Ward Identities in a Two-Dimensional Gravitational Model: Anomalous Amplitude Revisited Using a Completely Regularization-Independent Mathematical Strategy

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Abstract

We present a detailed investigation of the anomalous gravitational amplitude in a simple two-dimensional model with Weyl fermions. We employ a mathematical strategy that completely avoids any regularization prescription for handling divergent perturbative amplitudes. This strategy relies solely on the validity of the linearity of the integration operation and avoids modifying the amplitudes during intermediate calculations, unlike studies using regularization methods. Additionally, we adopt arbitrary routings for internal loop momenta, representing the most general analysis scenario. As expected, we show that surface terms play a crucial role in both preserving the symmetry properties of the amplitude and ensuring the mathematical consistency of the results. Notably, our final perturbative amplitude can be converted into the form obtained using any specific regularization prescription. We consider three common scenarios, one of which recovers the traditional results for gravitational anomalies. However, we demonstrate that this scenario inevitably breaks the linearity of integration, leading to an undesirable mathematical situation. This clean and transparent conclusion, enabled by the general nature of our strategy, would not be apparent in similar studies using regularization techniques.

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I. INTRODUCTION

In the late 1960s, the study of neutral electromagnetic pion decay revealed one of the most remarkable, subtle, and intriguing aspects of quantum field theory (QFT): the anomaly phenomenon. Specifically, this anomaly is known as the Adler-Bell-Jackiw (ABJ) anomaly or triangular anomaly, named after the type of Feynman diagram involved [1–3]. Its implications extend beyond simply mimicking experimental data; they play a fundamental role in the structure of QFT itself. For instance, the existence of three families of six quarks and six leptons as fundamental constituents in the Standard Model (SM) can be understood as a direct consequence of anomalies. When anomalous amplitudes are present in a theory, it means that not all Ward identities associated with different symmetries can be simultaneously satisfied for those amplitudes. If the broken symmetry is internal, it leads to internal inconsistencies in the theory, potentially destroying renormalizability and violating the unitarity of the S-matrix [4]. Consequently, the theory's renormalizability can be rescued if an anomaly cancellation mechanism exists, where specific combinations of 1/2-spin fermions cancel the violations arising from different sectors of the theory. Such a mechanism, consistent with the SM's structure, necessitates the existence of six quarks and six leptons [5].

Following the discovery of the ABJ anomaly, numerous other forms of anomalies have been explored using diverse methods and approaches, both perturbative and nonperturbative. These include the heat kernel method [6, 7], the path integral approach by Fujikawa [8, 9], and formal techniques like differential geometry and cohomology [10–13, 15, 16, 31]. While these provide elegant formulations, additional tools are often needed to extract momentum dependence for physical processes. This involves explicitly evaluating anomalous Feynman diagrams, which are odd tensors (in all even-dimensional spacetimes) with an odd number of axial-vector vertices and the remaining vertices being vectors with minimal internal fermionic propagators. Having at least two Lorentz indices, these tensors cannot simultaneously preserve all Ward identities (chiral anomalies) while reaching the expected low-energy limits [17–19]. Such amplitudes share the characteristic that, during dominant-order perturbative calculations, loop contributions render them divergent. Consequently, their explicit evaluation relies heavily on the chosen prescription and intermediate choices, such as internal loop momenta labeling. This presents a dilemma: either accept the results' dependence on these choices and adjust ambiguous terms later to achieve desired outcomes, or seek universal procedures for choice-independent results despite the inherent mathematical ambiguity, recognizing the anomaly as a fundamental QFT phenomenon.

In fact, a similar universal procedure already exists, proposed in the early 2000s by one of the authors of this work in his doctoral thesis [20]. This method arose from an effort to develop a divergence-handling strategy for QFTs that is free from limitations and widely consistent, allowing tensors and pseudotensors to be treated identically. The strategy, based on a remarkably simple idea, avoids integrating ill-defined integrals. Instead, it extracts the physical content by rewriting the integrand as a sum of finite integrals, surface terms, and purely divergent objects. Within this framework, divergent quantities lack physical parameters. Only finite integrals are calculated, while divergent pieces are regrouped into scalar objects and surface terms. This approach preserves the original properties of the integrals, enabling broader analysis of relevant physical processes. This often provides an advantage, allowing sound conclusions in situations where traditional regularization methods encounter difficulties. This method is particularly useful when surface terms play a significant role, as in the case of anomalous perturbative amplitudes, such as the gravitational anomalies considered in this contribution.

Similar to chiral anomalies in gauge theories, anomalies might arise in the context of gravitation when fermionic fields couple to the external gravitational field through the energy-momentum tensor [21–25]. In a seminal work, Alvarez-Gaumé and Witten [26] comprehensively studied gravitational anomalies in various field theories. They revealed the structure of these anomalies in higher dimensions and imposed restrictions on theories compatible with gravity, assuming anomaly cancellation. Specifically, two-dimensional Weyl spinors exhibit Lorentz and gravitational anomalies [24, 26, 27]. More recently, Bertlmann and Kohlprath [28, 29] employed the dispersion relations approach in two-dimensional spacetime to investigate Einstein and Weyl anomalies. They calculated the one-loop Feynman diagram of a Weyl fermion in a linearized gravitational background, offering a unique perspective on anomalies compared to ultraviolet regularization methods. Inspired by this valuable work and the critical nature of the issues raised, we revisit this intriguing and significant problem in this study. We believe the adopted procedure can unlock new avenues for analysis. This approach allows us to obtain results untainted by specific choices typically made during intermediate calculation steps. In particular, we can clearly examine the role of arbitrariness associated with internal loop momentum routing in loop-perturbative amplitudes. It is well-known that shifting the integration variable for linearly divergent integrals requires compensating with a corresponding surface term to maintain equality. Therefore, the results for such amplitudes are expected to depend on chosen internal momentum routings. Any analysis where these routings are treated as specific combinations of physical external momenta risks being compromised, as different choices can lead to different results. This aspect, intimately linked to the role of surface terms in perturbative calculations, will be demonstrably clarified in this investigation. Given the absence of these considerations in previous works and their crucial impact on conclusions, this contribution is warranted.

Building upon the work presented in Ref. [30], this work offers an alternative calculation of the gravitational amplitude described in Bertlmann and Kohlprath's studies [28, 29]. We treat the internal loop momenta as arbitrary and avoid assigning specific values to surface terms during intermediate steps. This approach directly reveals the structure of ambiguity associated with these terms and their impact on the qualitative and quantitative interpretation of results. Surface terms, whose values can vary between methods, are a key factor in regularization-dependent results. We analyze three commonly encountered choices associated with different regularization procedures. We demonstrate that while specific choices allow us to recover traditional results for gravitational anomalies, these choices inevitably break the linearity of the integration operation, a fact hidden within traditional methods. Another notable aspect of our investigation is the connection between $2D$ gravitational anomalies and the $2D$ chiral anomaly. Our systematic approach with subamplitudes allows us to identify mathematical structures shared with simpler theories like $2D$ quantum electrodynamics (QED_2). This approach reveals universal aspects of $2D$ anomalies not accessible in traditional methods.

We organize the work as follows. In Section 2 we establish the theoretical foundation for our investigation by outlining the expected relationships among Green's functions (RAGFs) and Ward identities (WIs) associated with the gravitational amplitude. To facilitate comprehension and simplify the calculation, we decompose the gravitational amplitude into smaller, manageable components called subamplitudes. Notably, some of these subamplitudes align with typical perturbative amplitudes found in simpler QFTs like QED_2. In Section 3, we briefly explain the chosen method for handling the divergent Feynman integrals encountered during the calculation of the gravitational amplitude. The Section 4 focuses on analyzing the subamplitudes individually. We calculate each subamplitude and explicitly verify its corresponding RAGFs. Additionally, a set of conditions required for this purpose is identified. Leveraging the general results obtained in Section 4, in Section 5 we investigate the possibility of the gravitational amplitude simultaneously satisfying its WIs and RAGFs. We emphasize three representative scenarios for fixing the undefined quantities involved, including the scenario that generates the usual results for gravitational anomalies. Concluding remarks and a summary of the key findings are presented in the Section 6.

II. THE GRAVITATIONAL AMPLITUDE

In this work we adopt the same model discussed in the Refs. [28] and [29] as well as some of definitions and notations stated there.

A. The Model and Definitions

The background model of our discussions has a Lagrangian whose (linearized) interaction part may be written as [31]

$$\mathcal{L}_I^{lin} = -\frac{1}{2} h_{\mu\nu} T^{\mu\nu} \ , \tag{1}$$

where $T_{\mu\nu}$ is the (symmetric) energy-momentum tensor, explicitly given by

$$T^{\mu\nu} = \frac{i}{4} \left[E_a^{\nu} \overline{\psi} \gamma^a \left(\frac{1 \pm \gamma^3}{2} \right) \overleftrightarrow{\partial^{\mu}} \psi + E_a^{\mu} \overline{\psi} \gamma^a \left(\frac{1 \pm \gamma^3}{2} \right) \overleftrightarrow{\partial^{\nu}} \psi \right] \ . \tag{2}$$

Here E_a^{μ} is the inverse of zweibein e_{μ}^a, ψ is the fermion field, γ^a are the usual Dirac matrices and $h_{\mu\nu}$ is the linearized gravitational field, defined through the approximations

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \kappa h_{\mu\nu} \ , \qquad g^{\mu\nu} \approx \eta^{\mu\nu} - \kappa h^{\mu\nu} \ ,$$
$$e_{\mu}^a \approx \eta_{\mu}^a + \frac{1}{2}\kappa h_{\mu}^a \ , \qquad E^{a\mu} \approx \eta^{a\mu} - \frac{1}{2}\kappa h^{a\mu} \ ,$$

with $\eta^{\mu\nu}$ being the flat metric. This Lagrangian describe, in two space-time dimensions, the interaction of a Weyl fermion and a gravitational background field.

The full Green's function which we are interested in is the two-point function

$$G_{\mu\nu\rho\sigma}(p) = i \int d^2x \ e^{ip\cdot x} \langle 0 | T \left[T_{\mu\nu}(x) T_{\rho\sigma}(0) \right] | 0 \rangle \ , \tag{3}$$

which, at one-loop level, is written as

$$T^{G}_{\mu\nu\rho\sigma} = i \int \frac{d^2k}{(2\pi)^2} Tr \left\{ \Gamma^{G}_{\mu\nu} \frac{1}{[\not{k} + \not{k}_1 - m]} \Gamma^{G}_{\rho\sigma} \frac{1}{[\not{k} + \not{k}_2 - m]} \right\} \;, \tag{4}$$

where

$$\Gamma^{G}_{\mu\nu} = -\frac{i}{4} \left[\gamma_\mu \left((k+k_1)_\nu + (k+k_2)_\nu \right) + \gamma_\nu \left((k+k_1)_\mu + (k+k_2)_\mu \right) \right] \frac{(1 \pm \gamma_3)}{2} \;, \tag{5}$$

gives the Feynman rule for the vertex function. A diagrammatic representation of $T^{G}_{\mu\nu\rho\sigma}$ can be seen in the Fig. (1).

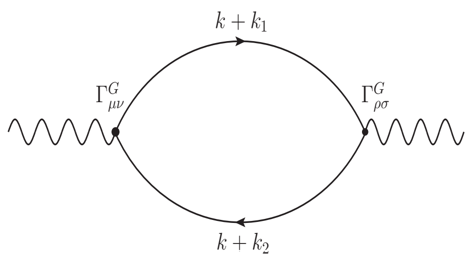

FIG. 1: One-loop diagrammatic representation for $T^{G}_{\mu\nu\sigma\rho}$.

Observe that we have adopted general labels for the internal propagators, namely $k+k_1$ and $k+k_2$, and, for convenience, taken the fermion as being massive. Given this routing, the external momentum is identified as $p = k_2 - k_1$. The adoption of arbitrary labels is an important attitude in perturbative calculations in general but is of special importance in the presently considered problem. Once the power counting of the loop momentum point out divergence degree higher than the logarithmic one, it is expected that the result is dependent on the routing adopted for the internal lines momenta. The arbitrary choice guarantee that such dependence can be identified in the final results. If the internal momenta are label in terms of external momenta such that the sum $k_1 + k_2$ is not zero, terms which would be nonphysical will be mixed with label independent terms, compromising then the analysis.

The explicit calculation of (4), especially with general labels for the internal propagators, is very long and tedious. However, the conclusions extracted from are strongly connected with the calculation details. In order to make an useful investigation, we need to adopt a systematic way to present such calculations. Having this in mind, in the present work, we adopt, for the sake of clarity, a particular systematization. We split out $T^{G}_{\mu\nu\rho\sigma}$ into three sets of amplitudes as

$$T^{G}_{\mu\nu\rho\sigma} = -\frac{i}{64} \left\{ T^{(V)}_{\mu\nu\rho\sigma} + T^{(AV)}_{\mu\nu\rho\sigma} + T^{(A)}_{\mu\nu\rho\sigma} \right\} \;, \tag{6}$$

where each set is composed by a sum of two-point subamplitudes, namely

$$T^{(V)}_{\mu\nu\rho\sigma} = 4\left[T^{VV}_{\nu\sigma;\mu\rho}\right] + 2p_\sigma\left[T^{VV}_{\nu;\mu\rho}\right] + 2p_\nu\left[T^{VV}_{\sigma;\mu\rho}\right] + p_\nu p_\sigma\left[T^{VV}_{\mu\rho}\right]$$
$$+ (\mu \longleftrightarrow \nu) + (\sigma \longleftrightarrow \rho)\ , \tag{7}$$

$$\pm T^{(AV)}_{\mu\nu\rho\sigma} = 4\left[T^{AV}_{\nu\sigma;\mu\rho}\right] + 2p_\sigma\left[T^{AV}_{\nu;\mu\rho}\right] + 2p_\nu\left[T^{AV}_{\sigma;\mu\rho}\right] + p_\nu p_\sigma\left[T^{AV}_{\mu\rho}\right]$$
$$+ 4\left[T^{VA}_{\nu\sigma;\mu\rho}\right] + 2p_\sigma\left[T^{VA}_{\nu;\mu\rho}\right] + 2p_\nu\left[T^{VA}_{\sigma;\mu\rho}\right] + p_\nu p_\sigma\left[T^{VA}_{\mu\rho}\right]$$
$$+ (\mu \longleftrightarrow \nu) + (\sigma \longleftrightarrow \rho)\ , \tag{8}$$

$$T^{(A)}_{\mu\nu\rho\sigma} = 4\left[T^{AA}_{\nu\sigma;\mu\rho}\right] + 2p_\sigma\left[T^{AA}_{\nu;\mu\rho}\right] + 2p_\nu\left[T^{AA}_{\sigma;\mu\rho}\right] + p_\nu p_\sigma\left[T^{AA}_{\mu\rho}\right]$$
$$+ (\mu \longleftrightarrow \nu) + (\sigma \longleftrightarrow \rho)\ . \tag{9}$$

The subamplitudes appearing in the above expressions are defined by

$$T^{ij}_{\rho\sigma} = \int \frac{d^2k}{(2\pi)^2} Tr\left\{[\Gamma_i]_\rho \frac{1}{\not k + \not k_1 - m}[\Gamma_j]_\sigma \frac{1}{\not k + \not k_2 - m}\right\}\ , \tag{10}$$

$$T^{ij}_{\mu;\rho\sigma} = \int \frac{d^2k}{(2\pi)^2} (k+k_1)_\mu\, Tr\left\{[\Gamma_i]_\rho \frac{1}{\not k + \not k_1 - m}[\Gamma_j]_\sigma \frac{1}{\not k + \not k_2 - m}\right\}\ , \tag{11}$$

$$T^{ij}_{\mu\nu;\rho\sigma} = \int \frac{d^2k}{(2\pi)^2} (k+k_1)_\mu (k+k_1)_\nu\, Tr\left\{[\Gamma_i]_\rho \frac{1}{\not k + \not k_1 - m}[\Gamma_j]_\sigma \frac{1}{\not k + \not k_2 - m}\right\}\ . \tag{12}$$

In the expressions above the quantities Γ_i are vertex operators belonging to the set $\Gamma_i = \{\Gamma_S, \Gamma_P, \Gamma_V, \Gamma_A\} = \{1, \gamma_3, \gamma_\alpha, \gamma_\alpha\gamma_3\}$. In addition to the calculation aspect, such introduced systematization will help us to verify the consistency of the obtained results in a wider sense. Note that the first set of subamplitudes appears in renormalizable theories like the QED_2 [32–34]. This allow us to add an additional aspect to the investigation.

B. Relations among Green's functions and Ward Identities

Along the difficult task of constructing a consistent interpretation of the perturbative amplitudes in QFT's, when the involved quantities are divergent, a special recourse can play a very important role. We denominated it as relations among Green's functions. Such relations can be stated always we have a Lorentz index attached to a perturbative amplitude. They are constructed by using simple ingredients like the Dirac algebra, cyclicity and linearity of the trace operation and, especially, the linearity of the integration operation. In particular, preservation of the linearity in the integration operation involving divergent Feynman integrals is not a trivial job, as we will see along this work.

Let us consider, in this section, the relevant RAGFs for all required subamplitudes of (4). In fact, to state the referred relations is a trivial task. In order to exemplify the procedure, we consider the algebraic identity

$$
\begin{aligned}
&(k_2-k_1)^{\nu}\left\{\gamma_{\mu}\frac{1}{[(\not{k}+\not{k}_1)-m]}\gamma_{\nu}\frac{1}{[(\not{k}+\not{k}_2)-m]}\right\} \\
&=\left\{\gamma_{\mu}\frac{1}{[(\not{k}+\not{k}_1)-m]}\right\}-\left\{\gamma_{\mu}\frac{1}{[(\not{k}+\not{k}_2)-m]}\right\}\ , \qquad (13)
\end{aligned}
$$

which is obtained through the ingredients cited above. In practical terms, through this operation is possible to cancel out an internal propagator. When the integration in the loop momentum k is taken, after taken the traces in both sides, this algebraic identity will be converted into a genuine RAGFs involving the contraction of the polarization tensor $T_{\mu\nu}^{VV}$ with the external momentum p^{ν} and two one-point vector amplitudes defined by

$$
T_{\mu}^{V}(k_i)=\int\frac{d^2k}{(2\pi)^2}Tr\left\{\gamma_{\mu}\frac{1}{[(\not{k}+\not{k}_i)-m]}\right\}\ . \qquad (14)
$$

Explicitly, we get the following RAGFs

$$
p^{\sigma}\left[T_{\sigma\rho}^{VV}(k_1,k_2)\right]=T_{\rho}^{V}(k_1)-T_{\rho}^{V}(k_2)\ , \qquad (15)
$$

and, in a similar way,

$$
p^{\rho}\left[T_{\sigma\rho}^{VV}(k_1,k_2)\right]=T_{\sigma}^{V}(k_1)-T_{\sigma}^{V}(k_2)\ . \qquad (16)
$$

If the $T_{\sigma\rho}^{VV}$ and T_{ρ}^{V} amplitudes are evaluated, through some particular procedure, in such a way that the final results are in disagreement with the RAGFs, it means, undoubtedly, that the linearity in the integration operation was violated through the operations made. Of course, this is not the adequate situation if one want to make predictions in perturbative treatments of a model or theory. In this sense, the RAGFs give us a powerful test of consistency of a method used to calculate divergent perturbative amplitudes.

Following this procedure it is possible to state two relevant RAGFs for $T_{\mu\nu\rho\sigma}^{G}$ through the contractions of (4) with the momentum p^{μ} and the metric $g^{\mu\nu}$. According to our previously introduced notation, we can write

$$
p^{\mu}T_{\mu\nu\rho\sigma}^{G}=-\frac{i}{64}\left\{\left[p^{\mu}T_{\mu\nu\rho\sigma}^{(V)}\right]+\left[p^{\mu}T_{\mu\nu\rho\sigma}^{(AV)}\right]+\left[p^{\mu}T_{\mu\nu\rho\sigma}^{(A)}\right]\right\}\ , \qquad (17)
$$

$$
g^{\mu\nu}T_{\mu\nu\rho\sigma}^{G}=-\frac{i}{64}\left\{\left[g^{\mu\nu}T_{\mu\nu\rho\sigma}^{(V)}\right]+\left[g^{\mu\nu}T_{\mu\nu\rho\sigma}^{(AV)}\right]+\left[g^{\mu\nu}T_{\mu\nu\rho\sigma}^{(A)}\right]\right\}\ , \qquad (18)
$$

which, in terms of the subamplitudes, means

$$\begin{aligned}
p^\mu \left[T^{(V)}_{\mu\nu\rho\sigma}\right] &= 4\left\{\left[p^\mu T^{VV}_{\mu\sigma;\nu\rho}\right] + \left[p^\mu T^{VV}_{\nu\sigma;\mu\rho}\right]\right\} \\
&+ 2p_\sigma \left\{\left[p^\mu T^{VV}_{\nu;\mu\rho}\right] + \left[p^\mu T^{VV}_{\mu;\nu\rho}\right]\right\} \\
&+ 2p_\nu \left[p^\mu T^{VV}_{\sigma;\mu\rho}\right] + p_\nu p_\sigma \left[p^\mu T^{VV}_{\mu\rho}\right] \\
&+ p^2 \left\{p_\sigma \left[T^{VV}_{\nu\rho}\right] + 2\left[T^{VV}_{\sigma;\nu\rho}\right]\right\} + (\sigma \leftrightarrow \rho) \ ,
\end{aligned} \tag{19}$$

$$\begin{aligned}
p^\mu \left[T^{(A)}_{\mu\nu\rho\sigma}\right] &= 4\left\{\left[p^\mu T^{AA}_{\nu\sigma;\mu\rho}\right] + \left[p^\mu T^{AA}_{\mu\sigma;\nu\rho}\right]\right\} \\
&+ 2p_\sigma \left\{\left[p^\mu T^{AA}_{\nu;\mu\rho}\right] + \left[p^\mu T^{AA}_{\mu;\nu\rho}\right]\right\} \\
&+ 2p_\nu \left[p^\mu T^{AA}_{\sigma;\mu\rho}\right] + p_\nu p_\sigma \left[p^\mu T^{AA}_{\mu\rho}\right] \\
&+ p^2 \left\{p_\sigma \left[T^{AA}_{\nu\rho}\right] - 2\left[T^{AA}_{\sigma;\nu\rho}\right]\right\} + (\sigma \leftrightarrow \rho) \ ,
\end{aligned} \tag{20}$$

$$\begin{aligned}
p^\mu \left[T^{(AV)}_{\mu\nu\rho\sigma}\right] &= \pm 4\left\{\left[p^\mu T^{VA}_{\nu\sigma;\mu\rho} + p^\mu T^{AV}_{\nu\sigma;\mu\rho}\right] + \left[p^\mu T^{VA}_{\mu\sigma;\nu\rho} + p^\mu T^{AV}_{\mu\sigma;\nu\rho}\right]\right\} \\
&\pm 2p_\sigma \left\{\left[p^\mu T^{VA}_{\nu;\mu\rho} + p^\mu T^{AV}_{\nu;\mu\rho}\right] + \left[p^\mu T^{VA}_{\mu;\nu\rho} + p^\mu T^{AV}_{\mu;\nu\rho}\right]\right\} \\
&\pm 2p_\nu \left[p^\mu T^{VA}_{\sigma;\mu\rho} + p^\mu T^{AV}_{\sigma;\mu\rho}\right] \pm p_\nu p_\sigma \left[p^\mu T^{VA}_{\mu\rho} + p^\mu T^{AV}_{\mu\rho}\right] \\
&\pm p^2 \left\{p_\sigma \left[T^{VA}_{\nu\rho} + T^{AV}_{\nu\rho}\right] + 2\left[T^{VA}_{\sigma;\nu\rho} + T^{AV}_{\sigma;\nu\rho}\right]\right\} + (\sigma \leftrightarrow \rho) \ ,
\end{aligned} \tag{21}$$

as well as

$$\begin{aligned}
g^{\mu\nu} \left[T^{(V)}_{\mu\nu\rho\sigma}\right] &= 8\left[g^{\mu\nu} T^{VV}_{\mu\sigma;\nu\rho}\right] + 4p^\sigma \left[g^{\mu\nu} T^{VV}_{\mu;\nu\rho}\right] \\
&+ 4\left[p^\mu T^{VV}_{\sigma;\mu\rho}\right] + 2p_\sigma \left[p^\mu T^{VV}_{\mu\rho}\right] + (\sigma \leftrightarrow \rho) \ ,
\end{aligned} \tag{22}$$

$$\begin{aligned}
g^{\mu\nu} \left[T^{(A)}_{\mu\nu\rho\sigma}\right] &= 8\left[g^{\mu\nu} T^{AA}_{\mu\sigma;\nu\rho}\right] + 4p^\sigma \left[g^{\mu\nu} T^{AA}_{\mu;\nu\rho}\right] \\
&+ 4\left[p^\mu T^{AA}_{\sigma;\mu\rho}\right] + 2p_\sigma \left[p^\mu T^{AA}_{\mu\rho}\right] + (\sigma \leftrightarrow \rho) \ ,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
g^{\mu\nu} \left[T^{(AV)}_{\mu\nu\rho\sigma}\right] &= \pm 8\left[g^{\mu\nu} T^{VA}_{\nu\sigma;\mu\rho} + g^{\mu\nu} T^{AV}_{\nu\sigma;\mu\rho}\right] \\
&\pm 4p_\sigma \left[g^{\mu\nu} T^{VA}_{\nu;\mu\rho} + g^{\mu\nu} T^{AV}_{\nu;\mu\rho}\right] \\
&\pm 4\left[p^\mu T^{VA}_{\sigma;\mu\rho} + p^\mu T^{AV}_{\sigma;\mu\rho}\right] \\
&\pm 2p_\sigma \left[p^\mu T^{VA}_{\mu\rho} + p^\mu T^{AV}_{\mu\rho}\right] + (\sigma \leftrightarrow \rho) \ .
\end{aligned} \tag{24}$$

In practical terms, we need to state the RAGFs for all subamplitudes defined in Eqs. (10), (11), and (12). Since the procedure to obtain such RAGFs presents no difficulties, we just list all of them in the appendix (A).

At this point it is interesting to see that the RAGFs are, strictly speaking, mathematical identities which are valid in a way independent of the particular context. Then, one would not expected that they were violated by any calculation procedure. On the other hand, one would not expect that the Ward identities must be satisfied automatically in the perturbative calculations due to the fact that they are stated by assuming translational invariance as an ingredient. Such property is not contained in the amplitudes constructed through the Feynman rules since, in cases where the divergence degree involved is higher than the logarithmic one, the result is dependent in the routing assumed for the loops internal lines momenta. Two distinct labels obeying energy-momentum conservation in all vertexes will generate results which can differ by terms that are proportional to surface terms. The coefficients of such terms are ambiguous combination of the internal lines momenta. In this way, the WIs are expected to be broken in situations where surface terms are involved. Within this context, it is the preservation that must be considered as a special accident and not the violation. So, we must, before considering the content of an explicit mathematical form of an amplitude, verify if the adopted procedure does not breaks the pertinent RAGFs. In order to satisfy the WIs a new ingredient must be added to the implication of Feynman rules. The usual one is the adoption of a regularization procedure. In the present investigation we adopt a procedure which does not modify the amplitudes in the intermediary steps of the calculations. The final form is a pure implication of the Feynman rules such that, from the results, the ones corresponding to other prescriptions can be obtained.

Given the symmetries of the considered model [28], the energy-momentum tensor $T_{\mu\nu}$ is expected to has the following three properties,

$$T_{\mu\nu} = T_{\nu\mu}, \quad \nabla^{\mu} T_{\mu\nu} = 0, \quad g^{\mu\nu} T_{\mu\nu} = 0 \ , \tag{25}$$

which imply, respectively, three canonical WIs (for massless fermions)

$$\begin{cases} T^{G}_{\mu\nu\rho\sigma} = T^{G}_{\nu\mu\rho\sigma} \ , \\ p^{\mu} T^{G}_{\mu\nu\rho\sigma} = 0 \ , \\ g^{\mu\nu} T^{G}_{\mu\nu\rho\sigma} = 0 \ . \end{cases} \tag{26}$$

As it is well-known, it is always possible to fulfill the first cited Ward identity ($T^{G}_{\mu\nu\rho\sigma} = T^{G}_{\nu\mu\rho\sigma}$) by imposing that the quantized energy-momentum tensor is symmetric [28]. Therefore, we need to investigate if the last two WIs can be satisfied also. It is expected that both are broken by anomalous terms known as Einstein and Weyl anomalies, respectively. In our investigation we will obtain, among other things, a set of conditions to be fulfilled in order to satisfy these properties.

The main task of next sections is to check, after the explicit calculations of $T^{G}_{\mu\nu\rho\sigma}$, if the obtained mathematical forms are, firstly, in accordance with the RAGFs to, after this, verify if it is possible to preserve the associated WIs. For the first task it is only required to be careful in the operations, in order to obey the mathematics, while for the second task it is expected that a set of additional conditions need to be identified in order to be imposed in addition to the application of the Feynman rules.

III. THE PROCEDURE FOR HANDLING DIVERGENT FEYNMAN INTEGRALS

Most of QFT's predictions are made through perturbative methods. The construction of the perturbative amplitudes, on the other hand, is systematized by the well-known Feynman rules. Within this context we find, invariably, a set of amplitudes at the loop level, corresponding to physical processes, which are divergent quantities. This requires the adoption of an adequate procedure in order to handle with this situation. Due to this reason, in this section we present the procedure which we adopt to handle the intrinsic mathematical problems of the perturbative series in QFTs. The mathematical strategy adopted play a crucial role in our investigation.

In a first step, by applying the Feynman rules, we construct the desirable perturbative amplitude, for one value of the loop momentum. Then, by a simple power counting, it is stated the superficial degree of divergence. Therefore, physical quantities, which are combinations of propagators and vertexes, may be in an integrand of a divergent integral when the integration is taken over the loop momentum, which, formally, corresponds to the implementation of the last Feynman rule. The usual procedure is, at this point, to adopt a regularization technique in order to make the integrals. This implies in to modify the amplitudes as they come from the corresponding Feynman rules. After all the operations are made, some limit is taken to, in principle, connect the obtained results to the initial situation, removing the effects of the mathematical modifications introduced. However, as it is well known, due to the divergent character of the modified integrals, the integration and the limit are not commuting operations such that the result is not unique and (which is bad) is dependent on the intermediary sequence of steps followed. Given this fact, some aspects of the perturbative calculations are prejudiced, especially those were surface terms are involved since, in the regularized expressions, they may assume prescription dependent values. In the dimensional regularization (DR) prescription [35–37], as an example, they are assumed as having vanishing values, allowing then shifts in the integrating momentum. On the other hand, in prescriptions where the regularization is made through distributions at a fixed space-time dimension, the value for the referred surface terms are not zero and the amplitudes became dependent in the particular routing adopted for the internal lines momenta of the loop. Both methods produce very different implications in qualitative and quantitative sense.

In order to avoid the previous described situation, it was developed a procedure which can circumvent the modifications of the perturbative amplitudes at the intermediary steps of the calculations such that all the ingredients are present at the final form obtained, like the aspects related to the surface terms involved. With this attitude a very rich analysis is allowed since, as we have said, a correspondence with all specific regularization technique is always possible.

The main idea is to assume that the linearity in the integration operation is a valid property for Feynman integrals, in such a way it is possible to write the expression for a perturbative amplitude in any mathematical form which is mathematically identical to that usually adopted by the Feynman rules, before implementing the last rule. Strictly speaking, there is an infinite number of equivalent mathematical forms for the Feynman amplitudes. This freedom allow us to choose the most convenient one for our purposes. We can assume, at this point a criterion for the choice; the most simple mathematical expression where *no physical parameters will be inside a divergent integral when the last rule is implemented.* Our next task is, therefore, to rewrite the propagators, where resides the dependence on the loop momentum, in a way which allows us to achieve this goal. In principle, any identity which generates a sequence of terms having a regressive power counting in the loop momentum can be adopted. Probably the most simple one is the identity [20]

$$\begin{aligned} \frac{1}{D_i} &= \frac{1}{[(k+k_i)^2 - m_i^2]} \\ &= \sum_{j=0}^{N} \frac{(-1)^j \left(k_i^2 + 2k_i \cdot k + \lambda^2 - m_i^2\right)^j}{(k^2-\lambda^2)^{j+1}} \\ &+ \frac{(-1)^{N+1} \left(k_i^2 + 2k_i \cdot k + \lambda^2 - m_i^2\right)^{N+1}}{(k^2-\lambda^2)^{N+1} \left[(k+k_i)^2 - m_i^2\right]} \ , \end{aligned} \tag{27}$$

where N need to be taken as equal to or greater than the superficial degree of divergence. On its turn, λ is an arbitrary parameter having dimension of mass and k_i is an internal (arbitrary) momentum. In practical terms, it is equivalent to say that the infinite forms allowed by the value of N, in the above expression, are completely equivalent to represent the required expression for a propagator in the application of Feynman rules. It is enough that the linearity in the integration operation is a valid mathematical property. In addition, it is required also that, in all steps, the complete independence of the arbitrary parameter λ is obtained.

The convenient use of the identity (27) make possible to split up any divergent Feynman integral into a sum of scalar (irreducible) divergent integrals, surface terms and finite functions of the external momenta. The set of divergent quantities is reduced to few objects which in our present investigation is composed by two irreducible (scalar) ones

$$I_{\log}^{(2)}\left(\lambda^2\right) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-\lambda^2)} \ , \tag{28}$$

$$I_{quad}^{(2)}\left(\lambda^2\right) = \int \frac{d^2k}{(2\pi)^2} \ln\left(\frac{k^2-\lambda^2}{k^2}\right) \ . \tag{29}$$

The masses within these objects (mass scales) can be changed freely through identities that are called scale relations and stated by

$$\left[I_{\log}^{(2)}\left(\lambda^2\right)\right] = \left[I_{\log}^{(2)}\left(\lambda_0^2\right)\right] + \frac{i}{4\pi} \ln\left(\frac{\lambda_0^2}{\lambda^2}\right) \ , \tag{30}$$

$$\begin{aligned} \left[I_{quad}^{(2)}\left(\lambda^2\right)\right] &= \left[I_{quad}^{(2)}\left(\lambda_0^2\right)\right] + \left(\lambda^2 - \lambda_0^2\right) \left[I_{\log}^{(2)}\left(\lambda_0^2\right)\right] \\ &+ \frac{i}{4\pi} \left[\lambda^2 - \lambda_0^2 + \lambda^2 \ln\left(\frac{\lambda_0^2}{\lambda^2}\right)\right] \ . \end{aligned} \tag{31}$$

On the other hand, we will find four quantities which can be recognized as being surface terms

$$\Delta_{1;\mu\nu}^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k_\mu} \left[k_\nu \left(2 - \ln \frac{k^2}{k^2-\lambda^2} \right) \right]$$
$$= \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{2k_\mu k_\nu}{(k^2-\lambda^2)} - g_{\mu\nu} \ln \left(\frac{k^2}{k^2-\lambda^2} \right) \right\} \ , \tag{32}$$

$$\Delta_{2;\mu\nu}^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k_\mu} \left(- \frac{k_\nu}{(k^2-\lambda^2)} \right)$$
$$= \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{2k_\mu k_\xi}{(k^2-\lambda^2)^2} - \frac{g_{\mu\xi}}{(k^2-\lambda^2)} \right\} \ , \tag{33}$$

$$\Box_{2;\mu\nu\alpha\beta}^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k_\mu} \left(- \frac{2k_\nu k_\alpha k_\beta}{(k^2-\lambda^2)} \right)$$
$$= \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{4k_\mu k_\nu k_\alpha k_\beta}{(k^2-\lambda^2)^2} - g_{\mu\nu} \frac{2k_\alpha k_\beta}{(k^2-\lambda^2)} \right.$$
$$\left. - g_{\mu\alpha} \frac{2k_\nu k_\beta}{(k^2-\lambda^2)} - g_{\mu\beta} \frac{2k_\nu k_\alpha}{(k^2-\lambda^2)} \right\} \ , \tag{34}$$

$$\Box_{3;\mu\nu\alpha\beta}^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k_\mu} \left(- \frac{2k_\nu k_\alpha k_\beta}{(k^2-\lambda^2)^2} \right)$$
$$= \int \left\{ \frac{d^2k}{(2\pi)^2} \frac{8k_\mu k_\nu k_\alpha k_\beta}{(k^2-\lambda^2)^3} - g_{\mu\nu} \frac{2k_\alpha k_\beta}{(k^2-\lambda^2)^2} \right.$$
$$\left. - g_{\mu\alpha} \frac{2k_\nu k_\beta}{(k^2-\lambda^2)^2} - g_{\mu\beta} \frac{2k_\nu k_\alpha}{(k^2-\lambda^2)^2} \right\} \ , \tag{35}$$

$$\Sigma_{4;\mu\nu\alpha\beta\xi\chi}^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k_\mu} \left(- \frac{8k_\nu k_\alpha k_\beta k_\xi k_\chi}{(k^2-\lambda^2)^3} \right)$$
$$= \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{48k_\mu k_\nu k_\alpha k_\beta k_\xi k_\chi}{(k^2-\lambda^2)^4} - g_{\mu\nu} \frac{8k_\alpha k_\beta k_\xi k_\chi}{(k^2-\lambda^2)^3} \right.$$
$$- g_{\mu\alpha} \frac{8k_\nu k_\beta k_\xi k_\chi}{(k^2-\lambda^2)^3} - g_{\mu\beta} \frac{8k_\nu k_\alpha k_\xi k_\chi}{(k^2-\lambda^2)^3}$$
$$\left. - g_{\mu\xi} \frac{8k_\nu k_\alpha k_\beta k_\chi}{(k^2-\lambda^2)^3} - g_{\mu\chi} \frac{8k_\nu k_\alpha k_\beta k_\xi}{(k^2-\lambda^2)^3} \right\} \ . \tag{36}$$

The convenience of this systematization will be clear in future discussions. In turn, the finite integrals arising can be integrated out without restrictions and the results written in terms of a set of finite functions defined through integral representations given in terms of

Feynman parameters [38]. In the present work such structures are defined by

$$\xi_k^{(-1)}\left(p^2,m^2\right)=\int_0^1 dx\frac{x^k}{Q\left(p^2,m^2;x\right)}\ , \tag{37}$$

$$\xi_k^{(0)}\left(p^2,m^2;\lambda^2\right)=\int_0^1 dx\ x^k\ln\left[\frac{Q\left(p^2,m^2;x\right)}{-\lambda^2}\right]\ , \tag{38}$$

with $k=0,1,2,...$ and the polynomial Q given by $Q\left(p^2,m^2;x\right)=p^2x(1-x)-m^2$. There are, obviously, relations between the functions corresponding to two different values of the index k , allowing us to reduce them to the $\xi_0^{(-1)}$ or $\xi_0^{(0)}$, which is particularly useful in RAGFs or WIs verification.

Observe that within a traditional regularization prescription the divergent objects would have a value attributed to them. For example, in DR all the above surface terms are taken as being zero and the $I_{\log}\left(\lambda^2\right)$ and $I_{quad}\left(\lambda^2\right)$ objects manifest themselves as poles, for specific values of the space-time dimension, in the amplitudes. In fact, one can always formulate a one-to-one map among our results and those produced by regularizations prescriptions, as will become clear in what follows. On the other hand, in our prescription, they remain untouched and are present in the final results, where their possible values could be considered and tested for consistency requirements. These aspects represents the heart of the analysis and conclusions made in this job.

IV. EXPLICITLY EVALUATION OF THE SUBAMPLITUDES AND THE VERIFICATION OF THEIR RAGFS

In order to explicitly calculate the gravitational amplitude $T^G_{\mu\nu\rho\sigma}$, we first calculate its subamplitudes, through the strategy described above, after that we check whether the results obtained are consistent with the corresponding RAGFs and, if it is the case, verify if it is possible to satisfy the WIs.

We start with the subamplitudes composing the vector sector given by

$$\begin{aligned}T^{(V)}_{\mu\nu\rho\sigma}&=4\left[T^{VV}_{\nu\sigma;\mu\rho}\right]+2p_\sigma\left[T^{VV}_{\nu;\mu\rho}\right]+2p_\nu\left[T^{VV}_{\sigma;\mu\rho}\right]+p_\nu p_\sigma\left[T^{VV}_{\mu\rho}\right]\\&+(\mu\longleftrightarrow\nu)+(\sigma\longleftrightarrow\rho)\ .\end{aligned} \tag{39}$$

These three kind of subamplitudes are defined by taking $\Gamma_i=\gamma_\mu$ and $\Gamma_j=\gamma_\nu$ in definitions (10), (11), and (12).

A. $T^{VV}_{\sigma\rho}$ amplitude

Let us consider first the $T^{VV}_{\mu\nu}$ (see Eq. (10)). This amplitude belongs to the spectrum of amplitudes arising in renormalizable theories like QED_2, where the gauge invariance plays a crucial role. In such a context it represents the one-loop polarization tensor. Therefore, its identification as a substructure of the $T^{G}_{\mu\nu\rho\sigma}$ amplitude may shine some light in our investigation about gravitational anomalies. We will consider, due to this, some details in the procedures.

We know that it is expected to identify the relations (15) and (16) as properties of its explicit form. Thus, it is relevant to know the corresponding expression for the one-point vector function in advance. Beside that, it is a good opportunity to exemplify the use of the procedure in a simple algebraic scenario. First, after taking the Dirac traces, we obtain the expression for one value of the loop momentum

$$t^{V}_{\mu}\left(k_1\right) = 2\left[\frac{k^{\alpha}}{D_1} + k_1^{\alpha}\frac{1}{D_1}\right] . \tag{40}$$

For the first term we adopt for the propagator the representation

$$\begin{aligned}\frac{k_\mu}{D_1} &= \frac{k_\mu}{(k^2-\lambda^2)} - \frac{\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)k_\mu}{(k^2-\lambda^2)^2} \\ &+ \frac{\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)^2 k_\mu}{(k^2-\lambda^2)^2\left[(k+k_1)^2-m^2\right]} ,\end{aligned} \tag{41}$$

which corresponds to adopt $N=1$ in (27). For the second term the same representation can be adopted. However, in order to avoid unnecessary algebraic efforts, one can take $N=0$,

$$\frac{1}{D_1} = \frac{1}{(k^2-\lambda^2)} - \frac{\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)}{(k^2-\lambda^2)\left[(k+k_1)^2-m^2\right]}. \tag{42}$$

Thus

$$\begin{aligned}\left[\frac{k_\alpha}{D_1} + k_{1\alpha}\frac{1}{D_1}\right]_{not\ odd} &= -k_1^{\beta}\left\{\frac{2k_\alpha k_\beta}{(k^2-\lambda^2)^2} - \frac{g_{\alpha\beta}}{(k^2-\lambda^2)}\right\} \\ &+ \frac{\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)^2}{(k^2-\lambda^2)^2\left[(k+k_1)^2-m^2\right]}k_\alpha \\ &- \frac{\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)}{(k^2-\lambda^2)\left[(k+k_1)^2-m^2\right]}k_{1\alpha} ,\end{aligned} \tag{43}$$

where an odd term was omitted since, after the integration, it will vanishes. Note that the dependence in the arbitrary internal momentum is located only in finite integrals. The divergent terms will not contain physical quantities since λ is an arbitrary parameter. After some reorganization, we can take the integration on both sides

$$\begin{aligned}
&\int \frac{d^2k}{(2\pi)^2}\left[t_\mu^V(k_1)+k_1^\beta\left\{\frac{2k_\alpha k_\beta}{(k^2-\lambda^2)^2}-\frac{g_{\alpha\beta}}{(k^2-\lambda^2)}\right\}\right] \\
&=\int \frac{d^2k}{(2\pi)^2}\frac{k_\alpha\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)^2}{(k^2-\lambda^2)^2\left[(k+k_1)^2-m^2\right]} \\
&-k_{1\alpha}\int \frac{d^2k}{(2\pi)^2}\frac{\left(k_1^2+2k\cdot k_1+\lambda^2-m^2\right)}{(k^2-\lambda^2)\left[(k+k_1)^2-m^2\right]}\ .
\end{aligned} \tag{44}$$

This expression is only consequence of the Feynman rules. On the right hand side, there are only finite integrals. They can be solved without any kind of concern since any reasonable regularization must give the same result for a finite integral. The integration reveals the identically zero value. A formal relation can be written by identifying a surface term like the one defined in (33),

$$T_\mu^V(k_1)=-2k_1^\alpha\left[\Delta_{2;\mu\alpha}^{(2)}\left(\lambda^2\right)\right]\ . \tag{45}$$

The result is proportional to the arbitrary momentum k_1 and to the surface term $\Delta_{2;\mu\alpha}^{(2)}$, whose argument is also an arbitrary quantity. The mathematical object $\Delta_{2;\mu\alpha}^{(2)}$ is prescription dependent in the sense that, in order to attribute a definite value to it, some particular mathematical procedure is required. It would be desirable that this perturbative amplitude gives a null result, as we can see below, but the Feynman rules do not imply that.

The same procedure can be used to evaluate the $T_{\sigma\rho}^{VV}$ amplitude. The result can be written as

$$T_{\sigma\rho}^{VV}=2\left[\Delta_{2;\sigma\rho}^{(2)}\right]+\frac{i}{\pi}\left(p_\sigma p_\rho-g_{\sigma\rho}p^2\right)\left[\xi_2^{(-1)}\left(p^2;m^2\right)-\xi_1^{(-1)}\left(p^2;m^2\right)\right]\ , \tag{46}$$

where $p=k_2-k_1$. In the above result we can see clearly the aforementioned organization through finite and (a priori) divergent objects. For our purposes, it is important to known if the above result is in accordance to the expected RAGFs (Eqs. (15) and (16)). The contraction of (46) with p^σ gives

$$p^\sigma T_{\sigma\rho}^{VV}=2\left(k_2-k_1\right)^\sigma\left[\Delta_{2;\sigma\rho}^{(2)}\right]\ , \tag{47}$$

which, given (45), can be recognized as being the RAGFs (15). So, this (vector) RAGFs is automatically fulfilled by (46). The same conclusion is also valid for p^ρ contraction.

This result is a good opportunity to illustrate our preceding comments about RAGFs and WIs. Since the $T_{\sigma\rho}^{VV}$ amplitude is proportional to the polarization tensor of QED_2, gauge invariance implies that it must have two conserved vector currents, i.e., $p^\sigma T^{VV}_{\sigma\rho} = p^\rho T^{VV}_{\sigma\rho} = 0$. It is easy to see that such requirements is not automatically satisfied by expression (46). The violating term is given by the (undefined) object $\Delta^{(2)}_{2;\sigma\rho}$, according to the expectations. The Feynman rules ended their job. The RAGFs are satisfied as it is required but the WIs satisfaction will depend on an additional ingredient. So, without additional assumptions, the unique way to obtain a polarization tensor $T^{VV}_{\sigma\rho}$ satisfying its WIs is in the absence of the object $\Delta^{(2)}_{2;\sigma\rho}$. As a surface term, this is exactly what would happens if we have used the DR prescription. This is, in fact, a necessary requirement for all regularizations which intend to be gauge preserving [39].

B. $T^{VV}_{\mu;\sigma\rho}$ amplitude

The next subamplitude to consider is $T^{VV}_{\mu;\sigma\rho}$ (see Eq. (11)). The result can be put in the form

$$\begin{aligned} T^{VV}_{\mu;\sigma\rho} &= -P^\alpha \left[\Box^{(2)}_{3;\alpha\mu\sigma\rho}\right] + P_\rho \left[\Delta^{(2)}_{2;\sigma\mu}\right] + P_\sigma \left[\Delta^{(2)}_{2;\rho\mu}\right] \\ &+ P^\alpha \left\{ g_{\sigma\rho} \left[\Delta^{(2)}_{2;\mu\alpha}\right] - g_{\mu\rho} \left[\Delta^{(2)}_{2;\alpha\sigma}\right] - g_{\mu\sigma} \left[\Delta^{(2)}_{2;\rho\alpha}\right] \right\} \\ &- \frac{p_\mu}{2} \left[T^{VV}_{\sigma\rho}\right] , \end{aligned} \tag{48}$$

where $P = k_2 + k_1$. The proposed systematization is, again, clear from the above expression. In the above equation we also see the polarization tensor $T^{VV}_{\sigma\rho}$ as being a substructure of $T^{VV}_{\mu;\sigma\rho}$. The crucial question is: does the above expression fulfill its expected RAGFs?

The contraction of the above result with p^σ reveals that the RAGFs (A1) is satisfied. The same happens for the p^μ contraction. These calculations can easily be done by observing the results for the one-point amplitude $T^V_{\mu;\nu}(k_i)$, given in Eq. (C10). On the other hand, the metric contraction gives

$$\begin{aligned} g^{\mu\sigma} \left[T^{VV}_{\mu;\sigma\rho}\right] &= \left[T^V_\rho (k_2)\right] + m \left[T^{SV}_\rho\right] \\ &- (k_2 + k_1)^\sigma \left\{ \left[g^{\mu\nu} \Box^{(2)}_{3;\mu\nu\sigma\rho}\right] - g_{\sigma\rho} \left[g^{\mu\nu} \Delta^{(2)}_{2;\mu\nu}\right] \right\} , \end{aligned} \tag{49}$$

where T^{SV}_ρ is given by Eq. (D2). The last term of the above equation shows that expression (48) for $T^{VV}_{\mu;\sigma\rho}$ does not (automatically) satisfy the expect RAGFs (A3). The spurious terms are composed by two surface terms. In order to not break this RAGFs we must have

$$g^{\mu\nu} \Box^{(2)}_{3;\mu\nu\sigma\rho} = g_{\sigma\rho} \left[g^{\mu\nu} \Delta^{(2)}_{2;\mu\nu}\right] . \tag{50}$$

Let us consider this in an explicit way. First we note that the integrand of $g^{\mu\nu}\Delta_{2;\mu\nu}^{(2)}$ satisfy a trivial algebraic identity

$$\frac{2k^2}{\left(k^2-m^2\right)^2}-\frac{2}{\left(k^2-m^2\right)}=\frac{2m^2}{\left(k^2-m^2\right)^2}, \tag{51}$$

which means that the integral is finite, as well as the linearity of the integration operation is assumed. Given that one obtain

$$g^{\mu\nu}\Delta_{2;\mu\nu}^{(2)}=-\frac{i}{2\pi}. \tag{52}$$

In a similar way, the integrand of the quantity $g^{\mu\nu}\left[\Box_{3;\mu\nu\sigma\rho}\right]$ can be rewritten as

$$\frac{8k^2k_\sigma k_\rho}{\left(k^2-m^2\right)^3}-\frac{8k_\sigma k_\rho}{\left(k^2-m^2\right)^2}=\frac{8m^2k_\sigma k_\rho}{\left(k^2-m^2\right)^3}, \tag{53}$$

such that the corresponding integral will be finite. Performing the integration we get

$$g^{\mu\nu}\left[\Box_{3;\mu\nu\sigma\rho}\right]=-\frac{i}{2\pi}g_{\sigma\rho}\ . \tag{54}$$

Given both results we obtain (50). This means that our procedure is consistent with the linearity in the integration operation. It is interesting to note that we have evaluated surface terms and the results obtained are nonzero. Here one can note that the condition (50) would also be satisfied by assuming $\Box_{3;\mu\nu\sigma\rho}^{(2)}=\Delta_{2;\mu\nu}^{(2)}=0$. If we had applied the DR to perform these calculations, such requirements would be fulfilled automatically since in the DR prescription surface terms are assumed to vanish.

C. $T^{VV}_{\mu\nu;\sigma\rho}$ amplitude

The last subamplitude composing the vector sector is $T^{VV}_{\mu\nu;\sigma\rho}$ (see Eq. (12)). After a long and tedious but a straightforward calculation, we obtain

$$\begin{aligned}
T^{VV}_{\mu\nu;\sigma\rho} &= S^{VV}_{\mu\nu;\sigma\rho} + \left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\mu}\right)\left[I^{(2)}_{quad}\left(m^2\right)\right] \\
&+ \left\{\frac{1}{3}\left(g_{\mu\nu}g_{\rho\sigma}p^2 - g_{\mu\nu}p_\rho p_\sigma - g_{\sigma\rho}p_\mu p_\nu\right)\right. \\
&\quad - \frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma}p^2 - g_{\mu\rho}p_\nu p_\sigma - g_{\nu\sigma}p_\mu p_\rho\right) \\
&\quad \left. -\frac{1}{6}\left(g_{\nu\rho}g_{\mu\sigma}p^2 - g_{\nu\rho}p_\mu p_\sigma - g_{\mu\sigma}p_\nu p_\rho\right)\right\}\left[I^{(2)}_{\log}\left(m^2\right)\right] \\
&+ \frac{i}{2\pi}\left\{\frac{p_\mu p_\nu p_\sigma p_\rho}{p^2} + \frac{p^2}{2}\left(g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)\right. \\
&\quad \left. -\frac{p_\nu}{2}\left(g_{\mu\sigma}p_\rho + g_{\mu\rho}p_\sigma\right) - \frac{p_\mu}{2}\left(g_{\nu\rho}p_\sigma + g_{\nu\sigma}p_\rho\right)\right\}\left[2\xi^{(0)}_2 - \xi^{(0)}_1\right] \\
&+ \frac{i}{2\pi}\frac{1}{p^2}\left(p_\sigma p_\rho - p^2 g_{\sigma\rho}\right)\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\xi^{(0)}_2 - \xi^{(0)}_1\right] + \frac{1}{4}p_\rho p_\sigma\left[T^{VV}_{\mu\nu}\right] \ ,
\end{aligned} \tag{55}$$

where $S^{VV}_{\mu\nu;\sigma\rho}$ represents a (ambiguous) combination of surface terms and is given explicitly in appendix (D). Note that in the expression above the divergent objects appear as functions of the physical mass (m) rather than an arbitrary scale mass (λ). From now on, we will adopt this simplified notation, because the mass scale, chosen for the divergent objects, will not play an important role in the discussions presented in this work. In addition, if needed for some reason, the mass scale can be changed freely using the scale relations shown in the Eqs. (30) and (31). The contraction of the above result with p^σ or p^μ shows that both expected RAGFs (see Eqs. (A2) and (A6)), are satisfied automatically while the contraction with the metric gives

$$\begin{aligned}
g^{\mu\sigma}\left[T^{VV}_{\mu\nu;\sigma\rho}\right] &= \left[T^{V}_{\nu;\rho}\left(k_2\right)\right] + m\left[T^{SV}_{\nu;\rho}\right] \\
&+ \frac{1}{3}\left(k^\xi_2 k^\chi_2 + k^\xi_1 k^\chi_2 + k^\xi_1 k^\chi_1\right)\left[g^{\mu\sigma}\Sigma^{(2)}_{4;\mu\sigma\nu\rho\xi\chi}\right] \\
&- \frac{1}{2}\left(k^2_1 + k^2_2\right)\left[g^{\mu\sigma}\Box^{(2)}_{3;\mu\sigma\nu\rho}\right] \\
&- \frac{1}{2}\left(k_2 + k_1\right)_\rho\left(k_2 + k_1\right)^\xi\left[g^{\mu\sigma}\Box^{(2)}_{3;\mu\sigma\nu\xi}\right] \\
&- \left(k_2 + k_1\right)^\xi k_{1\nu}\left[g^{\mu\sigma}\Box^{(2)}_{3;\mu\sigma\rho\xi}\right] + \left(k_2 + k_1\right)_\rho k_{1\nu}\left[g^{\mu\sigma}\Delta^{(2)}_{2;\mu\sigma}\right] \\
&+ \frac{i}{4\pi}\frac{1}{3}\left(p_\rho p_\nu - g_{\nu\rho}p^2\right) \ ,
\end{aligned} \tag{56}$$

where we have used the results (C10) and (D3). Now there are six potentially breaking terms for the RAGFs (A4). In order to fulfill this RAGFs we have to get fulfilled both the condition (50) and also

$$g^{\mu\sigma}\Sigma^{(2)}_{4;\mu\sigma\nu\rho\xi\chi} = -\frac{i}{2\pi}\left(g_{\nu\rho}g_{\xi\chi} + g_{\nu\xi}g_{\rho\chi} + g_{\nu\chi}g_{\rho\xi}\right) \ . \tag{57}$$

It is simple to see that this condition is satisfied by using the same sequence of steps used to obtain (50), i.e., by assuming the validity of the linearity in the integration operation. This result means that the RAGFs is preserved by the operations made.

D. $T^{AV}_{\sigma\rho}$ amplitude

In the axial-vector sector we can make a similar investigation of the pertinent set of subamplitudes, which are defined by taking $\Gamma_i = \gamma_\sigma\gamma_3$ and $\Gamma_j = \gamma_\rho$ in (10), (11), and (12).

Let us take the simplest amplitude of the set, namely $T^{AV}_{\sigma\rho}$. This subamplitude is a very interesting one for our present investigation since it is the well-known *anomalous* amplitude belonging to the chiral QED_2 [33, 34, 40–42]. Its evaluation can be made trivial if one note the relation

$$T^{AV}_{\sigma\rho} = -\varepsilon_{\sigma\alpha}g^{\alpha\beta}\left[T^{VV}_{\beta\rho}\right] \ , \tag{58}$$

such that, by using (46), we get

$$T^{AV}_{\sigma\rho} = -2\varepsilon_{\sigma\xi}\left[\Delta^{(2)}_{2;\xi\rho}\right] - \frac{i}{\pi}\varepsilon_{\sigma\xi}\left(p_\xi p_\rho - g_{\xi\rho}p^2\right)\left[\xi^{(-1)}_2\left(p^2;m^2\right) - \xi^{(-1)}_1\left(p^2;m^2\right)\right] \ . \tag{59}$$

There are two RAGFs expected to be satisfied by the above expression, which were stated in (A8) and (A9). The first one refers to the contraction with p^ρ. This contraction gives, immediately, the expected difference $T^A_\nu(k_1) - T^A_\nu(k_2)$ (see Eq. (C6)).

On the other hand, the contraction of (59) with the axial index (p^σ) reveals

$$\begin{aligned} p^\sigma T^{AV}_{\sigma\rho} &= -2\varepsilon_{\sigma\xi}\left(k_2-k_1\right)^\sigma g^{\xi\chi}\left[\Delta^{(2)}_{2;\chi\rho}\right] \\ &\quad - \frac{i}{\pi}\varepsilon_{\sigma\rho}p^\sigma\left[1 + m^2\xi^{(-1)}_0\left(p^2,m^2\right)\right] \ . \end{aligned} \tag{60}$$

Now, there is subtlety in order to identify the expected difference of two axial one-point functions in the right hand side of the equation above. This is a very important aspect of our investigation. For this, it is first necessary to change the position of the Lorentz indexes in the first term above. Through the Schouten identity

$$\varepsilon_{\sigma\xi}g^{\xi\chi}\left[\Delta_{2;\chi\rho}^{(2)}\right]=\varepsilon_{\rho\xi}g^{\xi\chi}\left[\Delta_{2;\chi\sigma}^{(2)}\right]-\varepsilon_{\sigma\rho}\left[g^{\xi\chi}\Delta_{2;\xi\chi}^{(2)}\right] \ , \tag{61}$$

such a change of index can be achieved. Given the PV amplitude (see Eq. (D4)), we can write

$$\begin{aligned} p^{\sigma}T_{\sigma\rho}^{AV} &= \left[T_{\rho}^{A}\left(k_{1}\right)\right]-\left[T_{\rho}^{A}\left(k_{2}\right)\right]+2m\left[T_{\rho}^{PV}\right] \\ &+2\varepsilon_{\rho\xi}p^{\xi}\left\{\frac{i}{2\pi}+\left[g^{\alpha\beta}\Delta_{2;\alpha\beta}^{(2)}\right]\right\} \ . \end{aligned} \tag{62}$$

From the expression above we see that the RAGFs (A9) is preserved, if and only if,

$$g^{\xi\chi}\Delta_{2;\xi\chi}^{(2)}=-\frac{i}{2\pi} \ , \tag{63}$$

which is the same result founded in (52). Again, the operations made are in accordance to the linearity in the integration operation. However, it must be noted that through the Schouten identity we have constructed two representations for the AV amplitude, such that, after the immediately above result, allow us to identify

$$p^{\rho}\left[T_{\sigma\rho}^{AV}\right]_{1}=T_{\nu}^{A}\left(k_{1}\right)-T_{\nu}^{A}\left(k_{2}\right) \ , \tag{64}$$

$$p^{\sigma}\left[T_{\sigma\rho}^{AV}\right]_{2}=\left[T_{\rho}^{A}\left(k_{1}\right)\right]-\left[T_{\rho}^{A}\left(k_{2}\right)\right]+2m\left[T_{\rho}^{PV}\right] \tag{65}$$

Both expressions $\left[T_{\sigma\rho}^{AV}\right]_{1}$ and $\left[T_{\sigma\rho}^{AV}\right]_{2}$ are identical from the mathematical point of view, as long as the Schouten identity is valid. Note that the referred identity relates a tensor with its trace.

It is possible to note an interesting aspect in the above results. In the amplitudes of the vector sector we evaluated the finite quantities $g^{\xi\chi}\Delta_{2;\xi\chi}^{(2)}$, $g^{\mu\nu}\Box_{3;\mu\nu\rho\xi}^{(2)}$, and $g^{\mu\sigma}\Sigma_{4;\mu\sigma\nu\rho\xi\chi}^{(2)}$. In those cases, the nonzero value found for these quantities put the results in accordance with the requirements but the null value fulfill the conditions as well. In the present case, the value $g^{\xi\chi}\Delta_{2;\xi\chi}^{(2)}=0$ breaks the linearity in the integration operation since the axial index contraction gives

$$p^{\sigma}T_{\sigma\rho}^{AV}=T_{\rho}^{A}\left(k_{1}\right)-T_{\rho}^{A}\left(k_{2}\right)+2m\left[T_{\rho}^{PV}\right]+\frac{i}{\pi}\varepsilon_{\rho\xi}p^{\xi} \ . \tag{66}$$

On the other hand, in order to fulfill the WIs

$$\begin{cases} p^{\rho}T_{\sigma\rho}^{AV}=0 \ , \\ p^{\sigma}T_{\sigma\rho}^{AV}=2m\left[T_{\rho}^{PV}\right] \ , \end{cases} \tag{67}$$

a necessary condition is

$$T_{\rho}^{A}\left(k_{1}\right)-T_{\rho}^{A}\left(k_{2}\right)=0\ , \tag{68}$$

which is guaranteed only by some prescription that attributes a null value for the object $\Delta_{2;\xi\chi}^{(2)}$, in the same way as it was required in the case of the WIs for the polarization tensor ($T_{\sigma\rho}^{VV}$). So, it seems that it is apparently possible to preserve the linearity in the integration operation and both WIs, simultaneously, with the conditions

$$\begin{cases} \Delta_{2;\xi\chi}^{(2)}=0\ , \\ g^{\alpha\beta}\Delta_{2;\alpha\beta}^{(2)}=-\frac{i}{2\pi}\ . \end{cases} \tag{69}$$

These two conditions are clearly not compatible with the Schouten identity which is necessary to generate the second representation $\left[T_{\sigma\rho}^{AV}\right]_{2}$ starting from the first one $\left[T_{\sigma\rho}^{AV}\right]_{1}$. So, in order to get this wonderful result it is necessary an illegal trick or to corrupt the mathematics. A consistent condition would be taken both objects as zero quantities, which gives us

$$p^{\sigma}T_{\sigma\rho}^{AV}=2m\left[T_{\rho}^{PV}\right]+\frac{i}{\pi}\varepsilon_{\rho\xi}p^{\xi}, \tag{70}$$

connecting us with the well-known anomalous phenomenon [33] in two-dimensions.

E. $T_{\mu;\sigma\rho}^{AV}$ amplitude

The second subamplitude of the axial-vector sector is $T_{\mu;\sigma\rho}^{AV}$. Through the relationship

$$T_{\mu;\sigma\rho}^{AV}=-\varepsilon_{\sigma\alpha}g^{\alpha\beta}\left[T_{\mu;\beta\rho}^{VV}\right]\ , \tag{71}$$

and the result (48) it is straightforward to get

$$\begin{aligned} T_{\mu;\sigma\rho}^{AV} &= \varepsilon_{\sigma\xi}P^{\chi}\left[\Box_{3;\mu\rho\chi\xi}^{(2)}\right]-\varepsilon_{\sigma\xi}P_{\rho}\left[\Delta_{2;\xi\mu}^{(2)}\right]-\varepsilon_{\sigma\xi}P^{\xi}\left[\Delta_{2;\rho\mu}^{(2)}\right] \\ &\quad -\varepsilon_{\sigma\xi}P^{\chi}\left\{g_{\xi\rho}\left[\Delta_{2;\mu\chi}^{(2)}\right]-g_{\mu\rho}\left[\Delta_{2;\chi\xi}^{(2)}\right]-g_{\mu\xi}\left[\Delta_{2;\rho\chi}^{(2)}\right]\right\} \\ &\quad +\frac{p_{\mu}}{2}\left[T_{\sigma\rho}^{AV}\right]\ . \end{aligned} \tag{72}$$

From this result it is expected that it should, by consistency, satisfies four RAGFs. The two RAGFs obtained for the contractions with p^{μ} and p^{ρ} are satisfied without additional hypothesis. In contrast, contracting (72) with p^{σ} and $g^{\mu\sigma}$ gives, respectively,

$$
\begin{aligned}
p^{\sigma}\left[T^{AV}_{\mu;\sigma\rho}\right] &= \left[T^{A}_{\mu;\rho}\left(k_1\right)\right] - \left[T^{A}_{\mu;\rho}\left(k_2\right)\right] + 2m\left[T^{PV}_{\mu;\rho}\right] \\
&\quad - \varepsilon_{\rho\sigma}p^{\sigma}p^{\xi}\left\{\left[g^{\alpha\beta}\square^{(2)}_{3;\alpha\beta\mu\xi}\right] - g_{\xi\mu}\left[g^{\alpha\beta}\Delta^{(2)}_{2;\alpha\beta}\right]\right\} \\
&\quad - \varepsilon_{\rho\sigma}p^{\sigma}p_{\mu}\left\{\frac{i}{2\pi} + \left[g^{\alpha\beta}\Delta^{(2)}_{2;\alpha\beta}\right]\right\} \ ,
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
g^{\mu\sigma}\left[T^{AV}_{\mu;\sigma\rho}\right] &= \left[T^{A}_{\rho}\left(k_2\right)\right] - m\left[T^{PV}_{\rho}\right] \\
&\quad - \varepsilon_{\rho\xi}p^{\xi}\left\{\frac{i}{2\pi} + \left[g^{\alpha\beta}\Delta^{(2)}_{2;\alpha\beta}\right]\right\} \ .
\end{aligned} \tag{74}
$$

We see clearly that the RAGFs, generated by such contractions, are not automatically satisfied. However, if one wants to preserve both of these RAGFs it is enough to fulfill the conditions (50) and (52) previously found.

Again, we note that the options $\square^{(2)}_{3;\mu\nu\rho\xi} = \Delta^{(2)}_{2;\mu\nu} = 0$ and $g^{\mu\nu}\square^{(2)}_{3;\mu\nu\rho\xi} = g_{\xi\rho}\left[g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}\right] = 0$ means violations of both RAGF's as happens in the case of the amplitude $T^{AV}_{\sigma\rho}$.

F. $T^{AV}_{\mu\nu;\sigma\rho}$ Amplitude

The last subamplitude of this set is $T^{AV}_{\mu\nu;\sigma\rho}$. Its relationship with $T^{VV}_{\mu\nu;\beta\rho}$, Eq. (55),

$$
T^{AV}_{\mu\nu;\sigma\rho} = -\varepsilon_{\sigma\alpha}g^{\alpha\beta}\left[T^{VV}_{\mu\nu;\beta\rho}\right] \ , \tag{75}
$$

makes its calculation immediate. The contraction of this result with p^{ρ} and p^{μ} reveals that the corresponding RAGFs are satisfied. On the other hand, the contractions with p^{σ} and $g^{\mu\sigma}$ reveal unexpected (violating) terms given explicitly below

$$
\begin{aligned}
p^{\sigma}\left[T^{AV}_{\mu\nu;\sigma\rho}\right] &= \left[T^{A}_{\mu\nu;\rho}\left(k_1\right)\right] - \left[T^{A}_{\mu\nu;\rho}\left(k_2\right)\right] + 2m\left[T^{PV}_{\mu\nu;\rho}\right] \\
&\quad + \frac{1}{3}\varepsilon_{\rho\sigma}p^{\sigma}\left(k_2^{\xi}k_2^{\chi} + k_1^{\xi}k_2^{\chi} + k_1^{\xi}k_1^{\chi}\right)\left[g^{\alpha\beta}\Sigma^{(2)}_{4;\alpha\beta\mu\nu\xi\chi}\right] \\
&\quad - \frac{1}{2}\varepsilon_{\rho\sigma}p^{\sigma}\left(k_1^2 + k_2^2\right)\left[g^{\alpha\beta}\square^{(2)}_{3;\alpha\beta\mu\nu}\right] \\
&\quad - \varepsilon_{\rho\sigma}p^{\sigma}p^{\xi}k_{1\mu}\left[g^{\alpha\beta}\square^{(2)}_{3;\alpha\beta\nu\xi}\right] - \varepsilon_{\rho\sigma}p^{\sigma}p^{\xi}k_{1\nu}\left[g^{\alpha\beta}\square^{(2)}_{3;\alpha\beta\mu\xi}\right] \\
&\quad + 2\varepsilon_{\rho\sigma}p^{\sigma}k_{1\mu}k_{1\nu}\left[g^{\alpha\beta}\Delta^{(2)}_{2;\alpha\beta}\right] + \frac{i}{4\pi}\frac{1}{3}\varepsilon_{\rho\sigma}p^{\sigma}\left[4p_{\mu}p_{\nu} - g_{\mu\nu}p^2\right] \ ,
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
g^{\mu\sigma}\left[T^{AV}_{\mu\nu;\sigma\rho}\right] = {} & \left[T^{A}_{\nu;\rho}\left(k_2\right)\right] - m\left[T^{PV}_{\nu;\rho}\right] \\
& + \frac{1}{2}\varepsilon_{\rho\xi}p^{\xi}\left(k_2+k_1\right)^{\sigma}\left[g^{\alpha\beta}\Box^{(2)}_{3;\alpha\beta\nu\sigma}\right] \\
& - \varepsilon_{\rho\xi}p^{\xi}k_{1\nu}\left[g^{\alpha\beta}\Delta^{(2)}_{2;\alpha\beta}\right] + \frac{i}{4\pi}\varepsilon_{\rho\xi}p^{\xi}p_{\nu}\ .
\end{aligned} \tag{77}
$$

Entirely similar to what occurred with $T^{VV}_{\mu\nu;\sigma\rho}$, in order to save both RAGFs it is imperative that

$$
\begin{cases}
g^{\mu\nu}\Box^{(2)}_{3;\mu\nu\rho\xi} = g_{\xi\rho}\left[g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}\right] = -\frac{i}{2\pi}g_{\xi\rho}\ , \\
g^{\mu\sigma}\Sigma^{(2)}_{4;\mu\sigma\nu\rho\xi\chi} = -\frac{i}{2\pi}\left(g_{\nu\rho}g_{\xi\chi}+g_{\nu\xi}g_{\rho\chi}+g_{\nu\chi}g_{\rho\xi}\right)\ ,
\end{cases} \tag{78}
$$

which are the same results previously obtained.

We should emphasize, again, that the results above can be easily checked by assuming the validity of the linearity in the integration operation, as we have pointed out before. The results for $T^{VA}_{\sigma\rho}$, $T^{VA}_{\mu;\sigma\rho}$, and $T^{VA}_{\mu\nu;\sigma\rho}$ are completely analogous to $T^{AV}_{\sigma\rho}$, $T^{AV}_{\mu;\sigma\rho}$, and $T^{AV}_{\mu\nu;\sigma\rho}$.

G. $T^{AA}_{\sigma\rho}$ Amplitude

Although the procedure is essentially the same to the one presented above, for completeness, we succinctly present the main results for the subamplitudes belonging to the axial sector, obtained through the substitutions: $\Gamma_i = \gamma_\sigma\gamma_3$ and $\Gamma_j = \gamma_\rho\gamma_3$ in (10), (11), and (12).

The first subamplitude is $T^{AA}_{\sigma\rho}$, which has the corresponding result

$$
\begin{aligned}
T^{AA}_{\sigma\rho} = {} & 2\left[\Delta^{(2)}_{2;\sigma\rho}\right] + 4\left[p_\sigma p_\rho - g_{\sigma\rho}p^2\right]\left[\xi^{(-1)}_{2}\left(p^2,m^2\right) - \xi^{(-1)}_{1}\left(p^2,m^2\right)\right] \\
& - 8m^2 g_{\sigma\rho}\left[\xi^{(-1)}_{1}\left(p^2,m^2\right)\right].
\end{aligned} \tag{79}
$$

Both expected RAGFs (see appendix A 3) are fulfilled by the above expression without any assumption about the object $\Delta^{(2)}_{2;\sigma\rho}$.

Again we observe that, similarly to $T^{VV}_{\sigma\rho}$ and $T^{AV}_{\sigma\rho}$ amplitudes, $T^{AA}_{\sigma\rho}$ belongs to the set of amplitudes associated with standard (renormalizable) theories. So, within such a context, this amplitude should satisfy additional constraints such as the two following WIs

$$
\begin{cases}
p^{\sigma}\left[T^{AA}_{\sigma\rho}\right] = 2m\left[T^{PA}_{\rho}\right]\ , \\
p^{\rho}\left[T^{AA}_{\sigma\rho}\right] = 2m\left[T^{AP}_{\sigma}\right]\ ,
\end{cases} \tag{80}
$$

representing the proportionality between the axial current divergence and the pseudoscalar one, for the case of massive fermions. In order to satisfy both WIs it is required that

$$
\left[T^{V}_{\rho}\left(k_1\right)\right] - \left[T^{V}_{\rho}\left(k_2\right)\right] = 0\ , \tag{81}
$$

in a completely similar way as in the case of the amplitude $T^{VV}_{\sigma\rho}$, as expected.

H. $T^{AA}_{\mu;\sigma\rho}$ Amplitude

The second subamplitude of this set is $T^{AA}_{\mu;\sigma\rho}$. We found

$$\begin{aligned}
T^{AA}_{\mu;\sigma\rho} &= -P^{\alpha}\left[\Box^{(2)}_{3;\alpha\mu\sigma\rho}\right] + P_{\rho}\left[\Delta^{(2)}_{2;\sigma\mu}\right] + P_{\sigma}\left[\Delta^{(2)}_{2;\rho\mu}\right] \\
&+ P^{\alpha}\left\{g_{\sigma\rho}\left[\Delta^{(2)}_{2;\mu\alpha}\right] - g_{\mu\rho}\left[\Delta^{(2)}_{2;\alpha\sigma}\right] - g_{\mu\sigma}\left[\Delta^{(2)}_{2;\rho\alpha}\right]\right\} \\
&- \frac{p_{\mu}}{2}\left[T^{AA}_{\sigma\rho}\right] \ .
\end{aligned} \tag{82}$$

When the RAGFs are checked, we find that those corresponding to the contractions $p^{\sigma}\left[T^{AA}_{\mu;\sigma\rho}\right]$ and $p^{\mu}\left[T^{AA}_{\mu;\sigma\rho}\right]$ (see appendix A 3) are satisfied, while that related to the contraction $g^{\mu\sigma}\left[T^{AA}_{\mu;\sigma\rho}\right]$ gives

$$\begin{aligned}
g^{\mu\sigma}\left[T^{AA}_{\mu;\sigma\rho}\right] &= \left[T^{V}_{\rho}\left(k_2\right)\right] - m\left[T^{PA}_{\rho}\right] \\
&- \left(k_2 + k_1\right)^{\xi}\left\{\left[g^{\alpha\beta}\Box^{(2)}_{3;\alpha\beta\rho\xi}\right] + g_{\rho\xi}\left[g^{\alpha\beta}\Delta^{(2)}_{2;\alpha\beta}\right]\right\}.
\end{aligned} \tag{83}$$

The conclusions are the same ones obtained for $T^{VV}_{\mu;\sigma\rho}$.

I. $T^{AA}_{\mu\nu;\sigma\rho}$ Amplitude

The last one to be calculated is $T^{AA}_{\mu\nu;\sigma\rho}$. The result can be put in the form

$$\begin{aligned}
T^{AA}_{\sigma\rho;\mu\nu} &= S^{VV}_{\sigma\rho;\mu\nu} + \left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\mu}\right)\left[I^{(2)}_{quad}\left(m^2\right)\right] \\
&+ \left\{\frac{1}{3}\left(g_{\mu\nu}g_{\rho\sigma}p^2 - g_{\mu\nu}p_{\rho}p_{\sigma} - g_{\sigma\rho}p_{\mu}p_{\nu}\right)\right. \\
&\quad - \frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma}p^2 - g_{\mu\rho}p_{\nu}p_{\sigma} - g_{\nu\sigma}p_{\mu}p_{\rho}\right) \\
&\quad \left. -\frac{1}{6}\left(g_{\nu\rho}g_{\mu\sigma}p^2 - g_{\nu\rho}p_{\mu}p_{\sigma} - g_{\mu\sigma}p_{\nu}p_{\rho}\right) - 2m^2 g_{\mu\nu}g_{\sigma\rho}\right\}\left[I^{(2)}_{\log}\left(m^2\right)\right] \\
&+ \frac{i}{2\pi}\left\{\frac{p_{\mu}p_{\nu}p_{\sigma}p_{\rho}}{p^2} + \frac{p^2}{2}\left(g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)\right. \\
&\qquad \left. -\frac{p_{\nu}}{2}\left(g_{\mu\sigma}p_{\rho} + g_{\mu\rho}p_{\sigma}\right) - \frac{p_{\mu}}{2}\left(g_{\nu\rho}p_{\sigma} + g_{\nu\sigma}p_{\rho}\right)\right\}\left[2\xi^{(0)}_{2} - \xi^{(0)}_{1}\right] \\
&+ \frac{i}{2\pi}\frac{1}{p^2}\left(p_{\sigma}p_{\rho} - p^2 g_{\sigma\rho}\right)\left(p_{\mu}p_{\nu} - p^2 g_{\mu\nu}\right)\left[\xi^{(0)}_{2} - \xi^{(0)}_{1}\right] \\
&- \frac{i}{2\pi}\frac{m^2}{p^2}g_{\mu\nu}\left(p_{\sigma}p_{\rho} - p^2 g_{\sigma\rho}\right)\left[\xi^{(0)}_{0}\right] + \frac{1}{4}p_{\rho}p_{\sigma}\left[T^{AA}_{\mu\nu}\right] \ .
\end{aligned} \tag{84}$$

The verification of the RAGFs shows that only the contraction $g^{\mu\sigma}T^{AA}_{\mu\nu;\sigma\rho}$ is not automatically satisfied. Instead, we find

$$\begin{aligned} g^{\mu\sigma}T^{AA}_{\mu\nu;\sigma\rho} &= \left[T^{V}_{\nu;\rho}\left(k_2\right)\right] - m\left[T^{PA}_{\nu;\rho}\right] \\ &+ \frac{1}{3}\left(k_2^{\xi}k_2^{\chi} + k_1^{\xi}k_2^{\chi} + k_1^{\xi}k_1^{\chi}\right)\left[g^{\mu\sigma}\Sigma^{(2)}_{4;\mu\sigma\nu\rho\xi\chi}\right] \\ &- \frac{1}{2}\left(k_1^2 + k_2^2\right)\left[g^{\mu\sigma}\Box^{(2)}_{3;\mu\sigma\nu\rho}\right] \\ &- \frac{1}{2}\left(k_2 + k_1\right)_{\rho}\left(k_2 + k_1\right)^{\xi}\left[g^{\mu\sigma}\Box^{(2)}_{3;\mu\sigma\nu\xi}\right] \\ &- k_{1\nu}\left(k_2 + k_1\right)^{\xi}\left[g^{\mu\sigma}\Box^{(2)}_{3;\mu\sigma\rho\xi}\right] \\ &+ k_{1\nu}\left(k_2 + k_1\right)_{\rho}\left[g^{\mu\sigma}\Delta^{(2)}_{2;\mu\sigma}\right] \\ &+ \frac{1}{3}\left[p_{\nu}p_{\rho} - g_{\nu\rho}p^2\right] \ . \end{aligned} \tag{85}$$

Clearly, the conditions that ensure this RAGFs (see Eq. (A21)) be fulfilled are the same ones required for the $T^{VV}_{\mu\nu;\sigma\rho}$ and $T^{AV}_{\mu\nu;\sigma\rho}$, as discussed above.

V. RAGFS VERSUS EINSTEIN AND WEYL GRAVITATIONAL ANOMALIES

In the section IV we shown that, in order to satisfy (simultaneously) all the RAGFs expected for the subamplitudes of $T^{(G)}_{\mu\nu\rho\sigma}$, it is required a set of conditions involving finite quantities. They represent necessary conditions for the calculation procedure be consistent with the linearity operation in the integrals. At this point we can ask ourselves: are the above requirements enough to guarantee also the maintenance of the RAGFs associated with $T^{(G)}_{\mu\nu\rho\sigma}$? Given the investigation about the subamplitudes made in the previous section, the answer to this query is immediate. This is because the gravitational amplitude $T^{(G)}_{\mu\nu\rho\sigma}$ was decomposed into a sum of subamplitudes. Thus, when we contract $T^{(G)}_{\mu\nu\rho\sigma}$ with p^{μ} or $g^{\mu\nu}$, we get contractions with these subamplitudes with p^{μ} or $g^{\mu\nu}$ also, and, each contraction generates a RAGFs for such subamplitudes, as we saw in the section IV. For instance, in the vector sector we have

$$\begin{aligned} p^{\mu}T^{(V)}_{\mu\nu\rho\sigma} &= 4\left\{\left[p^{\mu}T^{VV}_{\nu\sigma;\mu\rho}\right] + \left[p^{\mu}T^{VV}_{\mu\sigma;\nu\rho}\right]\right\} \\ &+ 2p_{\sigma}\left\{\left[p^{\mu}T^{VV}_{\nu;\mu\rho}\right] + \left[p^{\mu}T^{VV}_{\mu;\nu\rho}\right]\right\} \\ &+ 2p_{\nu}\left[p^{\mu}T^{VV}_{\sigma;\mu\rho}\right] + p_{\nu}p_{\sigma}\left[p^{\mu}T^{VV}_{\mu\rho}\right] \\ &+ 2p^{2}\left[T^{VV}_{\sigma;\nu\rho}\right] + p_{\sigma}p^{2}\left[T^{VV}_{\nu\rho}\right] \\ &+ (\sigma \longleftrightarrow \rho) \ , \end{aligned} \tag{86}$$

and

$$\begin{aligned} g^{\mu\nu}\left[T^{(V)}_{\mu\nu\rho\sigma}\right] &= 8\left[g^{\mu\nu}T^{VV}_{\mu\sigma;\nu\rho}\right] + 4p^{\sigma}\left[g^{\mu\nu}T^{VV}_{\mu;\nu\rho}\right] \\ &+ 4\left[p^{\mu}T^{VV}_{\sigma;\mu\rho}\right] + 2p_{\sigma}\left[p^{\mu}T^{VV}_{\mu\rho}\right] \\ &+ (\sigma \leftrightarrow \rho) \ . \end{aligned} \tag{87}$$

So, obviously, if all subamplitudes fulfill its RAGFs then $T^{(G)}_{\mu\nu\rho\sigma}$ fulfill its RAGFs as well. So far so good. However, this game becomes more complex when WIs are expected to be preserved too.

A. Ward Identities and Gravitational Anomalies

As we have argued along the work, the preservation of the RAGFs can be considered as a requirement of purely mathematical nature. From a physical point of view, the main question to be considered is about the WIs. Both aspects, however, seems to be coupled. Precisely due to this reason we reported the investigation in the way presented previously. As a summary, we saw that the (regularization independent) conditions for the RAGFs preservation for all subamplitudes are

$$\begin{cases} g^{\mu\nu}\Box^{(2)}_{3;\mu\nu\rho\xi} = g_{\xi\rho}\left[g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}\right] = -\frac{i}{2\pi}g_{\xi\rho} \ , \\ g^{\mu\nu}\Sigma^{(2)}_{4;\mu\nu\sigma\rho\xi\chi} = -\frac{i}{2\pi}\left(g_{\sigma\rho}g_{\xi\chi} + g_{\sigma\xi}g_{\rho\chi} + g_{\sigma\chi}g_{\rho\xi}\right) \ , \end{cases} \tag{88}$$

while the (regularization dependent) conditions for the WIs maintenance, in the two-Lorentz index amplitudes, are

$$\Sigma^{(2)}_{4;\mu\nu\sigma\rho\xi\chi} = \Box^{(2)}_{3;\mu\nu\rho\xi} = \Delta^{(2)}_{2;\mu\nu} = 0 \ . \tag{89}$$

The issue is that these conditions are conflicting ones.

With a few exceptions, all these calculations are usually performed after a regularization prescription is adopted. Within this context, in the most of time, it is not possible to see clearly that, after all calculations, the main difference among two distinct approaches resides on the value attributed to the aforementioned objects. Note that it is not relevant the way we write the scalar divergent objects. Relative to the possible values that can be attributed to the surface terms, it is reasonable to consider three different scenarios, which we will discuss in details in what follows.

In a first scenario one can adopt a prescription where all the surface terms defined above are null tensors, i.e.

$$\Sigma^{(2)}_{4;\mu\nu\sigma\rho\xi\chi} = \Box^{(2)}_{3;\mu\nu\rho\xi} = \Box^{(2)}_{2;\mu\nu\rho\xi} = \Delta^{(2)}_{2;\mu\nu} = \Delta^{(2)}_{1;\mu\nu} = 0 \ , \tag{90}$$

as well as their contractions with $g^{\mu\nu}$ (tensor traces)

$$g^{\mu\nu}\Sigma^{(2)}_{4;\mu\nu\sigma\rho\xi\chi} = g^{\mu\nu}\Box^{(2)}_{3;\mu\nu\rho\xi} = g^{\mu\nu}\Box^{(2)}_{2;\mu\nu\rho\xi} = g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu} = g^{\mu\nu}\Delta^{(2)}_{1;\mu\nu} = 0 \ . \tag{91}$$

These assumptions are consistent, in a trivial way, for example, with the Schouten identity

$$\varepsilon_{\sigma\xi} g^{\xi\chi}\left[\Delta^{(2)}_{2;\chi\rho}\right] - \varepsilon_{\rho\xi} g^{\xi\chi}\left[\Delta^{(2)}_{2;\chi\sigma}\right] = \varepsilon_{\sigma\rho}\left[g^{\xi\chi}\Delta^{(2)}_{2;\xi\chi}\right] \ , \tag{92}$$

the one that was required to verify whether the RAGFs for the pseudoamplitudes are preserved by the calculations made. As a consequence of these assumptions, all the one-point functions vanish identically, at least in the massless limit, which are physical desirable results. On the other side, they are not consistent with the linearity in the integration operation, as we have seen. We can look at properties (90) as a kind of (physical) consistency relations. The authors of the present work have shown, in different contexts involving perturbative calculations [39, 43–45] that the consistency relations are required in order to preserve gauge invariance as well as to eliminate ambiguous terms, as we have shown for the two-Lorentz index amplitudes in the section (IV). In general, the conditions (90) are satisfied by a class of regularizations called gauge preserving regularizations, of which DR is the most popular member.

By assuming the conditions above we can define what we may call the "physical" (sub)amplitudes:

$$\mathcal{T}^{VV}_{\sigma\rho} = \frac{i}{\pi}\left(p_\sigma p_\rho - g_{\sigma\rho}p^2\right)\left[\xi^{(-1)}_2 - \xi^{(-1)}_1\right] \ , \tag{93}$$

$$\mathcal{T}^{VV}_{\mu;\sigma\rho} = -\frac{p_\mu}{2}\left[\mathcal{T}^{VV}_{\sigma\rho}\right] \ , \tag{94}$$

$$
\begin{aligned}
\mathcal{T}^{VV}_{\mu\nu;\sigma\rho} &= \left(g_{\nu\sigma}g_{\mu\rho}+g_{\nu\rho}g_{\mu\sigma}\right)\left[I^{(2)}_{quad}\left(m^2\right)\right] \\
&\quad+\left\{\frac{1}{3}\left(g_{\sigma\rho}g_{\nu\mu}p^2-g_{\sigma\rho}p_\nu p_\mu-g_{\mu\nu}p_\sigma p_\rho\right)\right. \\
&\qquad-\frac{1}{6}\left(g_{\sigma\nu}g_{\rho\mu}p^2-g_{\sigma\nu}p_\rho p_\mu-g_{\rho\mu}p_\sigma p_\nu\right) \\
&\qquad\left.-\frac{1}{6}\left(g_{\rho\nu}g_{\sigma\mu}p^2-g_{\rho\nu}p_\sigma p_\mu-g_{\sigma\mu}p_\rho p_\nu\right)\right\}\left[I^{(2)}_{\log}\left(m^2\right)\right] \\
&\quad+\frac{i}{2\pi}\left\{\frac{p_\sigma p_\rho p_\mu p_\nu}{p^2}+\frac{p^2}{2}\left(g_{\rho\nu}g_{\sigma\mu}+g_{\rho\mu}g_{\sigma\nu}\right)\right. \\
&\qquad\left.-\frac{p_\rho}{2}\left(g_{\sigma\mu}p_\nu+g_{\sigma\nu}p_\mu\right)-\frac{p_\sigma}{2}\left(g_{\rho\nu}p_\mu+g_{\rho\mu}p_\nu\right)\right\}\left[2\xi^{(0)}_2-\xi^{(0)}_1\right] \\
&\quad+\frac{i}{2\pi}\frac{1}{p^2}\left(p_\mu p_\nu-p^2g_{\mu\nu}\right)\left(p_\sigma p_\rho-p^2g_{\sigma\rho}\right)\left[\xi^{(0)}_2-\xi^{(0)}_1\right]+\frac{1}{4}p_\nu p_\mu\left[\mathcal{T}^{VV}_{\sigma\rho}\right]\ ,
\end{aligned}
\tag{95}
$$

$$
\mathcal{T}^{AV}_{\mu\nu} = -\varepsilon_{\mu\alpha}g^{\alpha\beta}\left[\mathcal{T}^{VV}_{\beta\nu}\right]\ , \tag{96}
$$

$$
\mathcal{T}^{AV}_{\mu;\sigma\rho} = -\varepsilon_{\sigma\alpha}g^{\alpha\beta}\left[\mathcal{T}^{VV}_{\mu;\beta\rho}\right]\ , \tag{97}
$$

$$
\mathcal{T}^{AV}_{\mu\nu;\sigma\rho} = -\varepsilon_{\sigma\alpha}g^{\alpha\beta}\left[\mathcal{T}^{VV}_{\mu\nu;\beta\rho}\right]\ , \tag{98}
$$

$$
\begin{aligned}
\mathcal{T}^{AA}_{\sigma\rho} &= \frac{i}{\pi}\left\{\left(p_\sigma p_\rho-g_{\sigma\rho}p^2\right)\left[\xi^{(-1)}_2-\xi^{(-1)}_1\right]-g_{\sigma\rho}m^2\left[\xi^{(-1)}_0\right]\right\}\ , \\
\mathcal{T}^{AA}_{\mu;\sigma\rho} &= -\frac{p_\mu}{2}\left[\mathcal{T}^{AA}_{\sigma\rho}\right]\ ,
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
\mathcal{T}^{AA}_{\mu\nu;\sigma\rho} &= \left(g_{\nu\sigma}g_{\mu\rho}+g_{\nu\rho}g_{\mu\sigma}\right)\left[I^{(2)}_{quad}\left(m^2\right)\right] \\
&\quad+\left\{\frac{1}{3}\left(g_{\sigma\rho}g_{\nu\mu}p^2-g_{\sigma\rho}p_\nu p_\mu-g_{\mu\nu}p_\sigma p_\rho\right)\right. \\
&\qquad-\frac{1}{6}\left(g_{\sigma\nu}g_{\rho\mu}p^2-g_{\sigma\nu}p_\rho p_\mu-g_{\rho\mu}p_\sigma p_\nu\right) \\
&\qquad\left.-\frac{1}{6}\left(g_{\rho\nu}g_{\sigma\mu}p^2-g_{\rho\nu}p_\sigma p_\mu-g_{\sigma\mu}p_\rho p_\nu\right)-2m^2g_{\sigma\rho}g_{\mu\nu}\right\}\left[I^{(2)}_{\log}\left(m^2\right)\right] \\
&\quad+\frac{i}{2\pi}\left\{\frac{p_\sigma p_\rho p_\mu p_\nu}{p^2}+\frac{p^2}{2}\left(g_{\rho\nu}g_{\sigma\mu}+g_{\rho\mu}g_{\sigma\nu}\right)\right. \\
&\qquad\left.-\frac{p_\rho}{2}\left(g_{\sigma\mu}p_\nu+g_{\sigma\nu}p_\mu\right)-\frac{p_\sigma}{2}\left(g_{\rho\nu}p_\mu+g_{\rho\mu}p_\nu\right)\right\}\left[2\xi^{(0)}_2-\xi^{(0)}_1\right] \\
&\quad+\frac{i}{2\pi}\frac{1}{p^2}\left(p_\mu p_\nu-p^2g_{\mu\nu}\right)\left(p_\sigma p_\rho-p^2g_{\sigma\rho}\right)\left[\xi^{(0)}_2-\xi^{(0)}_1\right] \\
&\quad-\frac{i}{2\pi}\frac{m^2}{p^2}g_{\sigma\rho}\left(p_\mu p_\nu-p^2g_{\mu\nu}\right)\left[\xi^{(0)}_0\right]+\frac{1}{4}p_\mu p_\nu\left[\mathcal{T}^{AA}_{\sigma\rho}\right]\ .
\end{aligned}
\tag{100}
$$

Observe that all ambiguous terms were eliminated and all the one-point functions vanished (for massless case). From these redefined subamplitudes we get

$$
\begin{aligned}
p^{\mu}\mathcal{T}^{G}_{\mu\nu\rho\sigma} &= -4\left(p_{\sigma}g_{\rho\nu}+2p_{\nu}g_{\rho\sigma}+p_{\rho}g_{\nu\sigma}\right)\left[I^{(2)}_{quad}\left(m^{2}\right)\right] \\
&\quad + 2m^{2}\Big\{p_{\rho}g_{\nu\sigma}\left[I^{(2)}_{\log}\left(m^{2}\right)\right]+p_{\sigma}g_{\nu\rho}\left[I^{(2)}_{\log}\left(m^{2}\right)\right] \\
&\qquad + g_{\nu\rho}p^{\alpha}g_{\sigma\alpha}\left[I^{(2)}_{\log}\left(m^{2}\right)\right]+g_{\nu\sigma}p^{\alpha}g_{\rho\alpha}\left[I^{(2)}_{\log}\left(m^{2}\right)\right] \\
&\qquad + \frac{1}{64\pi}\left[p_{\rho}\left(p_{\nu}p_{\sigma}-g_{\nu\sigma}p^{2}\right)+p_{\sigma}\left(p_{\nu}p_{\rho}-g_{\nu\rho}p^{2}\right)\pm 2\varepsilon_{\beta\nu}p_{\beta}\left(p_{\sigma}p_{\rho}-g_{\sigma\rho}p^{2}\right)\right]\left[\xi^{(-1)}_{1}-2\xi^{(-1)}_{2}\right]\Big\} \\
&\quad \mp \frac{1}{96\pi}\varepsilon_{\beta\nu}p_{\beta}\left(p_{\sigma}p_{\rho}-g_{\sigma\rho}p^{2}\right)\ ,
\end{aligned}
\tag{101}
$$

or, by taking the massless limit

$$
p^{\mu}\mathcal{T}^{G}_{\mu\nu\rho\sigma}=\mp\frac{1}{96\pi}\varepsilon_{\beta\nu}p^{\beta}\left(p_{\sigma}p_{\rho}-g_{\sigma\rho}p^{2}\right)\ , \tag{102}
$$

which can be recognized as being the well-known Einstein's gravitational anomaly [29].

In the same way, the $g^{\mu\nu}T^{G}_{\mu\nu\rho\sigma}$ contraction gives

$$
\begin{aligned}
g^{\mu\nu}\mathcal{T}^{G}_{\mu\nu\rho\sigma} &= 4g_{\sigma\rho}\left\{3\left[I^{(2)}_{quad}\left(m^{2}\right)\right]-2m^{2}\left[I^{(2)}_{\log}\left(m^{2}\right)\right]\right\} \\
&\quad -\left(\frac{1}{8\pi}\right)m^{2}\left[\left(p_{\sigma}p_{\rho}-p^{2}g_{\rho\sigma}\right)\pm\frac{1}{2}\left(\varepsilon_{\mu\rho}p^{\mu}p_{\sigma}+\varepsilon_{\mu\sigma}p^{\mu}p_{\rho}\right)\right]\left[\xi^{(-1)}_{1}-2\xi^{(-1)}_{2}\right] \\
&\quad +\left(\frac{1}{24\pi}\right)\left[\left(p_{\sigma}p_{\rho}-p^{2}g_{\rho\sigma}\right)\mp\frac{1}{4}\left(\varepsilon_{\rho\lambda}p^{\lambda}p_{\sigma}+\varepsilon_{\sigma\lambda}p^{\lambda}p_{\rho}\right)\right]\ ,
\end{aligned}
\tag{103}
$$

or

$$
g^{\mu\nu}\mathcal{T}^{G}_{\mu\nu\rho\sigma}=\left(\frac{1}{24\pi}\right)\left[\left(p_{\sigma}p_{\rho}-p^{2}g_{\rho\sigma}\right)\mp\frac{1}{4}\left(\varepsilon_{\rho\lambda}p^{\lambda}p_{\sigma}+\varepsilon_{\sigma\lambda}p^{\lambda}p_{\rho}\right)\right]\ , \tag{104}
$$

for a massless fermion. This result is known as Weyl or trace gravitational anomaly. The main point of the preceding calculation is the demonstration that, from our general results shown in the section (IV), one can obtain the usual anomalies terms. The caveat, as should be clear, is that both anomalies are inevitably entangled to a violation of a basic mathematical property, the linearity in the integration operation. This aspect is, in fact, common to all anomaly phenomena in QFT.

A second possible track that one may follows, if a regularization get into the game, is to adopt a procedure where surface terms are taken as null objects, as above, but instead their contractions with $g^{\mu\nu}$

$$
\begin{cases}
g^{\mu\nu}\Box^{(2)}_{3;\mu\nu\rho\xi}=g_{\xi\rho}\left[g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}\right]=-\frac{i}{2\pi}g_{\xi\rho}\ , \\
g^{\mu\sigma}\Sigma^{(2)}_{4;\mu\sigma\nu\rho\xi\chi}=-\frac{i}{2\pi}\left(g_{\nu\rho}g_{\xi\chi}+g_{\nu\xi}g_{\rho\chi}+g_{\nu\chi}g_{\rho\xi}\right)\ ,
\end{cases}
\tag{105}
$$

being, in effect, non-null quantities. This situation may occurs indirectly depending on the step of calculations that a regularization is implemented. For instance, when one takes the surface terms as being zero after the use of the identity (61), this choice can be materialized. Then, within this paradoxical scenario, through a specific route followed in the calculations, it would be possible to get the linearity in the integration operation maintained and so no anomalous terms will survive. It seems that the best of the outcomes is achieved. One could propose a rule for the evaluation of perturbative amplitudes: for such referred amplitudes, when convenient, first use the identity (61) and calculate the quantities $g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}$, $g^{\mu\nu}\Box^{(2)}_{3;\mu\nu\rho\xi}$, $g^{\mu\sigma}\Sigma^{(2)}_{4;\mu\sigma\nu\rho\xi\chi}$ obtaining a non-zero value. After that, take the surface terms as being zero. The desirable results seems to be obtained and, at first sight, one can understand that, with this recipe, the anomalies are eliminated since all WIs can be fulfilled. However, if we use the Schouten identity, at least one of the expressions for the contracted tensors will contain terms like $g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}$. This situation reflects the mathematical impossibility of satisfying all the RAGFs without to use a Schouten identity like (61). So, the situation above, where all the WIs are preserved cannot, in fact, occur. In addition, the Schouten identity is violated since the tensors are null quantities and their traces are not.

A third possibility would be a choice where both, the surface terms as well as their traces are non-null tensors. For instance, if a regularization gives $\Delta^{(2)}_{2;\mu\nu}\Big|_{reg} = -\frac{i}{4\pi}g_{\mu\nu}$, then its trace is given by $g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu} = -\frac{i}{2\pi}$, the same value we found before. These assumptions are consistent with the preservation of the linearity in the integration operation and, consequently, with the uniqueness of the results since the Schouten identity is preserved also. One can say that this attitude represents the mathematical consistency. On the other hand, in this case the one-point functions are nonzero and ambiguous. This is, of course, undesirable just because the physical amplitudes are ambiguous. The WIs, therefore, are not preserved due to ambiguous terms. Of course, this class of regularizations yields a scenario which is not useful to make physical predictions in spite of being mathematically consistent. If we follow this path, some kind of procedure must be adopted, as an additional ingredient to the Feynman rules, in order to eliminate the ambiguous quantities arising.

VI. SUMMARY AND CONCLUSIONS

Across this paper, we have calculated the perturbative gravitational amplitude $T^{(G)}_{\mu\nu\rho\sigma}$. This amplitude was constructed through Feynman rules derived from a two-dimensional interaction Lagrangian where Weyl fermions couple to the gravitational field via the energy-momentum tensor. To organize the intermediate calculations and emphasize key aspects of the analysis, we decomposed the $T^{(G)}_{\mu\nu\rho\sigma}$ amplitude into sets of subamplitudes based on their tensor character. Each subamplitude was analyzed using a novel method designed to handle divergent Feynman integrals without limitations and treat both tensors and pseudotensors equally. Crucially, this method is not a regularization scheme, as no divergent integrals are calculated during intermediate steps. Instead, the undefined content of each amplitude is isolated and expressed as either surface terms or scalar objects devoid of physical parameters. For amplitudes with linear or higher divergence, the surface term coefficients capture all ambiguities arising from the chosen internal loop momenta. Conversely, the finite content is integrated directly and organized into convenient functions represented by Feynman parameter integrals. Section IV explicitly demonstrates these features for the calculated gravitational subamplitudes.

A fundamental question arose after applying the proposed method and obtaining the results: do they satisfy the expected RAGFs? This is crucial because failing to satisfy the RAGFs indicates a breaking of linearity in the integration operation, rendering the results unacceptable. For all subamplitudes to simultaneously satisfy their expected RAGFs, a set of conditions is necessary. These conditions involve finite quantities, interpretable as traces over objects identified as surface terms (105). Our direct calculations revealed that these conditions are universally satisfied, independent of the chosen prescription. This signifies that our procedure, despite dealing with undefined quantities, yields results consistent with the required linearity of integration. As such, our calculations align with the established mathematical requirements for the considered perturbative amplitudes.

The discussion now shifts to the physical interpretation of the results. While ensuring linearity in integration is crucial for mathematical consistency, satisfying WIs is essential for interpreting predictions as consequences of the theory's symmetries. For a universal analysis of perturbative amplitudes, the calculation method should be independent of the specific theory they originate from. Within the gravitational amplitude $T^{(G)}_{\mu\nu\rho\sigma}$, four subamplitudes

$T^{VV}_{\sigma\rho}$, $T^{AV}_{\sigma\rho}$, $T^{VA}_{\sigma\rho}$, and $T^{AA}_{\sigma\rho}$ are constrained to fulfill WIs if considered part of a gauge theory with conserved vector currents (e.g., QED_2). We found that satisfying all WIs for $T^{VV}_{\sigma\rho}$ and $T^{AA}_{\sigma\rho}$ requires $\Delta^{(2)}_{2;\xi\chi} = 0$. This condition also preserves the vector WI for $T^{AV}_{\sigma\rho}$ but violates the axial WI, exhibiting the expected $2D$ anomaly. However, setting $\Delta^{(2)}_{2;\xi\chi} = 0$ assigns a defined value to an undefined object, potentially through a regularization scheme. This directly contradicts the condition $g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu} = -\frac{i}{2\pi}$, necessary for preserving RAGFs involving pseudotensor amplitudes. This contradiction arises if $g^{\mu\nu}\Delta^{(2)}_{2;\mu\nu}$ is interpreted as the trace of $\Delta^{(2)}_{2;\mu\nu}$, implying a null tensor with a non-null trace. Additionally, this pair of values violates the Schouten identity. Even for this relatively simple problem, this scenario presents significant difficulties.

We observed that something similar occurred with the gravitational amplitude. We investigate three distinct scenarios corresponding to different choices for surface terms, recognizing that these choices effectively represent the selection of specific regularization schemes. In the first scenario, we made the assumption that there exists a procedure or regularization scheme capable of assigning a null value to the surface terms and their contractions with the metric. Consequently, all one-point functions vanish, and ambiguous terms connected to internal lines momenta are eliminated. This is because ambiguous combinations of internal lines momenta always act as coefficients of surface terms. Crucially, this approach preserves Schouten identities that involve surface terms, such as,

$$\varepsilon_{\sigma\xi}g^{\xi\chi}\left[\Delta^{(2)}_{2;\chi\rho}\right] - \varepsilon_{\rho\xi}g^{\xi\chi}\left[\Delta^{(2)}_{2;\chi\sigma}\right] = \varepsilon_{\sigma\rho}\left[g^{\xi\chi}\Delta^{(2)}_{2;\xi\chi}\right] \ , \tag{106}$$

ensuring compatibility with the RAGFs of pseudoamplitudes. Interestingly, this scenario aligns with the previously analyzed anomalous $T^{AV}_{\sigma\rho}$ amplitude. Within this context, the customary gravitational anomalies are recovered. From a physical perspective, it appears that these choices are suitable, demonstrating that our procedure can, once again, replicate the conventional results obtained through other techniques. However, from a purely mathematical standpoint, it introduces a contradiction. Presuming the contractions of surface terms to be null implicitly violates the principle of linearity in integration, which should hold true even for divergent integrals. While physically appealing, adopting null surface term contractions leads to a mathematically inconsistent outcome, unveiling a potential drawback associated with this particular choice.

Without questioning consistency, we supposed a second scenario that differs from the previous one by assuming non-null contractions of the surface terms with the metric (105), while retaining the assumption of null surface terms themselves. This choice, validated through simple calculations where the principle of linearity in integration holds, might seemingly allow satisfying all WIs for the $T^{(G)}_{\mu\nu\rho\sigma}$ amplitude. However, this apparent solution presents a fundamental obstacle. The Schouten identity, crucial for maintaining these WIs, becomes violated under this scenario. This violation exposes the inherent incompatibility of this choice with mathematical consistency, rendering the seemingly attainable solution physically implausible.

As the final possibility, we explore a scenario where neither the surface terms nor their contractions with the metric are assumed to be zero. This approach upholds the linearity of the integration operation, guaranteeing its mathematical consistency in this regard. However, this path comes at a cost. The resulting amplitudes exhibit broken symmetry relations and remain ambiguous quantities. This compromise in physical interpretation renders the obtained results unsuitable for predictive purposes.

This exploration of perturbative gravitational anomalies within a simple 2D model unveils some crucial insights regarding the limitations of regularizations. Notably, the results obtained are not unique, merely representing one possibility among many due to the inherent ambiguity introduced by regularization choices throughout the calculation process. Not surprisingly, surface terms, rather than purely divergent terms, play the primary role in this ambiguity. This finding underscores a fundamental challenge: existing regularizations cannot simultaneously achieve both mathematical consistency and physical meaning. It is not possible to find a regularization capable of resolving the involved dilemma for the following reasons:

1) **Setting surface terms and their traces to zero:** While this eliminates ambiguous terms often responsible for breaking symmetries, it does so by violating the linearity of integration. This, in turn, leads to non-unique results, undermining the ability to make genuine predictions.

2) **Retaining surface terms:** Although this preserves unique results and avoids violating the linearity of integration, it also retains the ambiguity and associated symmetry violations inherent in these terms. Consequently, the resulting predictions remain ambiguous and lack clear physical interpretation.

3) **Setting surface terms to zero but not their traces:** This approach appears to

offer a middle ground, but at the cost of violating the Schouten identity. As before, this renders the results non-unique and non-predictive, highlighting the impossibility of finding a "perfect" regularization that satisfies both conditions.

In essence, regularizations introduce ambiguity, resulting in non-unique outcomes. Furthermore, while surface terms hold the key to this ambiguity, no existing regularization can simultaneously deliver both mathematically consistent and physically meaningful results.

While the non-uniqueness and ambiguity caused by regularizations may be a general concern in perturbative QFT calculations, it becomes particularly critical in the context of anomalies. As demonstrated throughout this work, achieving consistent results through regularizations in anomaly calculations proves impossible across various examples. This necessitates exploring alternative strategies beyond traditional Feynman rules to circumvent these undesirable quantities and transform Feynman amplitudes into physically meaningful ones. This transformation, akin to the removal of infinities in the renormalization process without assuming them to be zero, requires a method that does not rely on regularizations.

Finally, it is important to emphasize that investigation was made in a completely regularization-free approach. Due to this it was possible to appreciate some aspects in our analisys which are not possible to do in contexts where regularizations are adopted.

Appendix A: Relations among Green functions

The RAGFs involving the subamplitudes defined in (10), (11), and (12) are presented in this appendix.

1. Vector sector

In the vector sector we have

$$p^{\sigma}\left[T^{VV}_{\mu;\sigma\rho}\right]=\left[T^{V}_{\mu;\rho}\left(k_1\right)\right]-\left[T^{V}_{\mu;\rho}\left(k_2\right)\right]\ , \tag{A1}$$

$$p^{\sigma}\left[T^{VV}_{\mu\nu;\sigma\rho}\right]=\left[T^{V}_{\mu\nu;\rho}\left(k_1\right)\right]-\left[T^{V}_{\mu\nu;\rho}\left(k_2\right)\right]\ , \tag{A2}$$

$$g^{\mu\sigma}\left[T^{VV}_{\mu;\sigma\rho}\right]=\left[T^{V}_{\rho}\left(k_2\right)\right]+m\left[T^{SV}_{\rho}\right]\ , \tag{A3}$$

$$g^{\mu\sigma}\left[T^{VV}_{\mu\nu;\sigma\rho}\right]=\left[T^{V}_{\nu;\rho}\left(k_2\right)\right]+m\left[T^{SV}_{\nu;\rho}\right]\ , \tag{A4}$$

$$
\begin{aligned}
p^{\rho}\left[T^{VV}_{\rho;\mu\nu}\right] = & -\frac{1}{2}g_{\mu\nu}p^{\alpha}\left[T^{V}_{\alpha}\left(k_1\right)+T^{V}_{\alpha}\left(k_2\right)\right] \\
& +\frac{1}{2}p_{\mu}\left[T^{V}_{\nu}\left(k_1\right)-T^{V}_{\nu}\left(k_2\right)\right] \\
& +\frac{1}{2}p_{\nu}\left[T^{V}_{\mu}\left(k_1\right)+T^{V}_{\mu}\left(k_2\right)\right] \\
& +\left[T^{V}_{\mu;\nu}\left(k_1\right)-T^{V}_{\mu;\nu}\left(k_2\right)\right] \\
& -\frac{1}{2}p^{2}\left[T^{VV}_{\mu\nu}\left(k_1,k_2\right)\right] \ , \qquad \text{(A5)}
\end{aligned}
$$

$$
\begin{aligned}
p^{\mu}\left[T^{VV}_{\mu\nu;\sigma\rho}\right] = & -\frac{1}{2}g_{\sigma\rho}p^{\xi}\left[T^{V}_{\nu;\xi}\left(k_1\right)+T^{V}_{\nu;\xi}\left(k_2\right)\right] \\
& +\frac{1}{2}p_{\sigma}\left[T^{V}_{\nu;\rho}\left(k_1\right)-T^{V}_{\nu;\rho}\left(k_2\right)\right] \\
& +\frac{1}{2}p_{\rho}\left[T^{V}_{\nu;\sigma}\left(k_1\right)+T^{V}_{\nu;\sigma}\left(k_2\right)\right] \\
& +\left[T^{V}_{\nu\rho;\sigma}\left(k_1\right)\right]-\left[T^{V}_{\nu\sigma;\rho}\left(k_2\right)\right] \\
& -\frac{1}{2}\left(k_2-k_1\right)^{2}\left[T^{VV}_{\nu;\sigma\rho}\right] \ . \qquad \text{(A6)}
\end{aligned}
$$

Substituting these RAGFs in (7) we get

$$
\begin{aligned}
p^{\mu}\left[T^{(V)}_{\mu\nu\rho\sigma}\right] = & \ 4\left[T^{V}_{\nu\sigma;\rho}\left(k_1\right)\right]-8\left[T^{V}_{\nu\sigma;\rho}\left(k_2\right)\right] \\
& +4\left[T^{V}_{\nu\rho;\sigma}\left(k_1\right)\right]-8\left[T^{V}_{\nu\rho;\sigma}\left(k_2\right)\right]+8\left[T^{V}_{\sigma\rho;\nu}\left(k_1\right)\right] \\
& -2g_{\nu\rho}p^{\xi}\left[T^{V}_{\sigma;\xi}\left(k_1\right)+T^{V}_{\sigma;\xi}\left(k_2\right)\right]-2g_{\nu\sigma}p^{\xi}\left[T^{V}_{\rho;\xi}\left(k_1\right)+T^{V}_{\rho;\xi}\left(k_2\right)\right] \\
& +4p_{\nu}\left[T^{V}_{\sigma;\rho}\left(k_1\right)-T^{V}_{\sigma;\rho}\left(k_2\right)\right]+4p_{\nu}\left[T^{V}_{\rho;\sigma}\left(k_1\right)-T^{V}_{\rho;\sigma}\left(k_2\right)\right] \\
& +2p_{\rho}\left[T^{V}_{\nu;\sigma}\left(k_1\right)-T^{V}_{\nu;\sigma}\left(k_2\right)\right]+2p_{\sigma}\left[T^{V}_{\nu;\rho}\left(k_1\right)-T^{V}_{\nu;\rho}\left(k_2\right)\right] \\
& +4p_{\rho}\left[T^{V}_{\sigma;\nu}\left(k_1\right)\right]+4p_{\sigma}\left[T^{V}_{\rho;\nu}\left(k_1\right)\right] \\
& -g_{\nu\rho}p_{\sigma}p^{\xi}\left[T^{V}_{\xi}\left(k_1\right)+T^{V}_{\xi}\left(k_2\right)\right]-g_{\nu\sigma}p_{\rho}p^{\xi}\left[T^{V}_{\xi}\left(k_1\right)+T^{V}_{\xi}\left(k_2\right)\right] \\
& +2p_{\sigma}p_{\nu}\left[T^{V}_{\rho}\left(k_1\right)-T^{V}_{\rho}\left(k_2\right)\right]+2p_{\rho}p_{\nu}\left[T^{V}_{\sigma}\left(k_1\right)-T^{V}_{\sigma}\left(k_2\right)\right] \\
& +2p_{\sigma}p_{\rho}\left[T^{V}_{\nu}\left(k_1\right)+T^{V}_{\nu}\left(k_2\right)\right] \ . \qquad \text{(A7)}
\end{aligned}
$$

2. Axial-Vector sector

The expected RAGFs for the subamplitudes in the axial-vector sector are

$$
p^{\rho}T^{AV}_{\sigma\rho}=\left[T^{A}_{\sigma}\left(k_1\right)\right]-\left[T^{A}_{\sigma}\left(k_2\right)\right] \ , \qquad \text{(A8)}
$$

$$
p^{\sigma}T^{AV}_{\sigma\rho}=\left[T^{A}_{\rho}\left(k_1\right)\right]-\left[T^{A}_{\rho}\left(k_2\right)\right]+2m\left[T^{PV}_{\rho}\right] \ , \qquad \text{(A9)}
$$

$$p^{\sigma}\left[T^{AV}_{\mu;\sigma\rho}\right]=\left[T^{A}_{\mu;\rho}\left(k_1\right)\right]-\left[T^{A}_{\mu;\rho}\left(k_2\right)\right]+2m\left[T^{PV}_{\mu;\rho}\right]\ , \tag{A10}$$

$$p^{\sigma}\left[T^{AV}_{\mu\nu;\sigma\rho}\right]=\left[T^{A}_{\mu\nu;\rho}\left(k_1\right)\right]-\left[T^{A}_{\mu\nu;\rho}\left(k_2\right)\right]+2m\left[T^{PV}_{\mu\nu;\rho}\right]\ ,$$

$$g^{\mu\sigma}\left[T^{AV}_{\mu;\sigma\rho}\right]=\left[T^{A}_{\rho}\left(k_2\right)\right]-m\left[T^{PV}_{\rho}\right]\ , \tag{A11}$$

$$g^{\mu\sigma}\left[T^{AV}_{\mu\nu;\sigma\rho}\right]=\left[T^{A}_{\nu;\rho}\left(k_2\right)\right]-m\left[T^{PV}_{\nu;\rho}\right]\ , \tag{A12}$$

$$\begin{aligned}p^{\mu}\left[T^{AV}_{\mu;\sigma\rho}\right]&=\frac{1}{2}\varepsilon_{\rho\xi}p^{\xi}\left[T^{V}_{\sigma}\left(k_1\right)+T^{V}_{\sigma}\left(k_2\right)\right]\\&+\frac{1}{2}p_{\rho}\left[T^{A}_{\sigma}\left(k_1\right)-T^{A}_{\sigma}\left(k_2\right)\right]\\&+\left[T^{A}_{\rho;\sigma}\left(k_1\right)\right]-\left[T^{A}_{\rho;\sigma}\left(k_2\right)\right]\\&-\frac{1}{2}p^{2}\left[T^{AV}_{\sigma\rho}\right]\ ,\end{aligned} \tag{A13}$$

$$\begin{aligned}p^{\mu}\left[T^{AV}_{\mu\nu;\sigma\rho}\right]&=-\frac{1}{2}\varepsilon_{\rho\xi}p^{\xi}\left[T^{V}_{\nu;\sigma}\left(k_1\right)+T^{V}_{\nu;\sigma}\left(k_2\right)\right]\\&+\frac{1}{2}p_{\rho}\left[T^{A}_{\nu;\sigma}\left(k_1\right)-T^{A}_{\nu;\sigma}\left(k_2\right)\right]\\&+\left[T^{A}_{\rho\nu;\sigma}\left(k_1\right)\right]-\left[T^{A}_{\rho\nu;\sigma}\left(k_2\right)\right]\\&-\frac{1}{2}p^{2}\left[T^{AV}_{\nu;\sigma\rho}\right]\ .\end{aligned} \tag{A14}$$

Adding them up to the Eq. (8) we get

$$\begin{aligned}p^{\mu}\left[\pm T^{(AV)}_{\mu\nu\rho\sigma}\right]&=4\left[T^{A}_{\nu\sigma;\rho}\left(k_1\right)-T^{A}_{\nu\sigma;\rho}\left(k_2\right)\right]+4\left[T^{A}_{\nu\rho;\sigma}\left(k_1\right)-T^{A}_{\nu\rho;\sigma}\left(k_2\right)\right]\\&+8\left[T^{A}_{\sigma\rho;\nu}\left(k_1\right)-T^{A}_{\sigma\rho;\nu}\left(k_2\right)\right]+4p_{\rho}\left[T^{A}_{\sigma;\nu}\left(k_1\right)-T^{A}_{\sigma;\nu}\left(k_2\right)\right]\\&+2p_{\rho}\left[T^{A}_{\nu;\sigma}\left(k_1\right)-T^{A}_{\nu;\sigma}\left(k_2\right)\right]+4p_{\sigma}\left[T^{A}_{\rho;\nu}\left(k_1\right)-T^{A}_{\rho;\nu}\left(k_2\right)\right]\\&+2p_{\sigma}\left[T^{A}_{\nu;\rho}\left(k_1\right)-T^{A}_{\nu;\rho}\left(k_2\right)\right]+2p_{\nu}\left[T^{A}_{\sigma;\rho}\left(k_1\right)-T^{A}_{\sigma;\rho}\left(k_2\right)\right]\\&+2p_{\nu}\left[T^{A}_{\rho;\sigma}\left(k_1\right)-T^{A}_{\rho;\sigma}\left(k_2\right)\right]+2p_{\sigma}p_{\rho}\left[T^{A}_{\nu}\left(k_1\right)-T^{A}_{\nu}\left(k_2\right)\right]\\&+p_{\nu}p_{\sigma}\left[T^{A}_{\rho}\left(k_1\right)-T^{A}_{\rho}\left(k_2\right)\right]+p_{\nu}p_{\rho}\left[T^{A}_{\sigma}\left(k_1\right)-T^{A}_{\sigma}\left(k_2\right)\right]\\&-2\varepsilon_{\rho\xi}p_{\xi}\left[T^{V}_{\sigma;\nu}\left(k_1\right)+T^{V}_{\sigma;\nu}\left(k_2\right)\right]-2\varepsilon_{\sigma\xi}p_{\xi}\left[T^{V}_{\rho;\nu}\left(k_1\right)+T^{V}_{\rho;\nu}\left(k_2\right)\right]\\&-\varepsilon_{\rho\xi}p_{\sigma}p_{\xi}\left[T^{V}_{\nu}\left(k_1\right)+T^{V}_{\nu}\left(k_2\right)\right]-\varepsilon_{\sigma\xi}p_{\rho}p_{\xi}\left[T^{V}_{\nu}\left(k_1\right)+T^{V}_{\nu}\left(k_2\right)\right]\\&+8m\left[T^{PV}_{\nu\sigma;\rho}+T^{PV}_{\nu\rho;\sigma}\right]+4mp_{\nu}\left[T^{PV}_{\sigma;\rho}+T^{PV}_{\rho;\sigma}\right]+4mp_{\sigma}\left[T^{PV}_{\nu;\rho}\right]\\&+4mp_{\rho}\left[T^{PV}_{\nu;\sigma}\right]+2mp_{\nu}p_{\sigma}\left[T^{PV}_{\rho}\right]+2mp_{\nu}p_{\rho}\left[T^{PV}_{\sigma}\right]\ ,\end{aligned} \tag{A15}$$

$$
\begin{aligned}
p^{\mu}\left[\pm T^{(VA)}_{\mu\nu\rho\sigma}\right] &= 8\left[T^{A}_{\nu\sigma;\rho}(k_1) - T^{A}_{\nu\sigma;\rho}(k_2)\right] + 8\left[T^{A}_{\nu\rho;\sigma}(k_1) - T^{A}_{\nu\rho;\sigma}(k_2)\right] \\
&+ 4p_{\nu}\left[T^{A}_{\sigma;\rho}(k_1) - T^{A}_{\sigma;\rho}(k_2) + T^{A}_{\rho;\sigma}(k_1) - T^{A}_{\rho;\sigma}(k_2)\right] \\
&+ 4p_{\sigma}\left[T^{A}_{\nu;\rho}(k_1) - T^{A}_{\nu;\rho}(k_2)\right] + 4p_{\rho}\left[T^{A}_{\nu;\sigma}(k_1) - T^{A}_{\nu;\sigma}(k_2)\right] \\
&+ 2p_{\sigma}p_{\nu}\left[T^{A}_{\rho}(k_1) - T^{A}_{\rho}(k_2)\right] + 2p_{\rho}p_{\nu}\left[T^{A}_{\sigma}(k_1) - T^{A}_{\sigma}(k_2)\right] \\
&- 2\varepsilon_{\nu\xi}p^{\xi}\left[T^{V}_{\sigma;\rho}(k_1) + T^{V}_{\sigma;\rho}(k_2) + T^{V}_{\rho;\sigma}(k_1) + T^{V}_{\rho;\sigma}(k_2)\right] \\
&- \varepsilon_{\nu\xi}p^{\xi}\left[p_{\sigma}T^{V}_{\rho}(k_1) + p_{\sigma}T^{V}_{\rho}(k_2) + p_{\rho}T^{V}_{\sigma}(k_1) + p_{\rho}T^{V}_{\sigma}(k_2)\right] \ . \qquad \text{(A16)}
\end{aligned}
$$

3. Axial sector

The expected RAGFs for the subamplitudes in the axial sector are

$$p^{\sigma}\left[T^{AA}_{\sigma\rho}\right] = \left[T^{V}_{\rho}(k_1)\right] - \left[T^{V}_{\rho}(k_2)\right] + 2m\left[T^{PA}_{\rho}\right] \ , \qquad \text{(A17)}$$

$$p^{\sigma}\left[T^{AA}_{\mu;\sigma\rho}\right] = \left[T^{V}_{\mu;\rho}(k_1)\right] - \left[T^{V}_{\mu;\rho}(k_2)\right] + 2m\left[T^{PA}_{\mu;\rho}\right] \ , \qquad \text{(A18)}$$

$$p^{\sigma}\left[T^{AA}_{\mu\nu;\sigma\rho}\right] = \left[T^{V}_{\mu\nu;\rho}(k_1)\right] - \left[T^{V}_{\mu\nu;\rho}(k_2)\right] + 2m\left[T^{PA}_{\mu\nu;\rho}\right] \ , \qquad \text{(A19)}$$

$$g^{\mu\sigma}\left[T^{AA}_{\mu;\sigma\rho}\right] = \left[T^{V}_{\rho}(k_2)\right] - m\left[T^{PA}_{\rho}\right] \ , \qquad \text{(A20)}$$

$$g^{\mu\sigma}\left[T^{AA}_{\mu\nu;\sigma\rho}\right] = \left[T^{V}_{\nu;\rho}(k_2)\right] - m\left[T^{PA}_{\nu;\rho}\right] \ , \qquad \text{(A21)}$$

$$
\begin{aligned}
p^{\mu}\left[T^{AA}_{\mu;\sigma\rho}\right] &= -\frac{1}{2}g_{\sigma\rho}p^{\xi}\left[T^{V}_{\xi}(k_1) + T^{V}_{\xi}(k_2)\right] \\
&+ \frac{1}{2}p_{\sigma}\left[T^{V}_{\rho}(k_1) - T^{V}_{\rho}(k_2)\right] \\
&+ \frac{1}{2}p_{\rho}\left[T^{V}_{\sigma}(k_1) + T^{V}_{\sigma}(k_2)\right] \\
&+ \left[T^{V}_{\rho;\sigma}(k_1)\right] - \left[T^{V}_{\sigma;\rho}(k_2)\right] \\
&- mg_{\sigma\rho}\left[T^{S}(k_1) - T^{S}(k_2)\right] \\
&- \frac{1}{2}p^2\left[T^{AA}_{\sigma\rho}\right] \ , \qquad \text{(A22)}
\end{aligned}
$$

$$p^{\mu}\left[T^{AA}_{\mu\nu;\sigma\rho}\right] = -\frac{1}{2}g_{\sigma\rho}p^{\xi}\left[T^{V}_{\nu;\xi}(k_1) + T^{V}_{\nu;\xi}(k_2)\right]$$
$$+\frac{1}{2}p_{\sigma}\left[T^{V}_{\nu;\rho}(k_1) - T^{V}_{\nu;\rho}(k_2)\right]$$
$$+\frac{1}{2}p_{\rho}\left[T^{V}_{\nu;\sigma}(k_1) + T^{V}_{\nu;\sigma}(k_2)\right]$$
$$+\left[T^{V}_{\nu\rho;\sigma}(k_1)\right] - \left[T^{V}_{\nu\sigma;\rho}(k_2)\right]$$
$$-mg_{\sigma\rho}\left[T^{S}_{\nu}(k_1) - T^{S}_{\nu}(k_2)\right]$$
$$-\frac{1}{2}p^2\left[T^{AA}_{\nu;\sigma\rho}\right] \ . \qquad (A23)$$

Replacing them into (9) gives

$$p^{\mu}\left[T^{(A)}_{\mu\nu\rho\sigma}\right] = 4\left[T^{V}_{\nu\sigma;\rho}(k_1)\right] - 8\left[T^{V}_{\nu\sigma;\rho}(k_2)\right] + 4\left[T^{V}_{\nu\rho;\sigma}(k_1)\right] - 8\left[T^{V}_{\nu\rho;\sigma}(k_2)\right]$$
$$+8\left[T^{V}_{\sigma\rho;\nu}(k_1)\right] - 2g_{\nu\rho}p_{\xi}\left[T^{V}_{\sigma;\xi}(k_1) + T^{V}_{\sigma;\xi}(k_2)\right]$$
$$-2g_{\nu\sigma}p_{\xi}\left[T^{V}_{\rho;\xi}(k_1) + T^{V}_{\rho;\xi}(k_2)\right]$$
$$+4p_{\nu}\left[T^{V}_{\sigma;\rho}(k_1) - T^{V}_{\sigma;\rho}(k_2) + T^{V}_{\rho;\sigma}(k_1) - T^{V}_{\rho;\sigma}(k_2)\right]$$
$$+2p_{\rho}\left[T^{V}_{\sigma;\nu}(k_1) + T^{V}_{\sigma;\nu}(k_2)\right] + 2p_{\rho}\left[T^{V}_{\sigma;\nu}(k_1) + T^{V}_{\nu;\sigma}(k_1)\right]$$
$$+2p_{\sigma}\left[T^{V}_{\rho;\nu}(k_1) + T^{V}_{\rho;\nu}(k_2)\right] + 2p_{\sigma}\left[T^{V}_{\rho;\nu}(k_1) + T^{V}_{\nu;\rho}(k_1)\right]$$
$$-4p_{\sigma}\left[T^{V}_{\nu;\rho}(k_2)\right] - 4p_{\rho}\left[T^{V}_{\nu;\sigma}(k_2)\right]$$
$$-\left(g_{\nu\rho}p_{\sigma} + g_{\nu\sigma}p_{\rho}\right)p^{\xi}\left[T^{V}_{\xi}(k_1) + T^{V}_{\xi}(k_2)\right]$$
$$+2p_{\nu}p_{\sigma}\left[T^{V}_{\rho}(k_1) - T^{V}_{\rho}(k_2)\right] + 2p_{\nu}p_{\rho}\left[T^{V}_{\sigma}(k_1) - T^{V}_{\sigma}(k_2)\right]$$
$$+2p_{\sigma}p_{\rho}\left[T^{V}_{\nu}(k_1) + T^{V}_{\nu}(k_2)\right]$$
$$-4mg_{\nu\rho}\left[T^{S}_{\sigma}(k_1) - T^{S}_{\sigma}(k_2)\right] - 2mg_{\nu\rho}p_{\sigma}\left[T^{S}(k_1) - T^{S}(k_2)\right]$$
$$-4mg_{\nu\sigma}\left[T^{S}_{\rho}(k_1) - T^{S}_{\rho}(k_2)\right] - 2mg_{\nu\sigma}p_{\rho}\left[T^{S}(k_1) - T^{S}(k_2)\right]$$
$$+8m\left[T^{PA}_{\nu\sigma;\rho} + T^{PA}_{\nu\rho;\sigma}\right] + 4mp_{\nu}\left[T^{PA}_{\sigma;\rho} + T^{PA}_{\rho;\sigma}\right] + 4mp_{\sigma}\left[T^{PA}_{\nu;\rho}\right]$$
$$+4mp_{\rho}\left[T^{PA}_{\nu;\sigma}\right] + 2mp_{\nu}p_{\sigma}\left[T^{PA}_{\rho}\right] + 2mp_{\nu}p_{\rho}\left[T^{PA}_{\sigma}\right] \ . \qquad (A24)$$

Appendix B: Integrals Results

In order to perform the required calculations to obtain the subamplitudes shown above, it is enough to use the integrals results which we list in the following. They are

$$I_2 = \frac{i}{(4\pi)}\left[\xi_0^{(-1)}\left(p^2, m^2\right)\right] \ , \qquad (B1)$$

$$I_{2\mu}=-\frac{i}{(4\pi)}p_{\mu}\left[\xi_{1}^{(-1)}\left(p^{2},m^{2}\right)\right]+\left(-k_{1}\right)_{\mu}\left[I_{2}\right] \ , \tag{B2}$$

$$\begin{aligned}
I_{2\mu\nu}=&\frac{1}{2}\left[\Delta_{2;\mu\nu}^{(2)}\left(\lambda^{2}\right)\right]+\frac{1}{2}g_{\mu\nu}\left[I_{\log}^{(2)}\left(\lambda^{2}\right)\right]\\
&-\frac{i}{(4\pi)}\frac{1}{2}g_{\mu\nu}\left[\xi_{0}^{(0)}\left(p^{2},m^{2};\lambda^{2}\right)\right]\\
&+\frac{i}{(4\pi)}p_{\mu}p_{\nu}\left[\xi_{2}^{(-1)}\left(p^{2},m^{2}\right)\right]\\
&+\left(-k_{1}\right)_{\mu}\left[I_{2\nu}\right]+\left(-k_{1}\right)_{\nu}\left[I_{2\mu}\right]\\
&-\left(k_{1}\right)_{\nu}\left(k_{1}\right)_{\mu}\left[I_{2}\right] \ ,
\end{aligned} \tag{B3}$$

$$\begin{aligned}
I_{2\mu\nu\lambda}=&-\frac{1}{4}\left(k_{2}+k_{1}\right)^{\xi}\left[\square_{3;\xi\mu\nu\lambda}^{(2)}\left(\lambda^{2}\right)\right]\\
&-\frac{1}{8}\left(k_{2}+k_{1}\right)^{\xi}\left\{g_{\mu\nu}\left[\Delta_{2;\lambda\xi}^{(2)}\left(\lambda^{2}\right)\right]+g_{\mu\lambda}\left[\Delta_{2;\nu\xi}^{(2)}\left(\lambda^{2}\right)\right]+g_{\nu\lambda}\left[\Delta_{2;\mu\xi}^{(2)}\left(\lambda^{2}\right)\right]\right\}\\
&-\frac{1}{8}\left(k_{2}+k_{1}\right)_{\mu}\left[\Delta_{2;\nu\lambda}^{(2)}\left(\lambda^{2}\right)\right]-\frac{1}{8}\left(k_{2}+k_{1}\right)_{\nu}\left[\Delta_{2;\mu\lambda}^{(2)}\left(\lambda^{2}\right)\right]-\frac{1}{8}\left(k_{2}+k_{1}\right)_{\lambda}\left[\Delta_{2;\mu\nu}^{(2)}\left(\lambda^{2}\right)\right]\\
&+\frac{1}{2}\left(k_{1}\right)_{\mu}\left[\Delta_{2;\nu\lambda}^{(2)}\left(\lambda^{2}\right)\right]+\frac{1}{2}\left(k_{1}\right)_{\nu}\left[\Delta_{2;\mu\lambda}^{(2)}\left(\lambda^{2}\right)\right]+\frac{1}{2}\left(k_{1}\right)_{\lambda}\left[\Delta_{2;\mu\nu}^{(2)}\left(\lambda^{2}\right)\right]\\
&-\frac{1}{4}\left[g_{\mu\nu}\left(k_{2}-k_{1}\right)_{\lambda}+g_{\mu\lambda}\left(k_{2}-k_{1}\right)_{\nu}+g_{\nu\lambda}\left(k_{2}-k_{1}\right)_{\mu}\right]\left[I_{\log}^{(2)}\left(\lambda^{2}\right)\right]\\
&+\frac{1}{2}g_{\mu\nu}p_{\lambda}\left[\xi_{1}^{(0)}\left(p^{2};m^{2};\lambda^{2}\right)\right]+\frac{1}{2}g_{\mu\lambda}p_{\nu}\left[\xi_{1}^{(0)}\left(p^{2};m^{2};\lambda^{2}\right)\right]\\
&+\frac{1}{2}g_{\nu\lambda}p_{\mu}\left[\xi_{1}^{(0)}\left(p^{2};m^{2};\lambda^{2}\right)\right]-p_{\mu}p_{\nu}p_{\lambda}\left[\xi_{3}^{(-1)}\left(p^{2};m^{2}\right)\right]\\
&+\left(-k_{1}\right)_{\mu}\left[I_{2\nu\lambda}\right]+\left(-k_{1}\right)_{\nu}\left[I_{2\mu\lambda}\right]+\left(-k_{1}\right)_{\lambda}\left[I_{2\mu\nu}\right]\\
&-\left(-k_{1}\right)_{\mu}\left(-k_{1}\right)_{\nu}\left[I_{2\lambda}\right]-\left(-k_{1}\right)_{\mu}\left(-k_{1}\right)_{\lambda}\left[I_{2\nu}\right]-\left(-k_{1}\right)_{\nu}\left(-k_{1}\right)_{\lambda}\left[I_{2\mu}\right]\\
&+\left(-k_{1}\right)_{\mu}\left(-k_{1}\right)_{\nu}\left(-k_{1}\right)_{\lambda}\left[I_{2}\right] \ .
\end{aligned} \tag{B4}$$

Appendix C: One-point Functions

The one-point functions which are used in this work are defined by

$$T_{\sigma}^{i}\left(k_{1}\right)=\int\frac{d^{2}k}{(2\pi)^{2}}Tr\left\{\left[\Gamma_{i}\right]_{\sigma}\frac{1}{\not k+\not k_{1}-m}\right\} \ , \tag{C1}$$

$$T_{\mu;\sigma}^{i}\left(k_{1}\right)=\int\frac{d^{2}k}{(2\pi)^{2}}\left(k+k_{1}\right)_{\mu}Tr\left\{\left[\Gamma_{i}\right]_{\sigma}\frac{1}{\not k+\not k_{1}-m}\right\} \ , \tag{C2}$$

$$T^{i}_{\mu\nu;\sigma}\left(k_1\right)=\int\frac{d^2k}{\left(2\pi\right)^2}\left(k+k_1\right)_\mu\left(k+k_1\right)_\nu Tr\left\{\left[\Gamma_i\right]_\sigma\frac{1}{\not{k}+\not{k}_1-m}\right\}\ , \tag{C3}$$

where Γ_i are vertex operators belonging to the set $\Gamma_i=\{\Gamma_S,\Gamma_P,\Gamma_V,\Gamma_A\}=\{1,\gamma_3,\gamma_\alpha,\gamma_\alpha\gamma_3\}$. By using the integrals shown above one obtain the following results,

$$T^{S}\left(k_1\right)=2m\left[I^{(2)}_{\log}\left(m^2\right)\right]\ , \tag{C4}$$

$$T^{S}_{\nu}\left(k_2\right)=2m\left\{-k_2^{\xi}\left[\Delta^{(2)}_{2;\nu\xi}\left(m^2\right)\right]-p_\nu\left[I^{(2)}_{\log}\left(m^2\right)\right]\right\}\ , \tag{C5}$$

$$T^{A}_{\sigma}=-\varepsilon_{\sigma\xi}g^{\xi\chi}T^{V}_{\chi}\ , \tag{C6}$$

$$T^{A}_{\mu;\sigma}=-\varepsilon_{\sigma\xi}g^{\xi\chi}T^{V}_{\mu;\chi}\ , \tag{C7}$$

$$T^{A}_{\mu\nu;\sigma}=-\varepsilon_{\sigma\xi}g^{\xi\chi}T^{V}_{\mu\nu;\chi}\ , \tag{C8}$$

$$T^{V}_{\mu}\left(k_i\right)=-2k_i^{\xi}\left[\Delta^{(2)}_{2;\mu\xi}\right]\ , \tag{C9}$$

$$\begin{aligned}T^{V}_{\nu;\mu}\left(k_2\right)=&\ g_{\mu\nu}\left[I^{(2)}_{quad}\left(m^2\right)\right]+\left[\Delta^{(2)}_{1;\mu\nu}\left(m^2\right)\right]\\&+k_2^{\xi}k_2^{\chi}\left\{\left[\Delta^{(2)}_{3;\mu\nu\xi\chi}\left(m^2\right)\right]+\frac{1}{2}g_{\mu\nu}\left[\Delta^{(2)}_{2;\xi\chi}\left(m^2\right)\right]\right\}\\&-\frac{1}{2}k_2^2\left[\Delta^{(2)}_{2;\mu\nu}\left(m^2\right)\right]-k_{2\mu}k_2^{\xi}\left[\Delta^{(2)}_{2;\nu\xi}\left(m^2\right)\right]\\&+\left(k_{2\nu}k_2^{\xi}-2k_{1\nu}k_2^{\xi}\right)\left[\Delta^{(2)}_{2;\mu\xi}\left(m^2\right)\right]\ ,\end{aligned} \tag{C10}$$

$$\begin{aligned}T^{V}_{\mu\nu;\sigma}\left(k_2\right)=&\left[\left(k_1-k_2\right)_\mu g_{\nu\sigma}+\left(k_1-k_2\right)_\nu g_{\mu\sigma}\right]\left[I^{(2)}_{quad}\left(m^2\right)\right]-\frac{1}{3}k_2^{\xi}k_2^{\chi}k_2^{\omega}\left[\Sigma^{(2)}_{4;\mu\nu\sigma\omega\xi\chi}\right]\\&-k_2^{\xi}\left[\Box^{(2)}_{2;\mu\nu\sigma\xi}\left(m^2\right)\right]-\left(k_2-k_1\right)_\mu k_2^{\xi}k_2^{\chi}\left[\Box^{(2)}_{3;\nu\sigma\xi\chi}\left(m^2\right)\right]+k_2^2k_2^{\xi}\left[\Box^{(2)}_{3;\mu\nu\sigma\xi}\left(m^2\right)\right]\\&+k_2^{\xi}k_2^{\chi}\left\{k_{2\sigma}\left[\Box^{(2)}_{3;\mu\nu\xi\chi}\left(m^2\right)\right]+k_{1\nu}\left[\Box^{(2)}_{3;\mu\sigma\xi\chi}\left(m^2\right)\right]\right.\\&\qquad\left.-\frac{1}{3}g_{\mu\nu}k_2^{\omega}\left[\Box^{(2)}_{3;\sigma\omega\chi\xi}\left(m^2\right)\right]-\frac{1}{3}g_{\mu\sigma}k_2^{\omega}\left[\Box^{(2)}_{3;\nu\omega\xi\chi}\left(m^2\right)\right]\right\}\\&+k_{2\sigma}\left[\Delta^{(2)}_{1;\mu\nu}\left(m^2\right)\right]+k_{1\nu}\left[\Delta^{(2)}_{1;\mu\sigma}\left(m^2\right)\right]-\left(k_2-k_1\right)_\mu\left[\Delta^{(2)}_{1;\nu\sigma}\left(m^2\right)\right]\\&-k_2^{\xi}\left\{g_{\mu\nu}\left[\Delta^{(2)}_{1;\sigma\xi}\left(m^2\right)\right]+g_{\mu\sigma}\left[\Delta^{(2)}_{1;\nu\xi}\left(m^2\right)\right]\right\}\\&-k_2^2k_{2\sigma}\left[\Delta^{(2)}_{2;\mu\nu}\left(m^2\right)\right]-k_2^2k_{1\mu}\left[\Delta^{(2)}_{2;\nu\sigma}\left(m^2\right)\right]+k_2^2\left(k_2-k_1\right)_\nu\left[\Delta^{(2)}_{2;\mu\sigma}\left(m^2\right)\right]\\&-2k_{1\mu}k_{2\sigma}k_2^{\xi}\left[\Delta^{(2)}_{2;\nu\xi}\left(m^2\right)\right]+\left[2\left(k_2-k_1\right)_\nu k_{2\sigma}+k_2^2g_{\nu\sigma}\right]k_2^{\xi}\left[\Delta^{(2)}_{2;\mu\xi}\left(m^2\right)\right]\\&+\left[k_2^2g_{\mu\nu}-2\left(k_2-k_1\right)_\mu\left(k_2-k_1\right)_\nu\right]k_2^{\xi}\left[\Delta^{(2)}_{2;\sigma\xi}\left(m^2\right)\right]\\&-\left[g_{\mu\sigma}\left(k_2-k_1\right)_\nu+g_{\nu\sigma}\left(k_2-k_1\right)_\mu\right]k_2^{\xi}k_2^{\chi}\left[\Delta^{(2)}_{2;\xi\chi}\left(m^2\right)\right]\ .\end{aligned} \tag{C11}$$

Appendix D: Two-point functions

Here we show some results for the two-point functions not shown explicitly in the main text. They are:

$$
\begin{aligned}
S^{VV}_{\mu\nu\sigma\rho} &= \left[\Box^{(2)}_{2;\mu\nu\sigma\rho}\left(m^2\right)\right] + \frac{1}{12}\left(3P^\xi P^\chi + p^\xi p^\chi\right)\left[\Sigma^{(2)}_{4;\mu\nu\sigma\rho\xi\chi}\left(m^2\right)\right] - \frac{1}{4}\left(P^2 + p^2\right)\left[\Box^{(2)}_{3;\mu\nu\sigma\rho}\left(m^2\right)\right] \\
&- \frac{1}{2}\left(P_\nu - p_\nu\right)P^\xi\left[\Box^{(2)}_{3;\mu\sigma\rho\xi}\left(m^2\right)\right] - \frac{1}{2}P_\sigma P^\xi\left[\Box^{(2)}_{3;\mu\nu\rho\xi}\left(m^2\right)\right] - \frac{1}{2}P_\rho P^\xi\left[\Box^{(2)}_{3;\mu\nu\sigma\xi}\left(m^2\right)\right] \\
&+ \frac{1}{12}\left(P_\mu P^\xi + p_\mu p^\xi - p_\mu P^\xi + P_\mu p^\xi\right)\left[\Box^{(2)}_{3;\nu\sigma\rho\chi}\left(m^2\right)\right] - \frac{1}{4}g_{\sigma\rho}\left(P^\xi P^\chi + p^\xi p^\chi\right)\left[\Box^{(2)}_{3;\mu\nu\xi\chi}\left(m^2\right)\right] \\
&+ \frac{1}{12}\left(3P^\xi P^\chi + p^\xi p^\chi\right)\left\{g_{\mu\nu}\left[\Box^{(2)}_{3;\sigma\rho\chi\xi}\left(m^2\right)\right] + g_{\mu\sigma}\left[\Box^{(2)}_{3;\nu\rho\xi\chi}\left(m^2\right)\right] + g_{\mu\rho}\left[\Box^{(2)}_{3;\nu\sigma\xi\chi}\left(m^2\right)\right]\right\} \\
&- g_{\sigma\rho}\left[\Delta^{(2)}_{1;\mu\nu}\left(m^2\right)\right] + g_{\mu\nu}\left[\Delta^{(2)}_{1;\sigma\rho}\left(m^2\right)\right] + g_{\mu\sigma}\left[\Delta^{(2)}_{1;\nu\rho}\left(m^2\right)\right] + g_{\mu\rho}\left[\Delta^{(2)}_{1;\nu\sigma}\left(m^2\right)\right] \\
&+ \left[\frac{1}{6}p_\mu p_\nu - \frac{1}{4}g_{\mu\nu}\left(P^2 + p^2\right)\right]\left[\Delta^{(2)}_{2;\sigma\rho}\left(m^2\right)\right] + \frac{1}{2}g_{\sigma\rho}\left(P_\nu - p_\nu\right)P^\xi\left[\Delta^{(2)}_{2;\mu\xi}\left(m^2\right)\right] \\
&+ \frac{1}{2}\left[g_{\sigma\rho}\left(P^2 + 3p^2\right) + P_\sigma P_\rho - p_\sigma p_\rho\right]\left[\Delta^{(2)}_{2;\mu\nu}\left(m^2\right)\right] \\
&- \frac{1}{4}g_{\mu\sigma}\left(P^2 + p^2\right)\left[\Delta^{(2)}_{2;\nu\rho}\left(m^2\right)\right] - \frac{1}{4}g_{\mu\rho}\left(P^2 + p^2\right)\left[\Delta^{(2)}_{2;\nu\sigma}\left(m^2\right)\right] \\
&+ \frac{1}{2}\left(P_\nu - p_\nu\right)P_\sigma\left[\Delta^{(2)}_{2;\mu\rho}\left(m^2\right)\right] + \frac{1}{2}\left(P_\nu - p_\nu\right)P_\rho\left[\Delta^{(2)}_{2;\mu\sigma}\left(m^2\right)\right] \\
&- \frac{1}{2}\left[g_{\mu\sigma}P_\rho P^\xi + g_{\mu\rho}P_\sigma P^\xi - g_{\sigma\rho}P_\mu P^\xi + g_{\sigma\rho}\left(P^\xi P_\mu + p^\xi p_\mu\right)\right]\left[\Delta^{(2)}_{2;\nu\xi}\left(m^2\right)\right] \\
&- \frac{1}{2}\left[g_{\mu\nu}P_\rho P^\xi - g_{\mu\rho}\left(-\frac{1}{2}P^\xi P_\nu + p_\nu P^\xi + \frac{1}{6}p^\xi p_\nu\right) - \frac{1}{3}g_{\nu\rho}p^\xi p_\mu\right]\left[\Delta^{(2)}_{2;\sigma\xi}\left(m^2\right)\right] \\
&+ \frac{1}{2}\left[\frac{1}{3}g_{\nu\sigma}p^\xi p_\mu - \frac{1}{6}g_{\mu\nu}\left(3P^\xi P_\sigma - p^\xi p_\sigma\right) + g_{\mu\sigma}\left(\frac{1}{3}p^\xi p_\nu + p_\nu P^\xi\right)\right]\left[\Delta^{(2)}_{2;\rho\xi}\left(m^2\right)\right] \\
&- \frac{1}{6}\left[g_{\mu\nu}g_{\sigma\rho}p^\xi p^\chi - \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right)\left(3P^\xi P^\chi + p^\xi p^\chi\right)\right]\left[\Delta^{(2)}_{2;\chi\xi}\left(m^2\right)\right] \ ,
\end{aligned} \tag{D1}
$$

$$
T^{SV}_{\sigma} = 0 \ , \tag{D2}
$$

$$
\begin{aligned}
T^{SV}_{\nu;\sigma} &= 2m\left\{\left[\Delta^{(2)}_{2;\nu\sigma}\left(m^2\right)\right] + g_{\nu\sigma}\left[I^{(2)}_{\log}\left(m^2\right)\right]\right\} \\
&- 2m\left\{g_{\nu\sigma}\left[\xi^{(0)}_0\left(p^2, m^2; m^2\right)\right] - p_\nu p_\sigma\left[2\xi^{(-1)}_2\left(p^2, m^2\right) - \xi^{(-1)}_1\left(p^2, m^2\right)\right]\right\} \ ,
\end{aligned} \tag{D3}
$$

$$
T^{PV}_{\sigma} = \frac{i}{2\pi}m\varepsilon_{\sigma\xi}p^\xi\left[\xi^{(-1)}_0\left(p^2, m^2\right)\right] \ , \tag{D4}
$$

$$
T^{PA}_{\sigma} = -2mp_\sigma\left[\xi^{(-1)}_0\left(p^2, m^2\right)\right] \ , \tag{D5}
$$

$$T_{\nu;\sigma}^{PV} = -\frac{1}{2} p_\nu \left[T_\sigma^{PV} \right] \ , \tag{D6}$$

$$T_{\mu;\sigma}^{PA} = -\frac{1}{2} p_\mu \left[T_\sigma^{PA} \right] \ , \tag{D7}$$

$$\begin{aligned} T_{\mu\nu;\sigma}^{PV} &= m \varepsilon_{\sigma\xi} p^\xi \left\{ \left[\Delta_{2;\mu\nu}^{(2)} \left(m^2 \right) \right] + g_{\mu\nu} \left[I_{\log}^{(2)} \left(m^2 \right) \right] \right\} \\ &+ m \varepsilon_{\sigma\xi} p^\xi \left(p_\mu p_\nu - g_{\mu\nu} p^2 \right) \left[2\xi_2^{(-1)} \left(p^2, m^2 \right) - \xi_1^{(-1)} \left(p^2, m^2 \right) \right] \\ &+ m \varepsilon_{\sigma\xi} p^\xi p_\mu p_\nu \left[\xi_1^{(-1)} \left(p^2, m^2 \right) \right] \ , \end{aligned} \tag{D8}$$

$$\begin{aligned} T_{\mu\nu;\sigma}^{PA} &= -m p_\sigma \left\{ \left[\Delta_{2;\mu\nu}^{(2)} \left(m^2 \right) \right] + g_{\mu\nu} \left[I_{\log}^{(2)} \left(m^2 \right) \right] \right\} \\ &+ m p_\sigma \left\{ \left(p_\mu p_\nu - g_{\mu\nu} p^2 \right) \left[\xi_1^{(-1)} \left(p^2, m^2 \right) - 2\xi_2^{(-1)} \left(p^2, m^2 \right) \right] \right. \\ &\qquad \left. - p_\mu p_\nu \left[\xi_1^{(-1)} \left(p^2, m^2 \right) \right] \right\} \ . \end{aligned} \tag{D9}$$

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