# LOCAL FACTORS AND CUNTZ-PIMSNER ALGEBRAS 

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#### Abstract

We recast the local factors of the Hasse-Weil zeta function at infinity in terms of the Cuntz-Pimsner algebras. The nature of such factors is an open problem studied by Deninger and Serre.


## 1. Introduction

Let $V$ be an $n$-dimensional smooth projective variety over a number field $k$ and let $V\left(\mathbf{F}_{q}\right)$ be a good reduction of $V$ modulo the prime ideal corresponding to $q=p^{r}$. Recall that the local zeta $Z_{q}(u):=\exp \left(\sum_{m=1}^{\infty}\left|V\left(\mathbf{F}_{q}\right)\right| \frac{u^{m}}{m}\right)$ is a rational function

$$
\begin{equation*}
Z_{q}(u)=\frac{P_{1}(u) \ldots P_{2 n-1}(u)}{P_{0}(u) \ldots P_{2 n}(u)} \tag{1.1}
\end{equation*}
$$

where $P_{0}(u)=1-u$ and $P_{2 n}(u)=1-q^{n} u$. Each $P_{i}(u)$ is the characteristic polynomial of the Frobenius endomorphism $F r_{q}^{i}:\left(a_{1}, \ldots, a_{n}\right) \mapsto\left(a_{1}^{q}, \ldots, a_{n}^{q}\right)$ acting on the $i$-th $\ell$-adic cohomology group $H^{i}(V)$ of variety $V$. The number of points on $V\left(\mathbb{F}_{q}\right)$ is given by the Lefschetz trace formula $\left|V\left(\mathbb{F}_{q}\right)\right|=\sum_{i=0}^{2 n}(-1)^{i} \operatorname{tr}\left(F r_{q}^{i}\right)$, where $t r$ is the trace of endomorphism $F r_{q}^{i}$ [Hartshorne 1977] [5, pp. 454-457]. The Hasse-Weil zeta function of $V$ is an infinite product

$$
\begin{equation*}
Z_{V}(s)=\prod_{p} Z_{q}\left(p^{-s}\right), \quad s \in \mathbf{C} \tag{1.2}
\end{equation*}
$$

where $p$ runs through all but a finite set of primes. Such a function encodes arithmetic of the variety $V$. For example, if $E$ is an elliptic curve over $\mathbf{Q}$ then $Z_{E}(s)=\frac{\zeta(s) \zeta(s-1)}{L(E, s)}$, where the order of zero of function $L(E, s)$ at $s=1$ is conjectured to be equal the rank of $E$.

Recall that a fundamental analogy between number fields and function fields predicts a prime $p=\infty$ in formula (1.1). It was a mystery how the factor $Z_{\infty}(u)$ looks like. The problem was studied by Serre who constructed local factors $\Gamma_{V}^{i}(s)$ realizing the analogy. The goal was achieved in terms of the $\Gamma$-functions attached to the Hodge structure on $V$ [Serre 1970] [10]. To define $\Gamma_{V}^{i}(s)$ in a way similar to finite primes, Deninger introduced an infinite-dimensional cohomology $H_{a r}^{i}(V)$ and an action of Frobenius endomorphism $F r_{\infty}^{i}: H_{a r}^{i}(V) \rightarrow H_{a r}^{i}(V)$, such that $\Gamma_{V}^{i}(s) \equiv \operatorname{char}^{-1} F r_{\infty}^{i}$, where char $F r_{\infty}^{i}$ is the characteristic polynomial of $F r_{\infty}^{i}$ [Deninger 1991] [2, Theorem 4.1].

The aim of our note is to recast $\Gamma_{V}^{i}(s)$ in terms of the Cuntz-Pimsner algebras [Pask \& Raeburn 1996] [9]. Namely, let $\mathscr{A}_{V}$ be the Serre $C^{*}$-algebra of $V$ [8, Section 5.3.1]. Recall [6, Lemma 4] that $\operatorname{tr}\left(F r_{q}^{i}\right)=\operatorname{tr}\left(M k_{q}^{i}\right)$, where $M k_{q}^{i}$ is the Markov endomorphism of a lattice $\Lambda_{i} \subseteq \tau_{*}\left(K_{0}\left(\mathscr{A}_{V} \otimes \mathcal{K}\right)\right) \subset \mathbf{R}$ defined by the

[^0]canonical trace $\tau$ on the $K_{0}$-group of stabilized $C^{*}$-algebra $\mathscr{A}_{V}$ [6, p.271]. Therefore $\left|V\left(\mathbb{F}_{q}\right)\right|=\sum_{i=0}^{2 n}(-1)^{i} \operatorname{tr}\left(M k_{q}^{i}\right)\left[6\right.$, Theorem 1] and the local zeta $Z_{q}(u)$ is a function of the endomorphisms $M k_{q}^{i}$, where $0 \leq i \leq 2 n$. On the other hand, $M k_{q}^{i} \in G L_{b_{i}}(\mathbf{Z})$ is given by a positive matrix, where $b_{i}$ is the $i$-th Betti number of $V$ [6, p. 274]. We shall denote by $\mathcal{O}_{M k_{q}^{i}}$ the Cuntz-Krieger algebra given by matrix $M k_{q}^{i}$ [Cuntz \& Krieger 1980] [1]. Thus the local factors $\Gamma_{V}^{i}(s)$ must correspond to the CuntzKrieger algebras given by the countably infinite matrices $A_{\infty}^{i} \in G L_{\infty}(\mathbf{Z})$. The $\mathcal{O}_{A_{\infty}^{i}}$ are called the Cuntz-Pimsner algebras [Pask \& Raeburn 1996] [9].

Each matrix $A_{\infty}^{i}$ is constructed as follows. Let $\operatorname{Mod}(V)$ be the moduli variety of $V$. Recall that an analog of $\mathscr{A}_{V}$ for $\operatorname{Mod}(V)$ is given by a cluster $C^{*}$-algebra $\mathbb{A}$, such that $\operatorname{Prim}(\mathbb{A}) \cong \operatorname{Mod}(V)$, where $\operatorname{Prim}(\mathbb{A})$ is the set of two-sided primitive closed ideals of $\mathbb{A}$ endowed with the Jacobson topology. Moreover, $\mathscr{A}_{V} \subset \mathbb{A} / I$ and $K_{0}\left(\mathscr{A}_{V}\right) \cong K_{0}(\mathbb{A} / I)$, where $I \in \operatorname{Prim}(\mathbb{A})$ [7, Theorem 2]. (Note that the construction is given for $n=1$ [7] but true for the dimensions $n \geq 1$.) In other words, one gets a short exact sequence of the abelian groups:

$$
\begin{equation*}
K_{0}(I) \stackrel{i}{\hookrightarrow} K_{0}(\mathbb{A}) \xrightarrow{p} K_{0}\left(\mathscr{A}_{V}\right), \tag{1.3}
\end{equation*}
$$

where $K_{0}(I) \cong K_{0}(\mathbb{A}) \cong \mathbf{Z}^{\infty}$. Since $K_{0}\left(\mathscr{A}_{V}\right) \cong K_{0}\left(\mathscr{A}_{V} \otimes \mathcal{K}\right)$, the $\mathbb{Z}$-modules $\Lambda_{i} \subseteq \tau_{*}\left(K_{0}\left(\mathscr{A}_{V} \otimes \mathcal{K}\right)\right)$ specified earlier, define a pull back of (1.3). Thus one gets an exact sequence of modules $\Lambda_{i}^{\infty} \stackrel{i_{*}}{\longrightarrow} \Lambda_{i}^{\infty} \xrightarrow{p_{*}} \Lambda_{i}$. Here the rank of cluster algebra $\tau^{-1}\left(\Lambda_{\infty}^{i}\right)$ is equal to the Betti number $b_{i}$ and $i_{*}$ is the injective homomorphism given by a matrix $A_{\infty}^{i} \in G L_{\infty}(\mathbf{Z})$ for each $0 \leq i \leq 2 n$. Our main result can be formulated as follows.

Theorem 1.1. For every smooth n-dimensional projective variety $V$ over a number field $k$ there exist the Cuntz-Pimsner algebras $\mathcal{O}_{A_{\infty}^{i}}$, such that the Hasse-Weil zeta function of $V$ is given by the formula:

$$
\begin{equation*}
Z_{V}(s)=\prod_{i=0}^{2 n}\left(\operatorname{char} A_{\infty}^{i}\right)^{(-1)^{i+1}} \tag{1.4}
\end{equation*}
$$

The paper is organized as follows. A brief review of the preliminary facts is given in Section 2. Theorem 1.1 proved in Section 3. An application of theorem 1.1 is considered in Section 4.

## 2. Preliminaries

We briefly review Deninger cohomology, Cuntz-Pimsner algebras and cluster $C^{*}$ algebras. We refer the reader to [Deninger 1991] [2], [7] and [Pask \& Raeburn 1996] [9] for a detailed exposition.
2.1. Deninger cohomology. The Hodge-Tate module is a $p$-adic generalization of the Hodge structure. Namely, let $G$ be the absolute Galois group of a $p$-adic field $\mathbf{Q}_{p}$ acting by continuity on the algebraic completion $C$ of $\mathbf{Q}_{p}$. If $\chi$ is a cyclotomic character of $G$, then a module generated by the integer powers of $\chi$ is called HodgeTate, see [Fontaine 1982][4, Section 1.1] for the details. Let $T:=\left(\lim _{\longleftarrow} \mu_{p^{n}}\right) \otimes \mathbf{Q}_{p}$, where $\mu_{m}$ is the $m$-th root of unity. The Hodge-Tate ring is defined as $B_{H T}:=$ $C\left[T^{ \pm 1}\right]$, where $G$ acts on $T^{i}$ by $\chi^{i}$. The Hodge filtration on the ring $B_{H T}$ is given by the formula $T^{i} C\left[T^{ \pm 1}\right]$. Using the multi-prime numbers $\left(p_{1}, \ldots, p_{n}\right)$, one can extend $B_{H T}$ to the multivaraible Laurent polynomials $C\left[T^{ \pm 1}\right]$, where $T=\left(x_{1}, \ldots, x_{n}\right)$.

Deninger's idea is to replace the ring $B_{H T}$ over $C$ by a ring $B_{a r}$ of the Laurent polynomials over the archimedian place $\mathbf{R}$ [Deninger 1991] [2, Section 3]. Deninger cohomology of a smooth projective variety $V$ is defined by the formula

$$
\begin{equation*}
H_{a r}^{i}(V)=\mathbb{D}\left(B_{a r}^{i}\right) \tag{2.1}
\end{equation*}
$$

where $B_{a r}^{i}$ is the $i$-th cohomology of $V$ viewed as a real Hodge structure and $\mathbb{D}$ is a functor from the category of Hodge structures to an additive category of modules defined by the derivation $\Theta=T \frac{d}{d T}$ on the ring $B_{a r}$. The following fundamental result relates the Deninger cohomology and the Serre local factors $\Gamma_{V}^{i}(s)$.

Theorem 2.1. ([2, Theorem 4.1]) The derivation $\Theta$ induces an endomorphism $F r_{\infty}^{i}: H_{a r}^{i}(V) \rightarrow H_{a r}^{i}(V)$, such that

$$
\begin{equation*}
\operatorname{char}^{-1} \operatorname{Fr}_{\infty}^{i} \equiv \Gamma_{V}^{i}(s) \tag{2.2}
\end{equation*}
$$

Remark 2.2. In what follows, all determinants are the regularized determinants of the countably infinite-dimensional matrices in the sense of [Deninger 1991] [2, Section 1]. Thus the polynomial char $F r_{\infty}^{i}:=\operatorname{det}\left(F r_{\infty}^{i}-s I\right)$ in (2.2) is well defined.
2.2. Cuntz-Pimsner algebras. Recall that the Cuntz-Krieger algebra $\mathcal{O}_{A}$ is a $C^{*}$-algebra generated by the partial isometries $s_{1}, \ldots, s_{n}$ which satisfy the relations

$$
\begin{cases}s_{1}^{*} s_{1} & =a_{11} s_{1} s_{1}^{*}+a_{12} s_{2} s_{2}^{*}+\cdots+a_{1 n} s_{n} s_{n}^{*}  \tag{2.3}\\ s_{2}^{*} s_{2} & =a_{21} s_{1} s_{1}^{*}+a_{22} s_{2} s_{2}^{*}+\cdots+a_{2 n} s_{n} s_{n}^{*} \\ & \cdots \\ s_{n}^{*} s_{n} & =a_{n 1} s_{1} s_{1}^{*}+a_{n 2} s_{2} s_{2}^{*}+\cdots+a_{n n} s_{n} s_{n}^{*}\end{cases}
$$

where $A=\left(a_{i j}\right)$ is a square matrix with $a_{i j} \in\{0,1,2, \ldots\}$. (Note that the original definition of $\mathcal{O}_{A}$ says that $a_{i j} \in\{0,1\}$ but is known to be extendable to all nonnegative integers [Cuntz \& Krieger 1980] [1].) Such algebras appear naturally in the study of local factors [6].

The Cuntz-Pimsner algebra is a generalization of $\mathcal{O}_{A}$ to the countably infinite matrices $A_{\infty} \in G L_{\infty}(\mathbf{Z})$ [Pask \& Raeburn 1996] [9]. Recall that the matrix $A_{\infty}$ is called row-finite, if for each $i \in \mathbf{N}$ the number of $j \in \mathbf{N}$ with $a_{i j} \neq 0$ is finite. The matrix $A$ is said to be irreducible, if some power of $A$ is a strictly positive matrix and $A$ is not a permutation matrix. It is known that if $A_{\infty}$ is row-finite and irreducible, then the Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}}$ is a well-defined and simple [Pask \& Raeburn 1996] [9, Theorem 1]. An AF-core $\mathscr{F} \subset \mathcal{O}_{A_{\infty}}$ is an Approximately Finite (AF-) $C^{*}-$ algebra defined by the closure of of the infinite union $\cup_{k, j} \cup_{i \in V_{k}^{j}} \mathscr{F}_{k}^{j}(i)$, where $\mathscr{F}_{k}^{j}(i)$ are finite-dimensional $C^{*}$-algebras built from matrix $A_{\infty}$, see [Pask \& Raeburn 1996] [ 9 , Definition 2.2.1] for the details. We shall denote by $\alpha: \mathcal{O}_{A_{\infty}} \rightarrow \mathcal{O}_{A_{\infty}}$ an automorphism acting on the generators $s_{i}$ of $\mathcal{O}_{A_{\infty}}$ by to the formula $\alpha_{z}\left(s_{i}\right)=z s_{i}$, where $z$ is a complex number of the absolute value $|z|=1$. Thus one gets an action of the abelian group $\mathbb{T} \cong \mathbf{R} / \mathbf{Z}$ on $\mathcal{O}_{A_{\infty}}$. It follows from the Takai duality [Pask \& Raeburn 1996] [9, p. 432] that:

$$
\begin{equation*}
\mathscr{F} \rtimes_{\hat{\alpha}} \mathbb{T} \cong \mathcal{O}_{A_{\infty}} \otimes \mathcal{K} \tag{2.4}
\end{equation*}
$$

where $\hat{\alpha}$ is the Takai dual of $\alpha$ and $\mathcal{K}$ is the $C^{*}$-algebra of compact operators. Using (2.4) one can calculate the the $K$-theory of $\mathcal{O}_{A_{\infty}}$.

Theorem 2.3. ([9, Theorem 3]) If $A_{\infty}$ is row-finite irreducible matrix, then there exists an exact sequence of the abelian groups:

$$
\begin{equation*}
0 \rightarrow K_{1}\left(\mathcal{O}_{A_{\infty}}\right) \rightarrow \mathbf{Z}^{\infty} \xrightarrow{1-A_{\infty}^{t}} \mathbf{Z}^{\infty} \xrightarrow{i_{*}} K_{0}\left(\mathcal{O}_{A_{\infty}}\right) \rightarrow 0, \tag{2.5}
\end{equation*}
$$

so that $K_{0}\left(\mathcal{O}_{A_{\infty}}\right) \cong \mathbf{Z}^{\infty} /\left(1-A_{\infty}^{t}\right) \mathbf{Z}^{\infty}$ and $K_{1}\left(\mathcal{O}_{A_{\infty}}\right) \cong \operatorname{Ker}\left(1-A_{\infty}^{t}\right)$, where $A_{\infty}^{t}$ is the transpose of $A_{\infty}$ and $i: \mathscr{F} \hookrightarrow \mathcal{O}_{A_{\infty}}$. Moreover, the Grothendieck semigroup $K_{0}^{+}(\mathscr{F}) \cong \xrightarrow[\longrightarrow]{\lim }\left(\mathbf{Z}^{\infty}, A_{\infty}^{t}\right)$.
2.3. Cluster $C^{*}$-algebras. The cluster algebra of rank $n$ is a subring $\mathcal{A}(\mathbf{x}, B)$ of the field of rational functions in $n$ variables depending on variables $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ and a skew-symmetric matrix $B=\left(b_{i j}\right) \in M_{n}(\mathbf{Z})$. The pair $(\mathbf{x}, B)$ is called a seed. A new cluster $\mathbf{x}^{\prime}=\left(x_{1}, \ldots, x_{k}^{\prime}, \ldots, x_{n}\right)$ and a new skew-symmetric matrix $B^{\prime}=\left(b_{i j}^{\prime}\right)$ is obtained from $(\mathbf{x}, B)$ by the exchange relations [Williams 2014] [11, Definition 2.22]:

$$
\begin{align*}
x_{k} x_{k}^{\prime} & =\prod_{i=1}^{n} x_{i}^{\max \left(b_{i k}, 0\right)}+\prod_{i=1}^{n} x_{i}^{\max \left(-b_{i k}, 0\right)} \\
b_{i j}^{\prime} & = \begin{cases}-b_{i j} & \text { if } i=k \text { or } j=k \\
b_{i j}+\frac{\left|b_{i k}\right| b_{k j}+b_{i k}\left|b_{k j}\right|}{2} & \text { otherwise }\end{cases} \tag{2.6}
\end{align*}
$$

The seed $\left(\mathrm{x}^{\prime}, B^{\prime}\right)$ is said to be a mutation of $(\mathbf{x}, B)$ in direction $k$. where $1 \leq k \leq n$. The algebra $\mathcal{A}(\mathbf{x}, B)$ is generated by the cluster variables $\left\{x_{i}\right\}_{i=1}^{\infty}$ obtained from the initial seed $(\mathbf{x}, B)$ by the iteration of mutations in all possible directions $k$. The Laurent phenomenon says that $\mathcal{A}(\mathbf{x}, B) \subset \mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]$, where $\mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]$ is the ring of the Laurent polynomials in variables $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ [Williams 2014] [11, Theorem 2.27]. In particular, each generator $x_{i}$ of the algebra $\mathcal{A}(\mathbf{x}, B)$ can be written as a Laurent polynomial in $n$ variables with the integer coefficients.

The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ has the structure of an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In other words, the $\mathcal{A}(\mathbf{x}, B)$ is a dimension group, see Section 2.1.6 or [8, Definition 3.5.2]. The cluster $C^{*}$-algebra $\mathbb{A}(\mathbf{x}, B)$ is an AF-algebra, such that $K_{0}(\mathbb{A}(\mathbf{x}, B)) \cong \mathcal{A}(\mathbf{x}, B)$.
2.3.1. Cluster $C^{*}$-algebra $\mathbb{A}\left(S_{g, n}\right)$. Denote by $S_{g, n}$ the Riemann surface of genus $g \geq 0$ with $n \geq 0$ cusps. Let $\mathcal{A}\left(\mathbf{x}, S_{g, n}\right)$ be the cluster algebra coming from a triangulation of the surface $S_{g, n}$ [Williams 2014] [11, Section 3.3]. We shall denote by $\mathbb{A}\left(S_{g, n}\right)$ the corresponding cluster $C^{*}$-algebra. Let $T_{g, n}$ be the Teichmüller space of the surface $S_{g, n}$, i.e. the set of all complex structures on $S_{g, n}$ endowed with the natural topology. The geodesic flow $T^{t}: T_{g, n} \rightarrow T_{g, n}$ is a one-parameter group of matrices $\operatorname{diag}\left(e^{t}, e^{-t}\right)$ acting on the holomorphic quadratic differentials on the Riemann surface $S_{g, n}$. Such a flow gives rise to a one parameter group of automorphisms $\sigma_{t}: \mathbb{A}\left(S_{g, n}\right) \rightarrow \mathbb{A}\left(S_{g, n}\right)$ called the Tomita-Takesaki flow on the AF-algebra $\mathbb{A}\left(S_{g, n}\right)$. Denote by Prim $\mathbb{A}\left(S_{g, n}\right)$ the space of all primitive ideals of $\mathbb{A}\left(S_{g, n}\right)$ endowed with the Jacobson topology. Recall ([7]) that each primitive ideal has a parametrization by a vector $\Theta \in \mathbf{R}^{6 g-7+2 n}$ and we write it $I_{\Theta} \in \operatorname{Prim} \mathbb{A}\left(S_{g, n}\right)$

Theorem 2.4. ([7]) There exists a homeomorphism $h: \operatorname{Prim} \mathbb{A}\left(S_{g, n}\right) \times \mathbf{R} \rightarrow$ $\left\{U \subseteq T_{g, n} \mid U\right.$ is generic $\}$ given by the formula $\sigma_{t}\left(I_{\Theta}\right) \mapsto S_{g, n}$; the set $U=T_{g, n}$ if and only if $g=n=1$. The $\sigma_{t}\left(I_{\Theta}\right)$ is an ideal of $\mathbb{A}\left(S_{g, n}\right)$ for all $t \in \mathbf{R}$ and the quotient algebra $\mathbb{A}\left(S_{g, n}\right) / \sigma_{t}\left(I_{\Theta}\right)$ is a non-commutative coordinate ring of the Riemann surface $S_{g, n}$.

## 3. Proof of theorem 1.1

For the sake of clarity, let us outline the main ideas. Let $F r_{\infty}^{i}$ be the Frobenius endomorphism of the Deninger cohomology $H_{a r}^{i}(V)$ as stated in Theorem 2.1. From (2.1) we recall that $H_{a r}^{i}(V)$ is the additive group of the ring of Laurent polynomilas $\mathbf{R}\left[\mathbf{x}^{ \pm 1}\right]$, where $\mathbf{x}=\left(x_{1}, \ldots, x_{b_{i}}\right)$. A restriction of $F r_{\infty}^{i}: \mathbf{R}\left[\mathbf{x}^{ \pm 1}\right] \rightarrow \mathbf{R}\left[\mathbf{x}^{ \pm 1}\right]$ to the Laurent polynomials $\mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]$ gives rise to a two-sided primitive ideal $I_{F}^{i}$ (called the $i$-th Fontaine ideal) in the cluster $C^{*}$-algebra $\mathbb{A}^{i}$, where $K_{0}\left(\mathbb{A}^{i}\right) \cong \mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]$ and $K_{0}\left(I_{F}^{i}\right) \cong F r_{\infty}^{i}\left(\mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]\right)$. We prove that the $A F$-algebras $\mathbb{A}^{i} / I_{F}^{i}$ and $\mathbb{A}_{V}^{i}$ are stably isomorphic, where $K_{0}\left(\mathbb{A}_{V}^{i}\right) \cong \Lambda_{i}\left(\right.$ Lemma 3.1). Next it is proved that matrix $A_{\infty}^{i}$ is conjugate to $F r_{\infty}^{i}$ in the group $G L_{\infty}(\mathbf{Z})$ (Lemma 3.3). In particular, char $A_{\infty}^{i} \equiv$ char $\mathrm{Fr}_{\infty}^{i}$ for all $0 \leq i \leq 2 n$ (Corollary 3.4). The rest of the proof follows from Theorem 2.1, see Lemma 3.5. Let us pass to a detailed argument.

Lemma 3.1. $\mathbb{A}^{i} / I_{F}^{i}$ and $\mathbb{A}_{V}^{i}$ are stably isomorphic $A F$-algebras, where $K_{0}\left(\mathbb{A}_{V}^{i}\right) \cong$ $\Lambda_{i}$.

Proof. (i) Let us show that if projective varieties $V, V^{\prime}$ are isomorphic over the number field $k$, then there exists a ring automorphism $\phi$ of $\mathbb{A}^{i}$ such that the corresponding Fontaine ideal $I_{F^{\prime}}^{i}=\phi\left(I_{F}^{i}\right)$. Indeed, let $V \rightarrow V^{\prime}$ be an isomorphism between projective varieties $V$ and $V^{\prime}$. The cohomology functor induces an isomorphism $\phi: H_{a r}^{i}(V) \rightarrow H_{a r}^{i}\left(V^{\prime}\right)$ of the corresponding Deninger cohomology groups. Recall that $H_{a r}^{i}(V) \cong \mathbf{R}\left[\mathbf{x}^{ \pm 1}\right]$ and since the isomorphism of $V$ is defined over a number field $k$, one gets an isomorphism $\phi: \mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right] \rightarrow \mathbf{Z}\left[\mathbf{y}^{ \pm 1}\right]$. (Note that group isomorphism $\phi$ extends to a ring isomorphism by choice of a monomial basis in the ring of the Laurent polynomials, and vice versa.) Recall that $K_{0}\left(\mathbb{A}^{i}\right) \cong \mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]$ and $K$-theory is a functor; thus one gets an an automorphism $\phi: \mathbb{A}^{i} \rightarrow \mathbb{A}^{i}$. It remains to notice that the endomorphism $F r_{\infty}^{i}: \mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right] \rightarrow \mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]$ commutes with $\phi$ and therefore $\phi\left(\operatorname{Fr}_{\infty}^{i}\left(\mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]\right)\right)=F r_{\infty}^{i}\left(\mathbf{Z}\left[\mathbf{y}^{ \pm 1}\right]\right)$. By definition $K_{0}\left(I_{F}^{i}\right) \cong F r_{\infty}^{i}\left(\mathbf{Z}\left[\mathbf{x}^{ \pm 1}\right]\right)$ and thus the Fontaine ideal $I_{F^{\prime}}^{i}=\phi\left(I_{F}^{i}\right)$.
(ii) Let $I_{F}^{i} \subset \mathbb{A}^{i}$ be an $i$-th Fontaine ideal. Since $I_{F}^{i}$ is a primitive two-sided ideal, the quotient $C^{*}$-algebra $\mathbb{A}^{i} / I_{F}^{i}$ is simple. It follows from item (i) that isomorphisms of $V$ over $k$ correspond to the $C^{*}$-isomorphisms of the algebra $\mathbb{A}^{i} / I_{F}^{i}$.
(iii) On the other hand, we have a lattice $\Lambda_{i} \subseteq \tau_{*}\left(K_{0}\left(\mathscr{A}_{V} \otimes \mathcal{K}\right)\right) \subset \mathbf{R}$, where the rank of $\Lambda_{i}$ is equal to the $i$-th Betti number $b_{i}$ of $V$ [6, p.271]. It is well known that if projective varieties $V, V^{\prime}$ are isomorphic over the number field $k$, then their Serre $C^{*}$-algebras $\mathscr{A}_{V}, \mathscr{A}_{V^{\prime}}$ must be stably isomorphic (even isomorphic) [8, Section 5.3.1]. Since the $K_{0}$-groups are invariant under the stable isomorphisms, the lattices $\tau_{*}\left(K_{0}\left(\mathscr{A}_{V} \otimes \mathcal{K}\right)\right) \equiv \tau_{*}\left(K_{0}\left(\mathscr{A}_{V^{\prime}} \otimes \mathcal{K}\right)\right)$ and $\Lambda_{i} \equiv \Lambda_{i}^{\prime}$ as subsets of the real line. By definition $K_{0}\left(\mathbb{A}_{V}^{i}\right)=\Lambda_{i}$, so that the $A F$-algebras $\mathbb{A}_{V}^{i}$ and $\mathbb{A}_{V^{\prime}}^{i}$ are isomorphic.
(iv) To finish the proof, it remains to compare the results of items (ii) and (iii). Indeed, we constructed two covariant functors $V \mapsto \mathbb{A}^{i} / I_{F}^{i}$ and $V \mapsto \mathbb{A}_{V}^{i}$ from smooth $n$-dimensional projective varieties $V$ to the category of $A F$-algebras. But all morphisms in the latter category are stable isomorphisms between the $A F$-algebras, i.e. $\left(\mathbb{A}^{i} / I_{F}^{i}\right) \otimes \mathcal{K} \cong \mathbb{A}_{V}^{i} \otimes \mathcal{K}$.

Lemma 3.1 is proved.

Corollary 3.2. Cluster algebra $K_{0}\left(\mathbb{A}^{i}\right)$ has rank equal to the $i$-th Betti number of $V$.

Proof. It is known that the rank of lattice $\Lambda_{i}$ is equal to the $i$-th Betti number $b_{i}$ of variety $V\left[6\right.$, p.271]. Since $K_{0}\left(\mathbb{A}_{V}^{i}\right) \cong \Lambda_{i}$ and $\left(\mathbb{A}^{i} / I_{F}^{i}\right) \otimes \mathcal{K} \cong \mathbb{A}_{V}^{i} \otimes \mathcal{K}$, we conclude that $K_{0}\left(\mathbb{A}^{i} / I_{F}^{i}\right) \cong \Lambda_{i}$ and thus the rank of $K_{0}\left(\mathbb{A}^{i}\right)$ is equal to $b_{i}$.

Lemma 3.3. There exists a simple Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}^{i}}$ such that:

$$
\begin{equation*}
\mathcal{O}_{A_{\infty}^{i}} \otimes \mathcal{K} \cong I_{F}^{i} \rtimes_{\hat{\alpha}^{i}} \mathbb{T} \tag{3.1}
\end{equation*}
$$

where $A_{\infty}^{i} \in G L_{\infty}(\mathbf{Z})$ is conjugate to the matrix $F r_{\infty}^{i}$ and $I_{F}^{i}$ is the $i$-th Fontaine ideal of $\mathbb{A}^{i}$.

Proof. (i) For an $i$-th Fontaine ideal $I_{F}^{i} \subset \mathbb{A}^{i}$, let us calculate the semi-group $K_{0}^{+}\left(I_{F}^{i}\right)$. It is easy to see, that $K_{0}\left(I_{F}^{i}\right) \cong \mathbf{Z}^{\infty}$ and the corresponding Grothendieck semigroup $K_{0}^{+}\left(I_{F}^{i}\right) \cong \underset{\longrightarrow}{\lim }\left(\mathbf{Z}^{\infty}, F r_{\infty}^{i}\right)$, where the injective limit is taken by the iterations of the endomorphism $F r_{\infty}^{i}$ acting on $\mathbf{Z}^{\infty}$.
(ii) On the other hand, if $\mathscr{F}^{i} \subset \mathcal{O}_{A_{\infty}^{i}}$ is the core $A F$-algebra of a Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}^{i}}$, then $K_{0}^{+}\left(\mathscr{F}^{i}\right) \cong \underset{\longrightarrow}{\lim }\left(\mathbf{Z}^{\infty},\left(A_{\infty}^{i}\right)^{t}\right)$, see Theorem 2.3.
(iii) We now define matrix $A_{\infty}^{i} \in G L_{\infty}(\mathbf{Z})$ so that:

$$
\begin{equation*}
K_{0}^{+}\left(\mathscr{F}^{i}\right) \cong K_{0}^{+}\left(I_{F}^{i}\right), \tag{3.2}
\end{equation*}
$$

where $\cong$ is an isomorphism of the Grothendieck semigroups, i.e. an order-isomorphism of the corresponding positive cones.
(iv) It follows from (3.2) that $I_{F}^{i} \rtimes_{\hat{\alpha}^{i}} \mathbb{T} \cong \mathcal{O}_{A_{\infty}^{i}} \otimes \mathcal{K}$, see formula (2.4). Moreover, an isomorphism $\underset{\longrightarrow}{\lim }\left(\mathbf{Z}^{\infty},\left(A_{\infty}^{i}\right)^{t}\right) \cong \underset{\longrightarrow}{\lim }\left(\mathbf{Z}^{\infty}, F r_{\infty}^{i}\right)$ implies that matrices $A_{\infty}^{i}$ and $F r_{\infty}^{i}$ are conjugate in $G L_{\infty}(\mathbf{Z})$.
(v) Since the determinant det $\left(F r_{\infty}^{i}-s I\right)$ is regular (Remark 2.2), we conclude that the conjugate matrix $A_{\infty}^{i}$ must be row-finite and irreducible, i.e. $\mathcal{O}_{A_{\infty}^{i}}$ is a correctly defined simple Cuntz-Pimsner algebra.

Lemma 3.3 is proved.
Corollary 3.4. char $A_{\infty}^{i} \equiv \operatorname{char} \mathrm{Fr}_{\infty}^{i}$.
Proof. The characteristic polynomial char $A_{\infty}^{i}=\operatorname{det}\left(A_{\infty}^{i}-s I\right)$ is invariant of the conjugacy class of matrix $A_{\infty}^{i}$. We conclude from Lemma 3.3 that char $A_{\infty}^{i} \equiv$ char $F r_{\infty}^{i}$. (The converse is false in general.) Corollary 3.4 is proved.

Lemma 3.5. $Z_{V}(s)=\prod_{i=0}^{2 n}\left(\operatorname{char} A_{\infty}^{i}\right)^{(-1)^{i+1}}$.
Proof. (i) Recall that

$$
\begin{equation*}
Z_{V}(s)=\prod_{i=0}^{2 n}\left(\Gamma_{V}^{i}(s)\right)^{(-1)^{i}} \tag{3.3}
\end{equation*}
$$

where $\Gamma_{V}^{i}(s)$ is the $i$-th Serre local factor at $p=\infty$ [Serre 1970] [10, Section 3]. In view of Deninger's Theorem 2.1 we can substitute $\Gamma_{V}^{i}(s) \equiv \operatorname{char}^{-1} \operatorname{Fr}_{\infty}^{i}$ so that the Hasse-Weil zeta function (3.3) becomes:

$$
\begin{equation*}
Z_{V}(s)=\prod_{i=0}^{2 n}\left(\operatorname{char} \operatorname{Fr}_{\infty}^{i}\right)^{(-1)^{i+1}} \tag{3.4}
\end{equation*}
$$

(ii) On the other hand, there exist Cuntz-Pimsner algebras $\mathcal{O}_{A_{\infty}^{i}}$, such that char $A_{\infty}^{i} \equiv$ char $\operatorname{Fr}_{\infty}^{i}$ (Corollary 3.4). Thus one can write the Hasse-Weil zeta function (3.4) in the form:

$$
\begin{equation*}
Z_{V}(s)=\prod_{i=0}^{2 n}\left(\operatorname{char} A_{\infty}^{i}\right)^{(-1)^{i+1}} \tag{3.5}
\end{equation*}
$$

Lemma 3.5 is proved.

Theorem 1.1 follows from Lemmas 3.3 and 3.5.

## 4. Riemann zeta function

Let us point out a relation between the Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}}$ and nontrivial zeroes of the Riemann zeta function $\zeta(s)$. If $V$ is a curve, then $n=1$ and formula (1.4) for the Hasse-Weil zeta function can be written as:

$$
\begin{equation*}
Z_{V}(s)=\prod_{i=0}^{2}\left(\operatorname{char} A_{\infty}^{i}\right)^{(-1)^{i+1}}=\frac{\operatorname{char} A_{\infty}^{1}}{\operatorname{char} A_{\infty}^{0} \operatorname{char} A_{\infty}^{2}} \tag{4.1}
\end{equation*}
$$

Moreover, char $A_{\infty}^{0}=\frac{s}{2 \pi}$ and char $A_{\infty}^{2}=\frac{s-1}{2 \pi}$ [Deninger 1992] [3, Section 3]. Thus one can write (4.1) in the form:

$$
\begin{equation*}
(2 \pi)^{-2} s(s-1) Z_{V}(s)=\operatorname{char} A_{\infty}^{1} \tag{4.2}
\end{equation*}
$$

On the other hand, the Hasse-Weil zeta function can be linked to the Riemann zeta function $\zeta(s)$ by the well known formula:

$$
\begin{equation*}
Z_{V}(s)=2^{-\frac{1}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{4.3}
\end{equation*}
$$

where $\Gamma\left(\frac{s}{2}\right)$ is the gamma function. We can use (4.3) to exclude $Z_{V}(s)$ from (4.2):

$$
\begin{equation*}
2^{-\frac{5}{2}} \pi^{\frac{-s-4}{2}} \Gamma\left(\frac{s}{2}\right) s(s-1) \zeta(s)=\operatorname{char} A_{\infty}^{1} \tag{4.4}
\end{equation*}
$$

It follows from (4.4) that non-trivial zeros of the Riemann zeta function coincide with the roots of characteristic polynomial of the matrix $A_{\infty}^{1}$ defining the CuntzPimsner algebra $\mathcal{O}_{A_{\infty}^{1}}$. Thus the row-finite and irreducible matrices are proper candidates for Hilbert's idea to settle the Riemann Hypothesis via spectra of the self-adjoint operators.

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