LOCAL FACTORS AND CUNTZ-PIMSNER ALGEBRAS

IGOR V. NIKOLAEV¹

ABSTRACT. We recast the local factors of the Hasse-Weil zeta function at infinity in terms of the Cuntz-Pimsner algebras. The nature of such factors is an open problem studied by Deninger and Serre.

1. INTRODUCTION

Let V be an n-dimensional smooth projective variety over a number field k and let $V(\mathbf{F}_q)$ be a good reduction of V modulo the prime ideal corresponding to $q = p^r$. Recall that the local zeta $Z_q(u) := \exp\left(\sum_{m=1}^{\infty} |V(\mathbf{F}_q)| \frac{u^m}{m}\right)$ is a rational function

$$Z_q(u) = \frac{P_1(u)\dots P_{2n-1}(u)}{P_0(u)\dots P_{2n}(u)},$$
(1.1)

where $P_0(u) = 1 - u$ and $P_{2n}(u) = 1 - q^n u$. Each $P_i(u)$ is the characteristic polynomial of the Frobenius endomorphism Fr_q^i : $(a_1, \ldots, a_n) \mapsto (a_1^q, \ldots, a_n^q)$ acting on the *i*-th ℓ -adic cohomology group $H^i(V)$ of variety V. The number of points on $V(\mathbb{F}_q)$ is given by the Lefschetz trace formula $|V(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i tr (Fr_q^i)$, where tr is the trace of endomorphism Fr_q^i [Hartshorne 1977] [5, pp. 454-457]. The Hasse-Weil zeta function of V is an infinite product

$$Z_V(s) = \prod_p Z_q(p^{-s}), \quad s \in \mathbf{C},$$
(1.2)

where p runs through all but a finite set of primes. Such a function encodes arithmetic of the variety V. For example, if E is an elliptic curve over **Q** then $Z_E(s) = \frac{\zeta(s)\zeta(s-1)}{L(E,s)}$, where the order of zero of function L(E,s) at s = 1 is conjectured to be equal the rank of E.

Recall that a fundamental analogy between number fields and function fields predicts a prime $p = \infty$ in formula (1.1). It was a mystery how the factor $Z_{\infty}(u)$ looks like. The problem was studied by Serre who constructed local factors $\Gamma_V^i(s)$ realizing the analogy. The goal was achieved in terms of the Γ -functions attached to the Hodge structure on V [Serre 1970] [10]. To define $\Gamma_V^i(s)$ in a way similar to finite primes, Deninger introduced an infinite-dimensional cohomology $H_{ar}^i(V)$ and an action of Frobenius endomorphism $Fr_{\infty}^i : H_{ar}^i(V) \to H_{ar}^i(V)$, such that $\Gamma_V^i(s) \equiv char^{-1} Fr_{\infty}^i$, where char Fr_{∞}^i is the characteristic polynomial of Fr_{∞}^i [Deninger 1991] [2, Theorem 4.1].

The aim of our note is to recast $\Gamma_V^i(s)$ in terms of the Cuntz-Pimsner algebras [Pask & Raeburn 1996] [9]. Namely, let \mathscr{A}_V be the Serre C^* -algebra of V [8, Section 5.3.1]. Recall [6, Lemma 4] that $tr (Fr_q^i) = tr (Mk_q^i)$, where Mk_q^i is the Markov endomorphism of a lattice $\Lambda_i \subseteq \tau_*(K_0(\mathscr{A}_V \otimes \mathcal{K})) \subset \mathbf{R}$ defined by the

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canonical trace τ on the K_0 -group of stabilized C^* -algebra \mathscr{A}_V [6, p.271]. Therefore $|V(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i tr (Mk_q^i)$ [6, Theorem 1] and the local zeta $Z_q(u)$ is a function of the endomorphisms Mk_q^i , where $0 \leq i \leq 2n$. On the other hand, $Mk_q^i \in GL_{b_i}(\mathbb{Z})$ is given by a positive matrix, where b_i is the *i*-th Betti number of V [6, p. 274]. We shall denote by $\mathcal{O}_{Mk_q^i}$ the Cuntz-Krieger algebra given by matrix Mk_q^i [Cuntz & Krieger 1980] [1]. Thus the local factors $\Gamma_V^i(s)$ must correspond to the Cuntz-Krieger algebras given by the countably infinite matrices $A_{\infty}^i \in GL_{\infty}(\mathbb{Z})$. The $\mathcal{O}_{A_{\infty}^i}$ are called the Cuntz-Pimsner algebras [Pask & Raeburn 1996] [9].

Each matrix A^i_{∞} is constructed as follows. Let Mod(V) be the moduli variety of V. Recall that an analog of \mathscr{A}_V for Mod(V) is given by a cluster C^* -algebra \mathbb{A} , such that $Prim(\mathbb{A}) \cong Mod(V)$, where $Prim(\mathbb{A})$ is the set of two-sided primitive closed ideals of \mathbb{A} endowed with the Jacobson topology. Moreover, $\mathscr{A}_V \subset \mathbb{A}/I$ and $K_0(\mathscr{A}_V) \cong K_0(\mathbb{A}/I)$, where $I \in Prim(\mathbb{A})$ [7, Theorem 2]. (Note that the construction is given for n = 1 [7] but true for the dimensions $n \geq 1$.) In other words, one gets a short exact sequence of the abelian groups:

$$K_0(I) \stackrel{i}{\hookrightarrow} K_0(\mathbb{A}) \stackrel{p}{\to} K_0(\mathscr{A}_V),$$
 (1.3)

where $K_0(I) \cong K_0(\mathbb{A}) \cong \mathbb{Z}^{\infty}$. Since $K_0(\mathscr{A}_V) \cong K_0(\mathscr{A}_V \otimes \mathcal{K})$, the \mathbb{Z} -modules $\Lambda_i \subseteq \tau_*(K_0(\mathscr{A}_V \otimes \mathcal{K}))$ specified earlier, define a pull back of (1.3). Thus one gets an exact sequence of modules $\Lambda_i^{\infty} \xrightarrow{i_*} \Lambda_i^{\infty} \xrightarrow{p_*} \Lambda_i$. Here the rank of cluster algebra $\tau^{-1}(\Lambda_{\infty}^i)$ is equal to the Betti number b_i and i_* is the injective homomorphism given by a matrix $A_{\infty}^i \in GL_{\infty}(\mathbb{Z})$ for each $0 \leq i \leq 2n$. Our main result can be formulated as follows.

Theorem 1.1. For every smooth n-dimensional projective variety V over a number field k there exist the Cuntz-Pimsner algebras $\mathcal{O}_{A^i_{\infty}}$, such that the Hasse-Weil zeta function of V is given by the formula:

$$Z_V(s) = \prod_{i=0}^{2n} \left(char \ A_{\infty}^i \right)^{(-1)^{i+1}}.$$
 (1.4)

The paper is organized as follows. A brief review of the preliminary facts is given in Section 2. Theorem 1.1 proved in Section 3. An application of theorem 1.1 is considered in Section 4.

2. Preliminaries

We briefly review Deninger cohomology, Cuntz-Pimsner algebras and cluster C^* algebras. We refer the reader to [Deninger 1991] [2], [7] and [Pask & Raeburn 1996] [9] for a detailed exposition.

2.1. **Deninger cohomology.** The Hodge-Tate module is a *p*-adic generalization of the Hodge structure. Namely, let *G* be the absolute Galois group of a *p*-adic field \mathbf{Q}_p acting by continuity on the algebraic completion *C* of \mathbf{Q}_p . If χ is a cyclotomic character of *G*, then a module generated by the integer powers of χ is called Hodge-Tate, see [Fontaine 1982][4, Section 1.1] for the details. Let $T := (\varprojlim \mu_{p^n}) \otimes \mathbf{Q}_p$, where μ_m is the *m*-th root of unity. The Hodge-Tate ring is defined as $B_{HT} := C[T^{\pm 1}]$, where *G* acts on T^i by χ^i . The Hodge filtration on the ring B_{HT} is given by the formula $T^i C[T^{\pm 1}]$. Using the multi-prime numbers (p_1, \ldots, p_n) , one can extend B_{HT} to the multivariable Laurent polynomials $C[T^{\pm 1}]$, where $T = (x_1, \ldots, x_n)$.

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Deninger's idea is to replace the ring B_{HT} over C by a ring B_{ar} of the Laurent polynomials over the archimedian place **R** [Deninger 1991] [2, Section 3]. Deninger cohomology of a smooth projective variety V is defined by the formula

$$H^i_{ar}(V) = \mathbb{D}(B^i_{ar}), \tag{2.1}$$

where B_{ar}^i is the *i*-th cohomology of V viewed as a real Hodge structure and \mathbb{D} is a functor from the category of Hodge structures to an additive category of modules defined by the derivation $\Theta = T \frac{d}{dT}$ on the ring B_{ar} . The following fundamental result relates the Deninger cohomology and the Serre local factors $\Gamma_V^i(s)$.

Theorem 2.1. ([2, Theorem 4.1]) The derivation Θ induces an endomorphism $Fr^i_{\infty}: H^i_{ar}(V) \to H^i_{ar}(V)$, such that

$$char^{-1} Fr^i_{\infty} \equiv \Gamma^i_V(s).$$
 (2.2)

Remark 2.2. In what follows, all determinants are the regularized determinants of the countably infinite-dimensional matrices in the sense of [Deninger 1991] [2, Section 1]. Thus the polynomial char $Fr_{\infty}^{i} := \det (Fr_{\infty}^{i} - sI)$ in (2.2) is well defined.

2.2. Cuntz-Pimsner algebras. Recall that the Cuntz-Krieger algebra \mathcal{O}_A is a C^* -algebra generated by the partial isometries s_1, \ldots, s_n which satisfy the relations

$$\begin{cases} s_1^* s_1 = a_{11} s_1 s_1^* + a_{12} s_2 s_2^* + \dots + a_{1n} s_n s_n^* \\ s_2^* s_2 = a_{21} s_1 s_1^* + a_{22} s_2 s_2^* + \dots + a_{2n} s_n s_n^* \\ \dots \\ s_n^* s_n = a_{n1} s_1 s_1^* + a_{n2} s_2 s_2^* + \dots + a_{nn} s_n s_n^*, \end{cases}$$

$$(2.3)$$

where $A = (a_{ij})$ is a square matrix with $a_{ij} \in \{0, 1, 2, ...\}$. (Note that the original definition of \mathcal{O}_A says that $a_{ij} \in \{0, 1\}$ but is known to be extendable to all non-negative integers [Cuntz & Krieger 1980] [1].) Such algebras appear naturally in the study of local factors [6].

The Cuntz-Pimsner algebra is a generalization of \mathcal{O}_A to the countably infinite matrices $A_{\infty} \in GL_{\infty}(\mathbf{Z})$ [Pask & Raeburn 1996] [9]. Recall that the matrix A_{∞} is called row-finite, if for each $i \in \mathbf{N}$ the number of $j \in \mathbf{N}$ with $a_{ij} \neq 0$ is finite. The matrix A is said to be irreducible, if some power of A is a strictly positive matrix and A is not a permutation matrix. It is known that if A_{∞} is row-finite and irreducible, then the Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}}$ is a well-defined and simple [Pask & Raeburn 1996] [9, Theorem 1]. An AF-core $\mathscr{F} \subset \mathcal{O}_{A_{\infty}}$ is an Approximately Finite (AF-) C^* algebra defined by the closure of of the infinite union $\cup_{k,j} \cup_{i \in V_k^j} \mathscr{F}_k^j(i)$, where $\mathscr{F}_k^j(i)$ are finite-dimensional C^* -algebras built from matrix A_{∞} , see [Pask & Raeburn 1996] [9, Definition 2.2.1] for the details. We shall denote by $\alpha : \mathcal{O}_{A_{\infty}} \to \mathcal{O}_{A_{\infty}}$ an automorphism acting on the generators s_i of $\mathcal{O}_{A_{\infty}}$ by to the formula $\alpha_z(s_i) = zs_i$, where z is a complex number of the absolute value |z| = 1. Thus one gets an action of the abelian group $\mathbb{T} \cong \mathbf{R}/\mathbf{Z}$ on $\mathcal{O}_{A_{\infty}}$. It follows from the Takai duality [Pask & Raeburn 1996] [9, p. 432] that:

$$\mathscr{F} \rtimes_{\hat{\alpha}} \mathbb{T} \cong \mathcal{O}_{A_{\infty}} \otimes \mathcal{K},$$

$$(2.4)$$

where $\hat{\alpha}$ is the Takai dual of α and \mathcal{K} is the C^* -algebra of compact operators. Using (2.4) one can calculate the K-theory of $\mathcal{O}_{A_{\infty}}$.

Theorem 2.3. ([9, Theorem 3]) If A_{∞} is row-finite irreducible matrix, then there exists an exact sequence of the abelian groups:

$$0 \to K_1(\mathcal{O}_{A_{\infty}}) \to \mathbf{Z}^{\infty} \xrightarrow{1-A_{\infty}^*} \mathbf{Z}^{\infty} \xrightarrow{i_*} K_0(\mathcal{O}_{A_{\infty}}) \to 0,$$
(2.5)

so that $K_0(\mathcal{O}_{A_{\infty}}) \cong \mathbf{Z}^{\infty}/(1-A_{\infty}^t)\mathbf{Z}^{\infty}$ and $K_1(\mathcal{O}_{A_{\infty}}) \cong Ker (1-A_{\infty}^t)$, where A_{∞}^t is the transpose of A_{∞} and $i: \mathscr{F} \hookrightarrow \mathcal{O}_{A_{\infty}}$. Moreover, the Grothendieck semigroup $K_0^+(\mathscr{F}) \cong \lim(\mathbf{Z}^{\infty}, A_{\infty}^t)$.

2.3. Cluster C^* -algebras. The cluster algebra of rank n is a subring $\mathcal{A}(\mathbf{x}, B)$ of the field of rational functions in n variables depending on variables $\mathbf{x} = (x_1, \ldots, x_n)$ and a skew-symmetric matrix $B = (b_{ij}) \in M_n(\mathbf{Z})$. The pair (\mathbf{x}, B) is called a seed. A new cluster $\mathbf{x}' = (x_1, \ldots, x'_k, \ldots, x_n)$ and a new skew-symmetric matrix $B' = (b'_{ij})$ is obtained from (\mathbf{x}, B) by the exchange relations [Williams 2014] [11, Definition 2.22]:

$$\begin{aligned}
x_k x'_k &= \prod_{i=1}^n x_i^{\max(b_{ik},0)} + \prod_{i=1}^n x_i^{\max(-b_{ik},0)}, \\
b'_{ij} &= \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.}
\end{aligned} \tag{2.6}$$

The seed (\mathbf{x}', B') is said to be a mutation of (\mathbf{x}, B) in direction k. where $1 \le k \le n$. The algebra $\mathcal{A}(\mathbf{x}, B)$ is generated by the cluster variables $\{x_i\}_{i=1}^{\infty}$ obtained from the initial seed (\mathbf{x}, B) by the iteration of mutations in all possible directions k. The Laurent phenomenon says that $\mathcal{A}(\mathbf{x}, B) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$, where $\mathbf{Z}[\mathbf{x}^{\pm 1}]$ is the ring of the Laurent polynomials in variables $\mathbf{x} = (x_1, \ldots, x_n)$ [Williams 2014] [11, Theorem 2.27]. In particular, each generator x_i of the algebra $\mathcal{A}(\mathbf{x}, B)$ can be written as a Laurent polynomial in n variables with the integer coefficients.

The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ has the structure of an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In other words, the $\mathcal{A}(\mathbf{x}, B)$ is a dimension group, see Section 2.1.6 or [8, Definition 3.5.2]. The cluster C^* -algebra $\mathbb{A}(\mathbf{x}, B)$ is an AF-algebra, such that $K_0(\mathbb{A}(\mathbf{x}, B)) \cong \mathcal{A}(\mathbf{x}, B)$.

2.3.1. Cluster C^* -algebra $\mathbb{A}(S_{g,n})$. Denote by $S_{g,n}$ the Riemann surface of genus $g \geq 0$ with $n \geq 0$ cusps. Let $\mathcal{A}(\mathbf{x}, S_{g,n})$ be the cluster algebra coming from a triangulation of the surface $S_{g,n}$ [Williams 2014] [11, Section 3.3]. We shall denote by $\mathbb{A}(S_{g,n})$ the corresponding cluster C^* -algebra. Let $T_{g,n}$ be the Teichmüller space of the surface $S_{g,n}$, i.e. the set of all complex structures on $S_{g,n}$ endowed with the natural topology. The geodesic flow $T^t : T_{g,n} \to T_{g,n}$ is a one-parameter group of matrices $\operatorname{diag}(e^t, e^{-t})$ acting on the holomorphic quadratic differentials on the Riemann surface $S_{g,n}$. Such a flow gives rise to a one parameter group of automorphisms $\sigma_t : \mathbb{A}(S_{g,n}) \to \mathbb{A}(S_{g,n})$ the space of all primitive ideals of $\mathbb{A}(S_{g,n})$ endowed with the Jacobson topology. Recall ([7]) that each primitive ideal has a parametrization by a vector $\Theta \in \mathbf{R}^{6g-7+2n}$ and we write it $I_{\Theta} \in Prim \mathbb{A}(S_{g,n})$

Theorem 2.4. ([7]) There exists a homeomorphism $h : Prim A(S_{g,n}) \times \mathbf{R} \to \{U \subseteq T_{g,n} \mid U \text{ is generic}\}\ given by the formula <math>\sigma_t(I_{\Theta}) \mapsto S_{g,n};\ \text{the set } U = T_{g,n}\ \text{if and only if } g = n = 1.$ The $\sigma_t(I_{\Theta})$ is an ideal of $A(S_{g,n})$ for all $t \in \mathbf{R}$ and the quotient algebra $A(S_{g,n})/\sigma_t(I_{\Theta})$ is a non-commutative coordinate ring of the Riemann surface $S_{g,n}$.

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3. Proof of theorem 1.1

For the sake of clarity, let us outline the main ideas. Let Fr_{∞}^{i} be the Frobenius endomorphism of the Deninger cohomology $H_{ar}^{i}(V)$ as stated in Theorem 2.1. From (2.1) we recall that $H_{ar}^{i}(V)$ is the additive group of the ring of Laurent polynomilas $\mathbf{R}[\mathbf{x}^{\pm 1}]$, where $\mathbf{x} = (x_{1}, \ldots, x_{b_{i}})$. A restriction of $Fr_{\infty}^{i} : \mathbf{R}[\mathbf{x}^{\pm 1}] \to \mathbf{R}[\mathbf{x}^{\pm 1}]$ to the Laurent polynomials $\mathbf{Z}[\mathbf{x}^{\pm 1}]$ gives rise to a two-sided primitive ideal I_{F}^{i} (called the *i*-th Fontaine ideal) in the cluster C^{*} -algebra \mathbb{A}^{i} , where $K_{0}(\mathbb{A}^{i}) \cong \mathbf{Z}[\mathbf{x}^{\pm 1}]$ and $K_{0}(I_{F}^{i}) \cong Fr_{\infty}^{i}(\mathbf{Z}[\mathbf{x}^{\pm 1}])$. We prove that the AF-algebras \mathbb{A}^{i}/I_{F}^{i} and \mathbb{A}_{V}^{i} are stably isomorphic, where $K_{0}(\mathbb{A}_{V}^{i}) \cong \Lambda_{i}$ (Lemma 3.1). Next it is proved that matrix A_{∞}^{i} is conjugate to Fr_{∞}^{i} in the group $GL_{\infty}(\mathbf{Z})$ (Lemma 3.3). In particular, char $A_{\infty}^{i} \equiv$ char Fr_{∞}^{i} for all $0 \leq i \leq 2n$ (Corollary 3.4). The rest of the proof follows from Theorem 2.1, see Lemma 3.5. Let us pass to a detailed argument.

Lemma 3.1. \mathbb{A}^i/I_F^i and \mathbb{A}_V^i are stably isomorphic AF-algebras, where $K_0(\mathbb{A}_V^i) \cong \Lambda_i$.

Proof. (i) Let us show that if projective varieties V, V' are isomorphic over the number field k, then there exists a ring automorphism ϕ of \mathbb{A}^i such that the corresponding Fontaine ideal $I_{F'}^i = \phi(I_F^i)$. Indeed, let $V \to V'$ be an isomorphism between projective varieties V and V'. The cohomology functor induces an isomorphism $\phi : H_{ar}^i(V) \to H_{ar}^i(V')$ of the corresponding Deninger cohomology groups. Recall that $H_{ar}^i(V) \cong \mathbf{R}[\mathbf{x}^{\pm 1}]$ and since the isomorphism of V is defined over a number field k, one gets an isomorphism $\phi : \mathbf{Z}[\mathbf{x}^{\pm 1}] \to \mathbf{Z}[\mathbf{y}^{\pm 1}]$. (Note that group isomorphism ϕ extends to a ring isomorphism by choice of a monomial basis in the ring of the Laurent polynomials, and vice versa.) Recall that $K_0(\mathbb{A}^i) \cong \mathbf{Z}[\mathbf{x}^{\pm 1}]$ and K-theory is a functor; thus one gets an an automorphism $\phi : \mathbb{A}^i \to \mathbb{A}^i$. It remains to notice that the endomorphism $Fr_{\infty}^i : \mathbf{Z}[\mathbf{x}^{\pm 1}] \to \mathbf{Z}[\mathbf{x}^{\pm 1}]$ commutes with ϕ and therefore $\phi(Fr_{\infty}^i(\mathbf{Z}[\mathbf{x}^{\pm 1}])) = Fr_{\infty}^i(\mathbf{Z}[\mathbf{y}^{\pm 1}])$. By definition $K_0(I_F^i) \cong Fr_{\infty}^i(\mathbf{Z}[\mathbf{x}^{\pm 1}])$ and thus the Fontaine ideal $I_{F'}^i = \phi(I_F^i)$.

(ii) Let $I_F^i \subset \mathbb{A}^i$ be an *i*-th Fontaine ideal. Since I_F^i is a primitive two-sided ideal, the quotient C^* -algebra \mathbb{A}^i/I_F^i is simple. It follows from item (i) that isomorphisms of V over k correspond to the C^* -isomorphisms of the algebra \mathbb{A}^i/I_F^i .

(iii) On the other hand, we have a lattice $\Lambda_i \subseteq \tau_*(K_0(\mathscr{A}_V \otimes \mathcal{K})) \subset \mathbf{R}$, where the rank of Λ_i is equal to the *i*-th Betti number b_i of V [6, p.271]. It is well known that if projective varieties V, V' are isomorphic over the number field k, then their Serre C^* -algebras $\mathscr{A}_V, \mathscr{A}_{V'}$ must be stably isomorphic (even isomorphic) [8, Section 5.3.1]. Since the K_0 -groups are invariant under the stable isomorphisms, the lattices $\tau_*(K_0(\mathscr{A}_V \otimes \mathcal{K})) \equiv \tau_*(K_0(\mathscr{A}_{V'} \otimes \mathcal{K}))$ and $\Lambda_i \equiv \Lambda'_i$ as subsets of the real line. By definition $K_0(\mathbb{A}_V^i) = \Lambda_i$, so that the AF-algebras \mathbb{A}_V^i and $\mathbb{A}_{V'}^i$ are isomorphic.

(iv) To finish the proof, it remains to compare the results of items (ii) and (iii). Indeed, we constructed two covariant functors $V \mapsto \mathbb{A}^i/I_F^i$ and $V \mapsto \mathbb{A}^i_V$ from smooth *n*-dimensional projective varieties *V* to the category of *AF*-algebras. But all morphisms in the latter category are stable isomorphisms between the *AF*-algebras, i.e. $(\mathbb{A}^i/I_F^i) \otimes \mathcal{K} \cong \mathbb{A}^i_V \otimes \mathcal{K}$.

Lemma 3.1 is proved.

Corollary 3.2. Cluster algebra $K_0(\mathbb{A}^i)$ has rank equal to the *i*-th Betti number of V.

Proof. It is known that the rank of lattice Λ_i is equal to the *i*-th Betti number b_i of variety V [6, p.271]. Since $K_0(\mathbb{A}_V^i) \cong \Lambda_i$ and $(\mathbb{A}^i/I_F^i) \otimes \mathcal{K} \cong \mathbb{A}_V^i \otimes \mathcal{K}$, we conclude that $K_0(\mathbb{A}^i/I_F^i) \cong \Lambda_i$ and thus the rank of $K_0(\mathbb{A}^i)$ is equal to b_i . \Box

Lemma 3.3. There exists a simple Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}^{i}}$ such that:

$$\mathcal{O}_{A^i_{\infty}} \otimes \mathcal{K} \cong I^i_F \rtimes_{\hat{\alpha}^i} \mathbb{T}, \tag{3.1}$$

where $A^i_{\infty} \in GL_{\infty}(\mathbf{Z})$ is conjugate to the matrix Fr^i_{∞} and I^i_F is the *i*-th Fontaine ideal of \mathbb{A}^i .

Proof. (i) For an *i*-th Fontaine ideal $I_F^i \subset \mathbb{A}^i$, let us calculate the semi-group $K_0^+(I_F^i)$. It is easy to see, that $K_0(I_F^i) \cong \mathbb{Z}^\infty$ and the corresponding Grothendieck semigroup $K_0^+(I_F^i) \cong \varinjlim (\mathbb{Z}^\infty, Fr_\infty^i)$, where the injective limit is taken by the iterations of the endomorphism Fr_∞^i acting on \mathbb{Z}^∞ .

(ii) On the other hand, if $\mathscr{F}^i \subset \mathcal{O}_{A^i_{\infty}}$ is the core AF-algebra of a Cuntz-Pimsner algebra $\mathcal{O}_{A^i_{\infty}}$, then $K^+_0(\mathscr{F}^i) \cong \lim_{\infty} (\mathbf{Z}^{\infty}, (A^i_{\infty})^t)$, see Theorem 2.3.

(iii) We now define matrix $A^i_{\infty} \in GL_{\infty}(\mathbf{Z})$ so that:

$$K_0^+(\mathscr{F}^i) \cong K_0^+(I_F^i), \tag{3.2}$$

where \cong is an isomorphism of the Grothendieck semigroups, i.e. an order-isomorphism of the corresponding positive cones.

(iv) It follows from (3.2) that $I_F^i \rtimes_{\hat{\alpha}^i} \mathbb{T} \cong \mathcal{O}_{A_{\infty}^i} \otimes \mathcal{K}$, see formula (2.4). Moreover, an isomorphism $\varinjlim(\mathbf{Z}^{\infty}, (A_{\infty}^i)^t) \cong \varinjlim(\mathbf{Z}^{\infty}, Fr_{\infty}^i)$ implies that matrices A_{∞}^i and Fr_{∞}^i are conjugate in $GL_{\infty}(\mathbf{Z})$.

(v) Since the determinant det $(Fr_{\infty}^{i} - sI)$ is regular (Remark 2.2), we conclude that the conjugate matrix A_{∞}^{i} must be row-finite and irreducible, i.e. $\mathcal{O}_{A_{\infty}^{i}}$ is a correctly defined simple Cuntz-Pimsner algebra.

Lemma 3.3 is proved.

Corollary 3.4. char $A^i_{\infty} \equiv char \ Fr^i_{\infty}$.

Proof. The characteristic polynomial char $A^i_{\infty} = \det(A^i_{\infty} - sI)$ is invariant of the conjugacy class of matrix A^i_{∞} . We conclude from Lemma 3.3 that char $A^i_{\infty} \equiv char \ Fr^i_{\infty}$. (The converse is false in general.) Corollary 3.4 is proved.

Lemma 3.5.
$$Z_V(s) = \prod_{i=0}^{2n} \left(char \ A^i_{\infty} \right)^{(-1)^{i+1}}$$
.

Proof. (i) Recall that

$$Z_V(s) = \prod_{i=0}^{2n} \left(\Gamma_V^i(s) \right)^{(-1)^i},$$
(3.3)

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where $\Gamma_V^i(s)$ is the *i*-th Serre local factor at $p = \infty$ [Serre 1970] [10, Section 3]. In view of Deninger's Theorem 2.1 we can substitute $\Gamma_V^i(s) \equiv char^{-1} Fr_{\infty}^i$ so that the Hasse-Weil zeta function (3.3) becomes:

$$Z_V(s) = \prod_{i=0}^{2n} \left(char \ Fr_{\infty}^i \right)^{(-1)^{i+1}}.$$
(3.4)

(ii) On the other hand, there exist Cuntz-Pimsner algebras $\mathcal{O}_{A^i_{\infty}}$, such that *char* $A^i_{\infty} \equiv char \ Fr^i_{\infty}$ (Corollary 3.4). Thus one can write the Hasse-Weil zeta function (3.4) in the form:

$$Z_V(s) = \prod_{i=0}^{2n} \left(char \ A^i_{\infty} \right)^{(-1)^{i+1}}.$$
(3.5)

Lemma 3.5 is proved.

Theorem 1.1 follows from Lemmas 3.3 and 3.5.

4. RIEMANN ZETA FUNCTION

Let us point out a relation between the Cuntz-Pimsner algebra $\mathcal{O}_{A_{\infty}}$ and nontrivial zeroes of the Riemann zeta function $\zeta(s)$. If V is a curve, then n = 1 and formula (1.4) for the Hasse-Weil zeta function can be written as:

$$Z_V(s) = \prod_{i=0}^{2} \left(char \ A_{\infty}^i \right)^{(-1)^{i+1}} = \frac{char \ A_{\infty}^1}{char \ A_{\infty}^0 \ char \ A_{\infty}^2}.$$
 (4.1)

Moreover, char $A_{\infty}^0 = \frac{s}{2\pi}$ and char $A_{\infty}^2 = \frac{s-1}{2\pi}$ [Deninger 1992] [3, Section 3]. Thus one can write (4.1) in the form:

$$(2\pi)^{-2}s(s-1)Z_V(s) = char \ A^1_{\infty}.$$
(4.2)

On the other hand, the Hasse-Weil zeta function can be linked to the Riemann zeta function $\zeta(s)$ by the well known formula:

$$Z_V(s) = 2^{-\frac{1}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$
(4.3)

where $\Gamma\left(\frac{s}{2}\right)$ is the gamma function. We can use (4.3) to exclude $Z_V(s)$ from (4.2):

$$2^{-\frac{5}{2}}\pi^{\frac{-s-4}{2}}\Gamma\left(\frac{s}{2}\right)s(s-1)\zeta(s) = char \ A^{1}_{\infty}.$$
(4.4)

It follows from (4.4) that non-trivial zeros of the Riemann zeta function coincide with the roots of characteristic polynomial of the matrix A^1_{∞} defining the Cuntz-Pimsner algebra $\mathcal{O}_{A^1_{\infty}}$. Thus the row-finite and irreducible matrices are proper candidates for Hilbert's idea to settle the Riemann Hypothesis via spectra of the self-adjoint operators.

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 1 Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, New York, NY 11439, United States.

Email address: igor.v.nikolaev@gmail.com