

Generic free fermions with nearest neighbour interactions

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Generic free fermions are free fermions with a single particle spectrum that satisfies the q no resonance condition, i.e., where equal sums of single-particle energies are unique. This property guaranties that they have no degeneracies and gives them relaxation properties more similar to those of generic systems. In this article we provide a minimal example of a generic free fermionic model with nearest neighbour interactions — a tight-binding model with complex hopping. Using some standard results from number theory we prove that this model fulfils the q no resonance condition when the number of lattice sites is prime. Whenever this is not the case one can recover the q no resonance condition by adding hopping terms between sites corresponding to the divisors of the number of sites. We further discuss its many-body spectral statistics and show that local probes — like the ratio of consecutive level spacings — look very similar to what is expected for the Poisson statistics. We however demonstrate that free fermion models can never have Poisson statistics with an analysis of the moments of the spectral form factor.

I. INTRODUCTION

In the last two decades there has been a large amount of progress classifying and understanding the dynamics generated by isolated quantum many body Hamiltonians. A rich literature is forming discussing properties of quantum systems in the context of equilibration, thermalization, and information spreading [1–10]. This also lead us to revisit concepts like quantum chaos, integrability, and localization in the quantum many-body setting [11–17].

The field has made several advances in terms of understanding late time dynamics and the corresponding equilibrium values of dynamical quantities such as expectation values, correlation functions, entanglement measures, out of time ordered correlators and more. Traditionally there have been two main approaches for characterising these quantities. The first consists in studying the problem directly in the thermodynamic limit and then explicitly take the limit of large times. This approach is particularly convenient for analytic studies and relies on some special simplifications occurring in the thermodynamic limit [5, 10]. The second approach instead consists of studying infinite time averages in finite systems. This second approach is more suited to numerical investigations and does not require any special simplification. Instead, late time properties are studied through dephasing arguments based on some typicality assumptions on the spectrum of the system and, sometimes, the functional form of the observables in the energy eigenbasis. Generally one assumes the q no-resonance condition [18–22]

$$\sum_{j \in M} E_j = \sum_{k \in N} E_k \quad \text{s.t.} \quad |M| = |N| \implies M = N, \quad (1)$$

where M, N are sets of indices with q entries, identifying the energy values of the model. This allows one to make

progress when taking the infinite time average. Such a simple assumption on the spectrum allows one to make strong statements on equilibration and quantum recurrence times [19] with fixed q assumptions being the most common in the literature [23–33]. Late time properties can typically be deduced with few additional assumptions like eigenstate typicality/thermalization, chaos, large inverse participation ratios etc [22, 32–35]. Eq. (1) is a strong assumption on the spectrum and is generally assumed to hold for quantum chaotic models. Recently the spectral statistics of (1) was studied, and indeed the local spectral statistics for many body chaotic (or random) Hamiltonians for the condition when $q > 1$ are Poissonian, indicating that violations of the conditions should be rare or non-existent [21]. Probing whether or not a given model violates the condition in Eq. (1) is however generically unfeasible, both analytically and numerically.

In this document we focus our attention on a more analytically tractable set of models: free fermionic systems, whose spectral properties have lately been at a centre of a resurgence of interest [36–40]. Specifically, we consider free fermion models defined by

$$\hat{H} = \sum_{n,m=1}^L M_{n,m} \hat{f}_n^\dagger \hat{f}_m, \quad (2)$$

with M Hermitian. One can always solve this model by diagonalising the matrix $M = O\epsilon O^\dagger$, where O is unitary and ϵ is a diagonal matrix with real entries. Therefore we can write

$$\hat{H} = \sum_{k=1}^L \epsilon_k d_k^\dagger d_k, \quad (3)$$

where $d_k^\dagger = \sum_{n=1}^L O_{n,k} \hat{f}_n^\dagger$. We make two demands of this model. First we ask it to have no non-trivial resonances in the eigenmode spectrum. This property is referred

to as the single particle q no resonance condition. More precisely, letting R_q and R'_q be two sets of exactly q eigenmodes ϵ_j , we say the model satisfies the single particle q no resonance condition if

$$\sum_{j \in R_q} \epsilon_j = \sum_{j \in R'_q} \epsilon_j \implies R_q = R'_q. \quad (4)$$

The second property we require from our free model is extensivity, i.e.,

$$O_{n,k} = \frac{c_{n,k}}{\sqrt{L}}, \quad (5)$$

with $c_{n,k} = \mathcal{O}(1)$. throughout this text we refer to models that satisfy these conditions as generic free fermion models.

These two properties imply the free model has exponentially long quantum recurrence times, and the concentration of equilibration is exponentially tight in system size [19]. This is to be compared to the typical tight binding model which has a quantum recurrence time that is linear in size system and a concentration around equilibrium that is a power law [24, 30, 41? ? -44]. Therefore, a model that satisfies Eqs. (4) and (5) has late time properties more similar to those of a generic system than typically studied free ones. Although generic free models of this kind can be constructed relatively easily if one allows for long range couplings, the open question is whether such a model can be found if we restrict to local interactions. In this article we resolve this question by providing the example of a “minimal” generic free fermion model with nearest neighbour interactions.

The model we construct has a non-degenerate many-body spectrum, a property that the tight binding model lacks, and a relevant question concerns the nature of its spectral correlations. First we show that under the ratio test its spectrum is compatible with an underlying Poissonian statistics of the energy levels. Therefore the system seems to follow the Berry Tabor conjecture [45] like interacting integrable systems. However, analysing its spectral form factor, and more precisely its higher moments, we show that this is not the case. The spectral statistics is Poissonian only at the level of two point correlations: higher point correlations can never be Poissonian in free systems.

The rest of this manuscript is laid out as follows. In Sec. II we construct the minimal generic free fermion model and show that it satisfies (4) and (5). In Sec. III we show that, contrary to the tight binding model, its level spacing distribution appears Poissonian. In Sec. IV we characterise its spectral statistics using the spectral form factor and its higher moments. Finally, Sec. V contains our conclusions.

II. MINIMAL MODEL

In this section we show that to achieve a generic free fermion model it is sufficient to add a parity-breaking

term to the tight binding model. Namely we consider

$$\hat{H} = \frac{\alpha + i\beta}{2} \sum_{n=1}^L \hat{f}_n^\dagger \hat{f}_{n+1} + \frac{\gamma}{2} \sum_{n=1}^L \hat{f}_n^\dagger \hat{f}_n + \text{h.c.} \quad (6)$$

In the spin language (i.e. via a standard Jordan Wigner transformation) this system is mapped into an XX chain with a magnetic field and a Dzyaloshinskii–Moriya interaction of the form

$$\propto \sum_j \sigma_j^x \sigma_{j+1}^y - \sigma_j^x \sigma_{j+1}^x. \quad (7)$$

Because of the translational invariance of (6), the diagonal form (3) is simply attained by Fourier transform, i.e. by choosing

$$O_{n,k} = \frac{e^{i\frac{2\pi}{L}kj}}{\sqrt{L}}, \quad (8)$$

and gives a dispersion relation of the form

$$\epsilon_k = \alpha \cos\left(\frac{2\pi}{L}k\right) + \beta \sin\left(\frac{2\pi}{L}k\right) + \gamma. \quad (9)$$

Note that the $O_{n,k}$ in Eq. (8) immediately satisfies our second condition (5). Instead, the validity of (4) is assessed by the following

Property 1. *When the couplings α, β and γ are incommensurate the dispersion relation (9) fulfils (4) iff L is prime.*

We begin by giving a more precise definition of incommensurate. Let

$$\mathcal{C}_L = \left\{ \cos\left(\frac{2\pi}{L}k_1\right) + \dots + \cos\left(\frac{2\pi}{L}k_n\right), \right. \\ \left. n \in \mathbb{Z}_L, \quad 1 \leq k_1 < \dots < k_n \leq L \right\}, \quad (10)$$

be the finite set of points achieved by summing any group of cosine terms generated by our finite quantised wave-numbers. Likewise we denote by \mathcal{S}_L set of sums of sines. We demand that α, β, γ be chosen such that for any choice of $x \in \mathcal{C}_L, y \in \mathcal{S}_L$ we have that $x\alpha/y\beta$ and $(x\alpha + y\beta)/\gamma$ are irrational.

This gives us two important properties. First, sums of cosine and sine terms will not cancel each other. Second, sums of different numbers of eigenmodes also cannot be equal. As we now show these properties are enough to ensure property (4). In fact they lead to an even stronger feature of the single-particle spectrum called *rational independence*. Namely they imply that

$$\sum_{k=1}^L a_k \epsilon_k = 0, \quad a_k \in \mathbb{Q}, \quad (11)$$

has no solutions except for the trivial case when $a_k = 0$ for all k .

To see this we note that, due to the incommensurability of α, β and γ , we can break Eq. (11) into the following three equations

$$\sum_{k=1}^L a_k \cos\left(\frac{2\pi}{L}k\right) = 0, \quad (12)$$

$$\sum_{k=1}^L a_k \sin\left(\frac{2\pi}{L}k\right) = 0, \quad (13)$$

$$\sum_{k=1}^L a_k = 0. \quad (14)$$

Focussing on the first two equations and combining them into one we arrive at

$$P(\xi) \equiv \sum_{k=0}^{L-1} a_k \xi^k = 0, \quad (15)$$

where we set $\xi \equiv \exp(i2\pi/L)$. The third equation is then rewritten as

$$P(1) = 0. \quad (16)$$

The key observation now is that Eq. (15) defines a polynomial with integer coefficients that has a zero at the first L -th root of unity. This family of polynomials is well studied in mathematics, with a particularly important role played by the so called L -th *cyclotomic polynomial* [46], which we denote by $\Phi_L(z)$. The latter is the unique polynomial with roots at *primitive* L -th roots of unity [47] that is irreducible in \mathbb{Q} , i.e. it cannot be factored as the product of two non-constant polynomials with rational coefficients.

Recalling some basic facts about the cyclotomic polynomial we have that it has degree $\phi(L) \leq L-1$, where $\phi(L)$ is Euler's totient function [46]. In particular, this means that its degree is strictly smaller than $L-1$ whenever L is not prime. Instead, for prime L one has [46]

$$\Phi_L(z) = 1 + z + \dots + z^{L-1}. \quad (17)$$

Since by definition $\Phi_L(z)$ must divide $P(z)$ and the degree of $P(z)$ is at most $L-1$, for L prime we have either $P(z) = 0$ or $P(z) = \Phi_L(z)$. Namely we either have $a_k = 0$ for all k or $a_k = 1$ for all k . Considering now (16) we see that it is only fulfilled by the former choice. This proves that, for L prime we have rational independence.

Next, let us show that Eq. (4) does not hold for non-prime L by providing an explicit counter example. Consider a system size $L = np$, with p prime and $n > 1$, and let $w = \xi^n$. Then w is a p -th primitive root of unity and, therefore, it is a zero of the cyclotomic polynomial $\Phi_p(z)$. Namely

$$\Phi_p(w) = w^{p-1} + w^{p-2} + \dots + 1 = 0. \quad (18)$$

Expressing this equation in terms of ξ , we can write it as $Q(\xi) = 0$, where we introduced the polynomial

$$Q(z) \equiv z^{n(p-1)} + z^{n(p-2)} + \dots + 1. \quad (19)$$

Since $Q(\xi) = 0$ we can multiply any $R(\xi)$ into $Q(\xi)$ giving $R(\xi)Q(\xi) = 0$. We can use this to construct suitable $P(z)$ fulfilling (15) and (16).

Take for example

$$R(z) = z - 1, \quad (20)$$

and set $P(z) = R(z)Q(z)$. This is a legitimate choice because $R(z)Q(z)$ has degree $L-n+1 \leq L-1$. Moreover, since $R(1) = 0$ we have that this polynomial also fulfils Eq. (16). \square

To recover rational independence for non-prime L one should add higher harmonics to the dispersion relation. In particular, for a generic L a rational independent dispersion relation takes the form

$$\epsilon_k = \sum_{d|L} \left(\alpha_d \cos\left(\frac{2\pi d}{L}k\right) + \beta_d \sin\left(\frac{2\pi d}{L}k\right) \right) + \gamma \quad (21)$$

where the sum is over all the divisors of L strictly smaller than L and $\{\alpha_d, \beta_d\}$, and γ are again incommensurable. The corresponding real space Hamiltonian is recovered by again taking the Fourier transform in Eq. (8). The resulting model can be written similarly to Eq. (6) with the addition of terms of the form

$$\hat{H}_d = \frac{\alpha_d + i\beta_d}{2} \sum_{n=1}^L \hat{f}_n^\dagger \hat{f}_{n+d} + \text{h.c.} \quad (22)$$

This point is discussed more extensively in Appendix A where we provide the explicit example of $L = pq$ with p and q primes.

III. LEVEL SPACING STATISTICS OF THE MANY BODY SPECTRUM

In this section we investigate the level spacing statistics of the many body spectrum $E_n = \tilde{n} \cdot \epsilon$ where we take ϵ from our minimal model in Eq. (9). In particular we perform the ratio test introduced in Ref. [48]. This method avoids unfolding the spectrum while still probing the spectrum's similarity to random matrix theory. Suppose we take E_n and re-order it such that $E_k < E_{k+1}$ for all k . Then we define gaps in the spectrum as $s_k = E_k - E_{k-1}$. The ratio test is then concerned with the quantity

$$r_k = \frac{\min\{s_k, s_{k+1}\}}{\max\{s_k, s_{k+1}\}}. \quad (23)$$

A spectrum obeying Poisson statistics will follow the distribution [49]

$$p(r) = \frac{2}{(1+r)^2}. \quad (24)$$

and have the mean $\langle r \rangle = 2 \ln 2 - 1 \approx 0.38629436112$.

In Fig. 1 we see the surprising result that this model looks approximately Poisson both when calculating $\langle r \rangle$

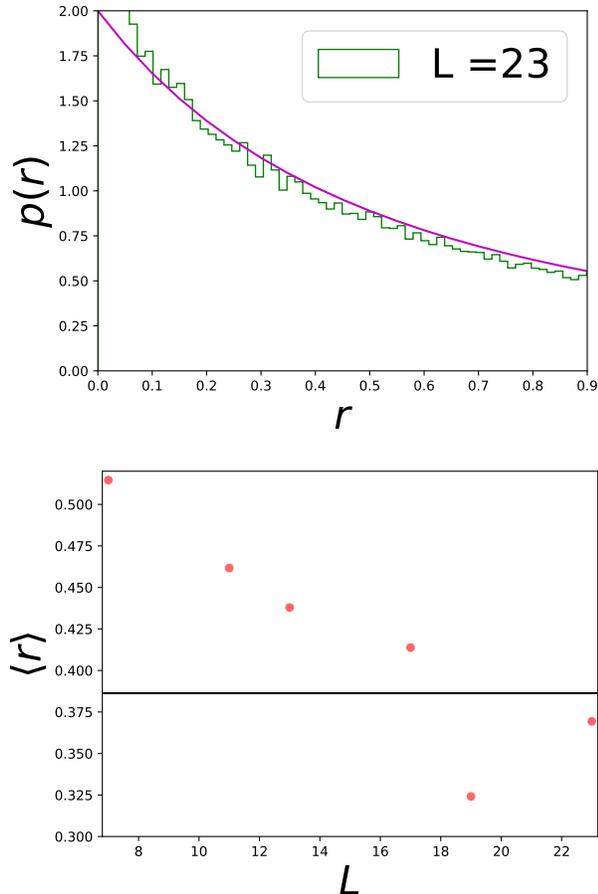


FIG. 1. In the top figure we have a histogram of the ratio test with our generic single particle spectrum ϵ_k for $L = 23$. In purple we plot the Poisson distribution. On the bottom figure we print the average of the ratios for various prime system sizes.

and when overlaying its distribution $p(r)$. The model clearly experiences some form of level attraction but the gaps do not cluster around zero as one expects for a typical tight binding model. At $L = 23$ we observe $\langle r \rangle = 0.36936$, a value somewhat similar the expected Poisson result. One may be tempted to conclude that the model has Poisson spectral statistics and is just experiencing relatively slow convergence to that behaviour. We demonstrate in the next section that this cannot be true by presenting a more extensive probe of the spectral statistics. For comparison we plot the ratio test for $L = 20$, a non-prime system size, in Fig. 2. The ratios cluster around zero with a large amount of exact degeneracies. Note, in the case of the tight binding model, the vast majority of gaps are zero and we would have an approximate delta function at 0.

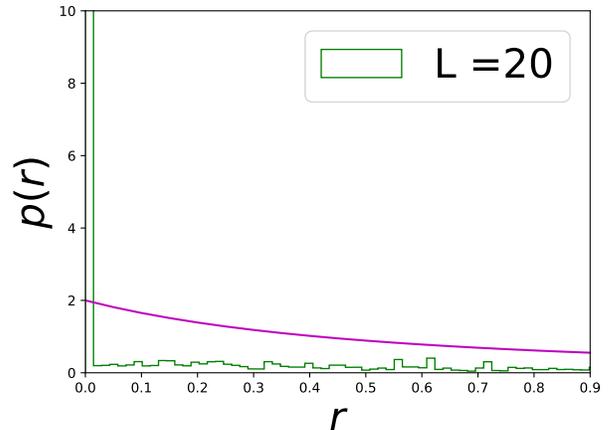


FIG. 2. In this figure we have a histogram of the ratio test with our generic single particle spectrum ϵ_k for $L = 20$. In purple we plot the Poisson distribution.

IV. MOMENTS OF THE SPECTRAL FORM FACTOR

A more robust probe of the spectral statistics is given by the so called spectral form factor (SFF) [13, 40, 50–61], which is the Fourier transform of the spectral two-point function. The SFF is expressed as

$$K(t, L) = \mathbb{E}[|\text{Tr}[(U_t)]|^2], \quad (25)$$

where $U_t = e^{i\hat{H}t}$ and $\mathbb{E}[\cdot]$ is an average (either over an ensemble of similar systems or in time) that removes non-universal features [62]. Contrary to the level spacing statistics this quantity tests spectral correlations over all energy scales, not only over short ones.

To probe also higher point correlations among energy levels, here we consider arbitrary higher moments of the SFF. Namely

$$K_q(t, L) = \mathbb{E}[|\text{Tr}[(U_t)]|^{2q}], \quad (26)$$

and take $\mathbb{E}[\cdot]$ to be the moving time-average over a time window of size τ

$$\mathbb{E}[f(t)] = \frac{1}{\tau} \int_t^{t+\tau} f(s) ds. \quad (27)$$

In fact, to make the expressions more readable we will always consider the limit of large τ [63].

Let us now compute $K_q(t, L)$ in our minimal generic free fermionic model and compare our results with the predictions of the Poissonian statistics, which gives the following SFF higher moments for large L [64]

$$K_q(t, L) = 2^{qL} q!. \quad (28)$$

We begin by considering $q = 1$ and noting that the trace can be expressed as

$$\text{Tr}[(U_t)] = \sum_{m=1}^{2^L} e^{it\tilde{m}\cdot\vec{\epsilon}}, \quad (29)$$

where \vec{m} is the L -bit binary representation of m and $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_L)$. This means that we can write

$$|\text{Tr}[(U_t)]|^2 = \sum_{m,n} e^{it(\vec{m}-\vec{n})\cdot\vec{\epsilon}}. \quad (30)$$

A term survives the time average and the $\tau \rightarrow \infty$ limit iff

$$(\vec{m} - \vec{n}) \cdot \vec{\epsilon} = 0. \quad (31)$$

This is clearly fulfilled when $\vec{m} = \vec{n}$. To see that this is the only case we expand it as

$$(\vec{m} - \vec{n}) \cdot \vec{\epsilon} = \sum_k (m_k - n_k) \epsilon_k = 0. \quad (32)$$

$m_k - n_k \in \mathbb{Z}$. It then follows that our only choice is for $m_k - n_k = 0$ because of the rational independence of ϵ_k . Putting all together we then have

$$K_1(t, L) = \prod_{k=1}^L \sum_{m_k} 1 = 2^L. \quad (33)$$

Therefore our spectrum gives the first moment expected for the Poisson ensemble.

Considering now arbitrary higher moments we have

$$|\text{Tr}[(U_t)]|^{2q} = \sum_{m_i, n_i=1}^{2^L} e^{it\epsilon \cdot \sum_{i=1}^q (\vec{m}_i - \vec{n}_i)}. \quad (34)$$

We now wish to understand which terms survive time average and infinite τ limit. This is again when the argument of the exponential is zero

$$\epsilon \cdot \sum_{i=1}^q (\vec{m}_i - \vec{n}_i) = 0, \quad (35)$$

writing this component-wise gives

$$\sum_{k=1}^L \epsilon_k \sum_{i=1}^q (m_{ik} - n_{ik}) = 0. \quad (36)$$

The number $\sum_{i=1}^q (m_{ik} - n_{ik})$ takes integer values in $[-q, q]$. Therefore, the rational independence of $\{\epsilon_k\}$ tells us that the only solution to this equation is

$$\sum_{i=1}^q (m_{ik} - n_{ik}) = 0, \quad \forall k. \quad (37)$$

This means that there are precisely $q!$ ways for this term to survive the average and we obtain

$$K_q(L) = \prod_{k=1}^L \sum_{m_k^{(i)}} q! = (q!2^q)^L. \quad (38)$$

This result agrees with Eq. (28) for $q = 1$ but disagrees with it for $q > 1$. Note in particular that the disagreement comes because Eq. (38) is larger than Eq. (28).

This happens because, even though the Hamiltonian in Eq. (6) has a non-degenerate spectrum for α, β, γ incommensurate and L prime, the spectrum of

$$\hat{H}_q = \sum_{j=0}^{q-1} \overbrace{I \otimes \dots \otimes I}^j \otimes \hat{H} \otimes I \otimes \dots \otimes I, \quad (39)$$

has far more degeneracies than

$$\hat{H}_{\text{int},q} = \sum_{j=0}^{q-1} \overbrace{I \otimes \dots \otimes I}^j \otimes \hat{H}_{\text{int}} \otimes I \otimes \dots \otimes I, \quad (40)$$

where \hat{H}_{int} is an Hamiltonian with a generic Poisson distributed spectrum, e.g. that of a Bethe Ansatz integrable model. Indeed, whereas $\hat{H}_{\text{int},q}$ has an obvious $q!$ degeneracy due to permutation symmetry, \hat{H}_q has a degeneracy of $(q!)^L$ because it is invariant under permutations of each separate mode. Therefore, free fermionic models cannot display Poisson statistics in the many body spectrum. However, thanks to the rational independence of $\{\epsilon_k\}$, one can claim that our model produces the smallest possible SFF moments for a free fermionic system.

V. CONCLUSION

In this article we have identified a locally interacting free fermionic system — a tight binding model with a parity breaking term — for which we could prove rational independence of the single-particle spectrum, i.e., that for fixed system size its energy eigenvalues are linearly independent over rational numbers. Free fermionic systems with this property are known to have special late time behaviour including “interacting-like” concentration of equilibration and quantum recurrence times and are sometimes referred to as generic free fermions: Here we presented the first example of such systems that also has local interactions.

We have also shown that our local generic free fermion model has a non-degenerate many body spectrum that under local probes of the spectrum (the ratio test [48]) looks strikingly Poissonian — this is not the case when rational independence is lifted as seen in Fig. 1. Studying the higher moments of the spectral form factor, however, we have shown that the spectral statistics of our system agrees with the Poissonian prediction only at the level of two-point spectral correlations. In fact, this is the closest any free system can ever be to have Poissonian statistics. This shows that the predictions of local probes should be taken with care.

An immediate direction for future research is to identify other dynamical signatures of rational independent single-particle spectrum in the context of local interactions. In particular, a fascinating question concerns the interplay between rational independence and interacting perturbations. Since rational independence reduces the number of thermalising channels — for a fixed size there

are no non-trivial energy conserving scattering processes — systems with rational independence should anomalously long prethermalization plateaus in finite volume.

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Appendix A: Higher order harmonics

In this appendix we discuss how the inclusion of higher harmonics helps achieving rational independence also for non-prime sizes.

First we note that our counter example from Sec. II does not apply if one adds higher harmonics. To see this consider again $L = np$ with p prime and add the term

$$\delta \cos\left(\frac{2\pi p}{L}k\right) + \eta \sin\left(\frac{2\pi p}{L}k\right) \quad (\text{A1})$$

to the dispersion relation (9). Here δ and η are chosen to retain incommensurability of the individual terms in the dispersion. This means that we now have the following three equations

$$P(1) = 0, \quad P(\xi) = 0, \quad P(\xi^p) = 0. \quad (\text{A2})$$

where we again set $\xi = \exp(i2\pi/L)$. Our choice $P(z) = (z-1)Q(z)$ in the main text, with $Q(z)$ given in Eq. (19), does not satisfy the third equation. Indeed, we have $w^p = \xi^{np} = 1$ and $Q(\xi^p) = p \neq 0$.

Second, we show that when n is also prime we have rational independence if we add harmonics of order p and

n . This can be seen as follows. Assuming again incommensurability, we obtain that (11) can be decomposed into the following four equations

$$P(1) = 0, \quad P(\xi) = 0, \quad P(\xi^p) = 0, \quad P(\xi^n) = 0. \quad (\text{A3})$$

Considering the last three equations we have that $P(z)$ has zeros at the primitive roots of unity ξ , ξ^p , and ξ^n . This means that it must be proportional to the cyclotomic polynomials $\Phi_{np}(z)$, $\Phi_n(z)$, and $\Phi_p(z)$. Namely we have

$$P(z) = R_1(z)\Phi_{np}(z) = R_2(z)\Phi_n(z) = R_3(z)\Phi_p(z). \quad (\text{A4})$$

for some $R_j(z)$, $j = 1, 2, 3$. Since by definition each cyclotomic polynomial is irreducible the above equation implies

$$P(z) = R_4(z)\Phi_n(z)\Phi_p(z)\Phi_{np}(z). \quad (\text{A5})$$

for some $R_4(z)$.

We now recall another important property of products of cyclotomic polynomials [46]

$$\prod_{d|n} \Phi_d(z) = z^n - 1. \quad (\text{A6})$$

This tells us that $\Phi_n(z)\Phi_p(z)\Phi_{np}(z)$ is by definition

$$\Phi_n(z)\Phi_p(z)\Phi_{np}(z) = 1 + z + \dots + z^{pn-1}, \quad (\text{A7})$$

Since $pn-1$ is the maximal degree that $P(z)$ can have (cf. Eq. (15)) we then conclude $R_4(z) = 0$ or $R_4(z) = 1$. The first choice gives rational independence and the second is incompatible with the first of (A3). This concludes the proof.

Upon adding all d order harmonics such that the step in (A6) is reproduced, this argument can be directly extended to arbitrary products of powers of primes. Namely, to arbitrary $L \in \mathbb{N}$.

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