions indeed provide a natural supersymmetric generalization of ChPT, we now write down the component expression of the Lagrangian $\mathcal{L}_{\rm SChPT} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm WZW} + \mathcal{L}_{\rm GW}$. After some calculations, the bosonic part of the Lagrangian is found to be

$$\begin{aligned}
\mathcal{L}_{\rm SChPT} = {}& \frac{2}{\beta^2}{\rm tr}\Big[-\partial_m\Sigma^\dagger\partial^m\Sigma + F_\Sigma\bar{F}_\Sigma\Big] + \frac{2m}{\beta^2}{\rm tr}\Big[F_\Sigma + \bar{F}_\Sigma\Big] \\
& - 2{\rm tr}\Big[\Big(-\eta^{mn}\eta^{pq} + \eta^{mp}\eta^{nq} + \eta^{mq}\eta^{np} + i\varepsilon^{mnpq}\Big)A_m^B(\Sigma\partial_n\Sigma^\dagger)(\Sigma\partial_p\Sigma^\dagger)(\partial_q\Sigma\cdot\Sigma^\dagger) \\
& \qquad + A_m(\Sigma\partial^m\Sigma^\dagger)\Sigma\bar{F}_\Sigma F_\Sigma\Sigma^\dagger\Big] + {\rm h.c.} \\
& + \frac{i}{2}\Big(-A_m^B D\,{\rm tr}\big[\tau_3\Sigma^\dagger\partial^m\Sigma\big] - A_m^B F^{mn}{\rm tr}\big[\tau_3\Sigma^\dagger\partial_n\Sigma\big] \\
& \qquad - \frac{i}{2}\varepsilon^{mnpq}A_m^B F_{pq}{\rm tr}\big[\tau_3\Sigma^\dagger\partial_n\Sigma\big]\Big) + {\rm h.c.}
\end{aligned} \tag{3.7}$$

[2] More generally, the Kähler potential can be an arbitrary function f of two variables: $\int d^4\theta\, f\left({\rm tr}\big[\boldsymbol{\Sigma}^\dagger\boldsymbol{\Sigma}\big], {\rm tr}\big[\boldsymbol{\Sigma}^\dagger\boldsymbol{\Sigma}\big]^2\right)$. This corresponds to the degrees of freedom to deform noncompact directions corresponding to quasi-NG bosons, which cannot be fixed by the real symmetry SU(2) [109, 110, 112].

We find that this precisely contains the interaction terms in (2.11) and is a natural SUSY generalization of the ChPT in the external background gauge fields.

In order to find the interactions for the neutral pion, we now write

$$\mathbf{\Sigma} = e^{i\frac{\tau^3}{2}\beta\Phi} = \mathbf{1}_2 \cos\left(\frac{\beta\Phi}{2}\right) + i\tau_3 \sin\left(\frac{\beta\Phi}{2}\right) \tag{3.8}$$

where $\Phi(y,\theta) = \varphi + \sqrt{2}\theta\psi + \theta^2 F$ is the chiral superfield for the neutral pion ϕ. The real part of the complex scalar field φ corresponds to the neutral pion ϕ (up to the normalization) while the imaginary part is the quasi-NG boson. Then the superfield Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\text{SChPT}} = \frac{4}{\beta^2}\int d^4\theta \, \cos\left(\frac{\beta}{2}(\Phi - \Phi^\dagger)\right) + \left[\frac{4m}{\beta^2}\int d^2\theta \, \cos\left(\frac{\beta}{2}\Phi\right) + \text{h.c.}\right] \\ + \left[\frac{\beta^3}{4}\int d^4\theta \, V^B \sin\left(\frac{3}{2}\beta(\Phi - \Phi^\dagger)\right)(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\Phi^\dagger \bar{D}^{\dot{\alpha}}\Phi^\dagger D^\alpha\Phi + \text{h.c.}\right] \\ + \left[\frac{i\beta}{2}\int d^4\theta \, V^B W_\alpha \cos\left(\frac{\beta}{2}(\Phi - \Phi^\dagger)\right) D^\alpha\Phi + \text{h.c.}\right].\end{aligned} \tag{3.9}$$

It is straightforward to calculate the bosonic part of the Lagrangian. The result is

$$\begin{aligned}\mathcal{L}_{\text{SChPT}} = & -\frac{1}{2\beta}\sin\left(\frac{\beta}{2}(\varphi-\bar{\varphi})\right)\Box(\varphi-\bar{\varphi}) - \frac{1}{4}\cos\left(\frac{\beta}{2}(\varphi-\bar{\varphi})\right)\partial_m(\varphi+\bar{\varphi})\partial^m(\varphi+\bar{\varphi}) \\ & + \cos\left(\frac{\beta}{2}(\varphi-\bar{\varphi})\right)F\bar{F} - \frac{2m}{\beta}\left\{F\sin\left(\frac{\beta}{2}\varphi\right) + \bar{F}\sin\left(\frac{\beta}{2}\bar{\varphi}\right)\right\} \\ & - \frac{\beta^2}{4}\sin\left(\frac{3}{2}\beta(\varphi-\bar{\varphi})\right)\left[(A^B_m\partial^m\varphi)(\partial_n\bar{\varphi}\partial^n\bar{\varphi}) - (A_m\partial^m\bar{\varphi})(\partial_n\varphi\partial^n\varphi)\right] \\ & + \frac{\beta^2}{2}iF\bar{F}\sin\left(\frac{3}{2}\beta(\varphi-\bar{\varphi})\right)A^B_m\partial^m(\varphi-\bar{\varphi}) \\ & + \frac{\beta}{2}\cos\left(\frac{\beta}{2}(\varphi-\bar{\varphi})\right)\Big[iA^B_m D\,\partial^m(\varphi-\bar{\varphi}) - iA^B_m F^{mn}\partial_n(\varphi-\bar{\varphi}) \\ & \qquad + \frac{1}{2}\varepsilon^{mnpq}A^B_p F_{mn}\partial_q(\varphi+\bar{\varphi})\Big].\end{aligned} \tag{3.10}$$

We stress that even though the superfield Lagrangian involves derivative interactions, the auxiliary fields do not have spacetime derivative terms in the component Lagrangian. This fact is necessary in the sense that the equations of motion for the auxiliary fields are algebraic ones and we can write down the explicit interaction terms for φ. Indeed, the equation of motion for $\bar{F}$ is solved by

$$F = \frac{2m}{\beta}\sin\left(\frac{\beta}{2}\bar{\varphi}\right)\left\{\cos\left(\frac{\beta}{2}(\varphi-\bar{\varphi})\right) + \frac{\beta^2}{2}i\sin\left(\frac{3}{2}\beta(\varphi-\bar{\varphi})\right)A^B_m\partial^m(\varphi-\bar{\varphi})\right\}^{-1}. \tag{3.11}$$

After integrating out the auxiliary fields $F, \bar{F}$, we find that the potential term together with the derivative interactions appear in the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\rm SChPT} = & \frac{1}{\beta}\sinh(\beta\chi)\Box\chi - \cosh(\beta\chi)\partial_m\phi\partial^m\phi \\
& - \frac{4m^2}{\beta^2}\left\{\cos^2\left(\frac{\beta\phi}{2}\right)\sinh^2\left(\frac{\beta\chi}{2}\right) + \sin^2\left(\frac{\beta\phi}{2}\right)\cosh^2\left(\frac{\beta\chi}{2}\right)\right\}\times \\
& \qquad\qquad \times\left\{\cosh(\beta\chi) - \beta^2\sinh(3\beta\chi)A_m^B\partial^m\chi\right\}^{-1} \\
& + \frac{\beta^2}{2}\sinh(3\beta\chi)\Big[-2(A_m^B\partial^m\phi)(\partial_n\phi\partial^n\chi) + (A_m^B\partial^m\chi)(\partial_n\phi\partial^n\phi - \partial_n\chi\partial^n\chi)\Big] \\
& + \beta\cosh(\beta\chi)\Big[-D\,A_m^B\partial^m\chi + A_m^B F^{mn}\partial_n\chi + \frac{1}{2}\varepsilon^{mnpq}A_p^B F_{mn}\partial_q\phi\Big]. \qquad (3.12)
\end{aligned}
$$

Here, for later convenience, we have decomposed the scalar field into the real and the imaginary parts $\varphi = \phi + i\chi$.

Lorentz invariant vacua are found for the constant scalar fields ϕ and χ. They are specified by the zeros of the potential

$$
V = \frac{4m^2}{\beta^2}\left\{\cos^2\left(\frac{\beta\phi}{2}\right)\sinh^2\left(\frac{\beta\chi}{2}\right) + \sin^2\left(\frac{\beta\phi}{2}\right)\cosh^2\left(\frac{\beta\chi}{2}\right)\right\}\Big\{\cosh(\beta\chi)\Big\}^{-1} \qquad (3.13)
$$

and given by

$$
\varphi = \frac{2\pi n}{\beta}\ (n = 0, \pm 1, \pm 2, \ldots), \qquad \chi = 0. \qquad (3.14)
$$

When the external gauge fields vanish $A_m^B = D^B = 0$, $A_m = D = 0$, then the vacua preserve all SUSY. The energy density corresponding to these vacua is exactly zero $\mathcal{E} = 0$.

4 BPS solitons and chiral soliton lattice ground states

In this section, we construct BPS domain walls and show that the ground state is a soliton lattice consisting of an array of BPS soliton and anti-BPS soliton, either in a strong magnetic field and/or large chemical potential, or in the presence of a background fermion condensate.

4.1 BPS conditions

We are now in a position to examine the BPS states in the model (3.10). The vanishing conditions on the SUSY transformations of fermions provide the BPS equations:

$$
\begin{aligned}
\delta\psi_\alpha =& i\sqrt{2}(\sigma^3)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\left(\partial_3\varphi \mp e^{i\eta}F\right) = 0, \\
\delta\lambda_\alpha =& (\sigma^{mn})_\alpha{}^\beta\xi_\beta F_{mn} + i\xi_\alpha D = 0, \\
\delta\lambda_\alpha^B =& (\sigma^{mn})_\alpha{}^\beta\xi_\beta F_{mn}^B + i\xi_\alpha D^B = 0. \qquad (4.1)
\end{aligned}
$$

where we have assumed the half SUSY projection on the parameters $\xi_\alpha = \mp i e^{i\eta}(\sigma^3)_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}$. Here η is a phase factor which will be set to zero in the following. The first condition in Eq. (4.1) implies the following BPS equation:

$$\varphi' \mp \frac{2m}{\beta}\sin\left(\frac{\beta}{2}\bar\varphi\right)\left\{\cos\left(\frac{\beta}{2}(\varphi-\bar\varphi)\right)+\frac{\beta^2}{2}i\sin\left(\frac{3\beta}{2}(\varphi-\bar\varphi)\right)A^B_m\partial^m(\varphi-\bar\varphi)\right\}^{-1}=0, \tag{4.2}$$

where the prime stands for the differentiation with respect to $x^3 = z$. It is convenient to decompose the complex scalar field into its real and imaginary parts $\varphi = \phi + i\chi$. Then, Eq. (4.2) can be rewritten as follows:

$$\begin{aligned}
\phi' \mp \frac{2m}{\beta}\sin\left(\frac{\beta}{2}\phi\right)\cosh\left(\frac{\beta}{2}\chi\right)\left[\cosh(\beta\chi)-\beta^2\sinh(3\beta\chi)A^B_m\partial^m\chi\right]^{-1}&=0,\\
\chi' \mp \frac{2m}{\beta}\sinh\left(\frac{\beta}{2}\chi\right)\cos\left(\frac{\beta}{2}\phi\right)\left[\cosh(\beta\chi)-\beta^2\sinh(3\beta\chi)A^B_m\partial^m\chi\right]^{-1}&=0.
\end{aligned} \tag{4.3}$$

The last two conditions in Eq. (4.1) can be trivially solved by $A_m = D = 0$, $A^B_m = D^B = 0$. Then, the solution to Eq. (4.3) is given by $\chi = 0$ and ϕ satisfying the following equation:

$$\phi' \mp \frac{2m}{\beta}\sin\left(\frac{\beta}{2}\phi\right)=0. \tag{4.4}$$

This equation with the upper(lower) sign is nothing but the (anti-)BPS equation in the sine-Gordon model. The solution to this equation is a single sine-Gordon (anti-)kink:

$$\phi(z)=\frac{4}{\beta}\arctan\left[e^{\pm m(z-c)}\right], \tag{4.5}$$

where c is the integration constant. Since we have $\phi \to 0, \pm\frac{2\pi}{\beta}$ at $z \to \pm\infty$, these solutions connect the two adjacent vacua $n = 0$ and $n = \pm 1$ in Eq. (3.14). The corresponding energy can be calculated as

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} dz\left[\phi'^2+\frac{4m^2}{\beta^2}\sin^2\left(\frac{\beta\phi}{2}\right)\right]\\
&= \int_{-\infty}^{\infty} dz\left[\phi' \mp \frac{2m}{\beta}\sin\left(\frac{\beta}{2}\phi\right)\right]^2 \pm \frac{8m}{\beta^2}\int_{-\infty}^{\infty} dz\,\frac{d}{dz}\left\{\cos\left(\frac{\beta}{2}\phi\right)\right\}\\
&= \frac{16m}{\beta^2}.
\end{aligned} \tag{4.6}$$

4.2 Chiral soliton lattice ground state in background gauge fields

We next study ground states in the presence of the external background gauge fields. The equation of motion for the auxiliary field D is given by

$$\cosh(\beta\chi)\,A^B_m\partial^m\chi = 0. \tag{4.7}$$

A solution to this constraint is found to be $A_m^B = (\mu_{\rm B}, 0, 0, 0)$ and $\chi = 0$ where the constant $\mu_{\rm B}$ is the baryon chemical potential. It is evident that this external background breaks SUSY.

Now we look for ground states where the quasi-NG boson χ remains in the SUSY vacuum $\chi = 0$. We also assume that the field strength (magnetic field) of the external $\mathrm{U}(1)_{\rm em}$ gauge field is constant. Then, the effective Lagrangian for the neutral pion is

$$\mathcal{L}_{\rm SChPT}\Big|_{\phi} = -\partial_i\phi\partial_i\phi - \frac{4m^2}{\beta^2}\sin^2\left(\frac{\beta}{2}\phi\right) + \mu_{\rm B}\beta B_i\partial_i\phi, \tag{4.8}$$

where the sum over the index i is assumed. This is nothing but the sine-Gordon model in the external magnetic field, or the chiral sine-Gordon model.

Let us assume $B_i = (0, 0, B_3)$ and a one-dimensional dependence of the configurations. Then, the energy density is

$$\mathcal{E} = (\phi')^2 + \frac{4m^2}{\beta^2}\sin^2\left(\frac{\beta}{2}\phi\right) - \mu_{\rm B}\beta B_3\phi'. \tag{4.9}$$

The last term which is a total derivative term is induced by the external background. The equation of motion for ϕ is

$$\phi'' = \frac{m^2}{\beta}\sin(\beta\phi). \tag{4.10}$$

Note that the last term in Eq. (4.8) does not contribute to the equation of motion.

A single sine-Gordon (anti-)kink solution in Eq. (4.5) remains a solution. In this case, the tension (energy per unit area) is

$$E = \frac{16m}{\beta^2} \mp 2\pi\mu_{\rm B}B_3, \tag{4.11}$$

where $\mp$ corresponds to the sign of the (anti-)BPS kink. Thus, the second term contributes negatively (positively) to the tension for the (anti-)kink. Therefore for

$$\mu_{\rm B}|B_3| \geq \frac{8m}{\pi\beta^2}, \tag{4.12}$$

equivalent to Eq. (1.1) in this parameterization, the energy of the BPS solutions is negative (zero) and is thus less than (or equal to) that of the SUSY vacua.

More precisely, at the boundary of Eq. (4.12), a single kink has a zero tension and is a ground state degenerate with the SUSY vacua. When the inequality in Eq. (4.12) holds, the ground state is more general solution, given by the CSL

$$\phi(z) = \pm\frac{2}{\beta}\mathrm{am}\Big(mk^{-1}(z-c), k\Big) \tag{4.13}$$

where $\mathrm{am}(x,k)$ is the amplitude of the Jacobi elliptic function, c is the integration constant and $0 \le k \le 1$ is the elliptic modulus parameter, determined below.

The energy per unit volume is given by

$$E = \frac{m}{2kK(k)}\left[\frac{4m}{\beta^2}\left\{\frac{2E(k)}{k} + \left(k - \frac{1}{k}\right)K(k)\right\} \mp 2\pi\mu_{\mathrm{B}}|B_3|\right], \tag{4.14}$$

where $K(k)$ and $E(k)$ are the complete elliptic integral of the first and the second kinds, respectively. Again, we take the upper sign for the ground state. The extremization of the energy in the expression (4.14) gives the condition for k:

$$\frac{E(k)}{k} = \frac{\pi\mu_{\mathrm{B}}\beta^2}{4m}|B_3|. \tag{4.15}$$

Then, this condition actually give the minimum energy, given by

$$E = \frac{4m}{\beta^2}\left(k - \frac{1}{k}\right)K(k) < 0. \tag{4.16}$$

This is clearly less than (or equal to) the energy of the SUSY vacua. As a larger background $\mu_{\mathrm{B}}|B_3|$, the elliptic modulus k is smaller and the lattice spacing becomes shorter ($k \to 0$ as $\mu_{\mathrm{B}}|B_3| \to \infty$). At the boundary of the inequality (4.12), the elliptic modulus is $k = 1$ recovering a single soliton with $E = 0$.

4.3 Chiral soliton lattice ground state in fermion condensates

We next examine the effects of fermion contributions in the background vector multiplets. When we keep the fermions in the background multiplets V and V^B and drop the chiral fermion ψ in Φ, we find that the interaction Lagrangian gives

$$\begin{aligned}
&-\frac{\beta}{2}\int d^4\theta\,\cos\left[\frac{\beta}{2}(\Phi - \Phi^\dagger)\right] V^B W_\alpha D^\alpha \Phi + \text{h.c.} \\
&\sim \beta\cos(\beta\chi)\Bigg\{ - A^B_m D\partial^m\chi + A^B_m F^{mn}\partial_n\chi + \frac{1}{2}\varepsilon^{mnpq}A^B_p F_{mn}\partial_q\phi \\
&\qquad + \frac{i}{2}\left(\lambda^B\bar{\sigma}^m\lambda - \bar{\lambda}\bar{\sigma}^m\lambda^B\right)\partial_m\phi - \frac{1}{2}\left(\lambda^B\bar{\sigma}^m\lambda + \bar{\lambda}\bar{\sigma}^m\lambda^B\right)\partial_m\chi \\
&\qquad + \frac{1}{2}(\lambda^B\lambda)F + \frac{1}{2}(\bar{\lambda}\bar{\lambda}^B)F^\dagger\Bigg\}.
\end{aligned} \tag{4.17}$$

Assuming that the background fermion bilinear consisting of the gaugino λ and the superpartner λ^B of the baryon gauge field

$$j^m = \frac{i}{2}(\lambda^B\bar{\sigma}^m\lambda - \bar{\lambda}\bar{\sigma}^m\lambda^B) \tag{4.18}$$

gets a constant VEV $\langle j^i \rangle$, then the energy density for the static field ϕ is given by

$$\mathcal{E}|_\phi = (\partial_i\phi)^2 + \frac{4m^2}{\beta^2}\sin^2\left(\frac{\beta}{2}\phi\right) \mp \beta\langle j^i\rangle\partial_i\phi. \tag{4.19}$$

Since the third term is the total derivative, it never contributes to the equation of motion for ϕ. However, it does contribute to the energy. For example, for a single sine-Gordon (anti-)kink solution, we find its tension as

$$E = \frac{16m}{\beta^2} \mp 2\pi\langle j^3\rangle \tag{4.20}$$

where we have assumed $\langle j^i\rangle = \delta^{i3}\langle j^3\rangle$. The second term with upper (lower) sign apparently provides the negative contribution for the positive (negative) VEV. Therefore, when the inequality

$$\left|\frac{i}{2}(\lambda^B\bar{\sigma}^m\lambda - \bar{\lambda}\bar{\sigma}^m\lambda^B)\right| \geq \frac{8m}{\pi\beta^2} \tag{4.21}$$

holds, the CSL is the ground state in the presence of this fermion condensation. This is the SUSY counterpart of the baryon density and external magnetic field.

It is useful to consider the scale of the fermion condensation (4.18). A famous example is the gaugino condensation for which a fermion bilinear acquires a non-zero VEV due to the instanton effects [113]. In this case, the fermion bilinear should behave as Λ^3 by the dimensional analysis. Here Λ is the dynamical scale of the model. Other examples in which fermion bilinear develops a VEV can be found in studies of the early universe. For example, some kinds of dynamics have been studied to generate a VEV of fermionic current density in the locally AdS spacetime [114, 115]. It has been discussed that the temporal component of a fermion current dominates the energy density for the gauge field in the early (de Sitter) universe and it causes the cosmic inflation [116, 117]. In any event, it is therefore natural that the left-hand side in Eq. (4.21) behaves like Λ^3 and we expect that the CSL ground state happens at $\Lambda \gg \sqrt[3]{\frac{m}{\beta^2}}$.

5 Summary and Discussion

In this paper, we have constructed the neutral pion sector of a manifestly supersymmetric extension of the SU(2) chiral Lagrangian with the WZW term in magnetic field at finite density, which is the SUSY chiral sine-Gordon model. We then have constructed (anti-)BPS domain wall (chiral soliton) and shown that the tension of the domain wall becomes negative and the ground state is CSL where SUSY is broken, when the background magnetic field and/or chemical potential are large enough or when there is a background fermion condensates in the form of a fermion bilinear consisting of the superpartner of the baryon gauge field and the gaugino.

In this paper, we have been able to construct only the neutral pion sector of the SUSY chiral Lagrangian with the WZW term in the superfield formalism. Constructing the full Lagrangian including charged pions remain a future problem. Once the charged pions can be taken into account, one can also discuss domain-wall Skyrmions as were studied in QCD [57, 58]. The domain-wall Skyrmions are composite states of a domain wall and Skyrmions, initially introduced in the field theoretical models in 3+1 dimensions [118–123] and in 2+1 dimensions [124–126]. If a baby Skyrmion in 2+1 dimensions, supported by $\pi_2(S^2) \simeq \mathbb{Z}$, is absorbed into a domain wall, it becomes a sine-Gordon soliton supported by $\pi_1(S^1) \simeq \mathbb{Z}$ in the domain-wall world line. Similarly, a Skyrmion in 3+1 dimensions, supported by $\pi_3(S^3) \simeq \mathbb{Z}$, is absorbed into a chiral soliton to become a topological lump (or baby Skyrmion) supported by $\pi_2(S^2) \simeq \mathbb{Z}$ in the solitons's worldvolume [57]. SUSY extensions of these cases are worth to be investigated.[3] In particular, whether such a composite state can be a 1/4 BPS state [71–74, 129–131] is one of interesting direction to be explored.

In dense QCD, a CSL state is unstable due to the charged pion condensation in a region of higher density and/or stronger magnetic field, asymptotically expressed at B larger than $16\pi^4 f_\pi^4/\mu_{\rm B}^2$ [43] above which tachyon appears and the charged pions are condensed. Consequently, the CSL becomes unstable, where an Abrikosov's vortex lattice was proposed as a consequence of the charged pion condensation [62, 63]. It is an open question whether such an instability is present in our SUSY extension. Although SUSY is completely broken in the total configuration of CSL, each (anti-)soliton is an (anti-)BPS state preserving a half SUSY and is stable at least when individual solitons are well separated. We thus expect that the stability is enhanced in our SUSY extension.

We have discussed bosonic and fermionic backgrounds separately. If there are both kinds of backgrounds simultaneously in different spatial directions, say $B_i = (0, 0, B_3)$ and $\langle j^i \rangle = (0, j^2, 0)$, sine-Gordon solitons have two preferable directions. It is an open question whether the ground state is still a CSL in a certain direction linearly determined from the two directions or a soliton junction.

In this paper, we have discussed only the leading order $\mathcal{O}(p^2)$ of the ChPT. The higher order term of SUSY ChPT was discussed in Ref. [87] by using SUSY higher derivative terms free from ghost and auxiliary field problem [81–89]. SUSY may help us to consider ChPT in a more controllable manner and may eventually uncover a phase structure of non-SUSY QCD under extreme conditions such as strong magnetic field and/or rapid rotation.

As mentioned in the introduction, chiral magnets also admit solitonic ground states

[3] However, it is worth to mention that a supersymmetric extension of the Skyrme model does not contain quadratic derivative term [127], allowing only non-BPS solitons [128].

such as CSL and a Skyrmion lattice. This model can be written as a $\mathbb{C}P^1$ model with a background non-Abelian gauge field [132]. It can be made supersymmetric and can be embedded into string theory [24]. However, the bosonic part of the model was given only in terms of component fields and the fermionic part is not given [24], and so it is a future problem to construct the chiral magnets in the superfield formalism to study e. g. SUSY breaking. It is desirable to have a unified understanding of inhomogeneous ground states in SUSY field theories with background gauge fields.

Acknowledgments

This work is supported in part by JSPS KAKENHI [Grants No. JP22H01221 (MN)] and the WPI program "Sustainability with Knotted Chiral Meta Matter (SKCM^2)" at Hiroshima University (MN).

A Fayet-Iliopoulos term

In this appendix, we show that the introduction of the Fayet-Iliopoulos term in our model causes inconsistencies. The Lagrangian may be given by

$$\mathcal{L} = \mathcal{L}_{\text{SChPT}} + \xi \int d^4\theta\, V, \tag{A.1}$$

where ξ is the Fayet-Iliopoulos parameter. Then, the D-term condition is found to be

$$-\beta \cosh(\beta\chi) A_m^B \partial^m \chi + \frac{1}{2}\xi = 0. \tag{A.2}$$

For the given background $A_m^B = (\mu_{\text{B}}, 0, 0, 0)$, we have the condition

$$\beta\mu_{\text{B}} \cosh(\beta\chi)\dot{\chi} = -\frac{1}{2}\xi. \tag{A.3}$$

The solution to this equation is

$$\chi(t) = \frac{1}{4}\arctan\Big[-\frac{\xi t}{2\mu_{\text{B}}}t + c\Big], \tag{A.4}$$

where c is the integration constant. On the other hand, for $\phi = 0$, the equation of motion for χ is given by

$$\begin{aligned} &-\beta \sinh(\beta\chi)\partial_m\chi\partial^m\chi - \frac{2m^2}{\beta}\frac{\sinh(\beta\chi)}{\cosh^2(\beta\chi)} + \beta\sinh(\beta\chi)A_m^B F^{mn}\partial_n\chi \\ &-\partial_m\left[-2\cosh(\beta\chi)\partial^m\chi + \cosh(\beta\chi)A_m^B F^{mn}\partial_n\chi\right] = 0. \end{aligned} \tag{A.5}$$

Since the solution (A.4) explicitly depends on ξ but the equation of motion does not, it is obvious that the solution (A.4) never satisfies the equation of motion unless $\xi = c = 0$ and $\chi = 0$.

References

[1] P. Fulde and R. A. Ferrell, *Superconductivity in a Strong Spin-Exchange Field*, *Phys. Rev.* **135** (1964) A550.

[2] A. I. Larkin and Y. N. Ovchinnikov, *Nonuniform state of superconductors*, *Zh. Eksp. Teor. Fiz.* **47** (1964) 1136.

[3] K. Machida and H. Nakanishi, *Superconductivity under a ferromagnetic molecular field*, *Phys. Rev. B* **30** (1984) 122.

[4] R. Yoshii, S. Tsuchiya, G. Marmorini and M. Nitta, *Spin imbalance effect on Larkin-Ovchinnikov-Fulde-Ferrel state*, *Phys. Rev. B* **84** (2011) 024503 [1101.1578].

[5] W. P. Su, J. R. Schrieffer and A. J. Heeger, *Soliton excitations in polyacetylene*, *Phys. Rev. B* **22** (1980) 2099.

[6] H. Takayama, Y. R. Lin-Liu and K. Maki, *Continuum model for solitons in polyacetylene*, *Phys. Rev. B* **21** (1980) 2388.

[7] A. J. Niemi and G. W. Semenoff, *Fermion Number Fractionization in Quantum Field Theory*, *Phys. Rept.* **135** (1986) 99.

[8] G. Basar and G. V. Dunne, *Self-consistent crystalline condensate in chiral Gross-Neveu and Bogoliubov-de Gennes systems*, *Phys. Rev. Lett.* **100** (2008) 200404 [0803.1501].

[9] G. Basar and G. V. Dunne, *A Twisted Kink Crystal in the Chiral Gross-Neveu model*, *Phys. Rev. D* **78** (2008) 065022 [0806.2659].

[10] G. Basar, G. V. Dunne and M. Thies, *Inhomogeneous Condensates in the Thermodynamics of the Chiral NJL(2) model*, *Phys. Rev. D* **79** (2009) 105012 [0903.1868].

[11] E. Nakano and T. Tatsumi, *Chiral symmetry and density wave in quark matter*, *Phys. Rev. D* **71** (2005) 114006 [hep-ph/0411350].

[12] D. Nickel, *How many phases meet at the chiral critical point?*, *Phys. Rev. Lett.* **103** (2009) 072301 [0902.1778].

[13] R. Casalbuoni and G. Nardulli, *Inhomogeneous superconductivity in condensed matter and QCD*, *Rev. Mod. Phys.* **76** (2004) 263 [hep-ph/0305069].

[14] R. Anglani, R. Casalbuoni, M. Ciminale, N. Ippolito, R. Gatto, M. Mannarelli et al., *Crystalline color superconductors*, *Rev. Mod. Phys.* **86** (2014) 509 [1302.4264].

[15] M. Buballa and S. Carignano, *Inhomogeneous chiral condensates*, *Prog. Part. Nucl. Phys.* **81** (2015) 39 [1406.1367].

[16] Y. Hidaka, K. Kamikado, T. Kanazawa and T. Noumi, *Phonons, pions and quasi-long-range order in spatially modulated chiral condensates*, *Phys. Rev. D* **92** (2015) 034003 [1505.00848].

[17] T.-G. Lee, E. Nakano, Y. Tsue, T. Tatsumi and B. Friman, *Landau-Peierls instability in a Fulde-Ferrell type inhomogeneous chiral condensed phase*, *Phys. Rev. D* **92** (2015) 034024 [1504.03185].

[18] I. Dzyaloshinskii, *A Thermodynamic Theory of 'Weak' Ferromagnetism of Antiferromagnetics*, *J. Phys. Chem. Solids* **4** (1958) 241.

[19] T. Moriya, *Anisotropic Superexchange Interaction and Weak Ferromagnetism*, *Phys. Rev.* **120** (1960) 91.

[20] Y. Togawa, T. Koyama, K. Takayanagi, S. Mori, Y. Kousaka, J. Akimitsu et al., *Chiral magnetic soliton lattice on a chiral helimagnet*, *Phys. Rev. Lett.* **108** (2012) 107202.

[21] J. Kishine and A. Ovchinnikov, *Chapter one - theory of monoaxial chiral helimagnet*, vol. 66 of *Solid State Physics*, pp. 1–130, Academic Press, (2015), DOI.

[22] A. A. Tereshchenko, A. S. Ovchinnikov, I. Proskurin, E. V. Sinitsyn and J.-I. Kishine, *Theory of magnetoelastic resonance in a monoaxial chiral helimagnet*, *Phys. Rev. B* **97** (2018) 184303.

[23] C. Ross, N. Sakai and M. Nitta, *Exact ground states and domain walls in one dimensional chiral magnets*, *JHEP* **12** (2021) 163 [2012.08800].

[24] Y. Amari and M. Nitta, *Chiral magnets from string theory*, *JHEP* **11** (2023) 212 [2307.11113].

[25] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer et al., *Skyrmion lattice in a chiral magnet*, *Science* **323** (2009) 915.

[26] X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui et al., *Real-space observation of a two-dimensional skyrmion crystal*, *Nature* **465** (2010) 901.

[27] S. Heinze, K. von Bergmann, M. Menzel, J. Brede, A. Kubetzka, R. Wiesendanger et al., *Spontaneous atomic-scale magnetic skyrmion lattice in two dimensions*, *Nat. Phys.* **7** (2011) 713.

[28] U. K. Rossler, A. N. Bogdanov and C. Pfleiderer, *Spontaneous skyrmion ground states in magnetic metals*, *Nature* **442** (2006) 797.

[29] S.-Z. Lin, A. Saxena and C. D. Batista, *Skyrmion fractionalization and merons in chiral magnets with easy-plane anisotropy*, *Phys. Rev. B* **91** (2015) 224407.

[30] J. H. Han, J. Zang, Z. Yang, J.-H. Park and N. Nagaosa, *Skyrmion Lattice in Two-Dimensional Chiral Magnet*, *Phys. Rev. B* **82** (2010) 094429.

[31] C. Ross, N. Sakai and M. Nitta, *Skyrmion interactions and lattices in chiral magnets: analytical results*, *JHEP* **02** (2021) 095 [2003.07147].

[32] Y. J. Lin, R. L. Compton, K. Jiménez-García, J. V. Porto and I. B. Spielman, *Synthetic magnetic fields for ultracold neutral atoms*, *Nature* **462** (2009) 628.

[33] N. Goldman, G. Juzeliūnas, P. Öhberg and I. B. Spielman, *Light-induced gauge fields for ultracold atoms*, *Rept. Prog. Phys.* **77** (2014) 126401 [1308.6533].

[34] T. Kawakami, T. Mizushima and K. Machida, *Textures of $f = 2$ spinor bose-einstein condensates with spin-orbit coupling*, *Phys. Rev. A* **84** (2011) 011607.

[35] T. Kawakami, T. Mizushima, M. Nitta and K. Machida, *Stable Skyrmions in SU(2) Gauged Bose-Einstein Condensates*, *Phys. Rev. Lett.* **109** (2012) 015301 [1204.3177].

[36] K. Fukushima and T. Hatsuda, *The phase diagram of dense QCD*, *Rept. Prog. Phys.* **74** (2011) 014001 [1005.4814].

[37] H. Leutwyler, *On the foundations of chiral perturbation theory*, *Annals Phys.* **235** (1994) 165 [hep-ph/9311274].

[38] D. T. Son and A. R. Zhitnitsky, *Quantum anomalies in dense matter*, *Phys. Rev. D* **70** (2004) 074018 [hep-ph/0405216].

[39] D. T. Son and M. A. Stephanov, *Axial anomaly and magnetism of nuclear and quark matter*, *Phys. Rev. D* **77** (2008) 014021 [0710.1084].

[40] J. Goldstone and F. Wilczek, *Fractional Quantum Numbers on Solitons*, *Phys. Rev. Lett.* **47** (1981) 986.

[41] E. Witten, *Global Aspects of Current Algebra*, *Nucl. Phys. B* **223** (1983) 422.

[42] M. Eto, K. Hashimoto and T. Hatsuda, *Ferromagnetic neutron stars: axial anomaly, dense neutron matter, and pionic wall*, *Phys. Rev. D* **88** (2013) 081701 [1209.4814].

[43] T. Brauner and N. Yamamoto, *Chiral Soliton Lattice and Charged Pion Condensation in Strong Magnetic Fields*, *JHEP* **04** (2017) 132 [1609.05213].

[44] T. Brauner and S. V. Kadam, *Anomalous low-temperature thermodynamics of QCD in strong magnetic fields*, *JHEP* **11** (2017) 103 [1706.04514].

[45] T. Brauner and S. Kadam, *Anomalous electrodynamics of neutral pion matter in strong magnetic fields*, *JHEP* **03** (2017) 015 [1701.06793].

[46] T. Brauner, H. Kolešová and N. Yamamoto, *Chiral soliton lattice phase in warm QCD*, *Phys. Lett. B* **823** (2021) 136767 [2108.10044].

[47] T. Brauner and H. Kolešová, *Chiral soliton lattice at next-to-leading order*, *JHEP* **07** (2023) 163 [2302.06902].

[48] X.-G. Huang, K. Nishimura and N. Yamamoto, *Anomalous effects of dense matter under rotation*, *JHEP* **02** (2018) 069 [1711.02190].

[49] K. Nishimura and N. Yamamoto, *Topological term, QCD anomaly, and the η' chiral soliton lattice in rotating baryonic matter*, *JHEP* **07** (2020) 196 [2003.13945].

[50] H.-L. Chen, X.-G. Huang and J. Liao, *QCD phase structure under rotation*, *Lect. Notes Phys.* **987** (2021) 349 [2108.00586].

[51] M. Eto, K. Nishimura and M. Nitta, *Phases of rotating baryonic matter: non-Abelian chiral soliton lattices, antiferro-isospin chains, and ferri/ferromagnetic magnetization*, *JHEP* **08** (2022) 305 [2112.01381].

[52] M. Eto, K. Nishimura and M. Nitta, *Domain-wall Skyrmion phase in a rapidly rotating QCD matter*, *JHEP* **03** (2024) 019 [2310.17511].

[53] M. Eto, K. Nishimura and M. Nitta, *Non-Abelian chiral soliton lattice in rotating QCD matter: Nambu-Goldstone and excited modes*, *JHEP* **03** (2024) 035 [2312.10927].

[54] M. Eto and M. Nitta, *Quantum nucleation of topological solitons*, *JHEP* **09** (2022) 077 [2207.00211].

[55] T. Higaki, K. Kamada and K. Nishimura, *Formation of a chiral soliton lattice*, *Phys. Rev. D* **106** (2022) 096022 [2207.00212].

[56] Z. Qiu and M. Nitta, *Quasicrystals in QCD*, *JHEP* **05** (2023) 170 [2304.05089].

[57] M. Eto, K. Nishimura and M. Nitta, *How baryons appear in low-energy QCD: Domain-wall Skyrmion phase in strong magnetic fields*, 2304.02940.

[58] M. Eto, K. Nishimura and M. Nitta, *Phase diagram of QCD matter with magnetic field: domain-wall Skyrmion chain in chiral soliton lattice*, *JHEP* **12** (2023) 032 [2311.01112].

[59] M. Kawaguchi, Y.-L. Ma and S. Matsuzaki, *Chiral soliton lattice effect on baryonic matter from a skyrmion crystal model*, *Phys. Rev. C* **100** (2019) 025207 [1810.12880].

[60] S. Chen, K. Fukushima and Z. Qiu, *Skyrmions in a magnetic field and $\pi 0$ domain wall formation in dense nuclear matter*, *Phys. Rev. D* **105** (2022) L011502 [2104.11482].

[61] S. Chen, K. Fukushima and Z. Qiu, *Magnetic enhancement of baryon confinement modeled via a deformed Skyrmion*, *Phys. Lett. B* **843** (2023) 137992 [2303.04692].

[62] G. W. Evans and A. Schmitt, *Chiral anomaly induces superconducting baryon crystal*, *JHEP* **09** (2022) 192 [2206.01227].

[63] G. W. Evans and A. Schmitt, *Chiral Soliton Lattice turns into 3D crystal*, 2311.03880.

[64] T. Brauner, G. Filios and H. Kolešová, *Anomaly-Induced Inhomogeneous Phase in Quark Matter without the Sign Problem*, *Phys. Rev. Lett.* **123** (2019) 012001 [1902.07522].

[65] T. Brauner, G. Filios and H. Kolešová, *Chiral soliton lattice in QCD-like theories*, *JHEP* **12** (2019) 029 [1905.11409].

[66] H. P. Nilles, *Supersymmetry, Supergravity and Particle Physics*, *Phys. Rept.* **110** (1984) 1.

[67] H. E. Haber and G. L. Kane, *The Search for Supersymmetry: Probing Physics Beyond the Standard Model*, *Phys. Rept.* **117** (1985) 75.

[68] G. L. Kane, C. F. Kolda, L. Roszkowski and J. D. Wells, *Study of constrained minimal supersymmetry*, *Phys. Rev. D* **49** (1994) 6173 [hep-ph/9312272].

[69] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, Nucl. Phys. B **431** (1994) 484 [hep-th/9408099].

[70] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl. Phys. B **426** (1994) 19 [hep-th/9407087], [Erratum: Nucl.Phys.B 430, 485–486 (1994)].

[71] D. Tong, *TASI lectures on solitons: Instantons, monopoles, vortices and kinks*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory*, 6, 2005, hep-th/0509216.

[72] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Solitons in the Higgs phase: The Moduli matrix approach*, J. Phys. A **39** (2006) R315 [hep-th/0602170].

[73] M. Shifman and A. Yung, *Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories*, Rev. Mod. Phys. **79** (2007) 1139 [hep-th/0703267].

[74] M. Shifman and A. Yung, *Supersymmetric Solitons*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, 7, 2023, 10.1017/9781009402200.

[75] D. Tong, *Quantum Vortex Strings: A Review*, Annals Phys. **324** (2009) 30 [0809.5060].

[76] K. A. Intriligator, N. Seiberg and D. Shih, *Dynamical SUSY breaking in meta-stable vacua*, JHEP **04** (2006) 021 [hep-th/0602239].

[77] M. Nitta, S. Sasaki and R. Yokokura, *Supersymmetry Breaking in Spatially Modulated Vacua*, Phys. Rev. D **96** (2017) 105022 [1706.05232].

[78] S. Bjarke Gudnason, M. Nitta, S. Sasaki and R. Yokokura, *Supersymmetry breaking and ghost Goldstino in modulated vacua*, Phys. Rev. D **99** (2019) 045012 [1812.09078].

[79] M. Nitta, S. Sasaki and R. Yokokura, *Spatially Modulated Vacua in a Lorentz-invariant Scalar Field Theory*, Eur. Phys. J. C **78** (2018) 754 [1706.02938].

[80] S. B. Gudnason, M. Nitta, S. Sasaki and R. Yokokura, *Temporally, spatially, or lightlike modulated vacua in Lorentz invariant theories*, Phys. Rev. D **99** (2019) 045011 [1810.11361].

[81] J. Khoury, J.-L. Lehners and B. Ovrut, *Supersymmetric $P(X,\phi)$ and the Ghost Condensate*, Phys. Rev. D **83** (2011) 125031 [1012.3748].

[82] J. Khoury, J.-L. Lehners and B. A. Ovrut, *Supersymmetric Galileons*, Phys. Rev. D **84** (2011) 043521 [1103.0003].

[83] M. Koehn, J.-L. Lehners and B. A. Ovrut, *Higher-Derivative Chiral Superfield Actions Coupled to N=1 Supergravity*, Phys. Rev. D **86** (2012) 085019 [1207.3798].

[84] C. Adam, J. M. Queiruga, J. Sanchez-Guillen and A. Wereszczynski, *N=1 supersymmetric extension of the baby Skyrme model*, Phys. Rev. D **84** (2011) 025008 [1105.1168].

[85] F. Farakos and A. Kehagias, *Emerging Potentials in Higher-Derivative Gauged Chiral Models Coupled to N=1 Supergravity*, JHEP **11** (2012) 077 [1207.4767].

[86] M. Nitta and S. Sasaki, *BPS States in Supersymmetric Chiral Models with Higher Derivative Terms*, Phys. Rev. D **90** (2014) 105001 [1406.7647].

[87] M. Nitta and S. Sasaki, *Higher Derivative Corrections to Manifestly Supersymmetric Nonlinear Realizations*, Phys. Rev. D **90** (2014) 105002 [1408.4210].

[88] M. Nitta and S. Sasaki, *Classifying BPS States in Supersymmetric Gauge Theories Coupled to Higher Derivative Chiral Models*, Phys. Rev. D **91** (2015) 125025 [1504.08123].

[89] M. Nitta and S. Sasaki, *Higher Derivative Supersymmetric Nonlinear Sigma Models on Hermitian Symmetric Spaces, and BPS States Therein*, Phys. Rev. D **103** (2021) 025001 [2011.07973].

[90] P. Di Vecchia and S. Ferrara, *Classical Solutions in Two-Dimensional Supersymmetric Field Theories*, Nucl. Phys. B **130** (1977) 93.

[91] J. Hruby, *On the Supersymmetric Sine-Gordon Model and a Two-Dimensional Bag*, Nucl. Phys. B **131** (1977) 275.

[92] T. Marinucci and S. Sciuto, *Quantization of the Conservation Laws in the Supersymmetric Sine-Gordon Model*, Nucl. Phys. B **156** (1979) 144.

[93] M. Cassandro, F. Nicolo and B. Scoppola, *The (N=1) Supersymmetric Sine-Gordon Model in Two-Dimensions. 1*, Commun. Math. Phys. **122** (1989) 681.

[94] M. Cassandro, F. Nicolo and B. Scoppola, *The (N=1) Supersymmetric Sine-Gordon Model in Two-Dimensions. 2*, Commun. Math. Phys. **123** (1989) 123.

[95] D. Nemeschansky and R. Rohm, *Anomaly Constraints on Supersymmetric Effective Lagrangian*, Nucl. Phys. B **249** (1985) 157.

[96] M. Nitta, *A Note on supersymmetric WZW term in four dimensions*, Mod. Phys. Lett. A **15** (2000) 2327 [hep-th/0101166].

[97] S. J. Gates, Jr., *Why auxiliary fields matter: The Strange case of the 4-D, N=1 supersymmetric QCD effective action*, Phys. Lett. B **365** (1996) 132 [hep-th/9508153].

[98] S. J. Gates, Jr., *Why auxiliary fields matter: The strange case of the 4-D, N=1 supersymmetric QCD effective action. 2.*, Nucl. Phys. B **485** (1997) 145 [hep-th/9606109].

[99] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt and S. Penati, *The Superspace WZNW action for 4-D, N=1 supersymmetric QCD*, Phys. Lett. B **503** (2001) 349 [hep-ph/0012301].

[100] I. L. Buchbinder, S. M. Kuzenko and A. Y. Petrov, *Superfield chiral effective potential*, Phys. Lett. B **321** (1994) 372.

[101] I. L. Buchbinder, S. Kuzenko and Z. Yarevskaya, *Supersymmetric effective potential: Superfield approach*, Nucl. Phys. B **411** (1994) 665.

[102] A. T. Banin, I. L. Buchbinder and N. G. Pletnev, *On quantum properties of the four-dimensional generic chiral superfield model*, Phys. Rev. D **74** (2006) 045010 [hep-th/0606242].

[103] S. B. Gudnason, M. Nitta and S. Sasaki, *BPS pion domain walls in the supersymmetric chiral Lagrangian*, *Phys. Rev. D* **94** (2016) 025003 [1602.02520].

[104] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, Princeton, NJ, USA, 1992.

[105] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, *Structure of Nonlinear Realization in Supersymmetric Theories*, *Phys. Lett. B* **138** (1984) 94.

[106] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, *Nonlinear Realization in Supersymmetric Theories*, *Prog. Theor. Phys.* **72** (1984) 313.

[107] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, *Nonlinear Realization in Supersymmetric Theories. 2.*, *Prog. Theor. Phys.* **72** (1984) 1207.

[108] M. Bando, T. Kugo and K. Yamawaki, *Nonlinear Realization and Hidden Local Symmetries*, *Phys. Rept.* **164** (1988) 217.

[109] G. M. Shore, *Geometry of Supersymmetric σ Models*, *Nucl. Phys. B* **320** (1989) 202.

[110] A. C. W. Kotcheff and G. M. Shore, *Kahler σ Models From Supersymmetric Gauge Theories*, *Int. J. Mod. Phys. A* **4** (1989) 4391.

[111] K. Higashijima, M. Nitta, K. Ohta and N. Ohta, *Low-energy theorems in N=1 supersymmetric theory*, *Prog. Theor. Phys.* **98** (1997) 1165 [hep-th/9706219].

[112] M. Nitta, *Moduli space of global symmetry in N=1 supersymmetric theories and the quasiNambu-Goldstone bosons*, *Int. J. Mod. Phys. A* **14** (1999) 2397 [hep-th/9805038].

[113] N. M. Davies, T. J. Hollowood, V. V. Khoze and M. P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, *Nucl. Phys. B* **559** (1999) 123 [hep-th/9905015].

[114] S. Bellucci, A. A. Saharian, D. H. Simonyan and V. V. Vardanyan, *Fermionic currents in topologically nontrivial braneworlds*, *Phys. Rev. D* **98** (2018) 085020 [1808.01577].

[115] S. Bellucci, A. A. Saharian, H. G. Sargsyan and V. V. Vardanyan, *Fermionic vacuum currents in topologically nontrivial braneworlds: Two-brane geometry*, *Phys. Rev. D* **101** (2020) 045020 [1907.13379].

[116] S. Alexander, A. Marciano and D. Spergel, *Chern-Simons Inflation and Baryogenesis*, *JCAP* **04** (2013) 046 [1107.0318].

[117] S. Alexander, D. Jyoti, A. Kosowsky and A. Marciano, *Dynamics of Gauge Field Inflation*, *JCAP* **05** (2015) 005 [1408.4118].

[118] M. Nitta, *Correspondence between Skyrmions in 2+1 and 3+1 Dimensions*, *Phys. Rev. D* **87** (2013) 025013 [1210.2233].

[119] M. Nitta, *Matryoshka Skyrmions*, *Nucl. Phys. B* **872** (2013) 62 [1211.4916].

[120] S. B. Gudnason and M. Nitta, *Domain wall Skyrmions*, *Phys. Rev. D* **89** (2014) 085022 [1403.1245].

[121] S. B. Gudnason and M. Nitta, *Incarnations of Skyrmions*, *Phys. Rev. D* **90** (2014) 085007 [1407.7210].

[122] M. Eto and M. Nitta, *Non-Abelian Sine-Gordon Solitons: Correspondence between $SU(N)$ Skyrmions and $\mathbb{C}P^{N-1}$ Lumps*, *Phys. Rev. D* **91** (2015) 085044 [1501.07038].

[123] M. Nitta, *Relations among topological solitons*, *Phys. Rev. D* **105** (2022) 105006 [2202.03929].

[124] M. Nitta, *Josephson vortices and the Atiyah-Manton construction*, *Phys. Rev. D* **86** (2012) 125004 [1207.6958].

[125] M. Kobayashi and M. Nitta, *Sine-Gordon kinks on a domain wall ring*, *Phys. Rev. D* **87** (2013) 085003 [1302.0989].

[126] P. Jennings and P. Sutcliffe, *The dynamics of domain wall Skyrmions*, *J. Phys. A* **46** (2013) 465401 [1305.2869].

[127] S. B. Gudnason, M. Nitta and S. Sasaki, *A supersymmetric Skyrme model*, *JHEP* **02** (2016) 074 [1512.07557].

[128] S. B. Gudnason, M. Nitta and S. Sasaki, *Topological solitons in the supersymmetric Skyrme model*, *JHEP* **01** (2017) 014 [1608.03526].

[129] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *All exact solutions of a 1/4 Bogomol'nyi-Prasad-Sommerfield equation*, *Phys. Rev. D* **71** (2005) 065018 [hep-th/0405129].

[130] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Instantons in the Higgs phase*, *Phys. Rev. D* **72** (2005) 025011 [hep-th/0412048].

[131] M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, *1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations*, *Nucl. Phys. B* **752** (2006) 140 [hep-th/0506257].

[132] B. J. Schroers, *Gauged Sigma Models and Magnetic Skyrmions*, *SciPost Phys.* **7** (2019) 030 [1905.06285].