WELL-POSEDNESS FOR FRACTIONAL HARDY-HÉNON PARABOLIC EQUATIONS WITH FRACTIONAL BROWNIAN NOISE

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Abstract. We investigate the fractional Hardy-Hénon equation with fractional Brownian noise

$$\partial_t u(t) + (-\Delta)^{\theta/2} u(t) = |x|^{-\gamma} |u(t)|^{p-1} u(t) + \mu\, \partial_t B^H(t),$$

where $\theta > 0$, $p > 1$, $\gamma \geq 0$, $\mu \in \mathbb{R}$, and the random forcing B^H is the fractional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (0,1)$. We obtain the local existence and uniqueness of mild solutions under tailored conditions on the parameters of the equation.

1. Introduction

We deal with the initial value problem for the fractional Hardy-Hénon equation with fractional Brownian noise

$$\begin{cases} \partial_t u(t) + (-\Delta)^{\theta/2} u(t) = |x|^{-\gamma} |u(t)|^{p-1} u(t) + \mu\, \partial_t B^H(t), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\theta > 0$, $p > 1$, $\gamma \geq 0$, $\mu \in \mathbb{R}$, and the random forcing B^H is the fractional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (0,1)$. When $H = 1/2$, $B^{1/2}$ corresponds to the standard Brownian motion.

As is customary, we analyze (1.1) by considering the associated integral equation:

$$\begin{aligned} u(t) = \mathrm{e}^{-t(-\Delta)^{\theta/2}} u_0 &+ \int_0^t \mathrm{e}^{(t-s)(-\Delta)^{\theta/2}} \left(|x|^{-\gamma} |u(s)|^{p-1} u(s) \right) ds \\ &+ \mu \int_0^t \mathrm{e}^{(t-s)(-\Delta)^{\theta/2}}\, dB^H(s), \end{aligned} \tag{1.2}$$

where $\mathrm{e}^{t\Delta}$ is the linear heat semi-group.

Date: April 19, 2024.

2020 *Mathematics Subject Classification.* 60H15, 60H30, 35R60, 35K05.

Key words and phrases. Stochastic PDE's, fractional Hardy-Hénon parabolic equation, mild solution, fractional Brownian motion.

The initial value problem (1.1) without fractional noise (i.e., when $\mu = 0$):

$$\partial_t u + (-\Delta)^{\theta/2} u = |x|^{-\gamma} |u|^{p-1} u, \tag{1.3}$$

has garnered significant attention within the mathematical community. In particular, the case $\theta = 2$ and $\gamma = 0$ corresponds to the classical semi-linear heat equation, which has been extensively studied (see, for example, [6, 16, 27, 31, 32, 36–38]). For nonlinearities that grow faster than a power, further investigations can be found in [18–20, 23, 24]. When considering $\theta = 2$ and $\gamma > 0$, known as the Hardy-Hénon parabolic model, we recommend referring to [3, 34] and the references provided therein for more detailed discussions and analysis.

In [28], the global well-posedness of solutions to (1.3) for general values of θ was extensively examined. The study primarily focused on small initial data within pseudomeasure spaces. Furthermore, an unified method was introduced in [29] to handle the well-posedness of (1.3) in the case of $\gamma = 0$ for general nonlinearities.

It is worth noting that the case $\gamma > 0$ for $\theta = 2$ was studied in [3]. In our analysis, we will leverage the approach employed in [3], along with the properties of the fractional heat kernel as described in [29]. By combining these techniques, we aim to address our specific problem and gain valuable insights into its solution.

The development of stochastic calculus for fractional Brownian motion (fBm) has naturally paved the way for investigating stochastic partial differential equations (SPDEs) driven by fBm. The study of such SPDEs represents a significant research avenue within probability theory and stochastic analysis, and it has yielded numerous insightful findings. This line of research has stemmed from the extensive applications of fBm in various fields, motivating the exploration of SPDEs as a powerful tool to model and analyze complex systems.

In recent years, the investigation of stochastic partial differential equations (SPDEs) driven by fractional noise has garnered significant attention within the mathematical community. This growing interest stems from both theoretical considerations and the broad range of applications in fields such as physics, biology, hydrology, and other scientific disciplines. Of particular significance is the study of the well-posedness of semilinear stochastic parabolic equations driven by infinite-dimensional fractional noise, which has been a subject of special interest. Noteworthy contributions in this area can be found in works such as [14, 26, 30, 33].

Moreover, the exploration of alternative types of noise has also been undertaken in the literature, as evidenced by studies conducted in [5,7,9]. These investigations further enhance our understanding of the dynamics and behavior of SPDEs, offering valuable insights into their mathematical properties and practical applications.

In this context, we focus on studying the semilinear Hardy-Hénon equation driven by fractional Brownian noise given by (1.1). Our primary objective is to find local-in-time solutions to (1.1) for initial data belonging to a specific Lebesgue space. To achieve this, we introduce the concept of a mild solution for (1.1), which provides a suitable framework for characterizing the solution behavior.

Definition 1.1. *Let $T > 0$ and $1 \leq q \leq \infty$. A measurable function $u : \Omega \times [0, T] \to L^q$ is a mild solution of* (1.1) *if*

i) $u \in C([0, T]; L^q)$.

ii) *u satisfies* (1.2) *with probability one.*

We have successfully established the following result regarding the local well-posedness of the problem at hand.

Theorem 1.1. *Let $N \geq 1$, $p > 1$, and $\mu \neq 0$. Assume that*

$$\begin{gathered} 2\theta > 1, \quad 0 < \gamma < \min(\theta,\, N), \\ \max\left(\frac{1}{q},\, \frac{N}{2\theta},\, \frac{1}{2}\right) < H < 1, \quad \max\left(\frac{Np}{N-\gamma},\, \frac{N(p-1)}{\theta-\gamma}\right) < q < \infty. \end{gathered} \tag{1.4}$$

Then, for any $u_0 \in L^q$, there exists $T = T(\|\mathbf{u}_0\|_q, B^H) > 0$ such that problem (1.1) *possesses a unique mild solution on $[0, T]$.*

Remark 1.1.

(i) *The Cauchy problem* (1.1) *was studied in [8] for $\theta = 2$, $\gamma = 0$ and $\mu = 1$. The local existence was proved under restrictive assumptions.*

(ii) *In [22], the authors improved the results of [8] by relaxing the assumptions on γ, q.*

(iii) *The assumption $H > \frac{N}{2\theta}$ ensures that the solution is a space-time continuous random field. More precisely, the parabolic equation is formulated as a stochastic Cauchy problem within a suitable Lebesgue space, and its solution is sought in the mild form, specifically as a stochastic convolution integral. See [11] for further explanations.*

The paper is organized as follows. In Section 2, we offer a comprehensive review of the essential background information necessary for understanding the subsequent proofs. Specifically, we provide a detailed overview of the relevant concepts and mathematical tools that form the foundation of our analysis. Section 3 is devoted to the proof of our main result, namely Theorem 1.1.

2. Preliminaries

2.1. **Fractional Brownian motion.** In the subsequent discussion, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space. Fractional Brownian motion was initially introduced and investigated by Kolmogorov [21] within the framework of Hilbert spaces. Let $[0, T]$ represent a time interval with an arbitrary fixed $T > 0$. A fractional Brownian motion with a Hurst parameter $H \in (0, 1)$ is a centered Gaussian process denoted as B^H, possessing the following covariance structure:

$$R(s,t) := E(B^H(t)B^H(s)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

where $s, t \in [0, T]$. It is worth emphasizing that when the Hurst parameter takes the value of $H = 1/2$, the process $B^H(t)$ corresponds to the standard Brownian motion.

The fractional Brownian motion (fBm) can be alternatively defined as the unique self-similar Gaussian process exhibiting stationary increments

$$E[(B^H(t) - B^H(s))^2] = |t-s|^{2H},$$

and H-self similar

$$\left(\frac{1}{c^H} B^H(ct),\, t \geq 0\right) \stackrel{d}{=} {}^{1} \left(B^H(t),\, t \geq 0\right),$$

for all $c > 0$. Furthermore, the process B^H can be expressed using the following Wiener integral representation:

$$B^H(t) = \int_0^t K^H(t,s)\, dW(s). \tag{2.5}$$

In the above representation, $W = W(t);, t \in [0, T]$ denotes a Wiener process, and $K^H(t, s)$ represents the kernel defined as follows:

$$K^H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H(\frac{1}{2} - H) \int_s^t (r-s)^{H-\frac{3}{2}}(1 - (1 + (\frac{s}{r}))^{\frac{1}{2}-H})\, dr, \tag{2.6}$$

[1] The symbol $\stackrel{d}{=}$ denotes equality in the distributional sense.

where c_H is given by

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \right)^{\frac{1}{2}}.$$

From (2.6) we obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H(H - \frac{1}{2}) \left(\frac{s}{t}\right))^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

It is noteworthy that when $H > 1/2$, the kernel $K_H(t,s)$ exhibits regularity and can be expressed in a simpler form as follows:

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (r-s)^{H-\frac{3}{2}} r^{H-\frac{1}{2}} \, dr.$$

Let ε_H denote the linear space of step functions on $[0,T]$ with the following form:

$$\varphi(t) = \sum_{i=1}^{n} a_i 1_{(t_i,t_{i+1}]}(t), \tag{2.7}$$

where $t_1, ..., t_n \in [0,T]$, $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and by $\mathcal{H}$ the closure of ε_H with respect to the scalar product

$$\langle 1_{[0,t]}, \, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

For $\varphi \in \varepsilon_H$, characterized by the form (2.7), we define its Wiener integral with respect to the fractional Brownian motion as follows:

$$\int_0^T \varphi_s \, dB^H(s) = \sum_{i=1}^{n} a_i (B^H(t_{i+1}) - B^H(t_i)).$$

Evidently, the mapping

$$\varphi = \sum_{i=1}^{n} a_i 1_{(t_i,t_{i+1}]} \longrightarrow \int_0^T \varphi_s \, dB^H(s),$$

serves as an isometry between ε_H and the linear space span$\{B^H(t), \ t \in \mathbb{R}\}$ when regarded as a subspace of $L^2(\Omega)$. The image of an element $\Phi \in \mathcal{H}$ under this isometry is referred to as the Wiener integral of Φ with respect to B^H.

For any $s < T$, let us consider the operator $K_H^* : \mathcal{H} \to L^2([0,T])$ given by

$$(K_H^* \varphi)(s) = K^H(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K^H}{\partial r}(r,s) \, dr.$$

When $H > 1/2$, the operator K_H^* has the simpler expression

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K^H}{\partial r}(r,s) \, dr.$$

We refer to [1] for the proof that K_H^* constitutes an isometry between $\mathcal{H}$ and $L^2[0,T]$. Consequently, we can establish the following relationship between the Wiener process W and the fractional Brownian motion B^H:

$$\int_0^t \varphi(s)\, dB^H(s) = \int_0^t (K_H^*\varphi)(s)\, dW(s),$$

for every $t \in [0,T]$ and $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K_H^*\varphi \in L^2[0,T]$. Furthermore, it is worth recalling that for $\phi, \chi \in \mathcal{H}$ satisfying $\int_0^T \int_0^T |\phi(s)||\chi(t)||t-s|^{2H-2}\, dsdt < \infty$, their scalar product in $\mathcal{H}$ can be expressed as:

$$\langle \phi, \chi \rangle_H = H(2H-1) \int_0^T \int_0^T \phi(s)\chi(t)|t-s|^{2H-2}\, dsdt. \tag{2.8}$$

This formula establishes the inner product relationship between ϕ and χ in the space $\mathcal{H}$.

It is important to note that in general, the existence of the right-hand side of equation (2.8) requires careful justification (see [30]). However, since we will be exclusively working with Wiener integrals over Hilbert spaces, it is worth highlighting that if X is a Hilbert space and $u \in L^2([0,T]; X)$ is a deterministic function, the relation (2.8) holds. Moreover, the right-hand side of (2.8) is well-defined in $L^2(\Omega, X)$ if $K_H^* u$ belongs to $L^2([0,T] \times X)$.

2.2. **Cylindrical fractional Brownian motion.** Following the approach in [14], we define the standard cylindrical fractional Brownian motion in X as the formal series:

$$B^H(t) = \sum_{n=0}^{\infty} e_n b_n^H(t), \tag{2.9}$$

where $\{e_n,\, n \in \mathbb{N}\}$ represents a complete orthonormal basis in X, and b_n^H denotes a one-dimensional fBm.

It is widely recognized that the infinite series (2.9) does not converge in $L^2(\mathbb{P})$, which implies that $B^H(t)$ is not a well-defined X-valued random variable. However, for any Hilbert space $\mathcal{N}$ such that $X \hookrightarrow \mathcal{N}$, where the embedding of X into $\mathcal{N}$ is a Hilbert-Schmidt operator, the series (2.9) defines a $\mathcal{N}$-valued random variable. Consequently, $\{B^H(t),\, t \geq 0\}$ can be regarded as a $\mathcal{N}$-valued identity fractional Brownian motion (Id-fBm).

Following the approach introduced for cylindrical Brownian motion in [12], it is possible to define a stochastic integral of the form:

$$\int_0^T f(t) B^H(t), \tag{2.10}$$

where $f : [0,T] \mapsto \mathcal{L}(X,Y)$ and Y represents another real and separable Hilbert space. The expression (2.10) corresponds to a Y-valued random variable that remains independent of the choice of $\mathcal{N}$.

Consider a deterministic function f with values in $\mathcal{L}_2(X,Y)$, which represents the space of Hilbert-Schmidt operators from X to Y. We now introduce the following assumptions on f:

i) For each $x \in X$, $f(.)x \in L^p([0,T];Y)$, for $qH > 1$.
ii) $\displaystyle\int_0^T \int_0^T \|f(s)\|_{\mathcal{L}_2(X,Y)} \|f(t)\|_{\mathcal{L}_2(X,Y)} |s-t|^{2H-2}\, dsdt < \infty.$

The stochastic integral (2.10) is defined as:

$$\int_0^T f(t)B^H(t) := \sum_{n=1}^{\infty} \int_0^t f(s)e_n\, db_n^H(s) = \sum_{n=1}^{\infty} \int_0^t (K_H^* f e_n)\, db_n(s), \tag{2.11}$$

where b_n represents the standard Brownian motion associated with the fractional Brownian motion b_n^H through the representation formula (2.5). It is important to note that since $fe_n \in L^2([0,T];Y)$ for each $n \in \mathbb{N}$, the variables $\int_0^t fe_n\, db_n^H$ are mutually independent (see [14]). The series (2.11) is finite and can be expressed as:

$$\sum_{n=1}^{\infty} \|K_H^*(fe_n)\|^2\, db_n(s) = \sum_{n=1}^{\infty} \||| \, fe_n \, \|_{\mathcal{H}} \|_X^2 < \infty.$$

In the case where $X = Y = \mathcal{H}$, we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^t f(s)e_n db_n^H(s) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e_m \int_0^t \langle f(s)e_n, e_m\rangle_{\mathcal{H}}\, db_n^H(s) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e_m \int_0^t \langle K_H^*(f(s)e_n), e_m\rangle_{\mathcal{H}}\, db_n(s) \\
&= \sum_{n=1}^{\infty} \int_0^t K_H^*(f(s)e_n)\, db_n(s).
\end{aligned}$$

2.3. **Smoothing effect.** Consider the linear homogeneous equation associated to (1.1)

$$\partial_t u + (-\Delta)^{\theta/2} u = 0, \quad u(0) = u_0. \tag{2.12}$$

It is well known that the operator $(-\Delta)^{\theta/2}$, $\theta > 0$ is a generator of a semi-group $e^{-t(-\Delta)^{\theta/2}}$ whose kernel E_θ is smooth, radial and satisfies the scaling property

$$E_\theta(x,t) = t^{-\frac{N}{\theta}} E_\theta(t^{-\frac{1}{\theta}}x, 1) := t^{-\frac{N}{\theta}} K_\theta\left(t^{-\frac{1}{\theta}}x\right),$$

where

$$K_\theta(x) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{ix\cdot\xi}\, e^{-|\xi|^\theta}\, d\xi.$$

Hence, the solution of (2.12) may be formally realized via convolution by

$$u(t,x) = \left(e^{-t(-\Delta)^{\theta/2}}\right) u_0(x) = \left(t^{-\frac{N}{\theta}} K_\theta\left(t^{-\frac{1}{\theta}}\cdot\right) * u_0\right)(x) \tag{2.13}$$

whenever this representation makes sense. We recall the following point-wise estimate for the kernel K_θ.

Lemma 2.1. *[29, Lemma 2.1, p. 463] Let $\theta > 0$ and $N \geq 1$. Then, we have*

$$|K_\theta(x)| \lesssim (1+|x|)^{-N-\theta}, \quad x \in \mathbb{R}^N. \tag{2.14}$$

As a result, we get $K_\theta \in L^r(\mathbb{R}^N)$ for any $1 \leq r \leq \infty$.

Remark 2.1.

(i) The inequality (2.14) can be derived from [4, Theorem 2.1, p. 263] in the case where $\theta < 2$.

(ii) For $\theta = 2m$ with $m \geq 1$ being an integer, a more refined estimate than (2.14) is provided in [15, Proposition 2.1, p. 1325]. More precisely, the following estimate holds:

$$\left|\int_{\mathbb{R}^N} e^{ix\cdot\xi}\, e^{-|\xi|^{2m}}\, d\xi\right| \lesssim \exp\left(-\kappa_m |x|^{\frac{2m}{2m-1}}\right),$$

where

$$\kappa_m = (2m-1)(2m)^{-\frac{2m}{2m-1}} \sin\left(\frac{\pi}{4m-2}\right).$$

See also [2, 13, 17].

From (2.13), Lemma 2.1 and Young's inequality, we easily derive the following $L^p - L^q$ estimate. See [25, Proposition 6.1, p. 521] for $\theta = 4$.

Proposition 2.1. *Let $\theta > 0$ and $N \geq 1$. Then, there exists a positive constant $\mathcal{H} = \mathcal{H}(\theta, N)$ such that for all $1 \leq p \leq q \leq \infty$, we have*

$$\|e^{-t(-\Delta)^{\theta/2}}\varphi\|_{L^q} \leq \mathcal{H}\, t^{-\frac{N}{\theta}(\frac{1}{p}-\frac{1}{q})} \|\varphi\|_{L^p}, \quad t > 0. \tag{2.15}$$

Remark 2.2.

(i) By employing the majorizing kernel as derived in [15, Proposition 2.1, p. 1325], it becomes evident that the constant $\mathcal{H}$ can be chosen independently of both p and q when considering $\theta \in 2\mathbb{N}$. This insight has been emphasized and applied in previous studies, including the work presented in [25].

(ii) To establish that the constant $\mathcal{H}$ in (2.15) can be chosen independent with respect to both p and q, given any positive θ, we employ an interpolation inequality, allowing us to express it as follows:

$$\|K_\theta\|_{L^r} \leq \|K_\theta\|_{L^\infty}^{1-1/r}\|K_\theta\|_{L^1}^{1/r} \leq \|K_\theta\|_{L^\infty} + \|K_\theta\|_{L^1} := \mathcal{H}.$$

(ii) The semigroup $e^{-t(-\Delta)^{\theta/2}}$ exhibits a contractive property in Lebesgue spaces, which can be understood as follows:

$$\|e^{-t(-\Delta)^{\theta/2}}\|_{L^q - L^q} \leq 1.$$

For $N \geq 1$, $\theta > 0$ and $\gamma \in (0, N)$, define the operator

$$\mathbf{S}_{\theta,\gamma}(t) = e^{-t(-\Delta)^{\theta/2}}|\cdot|^{-\gamma}. \tag{2.16}$$

Proposition 2.2. *Let $N \geq 1$, $\theta > 0$, $\gamma \in (0, N)$ and $1 < p, q \leq \infty$ such that*

$$\frac{1}{q} < \frac{\gamma}{N} + \frac{1}{p} < 1. \tag{2.17}$$

There exits a positive constant $C := C(N, p, q, \theta, \gamma)$ such that

$$\|\mathbf{S}_{\theta,\gamma}(t)\varphi\|_{L^q} \leq Ct^{-\frac{N}{\theta}(\frac{1}{p}-\frac{1}{q})-\frac{\gamma}{\theta}}\|\varphi\|_{L^p}, \quad t > 0. \tag{2.18}$$

Remark 2.3.

(i) The proof of (2.18) for $\theta = 2$ was given in [3, Prposition 2.1, p. 121].

(ii) The estimate (2.18) for $\theta \in 2\mathbb{N}$ was derived in [35, Proposition 3.8].

Proof of Proposition 2.2. To enhance reader convenience, we provide a comprehensive proof for any $\theta > 0$. As one will notice, our proof incorporates arguments borrowed from [3, 35]. By employing a scaling argument, we can reduce the task of proving (2.18) to the case of $t = 1$. In other words, it is sufficient to establish the inequality

$$\|\mathbf{S}_{\theta,\gamma}(1)\varphi\|_{L^q} \leq C\|\varphi\|_{L^p}.$$

To achieve this, we employ the following decomposition of the singular weight $|x|^{-\gamma}$:

$$|x|^{-\gamma} = |x|^{-\gamma}\chi_{\{|x|<1\}} + |x|^{-\gamma}\chi_{\{|x|\geq 1\}} := f(x) + g(x).$$

It is evident that $f \in L^a$ holds for all $a < \frac{N}{\gamma}$ and $g \in L^b$ holds for all $b > \frac{N}{\gamma}$. Using (2.17), it becomes apparent that $\frac{N}{\gamma} - \frac{p}{p-1} > 0$ and $\frac{\gamma}{N} > \frac{1}{q} - \frac{1}{p}$. Consequently, one can select a sufficiently small $\varepsilon > 0$ such that:

$$\frac{\gamma}{N+\gamma\varepsilon} \geq \frac{1}{q} - \frac{1}{p} \quad \text{and} \quad \frac{\gamma}{N-\gamma\varepsilon} \leq 1 - \frac{1}{p}.$$

Let us consider r and s with $r, s \geq 1$ satisfying the condition:

$$\frac{1}{r} = \frac{\gamma}{N-\gamma\varepsilon} + \frac{1}{p}, \quad \frac{1}{s} = \frac{\gamma}{N+\gamma\varepsilon} + \frac{1}{p}.$$

By leveraging the smoothing effect (2.15) and applying Hölder's inequality, we obtain

$$\begin{aligned}
\|\mathbf{S}_{\theta,\gamma}(1)\varphi\|_{L^q} &\leq \|\mathbf{S}_{\theta,0}(1)(f\varphi)\|_{L^q} + \|\mathbf{S}_{\theta,0}(1)(g\varphi)\|_{L^q} \\
&\lesssim \|f\varphi\|_{L^r} + \|g\varphi\|_{L^s} \\
&\lesssim \left(\|f\|_{L^{\frac{N}{\gamma}-\varepsilon}} + \|g\|_{L^{\frac{N}{\gamma}+\varepsilon}}\right) \|\varphi\|_{L^p}.
\end{aligned}$$

This finishes the proof of Proposition 2.2. ◻

3. Proof of the main result

This section is devoted to the proof of Theorem 1.1.

Lemma 3.1. *Let N, θ, γ, p, q be as defined in (1.4). Then, there exists $1 < r < \infty$ such that:*

$$\frac{1}{p}\left(\frac{1}{q} - \frac{\gamma}{N}\right) < \frac{1}{r} < \frac{1}{q}. \tag{3.19}$$

Furthermore, the following properties hold

$$\begin{aligned}
&1 - p\sigma > 0, \\
&1 - \frac{N}{\theta}\left(\frac{p}{r} - \frac{1}{q}\right) - \frac{\gamma}{\theta} - p\sigma > 0, \\
&1 + (1-p)\sigma - \frac{N(p-1)}{r\theta} - \frac{\gamma}{\theta} > 0,
\end{aligned}$$

where

$$\sigma = \frac{N}{\theta}\left(\frac{1}{q} - \frac{1}{r}\right). \tag{3.20}$$

The proof of the aforementioned Lemma is straightforward. It is worth emphasizing that one can choose

$$r = \frac{2pq}{p+1}.$$

In addition, with any r satisfying (3.19), we have

$$\frac{1}{p}\max\left(\frac{1}{q}-\frac{\gamma}{N},\ \frac{p}{q}-\frac{\theta}{N}\right) < \frac{1}{r} < \frac{1}{p}\min\left(1-\frac{\gamma}{N},\ \frac{1}{q}-\frac{\gamma}{N}+\frac{\theta}{N},\ \frac{p}{q}\right).$$

3.1. **Proof of Theorem 1.1.** First, we consider the linear Cauchy problem

$$\begin{cases} \partial_t \mathbf{Z}(t) + (-\Delta)^{\theta/2}\mathbf{Z}(t) = \mu\, \partial_t B^H(t), \quad t>0, \\ \mathbf{Z}(0) = 0. \end{cases} \tag{3.21}$$

The mild solution of (3.21) is given by

$$\mathbf{Z}(t) = \mu \int_0^t \mathrm{e}^{-(t-s)(-\Delta)^{\theta/2}}\, dB^H(s).$$

Since $qH > 1$, $H > \frac{N}{2\theta}$ and $\frac{1}{2} < H < 1$, we know from [10] that the mild solution $\mathbf{Z}$ belongs to $C([0,T];L^q)$ for any $T>0$. For $\sigma > 0$ and $1 < r < \infty$ to be fixed later, define

$$\mathbf{K}(T) = \sup_{0\le t\le T}\|\mathbf{Z}(t)\|_q + \sup_{0\le t\le T}\left(t^\sigma\,\|\mathbf{Z}(t)\|_r\right).$$

Now we are ready to give the detailed proof of Theorem 1.1.

We will apply the Banach fixed-point theorem to the integral equation (1.2) with $\mu \neq 0$. Let us fix $M > 2\mathcal{H}\|u_0\|_q$ with $\mathcal{H}$ being given in (2.15). For $T>0$, let $\mathbf{X}_T$ be defined as

$$\mathbf{X}_T = \left\{ u \in C_T(L^q)\cap C_T(L^r); \quad \max\left(\sup_{0\le t\le T}\|u(t)\|_q,\ \sup_{0\le t\le T}\left(t^\sigma\,\|u(t)\|_r\right)\right) \le M \right\}.$$

We endow $\mathbf{X}_T$ with the metric

$$d(u,v) = \sup_{0\le t\le T}\|u(t)-v(t)\|_q + \sup_{0\le t\le T}\left(t^\sigma\|u(t)-v(t)\|_r\right),$$

where r, σ are as in Lemma 3.1. We also define the operator $\Phi : \mathbf{X}_T \to \mathbf{X}_T$ by

$$\Phi(u)(t) = e^{-t(-\Delta)^{\theta/2}}u_0 + \int_0^t \mathbf{S}_{\theta,\gamma}(t-s)(|u(s)|^{p-1}u(s))\,ds + \mathbf{Z}(t),$$

where $\mathbf{S}_{\theta,\gamma}$ is given by (2.16). We aim to demonstrate that $\Phi(\mathbf{X}_T) \subset \mathbf{X}_T$ and that Φ acts as a contraction under appropriate choice of $T>0$. First, let us show that Φ maps $\mathbf{X}_T$ into itself for suitable choice of T. For $u \in \mathbf{X}_T$, we have by using (2.15), (2.18) and Lemma 3.1

$$\begin{aligned}\|\Phi(u)(t)\|_q &\leq \|e^{-t(-\Delta)^{\theta/2}}u_0\|_q + \|\mathbf{Z}(t)\|_q + \int_0^t \left\|\mathbf{S}_{\theta,\gamma}(t-s)(|u(s)|^{p-1}u(s))\right\|_q\,ds\\ &\leq \mathcal{H}\|u_0\|_q + \mathbf{K}(T) + C\int_0^t (t-s)^{-\frac{N}{\theta}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\theta}}\,\||u(s)|^p\|_{\frac{r}{p}}\,ds\\ &\leq \mathcal{H}\|u_0\|_q + \mathbf{K}(T) + CM^p\int_0^t (t-s)^{-\frac{N}{\theta}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\theta}}\,s^{-p\sigma}\,ds\\ &\leq \mathcal{H}\|u_0\|_q + \mathbf{K}(T) + CM^p\,T^{a+b-1}\,\mathcal{B}(a,b),\end{aligned}$$

where $\mathcal{B}$ is the standard Beta function and

$$a = 1 - \frac{N}{\theta}\left(\frac{p}{r} - \frac{1}{q}\right) - \frac{\gamma}{\theta}, \quad b = 1 - p\sigma. \tag{3.22}$$

Arguing similarly as above, we obtain

$$t^\sigma\|\Phi(u)(t)\|_r \leq \mathcal{H}\|u_0\|_q + \mathbf{K}(T) + CM^p\,T^{a+b-1}\,\mathcal{B}(a-\sigma,b),$$

with a, b defined as in (3.22). From Lemma 3.1, we have $a > 0, b > 0$, $a + b - 1 > 0$ and $a - \sigma > 0$. Therefore, for $T > 0$ small enough, we get $\Phi(\mathbf{X}_T) \subset \mathbf{X}_T$. Next, we aim to show that Φ acts as a strict contraction for $T > 0$ small enough. Let $u, v \in \mathbf{X}_T$. Observing that

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \lesssim |u-v|\left(|u|^{p-1} + |v|^{p-1}\right),$$

and using Proposition 2.2 together with Hölder's inequality, we infer

$$\begin{aligned}\|\Phi(u)(t) - \Phi(v)(t)\|_q &\leq C\int_0^t (t-s)^{a-1-p\sigma}\left\||u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\right\|_{\frac{q}{p}}\,ds\\ &\leq C\int_0^t (t-s)^{a-1-p\sigma}\,\|u(s)-v(s)\|_q\big(\|u(s)\|_q^{p-1} + \|v(s)\|_q^{p-1}\big)\,ds\\ &\leq CM^{p-1}\,T^{a-p\sigma}\,d(u,v),\end{aligned} \tag{3.23}$$

where σ and a are given by (3.20) and (3.22) respectively. Likewise as above, we also obtain

$$t^\sigma\|\Phi(u)(t) - \Phi(v)(t)\|_r \leq CM^{p-1}\,T^{a+\sigma}\,d(u,v). \tag{3.24}$$

Plugging estimates (3.23) and (3.24) together, we get

$$d(\Phi(u),\Phi(v)) \leq CM^{p-1}\left(T^{a-p\sigma} + T^{a+\sigma}\right)d(u,v). \tag{3.25}$$

Given that $a > 0$, $\sigma > 0$, and $a - p\sigma > 0$, we can deduce from (3.25) that Φ exhibits contraction properties for sufficiently small $T > 0$. By invoking the Banach fixed point theorem, we successfully conclude the proof of Theorem 1.1.

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