ON STRUCTURES OF BIHOM-SUPERDIALGEBRAS AND THEIR DERIVATIONS

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ABSTRACT. BiHom-superdialgebras are clear generalization of Hom-superdialgebras. The purpose of this note is to describe and to survey structures of BiHom-superdialgebras. Then we derive derivations of BiHom-superdialgebras.

1. Introduction

The associative superdialgebras are also known as diassociative superalgebras. In 2015, C. Wang, Q. Zhang and Z. Weiz described them as a generalization of associative superalgebras. These authors developed many results on associative superdialgebra in the paper [2]. As a generalization of associative superalgebras, they have two associative products and also they comply three other statements. On the other hand, many results of superdialgebras were investigated in [4, 5, 6]. There are many application areas of these algebras, for instance, physics, classical geometry and non-commutative geometry. For more details (see [3, 7, 8, 9, 10, 11]). The principal beginning point of this paper is based on the concept of BiHom-dialgebras introduced by Zahari and Bakayoko in [1]. In this paper, the authors studied BiHom-associative dialgebras and they gave the central extensions and the classifications of BiHom-associative dialgebras with dimension $n \geq 4$. By following their approach in [1], we present the concepts on BiHom-superdialgebras. Moreover, we state some main properties on such algebras.

2. Structure of BiHom-superdialgebras

By using some definitions on superdialgebras and Hom-dialgebras in [6, 12], we introduce some structure of BiHom-superdialgebras. Furthermore, we survey some important consequences on BiHom-superdialgebras which are similar to some statements on Hom-superdialgebras [1].

Definition 2.1. Let H be a superspace. If two even bilinear mapings $\dashv, \vdash \colon H \times H \longrightarrow H$ satisfy the next statements

$$\begin{aligned}
&\text{(i)} \quad p \vdash (q \dashv r) = (p \vdash q) \dashv r,\\
&\text{(ii)} \quad p \dashv (q \dashv r) = (p \dashv q) \dashv r = p \dashv (q \vdash r),\\
&\text{(iii)} \quad p \vdash (q \vdash r) = (p \vdash q) \vdash r = (p \dashv q) \vdash r
\end{aligned}$$

for every homogeneous elements $p, q, r \in H$, then a 3-tuple $(H, \dashv, \vdash)$ is called a superdialgebra.

Definition 2.2. Given a superspace H. If two even bilinear mapings $\dashv, \vdash \colon H \times H \longrightarrow H$ and an even superspace homomorphism $\alpha : H \longrightarrow H$ satisfy the next properties

$$\begin{aligned}
&\text{(i)} \quad \alpha(p \dashv q) = \alpha(p) \dashv \alpha(q), \quad \alpha(p \vdash q) = \alpha(p) \vdash \alpha(q),\\
&\text{(ii)} \quad \alpha(p) \dashv (q \dashv r) = (p \dashv q) \dashv \alpha(r) = \alpha(p) \dashv (q \vdash r),\\
&\text{(iii)} \quad \alpha(p) \vdash (q \vdash r) = (p \vdash q) \vdash \alpha(r) = (p \dashv q) \vdash \alpha(r),
\end{aligned}$$

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$$\text{(iv)} \quad \alpha(p) \vdash (q \dashv r) = (p \vdash q) \dashv \alpha(r)$$

for every homogeneous elements $p, q, r \in H$, then a tuple $(H, \dashv, \vdash \alpha)$ is said to be a Hom-superdialgebra.

Definition 2.3. Given a superspace H. If an even bilinear map $\cdot : H \times H \longrightarrow H$ and two even superspace homomorphisms $\alpha, \epsilon : H \longrightarrow H$ hold the next properties

$$\begin{aligned} &\text{(i)} \quad \alpha \cdot \epsilon = \epsilon \cdot \alpha, \\ &\text{(ii)} \quad \alpha(p \cdot q) = \alpha(p) \cdot \alpha(q),\ \epsilon(p \cdot q) = \epsilon(p) \cdot \epsilon(q), \\ &\text{(iii)} \quad \alpha(p) \cdot (q \cdot r) = (p \cdot q) \cdot \epsilon(r) \end{aligned}$$

for every homogeneous elements $p, q, r \in H$, then a 4-tuple $(H, \cdot, \alpha, \epsilon)$ is said to be a BiHom-associative superalgebra.

Definition 2.4. Given a superspace H. If two even bilinear mapings $\dashv, \vdash : H \times H \longrightarrow H$ and two even superspace homomorphisms $\alpha, \epsilon : H \longrightarrow H$ satisfy the following axioms

$$\begin{aligned} &\text{(i)} \quad \alpha \circ \epsilon = \epsilon \circ \alpha, \\ &\text{(ii)} \quad \alpha(p \dashv q) = \alpha(p) \dashv \alpha(q), \quad \alpha(p \vdash q) = \alpha(p) \vdash \alpha(q), \\ &\text{(iii)} \quad \epsilon(p \dashv q) = \epsilon(p) \dashv \epsilon(q), \quad \epsilon(p \vdash q) = \epsilon(p) \vdash \epsilon(q), \\ &\text{(iv)} \quad (p \dashv q) \dashv \epsilon(r) = \alpha(p) \dashv (q \dashv r), \\ &\text{(v)} \quad (p \vdash q) \dashv \epsilon(r) = \alpha(p) \vdash (q \dashv r), \\ &\text{(vi)} \quad (p \dashv q) \vdash \epsilon(r) = \alpha(p) \dashv (q \vdash r), \\ &\text{(vii)} \quad (p \vdash q) \vdash \epsilon(r) = \alpha(p) \vdash (q \vdash r) \end{aligned}$$

for every homogeneous elements $p, q, r \in H$, then a 5-tuple $(H, \dashv, \vdash, \alpha, \epsilon)$ is called a BiHom-superdialgebra.

Here α and ϵ are called structure maps of H. Since the maps α, ϵ commute, for any integer m, n, we denote by

$$\alpha^m \epsilon^n = \underbrace{\alpha \circ \ldots \circ \alpha}_{(m-\text{times})} \circ \underbrace{\epsilon \circ \ldots \circ \epsilon}_{(n-\text{times})}.$$

Particularly, $\alpha^0 \epsilon^0 = Id_H$, $\alpha^1 \epsilon^1 = \alpha\epsilon$ and $\alpha^{-m} \epsilon^{-n}$ is the inverse of $\alpha^m \epsilon^n$.

Definition 2.5. Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$, when α and ϵ are bijection, then $(H, \dashv, \vdash, \alpha, \epsilon)$ is called a regular BiHom-superdialgebra.

Remark 2.1. If $p \vdash q = p \dashv q = p \cdot q$ for any $p, q \in H$, then each BiHom-associative superalgebra H is a BiHom-superdialgebra.

Example 2.6. By taking $\epsilon = \alpha$, any Hom-superdialgebra becomes a BiHom-superdialgebra. By setting $\alpha = \epsilon = id$, any superdialgebra is a BiHom-superdialgebra.

Definition 2.7. Given two BiHom-superdialgebras $(H_1, \dashv, \vdash, \alpha, \epsilon)$ and $(H_2, \dashv', \vdash', \alpha', \epsilon')$ and let $g : H_1 \longrightarrow H_2$ be an even homomorphism. g is said to be a morphism of BiHom-superdialgebras if

$$g \circ \alpha = \alpha' \circ g, \quad g \circ \epsilon = \epsilon' \circ g$$

and

$$g(p) \dashv' g(q) = g(p \dashv q), \quad g(p) \vdash' g(q) = g(p \vdash q)$$

for every $p, q \in H_1$.

Definition 2.8. Let α and ϵ be two morphisms. A BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$ is said to be a multiplicative BiHom-superdialgebra.

Definition 2.9. if α and ϵ are bijective, we say that $(H, \dashv, \vdash, \alpha, \epsilon)$ is a regular BiHom-superdialgebra.

Theorem 2.10. *Let $(H, \dashv, \vdash, \alpha, \epsilon)$ be a BiHom-superdialgebra and $\alpha', \epsilon' : H \longrightarrow H$ be two commuting endomorphisms. Consider $\lhd = \dashv \circ(\alpha' \otimes \epsilon')$ and $\rhd = \vdash \circ(\alpha' \otimes \epsilon')$. Then, $H_{(\alpha',\epsilon')} = (H, \lhd, \rhd, \alpha\alpha', \epsilon\epsilon')$ is a BiHom-superdialgebra.*

Proof. Here, we will only prove one of conditions in Definition 2.4. For each $p, q, r \in H$, we obtain

$$\begin{aligned}(p \lhd q) \lhd \epsilon\epsilon'(r) - \alpha\alpha'(p) \lhd (q \rhd r) &= \alpha'(\alpha'(p) \dashv \epsilon'(q)) \dashv \epsilon'\epsilon\epsilon'(r) - \alpha'\alpha\alpha'(p) \dashv \epsilon'(\alpha'(q) \vdash \epsilon'(r)) \\ &= (\alpha'\alpha'(p) \dashv \alpha'\epsilon'(q)) \dashv \epsilon(\epsilon'\epsilon')(r) - \alpha\alpha'\alpha'(p) \dashv (\alpha'\epsilon'(q) \vdash \epsilon'\epsilon'(r))\end{aligned}$$

Since H is a BiHom-superdialgebra, by Using Definition 2.4 (v), we obtain

$$(p \lhd q) \lhd \epsilon\epsilon'(r) - \alpha\alpha'(p) \lhd (q \rhd r) = \alpha(\alpha'\alpha'(p)) \dashv (\alpha'\epsilon'(q) \vdash \epsilon'\epsilon'(r)) - \alpha\alpha'\alpha'(p) \dashv (\alpha'\epsilon'(q) \vdash \epsilon'\epsilon'(r)).$$

The left side is equal to zero, therefore, the proof of the part (iv) of Definition 2.4 is completed. By using the similar argument to other axioms, we prove the theorem. □

Consequently, the theorem derives the following sequence of corollaries.

Corollary 2.11. *Given the multiplicative BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$. Then, $(H, \dashv \circ(\alpha^n \otimes \epsilon^n), \vdash \circ(\alpha^n \otimes \epsilon^n), \alpha^{n+1}, \epsilon^{n+1})$ is a multiplicative BiHom-superdialgebra as well.*

Proof. It is clearly sufficient to take $\alpha' = \alpha^n$ and $\epsilon' = \epsilon^n$ in Theorem 2.10. Thus, this proves the corollary. □

Corollary 2.12. *Let $(H, \dashv, \vdash, \alpha)$ be a multiplicative Hom-superdialgebra and $\epsilon : H \longrightarrow H$ be an endomorphism of H. Then, $(H, \dashv \circ(\alpha \otimes \epsilon), \vdash \circ(\alpha \otimes \epsilon), \alpha^2, \epsilon)$ is also a BiHom-superdialgebra.*

Proof. By taking $\alpha' = \alpha$, $\beta = Id_H$ and $\epsilon' = \epsilon$ in Theorem 2.10, we prove the corollary. □

Corollary 2.13. *If $(H, \dashv, \vdash, \alpha, \epsilon)$ is a regular BiHom-superdialgebra, then $(H, \dashv \circ(\alpha^{-1} \otimes \epsilon^{-1}), \vdash \circ(\alpha^{-1} \otimes \epsilon^{-1}))$ is a superdialgebra.*

Proof. The claim follows in the case where $\alpha' = \alpha^{-1}$ and $\epsilon' = \epsilon^{-1}$ of Theorem 2.10. □

Corollary 2.14. *Given a superdialgebra $(H, \dashv, \vdash)$ and two commuting homomorphisms $\alpha, \epsilon : H \longrightarrow H$. Then, $(H, \dashv \circ(\alpha \otimes \epsilon), \vdash \circ(\alpha \otimes \epsilon), \alpha, \epsilon)$ is a BiHom-superdialgebra.*

Proof. By taking $\alpha = \epsilon = Id_H$ and replacing α' by α and ϵ' by ϵ in Theorem 2.10, we obtain the result as required. □

Definition 2.15. Let $(H, \dashv, \vdash, \alpha, \epsilon)$ be a BiHom-superdialgebra and S a subset of H. If S is stable under α and ϵ and $p \dashv q, p \vdash q \in S$ for every $p, q \in S$, S is called a BiHom-supersubalgebra of H.

Definition 2.16. Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$ and a BiHom-supersubalgebra T of H. For every $p \in T, q \in H$, we say T is a left BiHom-ideal of H if we have $p \dashv q, p \vdash q \in T$. If $q \dashv p$ and $q \vdash p$ are in T, then T is called a right BiHom-ideal of H and T is said to be a two sided BiHom-ideal of H if $p \dashv q$, $q \dashv p$, $p \vdash q$, $q \vdash p \in T$.

Example 2.17. Clearly, $T = \{0\}$ and $T = H$ are two-sided BiHom-ideals. If $\theta : H_1 \longrightarrow H_2$ is a morphism of BiHom-superdialgebras, then the kernel $Ker\theta$ is a two sided BiHom-ideal in H_1 and the image $Im\theta$ is a BiHom-supersubalgebra of H_2. Moreover, if T_1 and T_2 are two sided BiHom-ideals of H, then $T_1 + T_2$ is two sided BiHom-ideal as well.

Proposition 2.18. *Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$ and a two sided BiHom-ideal T of $(H, \dashv, \vdash, \alpha, \epsilon)$. Then, $(H/T, \overline{\dashv}, \overline{\vdash}, \overline{\alpha}, \overline{\epsilon})$ is a BiHom-superdialgebra where*

$$\overline{\alpha}(\overline{p}) = \overline{\alpha(p)}, \quad \overline{\epsilon}(\overline{p}) = \overline{\epsilon(p)}$$

and

$$\overline{p} \mathbin{\overline{\dashv}} \overline{q} = \overline{p \dashv q}, \quad \overline{p} \mathbin{\overline{\vdash}} \overline{q} = \overline{p \vdash q}$$

for all $\overline{p}, \overline{q} \in H/T$.

Proof. Here, We only prove right superassociativity. Similarly, the others are proved. For every $\overline{p}, \overline{q}, \overline{r} \in H/T$, we obtain

$$\begin{aligned}
(\overline{p} \mathbin{\overline{\vdash}} \overline{q}) \mathbin{\overline{\vdash}} \overline{\epsilon}(\overline{r}) - \overline{\alpha}(\overline{p}) \mathbin{\overline{\vdash}} (\overline{q} \mathbin{\overline{\vdash}} \overline{r}) &= \overline{(p \vdash q)} \mathbin{\overline{\vdash}} \overline{\epsilon(r)} - \overline{\alpha(p)} \mathbin{\overline{\vdash}} \overline{(q \vdash r)} \\
&= \overline{(p \vdash q) \vdash \epsilon(r) - \alpha(p) \vdash (q \vdash r)} \quad \text{(by Definition 2.4 (vii))} \\
&= \overline{0}.
\end{aligned}$$

Then, $(H/T, \overline{\dashv}, \overline{\vdash}, \overline{\alpha}, \overline{\epsilon})$ is BiHom-superdialgebra. □

3. Derivation of BiHom-superdialgebra

Some significant concepts corresponding to derivations of BiHom-superdialgebras will be presented in this section.

Definition 3.1. Let $(H, \mu, \alpha, \epsilon)$ be a BiHom-superalgebra. A linear maping $d : H \longrightarrow H$ is said to be an $\alpha^m \epsilon^n$-derivation of $(H, \mu, \alpha, \epsilon)$ if it holds

$$\alpha \circ d = d \circ \alpha \quad \text{and} \quad d \circ \epsilon = \epsilon \circ d,$$
$$d \circ \mu(p, q) = \mu(d(p), \alpha^m \epsilon^n(q)) + (-1)^{|p||d|} \mu(\alpha^m \epsilon^n(p), d(q))$$

for every $p, q \in H$ and for each integers m, n.

Definition 3.2. A linear maping $d : H \longrightarrow H$ on a BiHom-superalgebra is called a differential if

$$d(p.q) = dp.q + (-1)^{|p||d|} p.dq, \quad \text{for every} \quad p, q \in H, \quad d^2 = 0$$

and

$$d \circ \alpha = \alpha \circ d \quad d \circ \epsilon = \epsilon \circ d.$$

Proposition 3.3. *Given a differential BiHom-superalgebra $(H, ., \alpha, \epsilon, d)$ and the products $\dashv$ and $\vdash$ on H by*

$$p \dashv q = \alpha(p).dq \quad \textit{and} \quad p \vdash q = dp.\epsilon(q).$$

Then $(H, \dashv, \vdash, \alpha, \epsilon)$ is a BiHom-superdialgebra where α and ϵ are idempotant structure maps.

Proof. It is easy to show the BiHom-superassociativity. We have

$$\begin{aligned}\alpha(p) \dashv (q \dashv r) &= \alpha(p) \dashv (\alpha(q).d(r))\\ &= \alpha(\alpha(p)).d(\alpha(q).d(r))\\ &= \alpha(\alpha(p)).(d(\alpha(q)).d(r) + \alpha(q)d^2(r))\\ &= \alpha(\alpha(p)).(d(\alpha(q)).d(r))\\ &= (p \dashv q) \dashv \epsilon(r)\end{aligned}$$

for any $p, q, r \in H$. By using a similar way, we obtain

$$\begin{aligned}\alpha(p) \vdash (q \vdash r) &= \alpha(p) \vdash (d(q).\epsilon(r))\\ &= d(\alpha(p)).\epsilon(d(q).\epsilon(r))\\ &= \alpha(d(p)).(\epsilon(d(q)).\epsilon^2(r))\\ &= (d(p).\epsilon(d(q))).\epsilon^2(r)\\ &= (p \vdash q) \vdash \epsilon(r).\end{aligned}$$

□

Proposition 3.4. *Let $(H, \dashv, \vdash, \alpha, \epsilon)$ be a BiHom-superdialgebra such that $\alpha(r) = \epsilon(r)$ for all $r \in H$. For any $p, q \in H$, we describe the linear maping $ad_r^{(\alpha,\epsilon)} : H \longrightarrow H$ by*

$$ad_r^{(\alpha,\epsilon)}(p) = p \dashv \epsilon(p) - (-1)^{|p||r|}\alpha(r) \vdash x.$$

Then $ad_r^{(\alpha,\epsilon)}$ is a derivation of H.

Proof. We have

$$\begin{aligned}ad_r^{(\alpha,\epsilon)}(p) \dashv q + p \dashv ad_r^{(\alpha,\epsilon)}(q) &= (p \dashv \epsilon(r) - (-1)^{|p||r|}\alpha(r) \vdash p) \dashv q + p \dashv (q \dashv \epsilon(r) - (-1)^{|q||r|}\alpha(r) \vdash q)\\ &= (p \dashv \epsilon(r)) \dashv q - (-1)^{|p||r|}(\alpha(r) \vdash p) \dashv q\\ &+ p \dashv (q \dashv \epsilon(r)) - (-1)^{|q||p|}p \dashv (\alpha(r) \vdash q)\\ &= p \dashv (\epsilon(r) \dashv q) - (-1)^{|p||r|}\alpha(r) \vdash (p \dashv q)\\ &+ (p \dashv q) \dashv \epsilon(r) - (-1)^{|q||r|}p \dashv (\alpha(r) \vdash q)\\ &= (p \dashv q) \dashv \epsilon(r) - (-1)^{|p||r|}\alpha(r) \vdash (p \dashv q)\\ &= ad_r^{(\alpha,\epsilon)}(p \dashv q).\end{aligned}$$

As a result, $ad_r^{(\alpha,\epsilon)}$ is a derivation. □

Definition 3.5. Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$. A linear maping $d : H \longrightarrow H$ is called a $\alpha^m\epsilon^n$-derivation of H, if it holds

$$\begin{aligned}&\text{(i)} \quad d \circ \alpha = \alpha \circ d, \epsilon \circ d = \quad d \circ \epsilon,\\ &\text{(ii)} \quad d(p \dashv q) = \alpha^m\epsilon^n(p) \dashv d(q) + (-1)^{|p||d|}d(p) \dashv \alpha^m\epsilon^n(q),\\ &\text{(iii)} \quad d(p \vdash q) = \alpha^m\epsilon^n(p) \vdash d(q) + (-1)^{|p||d|}d(p) \vdash \alpha^m\epsilon^n(q)\end{aligned}$$

for each $p, q \in H$ and for each integers m, n.

The set of all $\alpha^m\epsilon^n$-derivation of H is represented by $Der_{\alpha^m\epsilon^n}(H)$ and the set of all derivation on BiHom-superdialgebra is represented by

$$Der(H) = \bigoplus_{m,n\geq 0} Der_{\alpha^m\epsilon^n}(H).$$

We define, for any $d, d' \in Der(H)$, the even bilinear maping $[,] : Der(H) \times Der(H) \longrightarrow Der(H)$ by

$$[d, d'] = d \circ d' - (-1)^{|d||d'|} d' \circ d.$$

Proposition 3.6. *For each $d \in (Der_{\alpha^m\epsilon^n}(H))_i$ and $d' \in (Der_{\alpha^m\epsilon^n}(H))_j$, then*

$$[d, d'] \in Der_{\alpha^{m+s}\epsilon^{n+t}}(H)_{|d|+|d'|},$$

where $m + s, n + t \geq -1$ and $(i, j) \in \mathbb{Z}_2^2$.

Proof. For each $p, q \in H$, we have

$$\begin{aligned}
[d, d'](\mu(p, q)) &= (d \circ d' - (-1)^{|d||d'|} d' \circ d)(\mu(p, q))\\
&= d \circ d'\mu(p, q) - (-1)^{|d||d'|} d' \circ d\mu(p, q)\\
&= d(\mu(d'(p), \alpha^s\epsilon^t(q)) + (-1)^{|p||d'|}\mu(\alpha^s\epsilon^t(p), d'(q)) - (-1)^{|d||d'|} d'(\mu(d(p), \alpha^k\epsilon^l(q))\\
&+ (-1)^{|d||p|}\mu(\alpha^s\epsilon^t(p), d(q)))\\
&= \mu(d \circ d'(p), \alpha^m\epsilon^n\alpha^s\epsilon^t(q)) + (-1)^{|d|(|d'|+|p|)}\mu(\alpha^m\epsilon^n d'(p), d(\alpha^s\epsilon^t(q)))\\
&+ (-1)^{|p||d'|}\mu(d(\alpha^s\epsilon^t(p), \alpha^m\epsilon^n d'(q))) + (-1)^{|p||d'|+|p||d|}\mu(\alpha^m\epsilon^n\alpha^s\epsilon^t(p), d \circ d'(q)))\\
&- (-1)^{|d||d'|}\mu(d' \circ d(p), \alpha^s\epsilon^t\alpha^m\epsilon^n(q))) - (-1)^{|p||d'|}\mu(\alpha^s\epsilon^t d(p), d'\alpha^m\epsilon^n(q))\\
&- (-1)^{|d||d'|+|p||d|}\mu(d'(\alpha^m\epsilon^n(p), \alpha^s\epsilon^t d(q)) - (-1)^{|d||d'|+|p||d|+|d||d'|}\mu(\alpha^s\epsilon^t(p)\alpha^m\epsilon^n(p), d' \circ d(q)).
\end{aligned}$$

Since any two of maps d, d', α, ϵ commute, we have

$$\begin{aligned}
d' \circ \epsilon^n = \epsilon^n \circ d', \quad d' \circ \alpha^m = \alpha^m \circ d',\\
d \circ \epsilon^t = \epsilon^t \circ d, \quad d \circ \alpha^s = \alpha^s \circ d.
\end{aligned}$$

Therefore, we have

$$[d, d'](\mu(p, q)) = \mu([d, d'](p), \alpha^{m+s}\epsilon^{n+t}(q)) + (-1)^{|p|(|d'|+|d|)}\mu(\alpha^{m+s}\beta^{n+t}(p), [d, d'](q)).$$

In addition, it is easy to see that

$$\begin{aligned}
[d, d'] \circ \alpha &= (d \circ d' - (-1)^{|d||d'|} d' \circ d) \circ \alpha\\
&= d \circ d' \circ \alpha - (-1)^{|d||d'|} d' \circ d \circ \alpha\\
&= \alpha \circ d \circ d' - (-1)^{|d||d'|} \alpha \circ d' \circ d\\
&= \alpha \circ [d, d']
\end{aligned}$$

and

$$\begin{aligned}
[d, d'] \circ \epsilon &= (d \circ d' - (-1)^{|d||d'|} d' \circ d) \circ \epsilon\\
&= d \circ d' \circ \epsilon - (-1)^{|d||d'|} d' \circ d \circ \epsilon\\
&= \epsilon \circ d \circ d' - (-1)^{|d||d'|} \epsilon \circ d' \circ d\\
&= \epsilon \circ [d, d']
\end{aligned}$$

which occurs that

$$[d, d'] \in (Der_{\alpha^{m+s},\epsilon^{n+t}}(D))_{(|d||d'|)}$$

by taking $\mu = \dashv$ and $\mu = \vdash$ respectively.
Consequently, for each integers k and l, represent by

$$Der(H) = (\bigoplus_{m,n} Der_{\alpha^m,\epsilon^n}(H))_0 + (\bigoplus_{m,n} Der_{\alpha^m,\epsilon^n}(H))_1.$$

□

Definition 3.7. Let $(H, \dashv, \vdash, \alpha, \epsilon)$ be a BiHom-superdialgebra and γ, δ, λ elements in $\mathbb{C}$. A linear maping $d : H \longrightarrow H$ is a generalized $\alpha^m\epsilon^n$-derivation or a $(\gamma, \delta, \lambda) - (\alpha^m\epsilon^n)$-derivation of H if for every $p, q \in H$, we have

$$\begin{aligned}
&\text{(i)} \quad d \circ \alpha = \alpha \circ d, d \circ \epsilon = \epsilon \circ d,\\
&\text{(ii)} \quad \gamma(d(p \dashv q)) = \delta(d(p) \dashv \alpha^m\epsilon^n(q)) + (-1)^{|p||d|}\lambda(\alpha^m\epsilon^n(p) \dashv d(q)),\\
&\text{(iii)} \quad \gamma(d(p \vdash q)) = \delta(d(p) \vdash \alpha^m\epsilon^n(q)) + (-1)^{|p||d|}\lambda(\alpha^m\epsilon^n(p) \vdash d(q)).
\end{aligned}$$

Denote by $Der^{\gamma,\delta,\lambda}(H)$ the set of $(\gamma, \delta, \lambda) - (\alpha^m\epsilon^n)$-derivations of the BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$.

Proposition 3.8. *Let* $d \in Der^{(\gamma,\delta,\lambda)}_{(\alpha^m,\epsilon^n)}(H)$ *and* $d' \in Der^{(\gamma',\delta',\lambda')}_{(\alpha^{m'},\epsilon^{n'})}(H)$. *Then*

$$[d, d'] \in Der^{(\gamma+\gamma',\delta+\delta',\lambda+\lambda')}_{(\alpha^{m+m'},\epsilon^{n+n'})}(H).$$

Proof. Applying the similar way to Proposition 3.6 completes the proof. □

Definition 3.9. Let $(H, \dashv, \vdash, \alpha, \epsilon)$ be a BiHom-superdialgebra. A linear maping $d : H \longrightarrow H$ is said to be an $\alpha^k\epsilon^l$-quasi derivation of the BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \epsilon)$ if there is a derivation $d' : H \longrightarrow H$

$$\begin{aligned}
&\text{(i)} \quad d \circ \alpha = \alpha \circ d, \epsilon \circ d = \quad d \circ \epsilon,\\
&\text{(ii)} \quad d' \circ \alpha = \alpha \circ d', \epsilon \circ d' = \quad d' \circ \epsilon\\
&\text{(iii)} \quad d'(p \dashv q) = d(p) \dashv \alpha^m\epsilon^n(q) + (-1)^{|p||d|}(\alpha^m\epsilon^n(p) \dashv d(q)),\\
&\text{(iv)} \quad d'(p \vdash q) = d(p) \vdash \alpha^m\epsilon^n(q) + (-1)^{|p||d|}(\alpha^m\epsilon^n(p) \vdash d(q))
\end{aligned}$$

for all homogeneous elements $p, q \in H$.

Denote by $QDer_{\alpha^m\epsilon^n}(H)$ the set of $\alpha^m\epsilon^n$-quasi derivations of H. The set of all Quasi-derivations of H is given by

$$QDer(H) = (\bigoplus_{m,n\geq 0} QDer_{\alpha^m,\epsilon^n}(H))_{\overline{0}} + (\bigoplus_{m,n\geq 0} QDer_{\alpha^m,\epsilon^n}(H))_{\overline{1}}.$$

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