

LIFTING MAPS BETWEEN GRAPHS TO EMBEDDINGS

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ABSTRACT. In this paper, we study conditions for the existence of an embedding $\tilde{f}: P \rightarrow Q \times \mathbb{R}$ such that $f = \text{pr}_Q \circ \tilde{f}$, where $f: P \rightarrow Q$ is a piecewise linear map between polyhedra. Our focus is on non-degenerate maps between graphs, where non-degeneracy means that the preimages of points are finite sets.

We introduce combinatorial techniques and establish necessary and sufficient conditions for the general case. Using these results, we demonstrate that the problem of the existence of a lifting reduces to testing the satisfiability of a 3-CNF formula. Additionally, we construct a counterexample to a result by V. Poénaru on lifting of smooth immersions to embeddings.

Furthermore, by establishing connections between the stated problem and the approximability by embeddings, we deduce that, in the case of generic maps from a tree to a segment, a weaker condition becomes sufficient for the existence of a lifting.

1. INTRODUCTION

1.1. Background. In this paper, we consider the following problem: given compact one-dimensional polyhedra P and Q , and a piecewise linear map $f: P \rightarrow Q$, what conditions must be satisfied for the existence of a piecewise linear embedding $\tilde{f}: P \rightarrow Q \times \mathbb{R}$ such that $f = \text{pr}_Q \circ \tilde{f}$? If such an embedding exists, we refer to it as a **(codimension one) lifting of f to an embedding**, and we say that **f lifts to an embedding**. This problem can also be formulated in other contexts, such as for smooth maps between manifolds or continuous maps between topological spaces. Additionally, it is possible to consider liftings of codimension k , namely embeddings $\tilde{f}: P \rightarrow Q \times \mathbb{R}^k$ such that $f = \text{pr}_Q \circ \tilde{f}$. In this case, maps that admit such lifting are sometimes referred to as k -prems (short for k -projected embedding), see [28, 1, 15].

In this paper, our focus is on non-degenerate maps. A piecewise linear map $f: P \rightarrow Q$ is called **non-degenerate** if the set $f^{-1}(q)$ is finite for every point $q \in Q$. Additionally, we call a simplicial map $f: K \rightarrow L$ **non-degenerate** if the corresponding piecewise linear map $|f|: |K| \rightarrow |L|$ is non-degenerate. It is easy to see that a simplicial map $f: K \rightarrow L$ between finite simplicial complexes is non-degenerate if and only if it is injective on each simplex; that is, a k -simplex $A \in K$ maps to a k -simplex $f(A) \in L$.

In the one-dimensional case, non-degenerate simplicial maps are synonymous with graph homomorphisms. This establishes an interesting connection between the problem we are addressing in this paper and certain problems in graph theory. In fact, a graph homomorphism $f: G \rightarrow H$ is known as an H -colouring of the graph G . Each lifting

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of the map f to an embedding introduces orders on sets of vertices of the same colour, which, in turn, determine the orders on the edges. These constructions, which involve colourings along with orders, are actively studied in graph theory, see e.g. [7]. For example, so-called track layouts can be viewed as liftings of graph homomorphisms to complete graphs.¹

It should be highlighted that the one-dimensional piecewise linear version of the lifting to an embedding problem plays a special role among other versions of this problem. For instance, the problem of liftability to an embedding of smooth immersions between manifolds can be reduced to the one-dimensional piecewise linear version of the problem [5, 21]. Also, as we will show below in Theorem 3, the multidimensional piecewise linear version of the problem reduces to the one-dimensional case.

1.2. Structure. Let us now outline the general structure of the paper.

The second section introduces several combinatorial concepts that are utilized throughout the rest of the paper. In particular, we prove the following necessary conditions for the existence of a lifting to an embedding:

Theorem 1. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map between finite simplicial complexes such that $|f|: |K| \rightarrow |L|$ lifts to an embedding. Then the following holds:*

- (1) *all the covering maps $p_n: |K_f^{(n)}| \rightarrow |\tilde{K}_f^{(n)}|$, $n > 1$ are trivial;*
- (2) *there are no n -obstructors for f for any $n > 1$.*

Here $K_f^{(n)}$ and $\tilde{K}_f^{(n)}$ denote simplicial models for the ordered and unordered configuration spaces of n points with the same image under f , and an n -obstructor is a path in $K_f^{(n)}$ that realises a cyclic permutation of points; the reader may refer to the next section for the formal definitions.

Further, we prove two necessary and sufficient conditions for the existence of a lifting.

Theorem 2. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map between finite simplicial complexes. Then the piecewise linear map $|f|: |K| \rightarrow |L|$ lifts to an embedding if and only if there is a collection of mutually compatible linear orders on the sets $f^{-1}(v)$, $v \in V(L)$.*

Furthermore, there is a bijection between such collections of linear orders and the isotopy classes of liftings.

Informally, the compatibility here means that the orders on the vertices induce orders on the simplices; again, we postpone the formal definition until the second section.

Theorem 4. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map between finite simplicial complexes. Assuming that the covering map $p_2: |K_f^{(2)}| \rightarrow |\tilde{K}_f^{(2)}|$ is trivial, there is a 3-CNF formula Γ_f ,² such that the piecewise linear map $|f|: |K| \rightarrow |L|$ lifts to an embedding*

¹It is worth noting that the author has not come across research on track layouts for other types of colouring.

²A boolean formula Γ is said to be in 3-CNF if it can be expressed as $\Gamma = \bigwedge_i (\alpha_i \vee \beta_i \vee \gamma_i)$, where each of α_i , β_i , and γ_i is either a variable or its negation.

if and only if Γ_f is satisfiable. Furthermore, there is a bijection between the assignments that satisfy Γ_f and the isotopy classes of liftings.

We will explicitly describe the formula Γ_f just before we state [Theorem 4](#) in the second section.

In the third section, we explore connections between the problem of lifting of maps between graphs and the problem of lifting of smooth immersions to embeddings. In particular we show that the assumptions in a result stated by V. Poénaru in [\[21\]](#) need to be strengthened by including the condition of the satisfiability of Γ_f . The main result in this section demonstrates that without this stronger assumption, the main result on the existence of liftings of immersions stated in [\[21\]](#) cannot hold:

Theorem 6. *Each 3-CNF formula Γ of a specific form, as described in the statement of [Theorem 6](#) on page 18, can be realised as Γ_f of a map $f: G \rightarrow H$ between graphs. Moreover, there exists a generic immersion $g: S \looparrowright B$ of a surface S with boundary into a handlebody B , such that H is the graph of multiple points of g , and f is the restriction of g on $f^{-1}(H)$.*

In the last section, we establish a connection between the problem of the existence of liftings of maps between graphs and the problem of approximation by embeddings. By using this connection, we deduce the main result of the section:

Theorem 11. *For the generic simplicial maps from a tree to a segment the non-existence of 2-obstructors is a necessary and sufficient condition for the existence of a lifting.*

1.3. Conventions. Throughout this paper, we adhere to the following notation and conventions.

We use the letters K and L to represent finite simplicial complexes, with $|K|$ and $|L|$ denoting their geometric realisations. All maps between simplicial complexes are assumed to be simplicial. Additionally, for a simplicial map $f: K \rightarrow L$, we denote the induced piecewise linear map between $|K|$ and $|L|$ by $|f|$.

Given a simplicial complex or a graph K , we use $V(K)$ and $E(K)$ to denote the set of vertices and the set of edges of K , respectively. Moreover, $\text{sk}_n K$ denotes the n -skeleton of K — a simplicial complex consisting of all simplices of K of dimension lower than or equal to n .

As previously noted, non-degenerate simplicial maps between one-dimensional simplicial complexes are essentially graph homomorphisms. Occasionally, particularly in examples, it will be more convenient to consider multigraph homomorphisms.

Let G and H be multigraphs.³ A pair (f_V, f_E) of maps $f_V: V(G) \rightarrow V(H)$ and $f_E: E(G) \rightarrow E(H)$ is called a **multigraph homomorphism** $f: G \rightarrow H$ if, for any edge $e \in E(G)$, the map f_V takes the endpoints of e to the endpoints of $f_E(e)$.

As a graph, a multigraph G has a geometric realisation denoted by $|G|$. Specifically, $|G|$ is the union of $|V(G)|$ points p_v representing the vertices $v \in V(G)$, and $|E(G)|$ segments

³Graphs that can contain loops and multiple edges.

I_e representing the edges $e \in E(G)$, where the segments are glued to the points according to the incidence relation.⁴ Clearly, $|G|$ is a one-dimensional polyhedron uniquely defined up to piecewise linear homomorphism.

Furthermore, a multigraph homomorphism $f: G \rightarrow H$ induces a piecewise linear map $|f|$ between the geometric realisations of G and H , which maps $p_v \in |G|$ to $p_{f_V(v)} \in |H|$ and homeomorphically maps I_e to $I_{f_E(e)}$. Thus, by taking triangulations of $|G|$ and $|H|$ in which $|f|$ is simplicial, we can obtain a simplicial map. Therefore, whenever multigraph homomorphisms are used in this text, they can be replaced by simplicial maps after subdividing the involved multigraphs. However, to simplify the notation, we choose to use multigraph homomorphisms in some places of the text.

2. COMBINATORICS OF LIFTINGS TO EMBEDDINGS

In this section, we will discuss combinatorial techniques that are useful for studying the existence of liftings to embeddings of maps between graphs.

Following our conventions, let K and L be simplicial complexes, and $f: K \rightarrow L$ be a non-degenerate simplicial map between them. Let us denote by $K_f^{(n)}$ a simplicial complex whose vertices are n -tuples of distinct vertices of K that map to the same vertex by f . Additionally, a set $\{(v_1^1, \dots, v_n^1), \dots, (v_1^{k+1}, \dots, v_n^{k+1})\}$ of vertices forms a k -simplex if for each $j = 1, \dots, n$ the set $\{v_j^1, \dots, v_j^{k+1}\}$ forms a k -simplex A_j in K . Note that $A_j \cap A_s = \emptyset$ for all $j \neq s$, and all the simplices A_j have the same image under f .

The symmetric group S_n acts naturally on $K_f^{(n)}$ by permuting the points in n -tuples. Thus, we can define the unordered version of $K_f^{(n)}$, which we denote by $\tilde{K}_f^{(n)} = K_f^{(n)} / S_n$. Clearly, S_n also acts on the geometric realisation $|K_f^{(n)}|$ of $K_f^{(n)}$. This action is properly discontinuous, so it induces a covering map $p_n: |K_f^{(n)}| \rightarrow |\tilde{K}_f^{(n)}|$ that forgets the order of points. Moreover, it is clear that p_n is a principal S_n -bundle.⁵

Definition (n -obstructor). An n -**obstructor** for $f: |K| \rightarrow |L|$ is a vertex $(x_1, x_2, \dots, x_n) \in K_f^{(n)}$ and a path from $(x_1, x_2, \dots, x_n)$ to $(x_n, x_1, x_2, \dots, x_{n-1})$ in $K_f^{(n)}$.

The reader may refer to [Example 1](#) below to see an example of 2-obstructor.

Lemma 1. *For any non-degenerate simplicial map $f: K \rightarrow L$ and $N > 1$, the following statements are equivalent:*

- (1) *all the covering maps $p_k: |K_f^{(k)}| \rightarrow |\tilde{K}_f^{(k)}|$, $1 < k \leq N$ are trivial;*
- (2) *there are no k -obstructors for f for $1 < k \leq N$.*

Proof. The implication (1) $\Rightarrow$ (2) is straightforward. Indeed, assume there is a k -obstructor that defines a path $\gamma: I \rightarrow |K_f^{(k)}|$. Then $p_k \circ \gamma$ is a loop in $|\tilde{K}_f^{(k)}|$, but γ is not a loop in $|K_f^{(k)}|$, implying that p_k is not trivial.

⁴Certainly, loops are represented by loops (or segments with both endpoints identical) in $|G|$.

⁵For a short introduction to this topic the reader may refer to [\[19\]](#).

Now, let us prove the implication (2) $\Rightarrow$ (1). Assume there are no k -obstructors for any $1 < k \leq N$. Let $|K_f^{(k)}| = \bigsqcup_i C_i$, where $\{C_i\}_i$ are connected components of $|K_f^{(k)}|$.⁶

Clearly, the action of S_k on $|K_f^{(k)}|$ induces an action of S_k on $\{C_i\}_i$. Let us prove that this action is free. Suppose it is not. Then there exists $\sigma \neq \text{id} \in S_k$ and a connected component C such that $\sigma(C) = C$. Let p be a prime divisor of $\text{ord}(\sigma)$, and define $\sigma' = \sigma^{\text{ord}(\sigma)/p}$. Clearly, $\sigma'(C) = \sigma^{\text{ord}(\sigma)/p}(C) = C$. Moreover, since $\text{ord}(\sigma') = p$, the cycle decomposition of σ' consists of cycles of length one, and one or more disjoint cycles of length p . Let $(s_1 \ s_2 \ \dots \ s_p)$ be one of them.

Now consider any vertex $x \in C$. As $\sigma'(x) \in C$, there exists a path γ in C connecting x and $\sigma'(x)$. In fact, it defines k paths in K

$$\begin{aligned} \gamma_1: x_1 &\rightarrow y_1^1 \rightarrow \dots \rightarrow y_1^l \rightarrow x_{\sigma'(1)} \\ \gamma_2: x_2 &\rightarrow y_2^1 \rightarrow \dots \rightarrow y_2^l \rightarrow x_{\sigma'(2)} \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ \gamma_k: x_k &\rightarrow y_k^1 \rightarrow \dots \rightarrow y_k^l \rightarrow x_{\sigma'(k)} \end{aligned}$$

such that $y_i^k \neq y_j^k$ for $i \neq j$, and $f(y_i^k) = f(y_j^k)$. By taking the paths $\gamma_{s_1}, \dots, \gamma_{s_p}$, we obtain a path in $K_f^{(p)}$. It can be seen that this path defines a p -obstructor.

Therefore, the action of S_n on $\{C_i\}_i$ is free. Let $\{O_\alpha\}_\alpha$ be the set of orbits of the action. For each orbit O_α , fix an arbitrary connected component $C_\alpha \in O_\alpha$, and let $T = \bigsqcup_\alpha C_\alpha$.

For every point $x \in |\tilde{K}_f^{(k)}|$, the preimages $p_k^{-1}(x)$ form an orbit of the action of S_n on $|K_f^{(k)}|$. Thus, the connected components containing these preimages also form an orbit. Moreover, since the action of S_n on them is free, two distinct preimages cannot lie in the same connected component. The construction of T ensures that exactly one of the points in $p_k^{-1}(x)$ lies in some of $C_\alpha \subset T$. Therefore, the restriction $p_k|_T$ is bijective, and we can define a map $s: |\tilde{K}_f^{(k)}| \rightarrow |K_f^{(k)}|$ by putting $s(x) = (p_k|_T)^{-1}(x)$.

Note that p_k is an open map; indeed, for any open set $U \subset |K_f^{(k)}|$ we have $p_k^{-1}(p_k(U)) = \bigcup_{\sigma \in S_k} \sigma(U)$, and all the sets $\sigma(U)$ are open. Thus, for an open $U \subseteq |K_f^{(k)}|$, we have $s^{-1}(U) = s^{-1}(U \cap T) = p_k(U \cap T)$ which is open since p_k is open, and both U and T are open.

Therefore, s is a continuous map defining a section of a principal S_k -bundle p_k , which implies that p_k is trivial by [19, Proposition 2.1]. $\square$

Now, we will state necessary conditions for the existence of a lifting. It is worth mentioning that the necessity of (2), for the case of generic immersions between manifolds and in a slightly different formulation, is proven in [21, Lemme 1.1]. The necessity of (1) for $n = 2$ was discussed in [2, §3.1] for maps to $\mathbb{R}^N$ and proven in [10, Corollary 3] for generic immersions of surfaces into $\mathbb{R}^3$.

⁶Since the polyhedra are locally path-connected spaces, connectedness and path-connectedness are equivalent in this case.

Theorem 1. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map such that $|f|: |K| \rightarrow |L|$ lifts to an embedding. Then the following holds:*

- (1) *all the covering maps $p_n: |K_f^{(n)}| \rightarrow |\tilde{K}_f^{(n)}|$, $n > 1$ are trivial;*
- (2) *there are no n -obstructors for f for any $n > 1$.*

Proof. Let us begin by noting that the points of $|K_f^{(n)}|$ and $|\tilde{K}_f^{(n)}|$ can be considered as ordered and unordered n -tuples of distinct points of $|K|$ with the same image.

Suppose there is a lifting $\widetilde{|f|}: |K| \rightarrow |L| \times \mathbb{R}$. Let $h = \text{pr}_R \widetilde{|f|}: |K| \rightarrow \mathbb{R}$. This induces a continuous map that takes $(x_1, \dots, x_n)$ to $(h(x_1), \dots, h(x_n))$. Since $\widetilde{|f|}$ is an embedding, all $h(x_i)$'s are distinct. Thus, it actually maps $|K_f^{(n)}|$ to the configuration space $\text{Conf}_n(\mathbb{R})$ of n distinct points of $\mathbb{R}$. Moreover, it is equivariant with respect to the actions of S_n on both spaces.⁷ It is easy to see that $\text{Conf}_n(\mathbb{R})$ equivariantly deformation retracts to the discrete space $\text{Conf}_n(\{1, 2, \dots, n\})$ of all $n!$ permutations of the numbers $\{1, \dots, n\}$ with the natural action of S_n . By taking an equivariant isomorphism that sends the n -tuple $(1, 2, \dots, n)$ to $\text{id} \in S_n$, we can identify $\text{Conf}_n(\{1, 2, \dots, n\})$ with S_n .

Therefore, we have an equivariant map $H: |K_f^{(n)}| \rightarrow S_n$. Now define $g: |K_f^{(n)}| \rightarrow S_n \times |\tilde{K}_f^{(n)}|$ as $g(x) = (H(x), p_n(x))$. Clearly, g is a morphism between the principal S_n -bundles p_n and the trivial bundle $\text{pr}_{|\tilde{K}_f^{(n)}|}: S_n \times |\tilde{K}_f^{(n)}| \rightarrow |\tilde{K}_f^{(n)}|$, thus, it is an isomorphism by [19, Proposition 2.1]. Therefore, p_n is trivial.

The non-existence of n -obstructors then follows from Lemma 1. $\square$

Example 1. This example, along with the description of a 2-obstructor, is presented in [2, §3.1]. The map $f: G \rightarrow H$ first appeared in the proof of Theorem 2.1 in [26].

Consider the graphs G and H and the simplicial map $f: G \rightarrow H$ as in Fig. 1. As shown in the figure, there is a 2-obstructor for f given by the path in $G_f^{(2)}$ defined by the two paths in G :

$$\begin{aligned} \gamma_1: a_3 &\rightarrow b_4 \rightarrow c_4 \rightarrow d_3 \rightarrow c_3 \rightarrow b_3 \rightarrow a_1 \rightarrow b_1 \rightarrow c_1 \rightarrow d_1 \rightarrow c_1 \rightarrow \\ &\rightarrow b_1 \rightarrow a_1 \rightarrow b_3 \rightarrow c_3 \rightarrow b_2 \rightarrow a_2 \\ \gamma_2: a_2 &\rightarrow b_2 \rightarrow c_2 \rightarrow d_2 \rightarrow c_2 \rightarrow b_2 \rightarrow a_2 \rightarrow b_2 \rightarrow c_3 \rightarrow d_3 \rightarrow c_4 \rightarrow \\ &\rightarrow b_4 \rightarrow a_3 \rightarrow b_4 \rightarrow c_4 \rightarrow b_4 \rightarrow a_3 \end{aligned}$$

Therefore, f does not lift to an embedding.

Theorem 1 provides only the necessary conditions for the existence of a lifting (see Example 2 and Example 3). The next two theorems, however, provide conditions that are both necessary and sufficient.

Definition (admissible collection of linear orders). Let $f: K \rightarrow L$ be a non-degenerate simplicial map between finite simplicial complexes. For each simplex $C \in L$, let K_C be

⁷ $F: X \rightarrow Y$ is equivariant with respect to actions of a group G on X and Y if $F(g(x)) = g(F(x))$ for any $x \in X$ and $g \in G$.

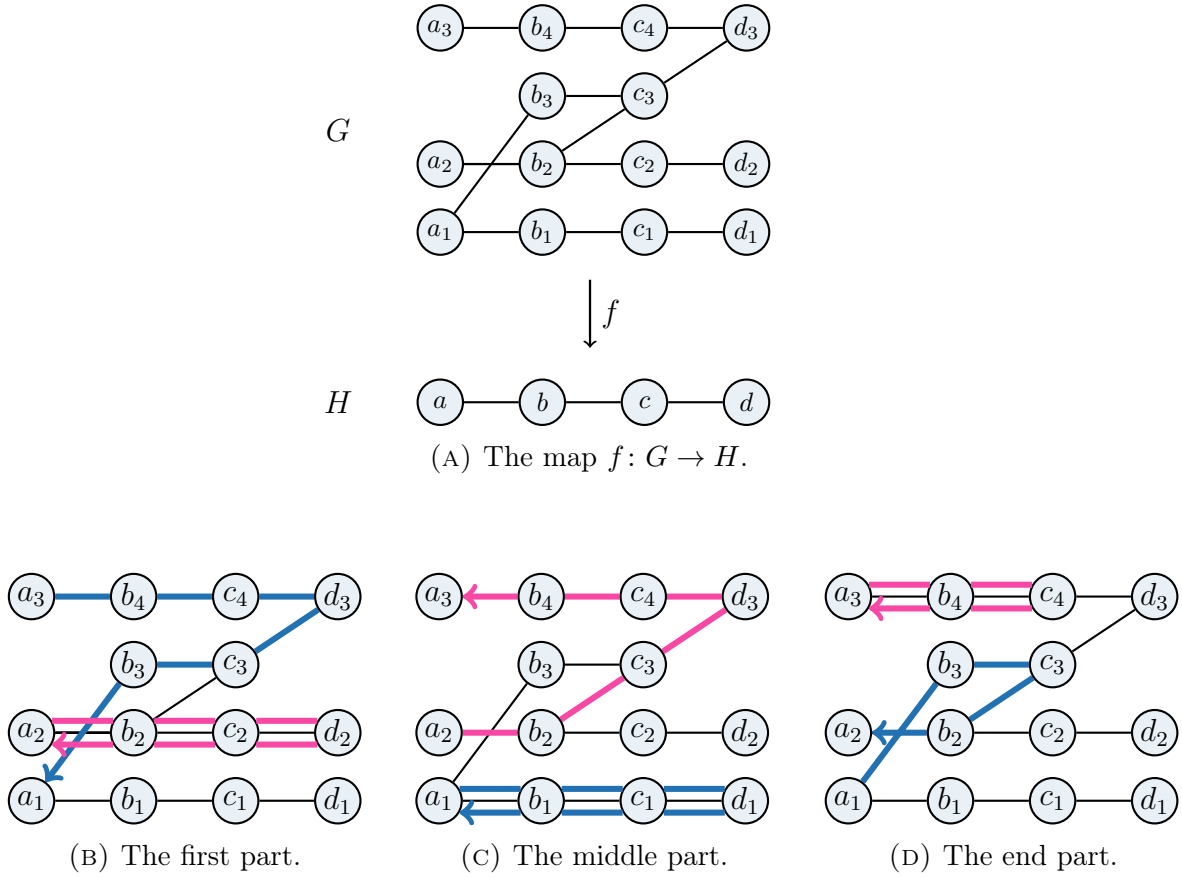


FIGURE 1. Sieklucki's example and the 2-obstructor for it shown as a pair of paths in G (in three stages).

the set of simplices $A \in K$ that map to C under f . Note that since f is non-degenerate, all the simplices in K_C have the same dimension as C . Furthermore, let $V(L)$ be the set of vertices of L .

We call a collection $\{(\prec_v, K_v)\}_{v \in V(L)}$ of linear orders on the sets $K_v = f^{-1}(v)$ **admissible** if it induces linear orders on all the sets K_C , $C \in L$.

Specifically, given a pair of simplices $A, B \in K_C$, let $V_f(A, B) \subset V(A) \times V(B)$ represent all pairs of vertices (v, w) satisfying $f(v) = f(w)$. Then it is required that either $v \preceq_{f(v)} w$ holds for all $(v, w) \in V_f(A, B)$, or $v \succeq_{f(v)} w$ holds for all $(v, w) \in V_f(A, B)$.

Theorem 2. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map between finite simplicial complexes. Then the piecewise linear map $|f|: |K| \rightarrow |L|$ lifts to an embedding if and only if there exists an admissible collection of linear orders on the sets K_v , $v \in V(L)$.*

Furthermore, there exists a bijection between the admissible collections of linear orders and the isotopy classes of liftings.

The reader may refer to [10, Proposition 4], [21, Lemme 1.4], and [16, Theorem 5] for variants of the stated theorem.

Proof. Let us first, by using a lifting of $|f|$, construct an admissible collection of linear orders. Assume there is a lifting $\widetilde{|f|}: |K| \rightarrow |L| \times \mathbb{R}$. It defines a collection of linear orders as follows: for vertices $v, w \in V(K)$ such that $f(v) = f(w)$, we define $v \prec_f w$ if $h(v) < h(w)$, where $h = \text{pr}_{\mathbb{R}} \circ \widetilde{|f|}$. Let us prove that this collection is admissible.

Suppose it is not. It implies that there exist simplices $A, B \in K$ with the same image $f(A) = f(B)$ and two pairs of vertices $f(v) = f(w), f(v') = f(w')$, where $v, v' \in A, w, w' \in B$, such that $v \prec_f w$ but $v' \succ_f w'$.

Clearly, we have $f(\{v, v'\}) = f(\{w, w'\}) = \{f(v), f(w)\}$ where $\{a, b\}$ denotes an edge with endpoints a and b . Let $d: [0, 1] \rightarrow \mathbb{R}$ be the map defined as $d(t) = h((1-t)v + tv') - h((1-t)w + tw')$. Since $v \prec_f w$ and $v' \succ_f w'$, we have $d(0) = h(v) - h(w) < 0$ and $d(1) > 0$. Therefore, there exists $t' \in [0, 1]$ such that $d(t') = 0$. However, since $|f|$ is linear on the simplices of K , this implies that $\widetilde{|f|}((1-t')v + t'v') = \widetilde{|f|}((1-t')w + t'w')$, which contradicts the fact that $\widetilde{|f|}$ is an embedding.

Now suppose we have an admissible collection of linear orders $\{(\prec_v, K_v)\}_{v \in V(L)}$. Let us construct a lifting $\widetilde{|f|}$ of $|f|$ based on this collection. First, we define a map $h: V(K) \rightarrow \mathbb{N}$ as follows: for each K_v , we let $h|_{K_v}$ be the order isomorphism $K_v \rightarrow \{1, \dots, |K_v|\}$ that takes the k -th vertex in K_v (in ascending order with respect to $\prec_v$) to k . By extending the map h linearly to the entire $|K|$, we obtain the map $h: |K| \rightarrow \mathbb{R}$. Take $\widetilde{|f|} = |f| \times h$. Now, let us prove that $\widetilde{|f|}$ is an embedding.

Suppose it is not. Then there exist points $v, w \in |K|$ such that $\widetilde{|f|}(v) = \widetilde{|f|}(w)$. Since v and w cannot be vertices of K by the definition of h , they lie in the interior of some simplices A and B of K , respectively, with $\dim A = \dim B > 0$.⁸ The last equality follows from the fact that f is non-degenerate and hence injective on simplices, so $f(A) = f(B)$ and the simplices A, B , and $f(A)$ have the same dimension. Let $a_0, \dots, a_n$ denote the vertices of A , and $b_0, \dots, b_n$ denote the corresponding vertices of B , such that $f(a_i) = f(b_i)$ for each i .

Now, let $v = \sum_{i=0}^n \alpha_i a_i$ where the α_i 's are the barycentric coordinates of v in A . Since $\widetilde{|f|}(v) = \widetilde{|f|}(w)$, we have $w = \sum_{i=0}^n \alpha_i b_i$. Therefore, we have $h(v) = \sum_{i=0}^n \alpha_i h(a_i) = \sum_{i=0}^n \alpha_i h(b_i) = h(w)$.

Because h was defined using the admissible collection of linear orders, we have either $h(a_i) \leq h(b_i)$ or $h(a_i) \geq h(b_i)$ for all i . Moreover, since $A \neq B$, at least one of the inequalities is strict. This implies that either $h(v) < h(w)$ or $h(v) > h(w)$, which contradicts the assumption that $\widetilde{|f|}(v) = \widetilde{|f|}(w)$, where $\widetilde{|f|} = |f| \times h$.

To establish the bijection between admissible collections of linear orders and isotopy classes of liftings, we observe that each lifting $\widetilde{|f|}$ is isotopic to the lifting $\widetilde{|f|}'$ constructed in the previous step using the admissible collection of linear orders induced by $\widetilde{|f|}$. The isotopy $F: |K| \times [0, 1] \rightarrow |L| \times \mathbb{R}$ is defined as $F(x, t) = (|f|(x), (1-t)h(x) + th'(x))$

⁸In this context, we define the interior of a simplex Δ as $|\Delta| \setminus \bigcup_A |A|$, where A ranges over all proper faces of Δ .

where $h = \text{pr}_{\mathbb{R}} |\widetilde{f}|$ and $h' = \text{pr}_{\mathbb{R}} |\widetilde{f}'|$. The proof that this is an actual isotopy is left to the reader. $\square$

The next theorem demonstrates that the problem of determining the existence of a lifting for a piecewise linear non-degenerate map between polyhedra can be reduced to the case of graphs.

Theorem 3. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map. Then the corresponding piecewise linear map $|f|: |K| \rightarrow |L|$ lifts to an embedding if and only if $|f|_{|\text{sk}_1 K|}: |\text{sk}_1 K| \rightarrow |\text{sk}_1 L|$ lifts to an embedding. Here, $\text{sk}_1 K$ and $\text{sk}_1 L$ denote the one-dimensional skeletons of K and L , respectively. Moreover, there is a bijection between the isotopy classes of liftings of $|f|$ and of $|f|_{|\text{sk}_1 K|}$.*

Proof. Based on [Theorem 2](#), it suffices to show that a collection $\{(\prec_v, K_v)\}_{v \in V(K)}$ of linear orders is admissible with respect to f if and only if it is admissible with respect to $f|_{|\text{sk}_1 K|}$.

It is clear that once it is admissible with respect to f , it is admissible with respect to $f|_{|\text{sk}_1 K|}$ as well, according to the definition.

Suppose a collection is admissible with respect to $f|_{|\text{sk}_1 K|}$, but not admissible with respect to f . This implies that there exist simplices $A, B \in K_C$ and two pairs of vertices $f(v) = f(w), f(v') = f(w')$, where $v, v' \in A, w, w' \in B$, such that $v \prec_{f(v)} w$ but $v' \succ_{f(v')} w'$. The edges $\{v, v'\}, \{w, w'\}$ lie in $\text{sk}_1 K$ and have the same image, but their endpoints are in the opposite order. This leads to a contradiction. $\square$

Note that, while studying the existence of a lifting of a piecewise linear map f , it suffices to focus attention only on the subpolyhedron of the preimages of the multiple points of f . In view of [Theorem 3](#), after setting f' to be a restriction of f on the set $\{x \in \text{sk}_1 K \mid |f^{-1}(f(x))| > 1\} \subseteq \text{sk}_1 K$, we have that f lifts if and only if f' lifts.

For instance, in the case of generic immersions of surfaces into $\mathbb{R}^3$, a similar observation allows us to narrow down the question of liftability of an immersion to the liftability of its restriction on the preimage of the set of multiple points, see [\[6, Theorem 3.2\]](#). Note that this restriction is, in fact, a map between graphs.

The next theorem demonstrates that the liftability of a non-degenerate piecewise linear map $|f|: |K| \rightarrow |L|$ corresponding to a simplicial map $f: K \rightarrow L$, can be encoded as the satisfiability of a 3-CNF boolean formula, which we will define below.

The key idea is as follows: assuming the covering map $p_2: |K_f^{(2)}| \rightarrow |\widetilde{K}^{(2)}|$ is trivial, each trivialization of it yields two mutually inverse collections of binary relations on the preimages of the vertices of L . Specifically, by selecting a section $s: |\widetilde{K}^{(2)}| \rightarrow |K_f^{(2)}|$ of p_2 , we define the relation as $x R y$ when (x, y) lies in $s(|\widetilde{K}^{(2)}|)$. This relation is antisymmetric and irreflexive; therefore, to be a (strict) order, it only needs to satisfy the transitivity

property. Transitivity, in turn, can be represented by a set of conditions of the form $x R y \wedge y R z \rightarrow x R z$.⁹

More precisely, let us define a boolean formula Γ_f . Denote by $\mathcal{C}$ the set of connected components of $|K_f^{(2)}|$. The involution τ on $|K_f^{(2)}|$ induces an involution on $\mathcal{C}$. Let us assume that the covering map p_2 is trivial, therefore, the latter involution is fixed-point free.

Let $O_1, O_2, \dots, O_n$ be the orbits of the involution on $\mathcal{C}$. For each orbit O_i , pick an element $C_i \in O_i$ in the orbit, so $O_i = \{C_i, \tau(C_i)\}$, and associate a boolean variable x_i with C_i . Next, we associate the literal $\neg x_i$ with each $\tau(C_i)$. As a result, each connected component C in $\mathcal{C}$ is associated with either a variable or its negation. We refer to the literal corresponding to C as α_C .

Let T be the set of ordered triples (C, D, E) of connected components, with $C \neq E$ and $D \neq E$, such that there exist three vertices $a, b, c \in V(K)$ satisfying $(a, b) \in C$, $(b, c) \in D$, $(a, c) \in E$. Let Γ_f be the boolean function

$$\Gamma_f(x_1, \dots, x_n) = \bigwedge_{(C,D,E) \in T} ((\alpha_C \wedge \alpha_D) \rightarrow \alpha_E) = \bigwedge_{(C,D,E) \in T} (\neg \alpha_C \vee \neg \alpha_D \vee \alpha_E)$$

Note that, aside from reordering the clauses, literals, and renaming variables, the only source of variability in the form of Γ_f arises from the initial choice of representatives for the orbits O_i . However, it is easy to check that once Γ_f contains a clause $\neg \alpha_C \vee \neg \alpha_D \vee \alpha_E$, then it must also contain the clause $\alpha_C \vee \alpha_D \vee \neg \alpha_E$.¹⁰ This is because, once we have $(a, b) \in C$, $(b, c) \in D$, and $(a, c) \in E$, we have $(c, b) \in \tau(D)$, $(b, a) \in \tau(C)$, and $(c, a) \in \tau(E)$.

Thus, the form of Γ_f does not actually depend on the choice of representatives, and, therefore, Γ_f is uniquely defined up to reordering the clauses and the literals within clauses, and renaming the variables.

Theorem 4. *Let $f: K \rightarrow L$ be a non-degenerate simplicial map. Then the corresponding piecewise linear map $|f|: |K| \rightarrow |L|$ lifts to an embedding if and only if the following conditions hold:*

- (1) *the covering map $p_2: |K_f^{(2)}| \rightarrow |\tilde{K}_f^{(2)}|$ is trivial,*
- (2) *Γ_f is satisfiable.*

Furthermore, there exists a bijection between the assignments that satisfy Γ_f and the isotopy classes of liftings.

Proof. Let us assume that there exists a lifting $\widetilde{|f|}: |K| \rightarrow |L| \times \mathbb{R}$ of $|f|$. By Theorem 1, the covering map p_2 is trivial. Moreover, as seen in the proof of Theorem 1, we can obtain

⁹An alternative approach to expressing transitivity in terms of the covering maps p_k 's is to find a pair of "compatible" trivializations for $p_2: |K_f^{(2)}| \rightarrow |\tilde{K}^{(2)}|$ and $p_3: |K_f^{(3)}| \rightarrow |\tilde{K}^{(3)}|$, refer to [16, Theorem 5] for further details.

¹⁰After double negation elimination, that is $\neg \neg x = x$.

a continuous equivariant map H from $|K_f^{(2)}|$ to S_2 by composing $h: |K_f^{(2)}| \rightarrow \text{Conf}_2(\mathbb{R})$ defined as $h((x, y)) = (\text{pr}_{\mathbb{R}} \circ \widetilde{|f|}(x), \text{pr}_{\mathbb{R}} \circ \widetilde{|f|}(y))$, with the retraction $\text{Conf}_2(\mathbb{R}) \rightarrow S_2$.

Let us define the variable assignment by putting $x_i = 1$ if $H(C_i) = \text{id} \in S_2$, and $x_i = 0$ otherwise. This assignment is well-defined because H is constant on the connected components of $|K_f^{(2)}|$. Moreover, under this assignment, $\alpha_C = 1$ if and only if $H(C) = \text{id}$.

Let us consider an implication $(\alpha_C \wedge \alpha_D) \rightarrow \alpha_E$ in Γ_f . According to the definition of Γ_f , there exist pairs $(a, b) \in C$, $(b, c) \in D$, and $(a, c) \in E$. Once $\alpha_C = 1$ and $\alpha_D = 1$, we have $\alpha_E = 1$ by the transitivity of $\prec_{f(a)}$. Therefore, the defined assignment satisfies the disjunction, and consequently, it satisfies the entire formula Γ_f .

Let us prove the reverse implication. Assume we have an assignment $\psi: \{x_i\}_{i=1}^n \rightarrow \{0, 1\}$ that satisfies Γ_f . We denote by $\psi(\alpha_C)$ the value of α_C under the assignment ψ . Now, we define a binary relation R on $V(K_f^{(2)})$ as follows: $a R b$ if $\psi(\alpha_C) = 1$ for the connected component C containing the pair (a, b) , if it exists.

It is easy to see that the relation R is antisymmetric and irreflexive. Moreover, it is transitive. Indeed, suppose that pairs (a, b) and (b, c) lie in connected components C and D , respectively, and $a R b$ and $b R c$, or, in other words, $\psi(\alpha_C) = \psi(\alpha_D) = 1$. Denote by E the connected component containing the pair (a, c) . If $E = C$ or $E = D$, then $\psi(\alpha_E) = \psi(\alpha_C) = 1$ or $\psi(\alpha_E) = \psi(\alpha_D) = 1$, respectively, hence $a R c$. Otherwise, we have a disjunction $\neg\alpha_C \vee \neg\alpha_D \vee \alpha_E$ in Γ_f confirming that $\psi(\alpha_E) = 1$ and hence $a R c$. Thus, R defines a collection of linear orders on the sets K_v by putting $a \prec_{f(a)} b$ if $a R b$.

Note that if $(a, b), (c, d) \in V(K_f^{(2)})$ lie in the same connected component of $K_f^{(2)}$, then, according to the definition of R , either $a \prec_{f(a)} b$ and $c \prec_{f(c)} d$, or $a \succ_{f(a)} b$ and $c \succ_{f(c)} d$.

Next, let us prove that this collection is admissible. To do this, we need to show that, given simplices A and B with the same image, we have either $a \prec_{f(a)} b$ or $a \succ_{f(a)} b$ simultaneously for all pairs of distinct vertices $(a, b) \in V(A) \times V(B)$ with $f(a) = f(b)$. However, all such pairs lie in the same connected component of $K_f^{(2)}$, which provides the desired result.

Thus, we have constructed an admissible collection of linear orders on the sets K_v , and, by [Theorem 2](#), it defines a lifting of f .

Note that, in the above proof, we established a bijective correspondence between the assignments satisfying Γ_f and the collections of admissible orders. Therefore, by [Theorem 2](#), this establishes a bijection between the assignments and the isotopy classes of liftings. This observation completes the proof. $\square$

Remark. Both [Theorem 2](#) and [Theorem 4](#) are proven under the assumption that f is a simplicial map. However, it is easy to see that the statements of these theorems remain true if we replace simplicial complexes with multigraphs, and simplicial maps with homomorphisms of multigraphs.

Example 2 (Giller's example, see [\[10\]](#)). The provided example, known as Giller's example, illustrates an immersion of the 2-sphere S^2 into 3-dimensional Euclidean space $\mathbb{R}^3$ that cannot be lifted to an embedding. The example is discussed in the paper by

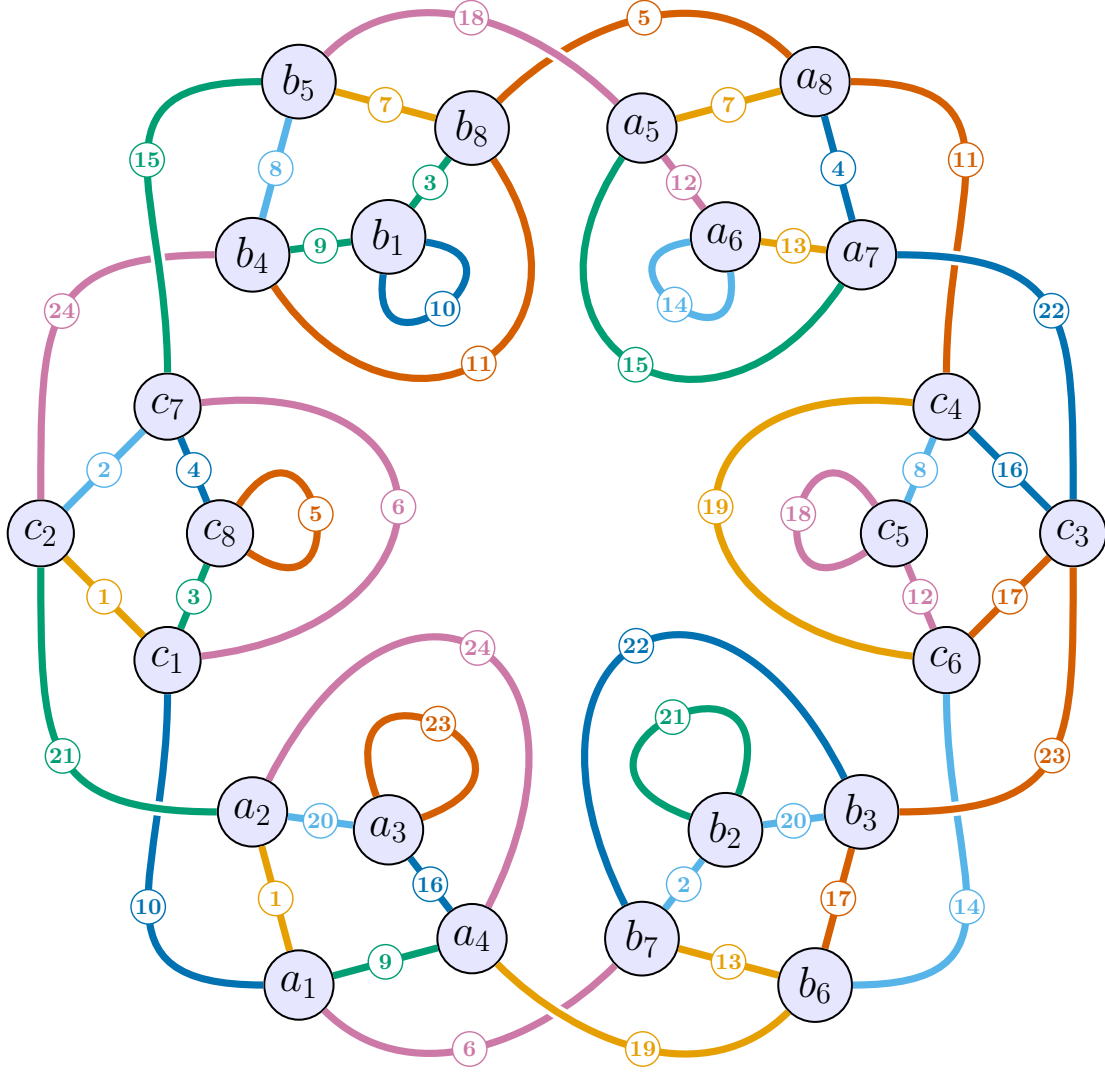


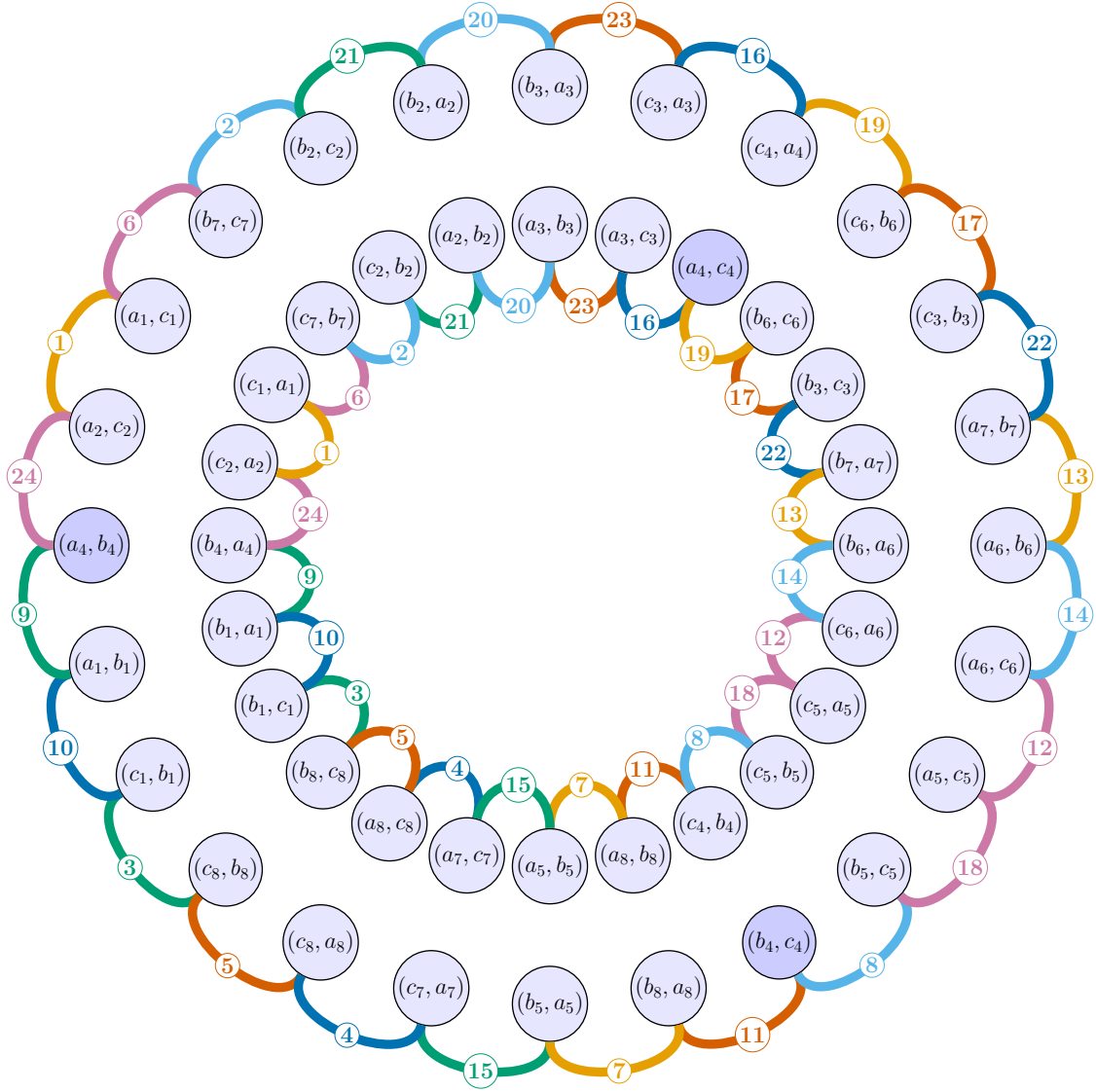
FIGURE 2. Giller’s example: graph of multiple point preimages. Vertices with the same subscript map to the same vertex, and identically labelled edges have the same image. In other words, $f(x_i) = f(y_j)$ if and only if $i = j$, and $f(e) = f(g)$ if and only if e and g share the same label.

Giller [10], and we refer the reader to it for details. We focus only on the graph of the multiple point preimages to understand its liftability. We denote by f the restriction of the immersion on this graph.

The multigraph of the multiple point preimages is shown in Fig. 2.

Additionally, Fig. 3 shows the corresponding graph $G_f^{(2)}$, which is the union of two circles. Note that the covering map p_2 is trivial in this case.

However, one can see that the pairs (a_4, b_4) and (b_4, c_4) lie in the outer circle of Fig. 3, while (a_4, c_4) lies in the inner circle. As a result, Γ_f contains clauses $(x \vee x \vee x) \wedge (\neg x \vee$

FIGURE 3. Giller's example: the graph $G_f^{(2)}$.

$\neg x \vee \neg x$), where x is a variable corresponding to the inner circle. Since this subformula is equivalent to $x \wedge \neg x$, the entire formula Γ_f is not satisfiable. Therefore, both f and the initial immersion do not lift to an embedding.

Let us now assume that for an arbitrary map f the formula Γ_f contains a clause corresponding to a triple (C, C, E) . Thus, it contains a clause corresponding to a triple $(\tau(C), \tau(C), \tau(E))$ as well, and, therefore,

$$\Gamma_f = (\neg\alpha_C \vee \neg\alpha_C \vee \alpha_E) \wedge (\alpha_C \vee \alpha_C \vee \neg\alpha_E) \wedge \Gamma' = (\alpha_C = \alpha_E) \wedge \Gamma'$$

This allows us to simplify the formula Γ_f by “gluing” connected components C and E (that is, defining an equivalence relation on the points of $|K_f^{(2)}|$ that is coarser than the one defined by the partition into connected components) once we meet a triple of pairs

$(a, b) \in C$, $(b, c) \in C$, and $(a, c) \in D$. This idea is implemented in [22, 21]; we will discuss it, and disprove some of the results stated in these papers, in the next section.

3. LIFTINGS OF SMOOTH IMMERSIONS

In this section, we explore connections between lifting maps between graphs and lifting generic immersions between smooth manifolds. We provide a counterexample to a result stated in [22, 21]. Additionally, we show that for each Γ that potentially could be a formula from Theorem 4, there exists a generic smooth immersion $f: X \looparrowright Y$ of a surface X with boundary into a 3-manifold Y with boundary such that the restriction f' of f onto the preimage of the multiple point set forms a map between graphs with $\Gamma_{f'} = \Gamma$.

We will use the word “generic” in the same sense in which it is used in [22]. Specifically, we say that $f: X \looparrowright Y$ is **generic** when

- (1) $f^{-1}(\partial Y) = \partial X$,
- (2) for $y \in Y \setminus \partial Y$ and $f^{-1}(y) = \{x_i\}_{i=1}^n$, the subspaces $df_{x_i}(T_{x_i}X)$ are in general position in $T_y Y$, that is

$$\text{codim} \bigcap_{i=1}^n df_{x_i}(T_{x_i}X) = \sum_{i=1}^n \text{codim} df_{x_i}(T_{x_i}X),$$

- (3) for $y \in \partial Y$ and $f^{-1}(y) = \{x_i\}_{i=1}^n$, the subspaces $T_y \partial Y$ and $df_{x_i}(T_{x_i}X)$ are in general position in $T_y Y$.

Let $f: X \looparrowright Y$ be a generic immersion. In [22] and [21, Théorème 1], Poénaru claimed that for the existence of a lifting $\tilde{f}: X \rightarrow Y \times \mathbb{R}$ to an embedding, it is necessary and sufficient that $\mu_2(f) = \nu_3(f) = 0$.

Here, the condition $\mu_2(f) = 0$ is equivalent to the triviality of the covering $P_2: \text{Conf}_f(X, 2) \rightarrow \widetilde{\text{Conf}}_f(X, 2)$, where $\text{Conf}_f(X, 2)$ represents the configuration space of ordered pairs of distinct points with the same image under f , and $\widetilde{\text{Conf}}_f(X, 2)$ is its unordered version. For non-degenerate piecewise linear maps between graphs K and L , this condition is equivalent to the first condition in Theorem 4. Indeed, both $\text{Conf}_f(X, 2)$ and $\widetilde{\text{Conf}}_f(X, 2)$ equivariantly deformation retract onto $|K_f^{(2)}|$ and $|\tilde{K}_f^{(2)}|$, respectively. Therefore, the triviality of P_2 is equivalent to the triviality of $p_2: |K_f^{(2)}| \rightarrow |\tilde{K}_f^{(2)}|$.

Now, let us consider the condition $\nu_3(f) = 0$. Following Poénaru’s approach, we define a set $S_{2,3} \subseteq \text{Conf}_f(X, 2) \times \widetilde{\text{Conf}}_f(X, 3)$ that includes points $(a, b) \times \{c, d, e\}$ where $\{a, b\} \subset \{c, d, e\}$. On $S_{2,3}$, we introduce a collection $\mathbb{T}$ of all equivalence relations satisfying the following properties (denoting unordered triples as T_1 and T_2 for simplicity):

- (1) if $(a, b) \times T_1 \sim (c, d) \times T_2$, then $(b, a) \times T_1 \sim (d, c) \times T_2$,
- (2) if the points $(a, b), (c, d)$ lie in the same connected component of $\text{Conf}_f(X, 2)$, then $(a, b) \times T_1 \sim (c, d) \times T_2$,
- (3) once $(a, b) \times \{a, b, c\} \sim (b, c) \times \{a, b, c\}$, we have $(a, b) \times \{a, b, c\} \sim (b, c) \times \{a, b, c\} \sim (a, c) \times \{a, b, c\}$.

Since the intersection of two equivalence relations in $\mathbb{T}$ belongs to $\mathbb{T}$, the set $\mathbb{T}$ forms a lower subsemilattice of the lattice $\text{Eq}(S_{2,3})$ consisting of all equivalence relations on the set $S_{2,3}$. Therefore, let $\sim_\nu$ denote the smallest (finest) equivalence relation in $\mathbb{T}$. We say that $\nu_3(f) = 0$ if there are no points such that $(a, b) \times \{a, b, c\} \sim_\nu (b, c) \times \{a, b, c\} \sim_\nu (c, a) \times \{a, b, c\}$.

Considering $(a, b) \times T_1 \sim (a, b) \times T_2$ for any $\sim$ from $\mathbb{T}$ and for any points $T_1, T_2 \in \widetilde{\text{Conf}}_f(X, 3)$ with $\{a, b\} \subset T_1, T_2$, each equivalence relation $\sim$ from $\mathbb{T}$ defines the equivalence relation $\psi(\sim)$ on the set $\text{Conf}_f(X, 2)$ by taking $(a, b) \psi(\sim) (c, d)$ if $(a, b) \times T_1 \sim (c, d) \times T_2$ for some $T_1, T_2 \in \widetilde{\text{Conf}}_f(X, 3)$. One can check that $\psi: \mathbb{T} \hookrightarrow \text{Eq}(\text{Conf}_f(X, 2))$ defines an embedding of the semilattice $\mathbb{T}$ into the lattice $\text{Eq}(\text{Conf}_f(X, 2))$ consisting of all the equivalence relations on $\text{Conf}_f(X, 2)$.

It can be verified that $\psi(\mathbb{T}) \subset \text{Eq}(\text{Conf}_f(X, 2))$ can be defined using the three conditions defining $\mathbb{T}$, where instead of the points of the form $(x, y) \times T$, we consider pairs (x, y) . Therefore, the second condition, saying that $(a, b) \sim (c, d)$ whenever (a, b) and (c, d) lie in the same connected component of $\text{Conf}_f(X, 2)$, ensures that all equivalence relations in $\psi(\mathbb{T})$ are coarser than the one defined by the partition of $\text{Conf}_f(X, 2)$ into connected components.

Moreover, as demonstrated in the proof of [21, Lemme 1.2], the equivalence relation $\psi(\sim_\nu)$ (and, thus, $\sim_\nu$) can be iteratively constructed as follows: we start with the equivalence relation on $\text{Conf}_f(X, 2)$ defined by the partition into connected components. We then sequentially join connected components C and D whenever we meet a triple of pairs $(a, b), (b, c) \in C$, and $(a, c) \in D$.

As discussed in the end of the preceding section, this process corresponds to simplifying the formula Γ_f by replacing terms of the form $(\neg\alpha_C \vee \neg\alpha_C \vee \alpha_D) \wedge (\alpha_C \vee \alpha_C \vee \neg\alpha_D)$ with equalities $\alpha_C = \alpha_D$. In view of Theorem 4, Poénaru's claim is equivalent to the fact that, for generic immersions between smooth manifolds, the part of the formula remaining after simplifications is always satisfiable.¹¹ The next example together with the next theorem show that the latter statement is not true.

Example 3. Let us consider the multigraph homomorphism shown in Fig. 4a. The graph $G_f^{(2)}$ (Fig. 4b) consists of 6 connected components that split into pairs with respect to the involution. Consequently, $p_2: |G_f^{(2)}| \rightarrow |\tilde{G}_f^{(2)}|$ is trivial.

Note that for each triple (p_1, p_2, p_3) , where p stands for a, b, c , or d , all the pairs $(p_i, p_j), i \neq j$ lie in different connected components of $G_f^{(2)}$. Each such triple defines two

¹¹This remark should be understood as an informal comment, since formally, the formulas Γ_f are defined only for non-degenerate simplicial maps; however, the fact that the problem of lifting to embeddings of generic immersions between manifolds reduces to the problem of lifting maps between graphs (see [21, pages 1.21–1.22] and [5, Theorem 4.6]) makes this comment meaningful.

disjunctions in Γ_f .¹² Given that, we can express Γ_f as follows:

$$\begin{aligned}\Gamma_f(x_1, x_2, x_3) = & ((x_1 \wedge x_2) \rightarrow x_3) \wedge ((\neg x_1 \wedge \neg x_2) \rightarrow \neg x_3) \wedge \\ & ((x_1 \wedge x_2) \rightarrow \neg x_3) \wedge ((\neg x_1 \wedge \neg x_2) \rightarrow x_3) \wedge \\ & ((x_1 \wedge \neg x_3) \rightarrow x_2) \wedge ((\neg x_1 \wedge x_3) \rightarrow \neg x_2) \wedge \\ & ((x_1 \wedge \neg x_2) \rightarrow \neg x_3) \wedge ((\neg x_1 \wedge x_2) \rightarrow x_3)\end{aligned}$$

Here x_1 represents the connected component containing (a_1, a_2) , x_2 represents the connected component containing (a_2, a_3) , and x_3 represents the connected component containing (a_1, a_3) .

The first four clauses imply that either $x_1 = 1$ and $x_2 = 0$, or $x_1 = 0$ and $x_2 = 1$. In the former case, the next-to-last disjunction yields $x_3 = 0$, which contradicts $(x_1 \wedge \neg x_3) \rightarrow x_2$ as $x_2 = 0$. In the latter case, the last disjunction implies $x_3 = 1$, which contradicts $(\neg x_1 \wedge x_3) \rightarrow \neg x_2$ as $x_2 = 1$. Consequently, Γ_f is not satisfiable, and, as a result, f does not lift to an embedding.

In spite of this, it can be seen that in the formula Γ_f , there are no clauses in which literals are repeated. In light of the comments above, this means that equivalence classes under $\sim_\nu$, defined as it has been done for generic immersions before, coincide with the connected components of $|G_f^{(2)}|$.

Now, using the provided example, we will construct a counterexample to the claim of [21, Théorème 1] showing that its assumptions need to be strengthened in the sense of Theorem 4:

Theorem 5. *There exists a smooth generic immersion $j: T \looparrowright Y$ from a compact surface T to a compact 3-manifold Y that satisfies the conditions $\mu_2(j) = \nu_3(j) = 0$, but does not lift to an embedding.*

Proof. Let $f: G \rightarrow H$ be the map from Example 3. We add three multiple edges e_1, e_2, e_3 to H , going from a to d . We also add six edges to G and define f on these edges as follows:

- (1) $f(\{a_1, d_1\}) = f(\{a_2, d_2\}) = e_1$,
- (2) $f(\{a_2, d_3\}) = f(\{a_3, d_2\}) = e_2$,
- (3) $f(\{a_1, d_3\}) = f(\{a_3, d_1\}) = e_3$.

Let us denote the resulting graphs by the same letters G and H .

It can be observed that the covering map p_2 for the resulting map $f: G \rightarrow H$ remains trivial; moreover, the formula Γ_f also coincides with the formula of the original map.¹³

Now let us construct a surface S with boundary immersed into a handlebody B .

¹²Actually, each triple defines 6 disjunctions, one for each permutation of three points. However, after dropping equal disjunctions up to reordering terms, only two disjunctions remain.

¹³In fact, we have simply replaced the segments that make up the graph $G_f^{(2)}$ by joining their endpoints to form circles.

We represent each vertex a, b, c, d of H as an intersection of three squares with equal sides parallel to the coordinate planes, and label each square by the number from one to three. We denote the i -th square in the intersection corresponding to the vertex x by X_i . For example, the vertex $a \in V(H)$ is represented by the intersection of three squares A_1 , A_2 , and A_3 . Furthermore, each square X_i corresponds to a vertex x_i of G . For instance, the squares A_1 , A_2 , and A_3 correspond to the vertices a_1 , a_2 , and a_3 , respectively.

Thus, each vertex $x \in V(H)$ corresponds to a triple point $X_1 \cap X_2 \cap X_3$. From each such triple point, six half-edges start: two for each intersection $X_1 \cap X_2$, $X_1 \cap X_3$, and $X_2 \cap X_3$.

For each unordered pair of edges $\{x_i, y_j\}, \{x_k, y_l\} \in E(G)$ with the same image, where $f(x_i) = f(x_k) = x$ and $f(y_j) = f(y_l) = y$, we perform the following procedure.

We choose one of the half-edges from $X_i \cap X_k$ and one of the half-edges lying in $Y_j \cap Y_l$ that are currently “free” (that is, not chosen in previous steps). Then, we connect these half-edges by an X-shaped ribbon,¹⁴ so that one plane of the X-shaped ribbon connects the side of the square X_i with the side of the square Y_j , and the other one connects the sides of the squares X_k and Y_l , see Fig. 5.

Note that each vertex of $\tilde{G}_f^{(2)}$ has degree two. This ensures that there is always a free half-edge to choose. Moreover, after processing all the pairs of edges, each half-edge belongs to some X-shaped ribbon.

Observe that each X-shaped ribbon could be twisted while connecting. Additionally, when performing the steps described above, there is flexibility in choosing which half-edges to connect with each other. However, in our construction, this choice is not crucial; hence, we select free half-edges and twist the ribbons in an arbitrary way.

Next, we take small 3-balls around each triple point (that is, the points $X_1 \cap X_2 \cap X_3$ inside the 3-intersections of squares) and small 3-tubes around each curve of double points (that is, half-edges connected by the central curve of an X-ribbon). This construction gives us a handlebody B . The intersections of squares connected by the X-shaped ribbons within B form a generic surface S immersed into B . Denote the immersion by g . Note that the multiple points set of g is the graph H embedded into the interior of B , and its preimage is the graph G embedded into S . The boundary of S is a circle embedded into the boundary of B .

Furthermore, $g|_{|G|}$ acts exactly as f on the curves representing the edges of G embedded into S : g takes the curve between points x_i and y_j to the curve between points $f(x_i)$ and $f(y_j)$.

To obtain a closed 3-manifold Y , we take the double of B , which is two copies of B glued along their common boundary. The double of S , denoted as T , is then immersed into Y by the immersion j induced by $g \sqcup g: S \sqcup S \rightarrow B \sqcup B$. Clearly, the immersion j is in general position. The multiple point set of j is a graph $H \sqcup H$ embedded into Y , and its preimage is a graph $G \sqcup G$ embedded into T . Moreover, it is clear that $j|_{|G| \sqcup |G|}$ acts on the curves representing the edges of $G \sqcup G$ as $f \sqcup f$.

¹⁴That is, the product of two intersecting segments (forming the letter X) with another segment.

Let us note that the set $S_{2,3}$ in the definition of $\sim_\nu$ for the map g consists of 24 points. Each point of $S_{2,3}$ represents an ordered pair of vertices in G with the same image under g , or, equivalently, under f . In other words, the set $S_{2,3}$ coincides with the set of vertices of the graph $G_f^{(2)}$. Furthermore, it is straightforward to verify that the equivalence relation defined by the partition of the vertices of $G_f^{(2)}$ into connected components satisfies all the properties in the definition of $\sim_\nu$; therefore, these equivalence relations coincide. Given that, it becomes apparent that $\nu_3(g) = 0$. It is also evident that $\mu_2(g) = 0$ since $p_2: |G_f^{(2)}| \rightarrow |\tilde{G}_f^{(2)}|$ is trivial.

As all the multiple points of g lie in the interior of B , the set $S_{2,3}$ and the relation $\sim_\nu$ for j coincide with those of $g \sqcup g$. Therefore, since $\nu_3(g) = \mu_2(g) = 0$, we have $\nu_3(j) = \mu_2(j) = 0$.

If j lifts to an embedding then so does f . However, as shown in [Example 3](#), the formula Γ_f is not satisfiable, and consequently, f does not lift to an embedding. $\square$

It can be observed that [Example 3](#) also provides a counterexample to [\[21, Lemme 1.7\]](#). One can notice that the preimage of the cycle shown in [\[21, Fig. 1.3\]](#) does not necessarily have to be the two cycles as shown in [\[21, Fig. 1.4\]](#), as the paper claims, which is where the proof fails.

It is worth noting that the mentioned mistake only affects [\[21, Lemme 1.7\]](#), which states that a “labyrinth” (that is, a map f_j between graphs, obtained using a given immersion j of manifolds; see [\[21, Definition 3\]](#) for details) lifts if $\mu_2(f_j) = \nu_3(f_j) = 0$. However, the reduction from the considered case of immersions of manifolds to maps between graphs is itself correct. Thus, the immersion j lifts if and only if f_j lifts, or, equivalently, the covering map p_2 for f_f is trivial, and Γ_{f_j} is satisfiable. Our next goal is to show that these conditions cannot be weakened in the general case: we will demonstrate that for any suitable boolean formula Γ it is possible to construct a generic immersion j with $\Gamma_{f_j} = \Gamma$.

Notice that any Γ_f , as defined in the previous section, satisfies the following condition: once it contains a clause of the form $(\alpha \wedge \beta) \rightarrow \gamma$, it also contains a clause $(\neg \alpha \wedge \neg \beta) \rightarrow \neg \gamma$, up to double negation elimination. Additionally, let us note that the formulas that cannot be simplified by the algorithm described at the end of the previous section are those where, for each clause $(\alpha \wedge \beta) \rightarrow \gamma$, all variables constituting α , β , and γ are distinct.

The next theorem shows that any Γ of the form described above can be realised as Γ_f of a multigraph homomorphism f induced by a generic immersion of a surface into a handlebody, similar to what we have constructed in the previous theorem.

Theorem 6. *Let $\Gamma(x_1, \dots, x_n)$ be a boolean formula of the form*

$$\Gamma(x_1, \dots, x_n) = \bigwedge_{j=1}^m ((\alpha_j \wedge \beta_j) \rightarrow \gamma_j)$$

where α_j , β_j , and γ_j are variables or their negations. Assume that for every j

- (1) the variables in the literals α_j , β_j , and γ_j are distinct,*

- (2) there exists j' such that $\alpha_{j'} = \neg\alpha_j$, $\beta_{j'} = \neg\beta_j$, and $\gamma_{j'} = \neg\gamma_j$ (after double negation elimination).

Then there exists a generic immersion $g: S \looparrowright B$ of a surface S with boundary into a handlebody B such that the restriction of g onto the preimage of the set of multiple points induces a homomorphism $f: G \rightarrow H$ of multigraphs. Additionally, the formula Γ_f is equivalent to Γ , differing only in variable renaming, the presence of duplicate clauses, and reordering of clauses and literals within clauses.

To prove the theorem, we will use the following lemma, which says, informally, that for each Γ as in [Theorem 6](#), we can construct a multigraph homomorphism $f: G \rightarrow H$, whose Γ_f is equivalent to Γ , and f is similar to that we used in the construction of [Theorem 5](#) which allows us, following this construction, to obtain the needed immersion $g: S \looparrowright B$.

Lemma 2. *Let $\Gamma(x_1, \dots, x_n)$ be the same boolean formula as the one in [Theorem 6](#).*

Then, there is a multigraph homomorphism $f: G \rightarrow H$, such that

- (1) $p_2: |G_f^{(2)}| \rightarrow |\tilde{G}_f^{(2)}|$ is trivial,
- (2) Γ_f is equivalent to Γ in the sense of [Theorem 6](#),
- (3) the edges of H have two preimages,
- (4) the vertices of H have three preimages,
- (5) the vertices of G have degree 4,
- (6) the vertices of H have degree 6,
- (7) the vertices of the graph $G_f^{(2)}$ have degree two, or, equivalently, $G_f^{(2)}$ is a disjoint union of circles.

Proof. First note that all disjunctions that constitute Γ split into pairs $(\alpha \wedge \beta) \rightarrow \gamma$, $(\neg\alpha \wedge \neg\beta) \rightarrow \neg\gamma$. After selecting only one disjunction from each pair, let us renumber them from 1 to $m/2$. Thus, we have $m/2$ disjunctions $(\alpha_1 \wedge \beta_1) \rightarrow \gamma_1, \dots, (\alpha_{m/2} \wedge \beta_{m/2}) \rightarrow \gamma_{m/2}$.

We will construct the graphs G and H and the map $f: G \rightarrow H$ iteratively. To begin with, let G and H be empty graphs.

For each disjunction $(\alpha_j \wedge \beta_j) \rightarrow \gamma_j$, add three vertices v_1^j, v_2^j , and v_3^j to the graph G , and add one vertex v^j to the graph H , and define a map $f(v_i^j) = v^j$ on the new vertices v_i^j .

Next, we define a set S_i for each variable x_i , which, at the end of the procedure described below, will contain all the vertices of the connected component associated with the variable x_i . For simplicity, if $y = x_i$ is a variable, denote the corresponding set S_i by $S(y)$.

Initially, let all $S(y)$ be empty. Then, for each of the $m/2$ disjunctions $(\alpha_j \wedge \beta_j) \rightarrow \gamma_j$, we perform the following steps:

- (1) If $\alpha_j = x$, add (v_1^j, v_2^j) to $S(x)$. Otherwise, if $\alpha_j = \neg x$, add (v_2^j, v_1^j) to $S(x)$.
- (2) Similarly, if $\beta_j = y$, add (v_2^j, v_3^j) to $S(y)$. If $\beta_j = \neg y$, add (v_3^j, v_2^j) to $S(y)$,
- (3) Finally, if $\gamma_j = z$, add (v_1^j, v_3^j) to $S(z)$. If $\gamma_j = \neg z$, add (v_3^j, v_1^j) to $S(z)$.

After performing these steps, for each j and for any $1 \leq i \neq k \leq 3$, exactly one of the pairs (v_i^j, v_k^j) and (v_k^j, v_i^j) belongs to some set S_l . Furthermore, $S_l \cap S_t = \emptyset$ for $l \neq t$, and $S_l \cap \tau(S_t) = \emptyset$ for any l and t . Here $\tau(S_t) = \{(y, x) \mid (x, y) \in S_t\}$ is the set of “reversed” pairs in S_t .

We proceed by choosing arbitrary linear orders on each set S_l . Using these orders, we will add edges to G and H as follows.

Let $S_l = \{(v_{i_1}^{j_1}, v_{k_1}^{j_1}), (v_{i_2}^{j_2}, v_{k_2}^{j_2}), \dots, (v_{i_s}^{j_s}, v_{k_s}^{j_s})\}$. For each $r = 1, \dots, s-1$ we add a new edge $e_{l,r}$ between $v_{i_r}^{j_r}$ and $v_{i_{r+1}}^{j_{r+1}}$ in H .¹⁵ We add two edges $e_{l,r}^1$ and $e_{l,r}^2$ connecting $v_{i_r}^{j_r}$ with $v_{i_{r+1}}^{j_{r+1}}$, and $v_{k_r}^{j_r}$ with $v_{k_{r+1}}^{j_{r+1}}$, respectively, to H , and set $f(e_{l,r}^1) = f(e_{l,r}^2) = e_{l,r}$.

Additionally, we add an edge $e_{l,s}$ between $v_{i_s}^{j_s}$ and $v_{i_1}^{j_1}$ to H , and two edges $e_{l,s}^1$ and $e_{l,s}^2$ between $v_{i_s}^{j_s}$ and $v_{i_1}^{j_1}$, and between $v_{k_s}^{j_s}$ and $v_{k_1}^{j_1}$, respectively, to G . We set $f(e_{l,s}^1) = f(e_{l,s}^2) = e_{l,s}$.

It is evident that at each step, two edges $((v_{i_r}^{j_r}, v_{k_r}^{j_r}), (v_{i_{r+1}}^{j_{r+1}}, v_{k_{r+1}}^{j_{r+1}}))$ and $((v_{k_r}^{j_r}, v_{i_r}^{j_r}), (v_{k_{r+1}}^{j_{r+1}}, v_{i_{r+1}}^{j_{r+1}}))$ are added to $G_f^{(2)}$. Thus, after processing a single set S_l , we obtain in $G_f^{(2)}$ two circles on the vertices of S_l and $\tau(S_l)$. Therefore, after processing all the sets S_l , the graph $G_f^{(2)}$ consists of a disjoint union of circles on the vertices of the sets S_l and $\tau(S_l)$.

It is clear that $p_2: |G_f^{(2)}| \rightarrow |\tilde{G}_f^{(2)}|$ is trivial because $S_l \cap \tau(S_l) = \emptyset$ for any S_l .

Additionally, if we associate each S_l to a variable x_l , it can be seen that each triple $\{v_1^j, v_2^j, v_3^j\}$ contributes two different disjunctions $(\alpha_j \wedge \beta_j) \rightarrow \gamma_j$ and $(\neg\alpha_j \wedge \neg\beta_j) \rightarrow \neg\gamma_j$ to Γ_f . Actually, it adds 6 disjunctions, one for each permutation of the triple, but four of them can be derived from the other two by permutation of their terms, so we consider them as duplicates.

Therefore, since these and only these disjunctions form the formula Γ , the formulas Γ_f and Γ become equivalent after elimination of the duplicates.

It follows from the construction that the vertices of H have 3 preimages, and the edges of H have 2 preimages. Furthermore, from the latter fact it follows that the map $\hat{f}: \tilde{G}_f^{(2)} \rightarrow H$ induced by $f: G \rightarrow H$ is injective on the edges; thus, because each vertex v of H have 3 preimages under $\hat{f}$ (that is, unordered pairs of $f^{-1}(v)$), and each such preimage have degree two,¹⁶ the vertices of H have degree six.

Finally, let us note that each vertex x of G is included into exactly 2 vertices of $\tilde{G}_f^{(2)}$ (recall that they are unordered pairs of distinct vertices of G with the same image), and both of them have degree two. The construction provides that each such vertex corresponds to a pair of edges in G incident with the vertex x . Therefore, the vertices of G have degree four. $\square$

¹⁵If there are edges between $v_{i_r}^{j_r}$ and $v_{i_{r+1}}^{j_{r+1}}$ in H , we add a new one.

¹⁶Recall that $G_f^{(2)}$ is a disjoint union of circles, so $\tilde{G}_f^{(2)}$ is as well.

Proof of Theorem 6. By using Lemma 2 we obtain a multigraph homomorphism $f: G \rightarrow H$ whose formula Γ_f is equivalent to Γ . To construct a generic immersion $g: S \looparrowright B$ let us follow the construction of $g: S \looparrowright B$ in Theorem 5.

To see that it is applicable in this case, let us note that each vertex v of H has degree 6, and each unordered pair of its distinct preimages form a vertex in $\tilde{G}_f^{(2)}$ of degree 2, hence it is possible to represent v as an intersection of 3 squares, and assign a half-edge coming from the triple point of this intersection to each edge in $\tilde{G}_f^{(2)}$ incident to the pair of preimages of v . $\square$

It is worth noting that B can be embedded in $\mathbb{R}^3$. Thus, a natural question arises: for which examples is it possible to glue the boundary of S to a surface embedded in $\mathbb{R}^3 \setminus B$ such that we obtain a generic immersion $S \looparrowright \mathbb{R}^3$ that realises Γ ?¹⁷

In some cases, such as the examples from Theorem 5 and certain examples obtained from the construction in Example 3 by joining the endpoints of the connected components of $G_f^{(2)}$ to obtain circles, it can be shown that it is not possible to glue the boundary of S in such a way. One can demonstrate this by checking that the boundary of S is not null-homologous in $\mathbb{R}^3 \setminus H$. With a computer, checking this is straightforward, using a sort of Wirtinger presentation for $\pi_1(\mathbb{R}^3 \setminus H, \mathbb{Z})$ as described in [4, Proposition 9.1.9].

¹⁷Note that this question is different from the [5, Proposition 4.19] and [13], where it is proven that an appropriate graph H can be realised as the multiple point set of a generic immersion $S \looparrowright \mathbb{R}^3$.

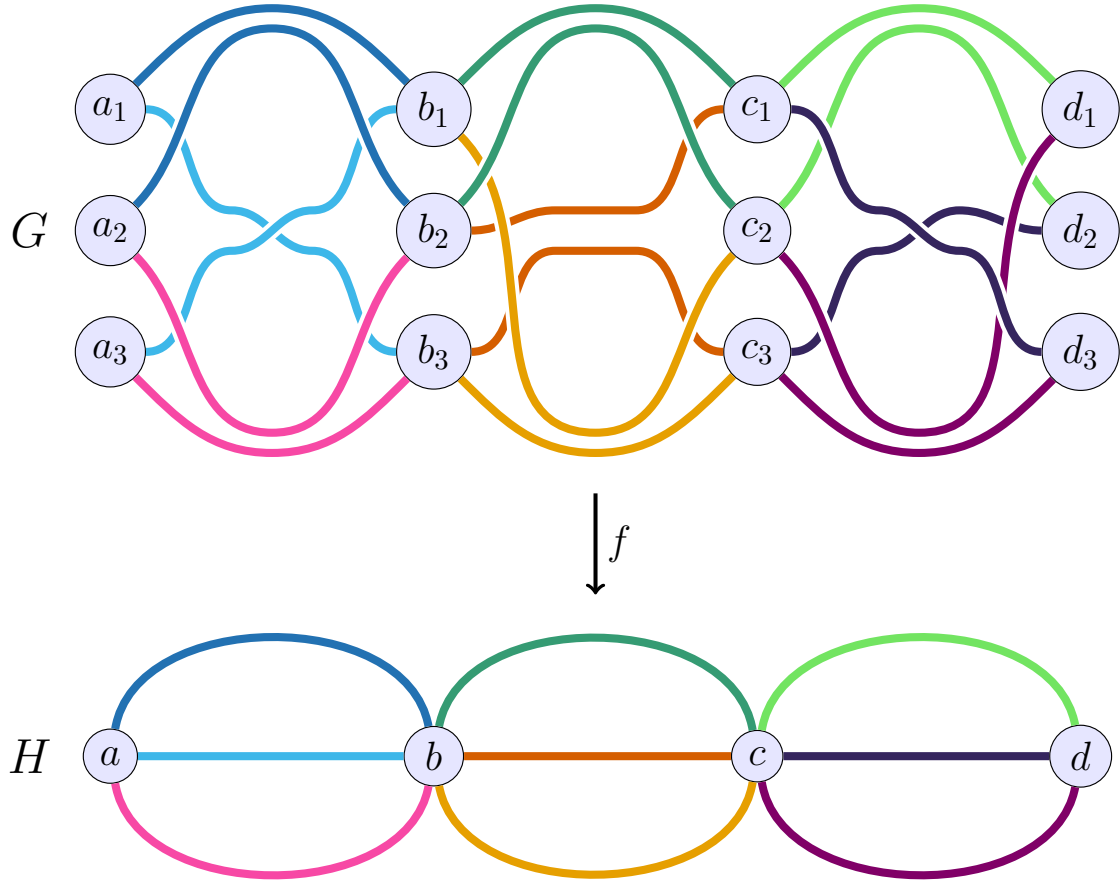
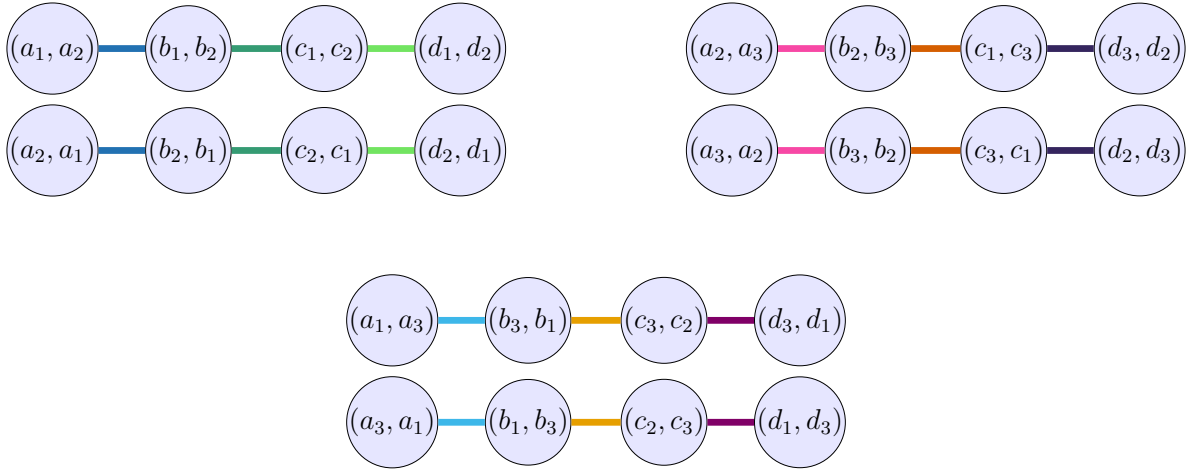
(A) The map $f: G \rightarrow H$.(B) The graph $G_f^{(2)}$.

FIGURE 4. An example of a map with “nontrivial” Γ_f . The edges of the graph G and their images in the graph H are coloured identically, the same holds for edges of $G_f^{(2)}$ and edges of G corresponding to them. Furthermore, edges of G with the same image are placed closely together, and the “upper” pairs of edges of G map to the “upper” edges of H ; the same holds for the “middle” and the “lower” pairs.

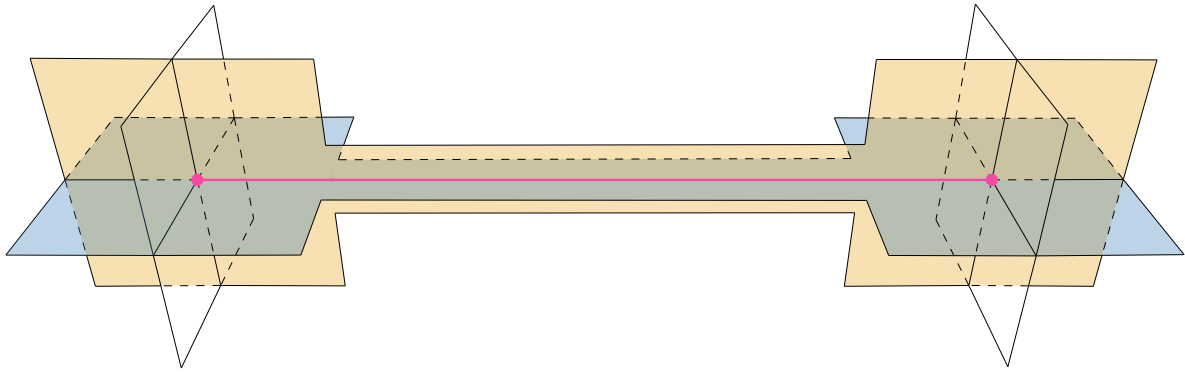


FIGURE 5. Two triples of intersecting squares, representing triple points, and an X-ribbon between them, representing a double edge. The preimages of the pink double edge are an edge lying on the blue squares and the blue ribbon between them, and an edge lying on the orange squares and the orange ribbon.

4. APPROXIMATION BY EMBEDDINGS

In this section, we will establish a connection between the existence of a lifting to an embedding and the concept of approximation by embeddings for generic maps. We will define the notion of generic maps in our context and show that the condition of the triviality of the covering map p_2 becomes sufficient for the existence of a lifting in the special case of generic maps from trees to segments.

Before we proceed, let us introduce some definitions.

Definition (approximability by embeddings). Let $f: G \rightarrow H$ be a simplicial map between graphs, and let $i: |H| \hookrightarrow S$ be a continuous embedding of $|H|$ into a surface S with a metric d . We say that f is **approximable by embeddings** if for every $\varepsilon > 0$, there exists a continuous embedding $j: |G| \hookrightarrow S$ such that $d(j(x), i(|f|(x))) < \varepsilon$ for every $x \in |G|$.

It is worth noting that the approximability of f generally depends on the embedding i . However, as we will demonstrate later, it does not depend on the choice of a metric on S . Moreover, as we will show later, the approximability of f is equivalent to the existence of a compatible with f embedding of $|G|$ into a neighborhood of $i(|H|)$ in S , and, therefore, depends only on the geometry of this neighborhood.

Definition (ribbon graph, see [8, Definition 1.5] or [18, Definition 2.4]). A **ribbon graph** $\mathcal{H}$ is a surface with boundary represented as the union of two sets of discs: a set $V(\mathcal{H})$ of vertices or 0-handles, and a set $E(\mathcal{H})$ of edges or 1-handles, satisfying the following conditions:

- (1) $D_v \cap D_w = \emptyset$ and $D_e \cap D_g = \emptyset$ for different vertices $v, w \in V(\mathcal{H})$ and different edges $e, g \in E(\mathcal{H})$,
- (2) for each edge D_e there are exactly two vertices D_v and D_w that intersect D_e ; moreover, $D_e \cap D_v$ and $D_e \cap D_w$ are two disjoint arcs lying in the boundary of D_e , and in the boundaries of D_v and D_w , respectively.

By [12, Theorem 1.3], an embedded graph $f: |H| \rightarrow S$ lies in a triangulation of S ; that is, there are a homeomorphism $h_s: S \rightarrow |K_s|$, a piecewise linear homeomorphism $g_s: |H| \rightarrow |H_s|$, where H_s subdivides H , and a simplicial embedding $f_s: H_s \hookrightarrow K_s$, such that $f = h_s^{-1} \circ |f_s| \circ g_s$.¹⁸ Recall that the surface S admits a unique piecewise linear structure.¹⁹ Thus, $|f_s|$ is uniquely defined by f up to piecewise linear homeomorphisms of $|H_s|$ and $|K_s|$.

Therefore, given an embedding $f: |H| \rightarrow S$, let the surface $\mathcal{H}$ be a regular neighborhood of $f(|H|)$ in S , decomposed into discs around the images of the vertices of H (these discs

¹⁸In fact, by applying [12, Theorem 1.3] we obtain not a simplicial complex, but a polyhedron with a graph embedded as a 1-subpolyhedron; however, it is easy to obtain a piecewise linear homeomorphism $g_s: |H| \rightarrow |H_s|$ and a simplicial embedding $f_s: H_s \rightarrow K_s$ by taking suitable triangulations of these polyhedra.

¹⁹See, for example, [20, Theorem 5, Chapter 8].

form the vertices of the ribbon graph $\mathcal{H}$), and strips around the edges of H (these strips, similarly, form the edges of $\mathcal{H}$).²⁰

Explicitly, assume we have a simplicial embedding $f_s: H_s \hookrightarrow K_s$. Take the second derived subdivisions H_s'' and K_s'' of H_s and K_s , respectively, and denote by f'' the induced simplicial embedding $f'': H_s'' \hookrightarrow K_s''$. Denote by $N(H_s'', K_s'')$ the simplicial neighborhood of $f''(H_s'')$ in K_s'' , that is, the simplicial complex consisting of the simplices that meet $f''(H_s'')$, with their faces. Hence $|N(H_s'', K_s'')|$ is a regular neighborhood of $f(H)$ in S . As discussed earlier, we define $\mathcal{H}$ to be $|N(H_s'', K_s'')|$.

To obtain a decomposition of $\mathcal{H}$ into edges and vertices, for each vertex $v \in V(H)$, let us consider its image under the composition $F = |f''| \circ g: |H| \rightarrow |H_s''| \hookrightarrow |K_s''|$, where g is a homeomorphism $|H| \cong |H_s''|$, and take the star of $F(v)$ in $N(H_s'', K_s'')$ as the disc $D_v \in V(\mathcal{H})$.²¹

It can be seen that these discs split $\mathcal{H}$ into a set of strips, with each strip intersecting $f(|H|)$ along exactly one curve, which is a portion of the image of one of the edges of H . Thus, each edge e of the graph H corresponds to a strip D_e containing a portion of its image. Let us take these strips as edges of the ribbon graph $\mathcal{H}$.

Since a regular neighborhood of a subpolyhedron is uniquely defined up to piecewise linear homeomorphism, see [24, Theorem 3.8], a ribbon graph of a graph embedded into a surface is also uniquely defined.

Furthermore, it follows that there is a natural bijection between the vertices and the edges of a graph H embedded into a surface S , and the vertices and the edges of an induced ribbon graph $\mathcal{H}$. Thus, taking this into account, we will label the vertices and the edges of H and $\mathcal{H}$ by the same letters; for example, a vertex $v \in V(H)$ corresponds to a vertex $D_v \in V(\mathcal{H})$, and an edge $e = \{v, w\} \in E(H)$ corresponds to an edge $D_e = D_{\{v, w\}} \in E(\mathcal{H})$.

It is also important to note that, although our definition of a ribbon graph induced by an embedding differs from the definition of the normal neighborhood in [18], the ribbon graph $\mathcal{H}$ in our construction is also a normal neighborhood in the sense of [18, Definition 2.4]; indeed, all conditions from the latter definition, except the second one, are automatically satisfied. However, it is easy to see that for each edge e incident to a vertex v of H , its image $f(|e|)$ intersects the boundary of D_v at exactly one point. Therefore, it follows that the condition (2) from [18, Definition 2.4] is also satisfied. This allows us to use results from [18], using ribbon graphs in place of normal neighborhoods.

Definition ([18, Definition 2.7]). Let $f: G \rightarrow H$ be a simplicial map between graphs, and $\mathcal{H}$ be a ribbon graph corresponding to an embedding of $|H|$ into a surface.

We say that f is **R-approximable by an embedding** if there exists a continuous embedding $j: |G| \hookrightarrow \text{Int } \mathcal{H}$ called an **R-approximation**, satisfying the following conditions:

- (1) for a vertex $v \in V(G)$, $j(v) \in D_{f(v)} \in V(\mathcal{H})$,

²⁰For the definitions of regular neighborhoods, derived subdivisions, and simplicial neighborhoods, the reader may refer to [24].

²¹Note that the image $F(v)$ is a vertex of K_s'' since H_s'' subdivides H .

- (2) if $f(\{v, w\})$ is a vertex of H , then $j(|\{v, w\}|) \subset D_{f(\{v, w\})}$, where $|\{v, w\}|$ is a segment representing the edge $\{v, w\}$ in $|G|$,
- (3) if $f(\{v, w\})$ is an edge of H , then $j(|\{v, w\}|)$ is a simple arc intersecting $D_{f(v)} \cap D_{\{f(v), f(w)\}}$ at a single point p and $D_{f(w)} \cap D_{\{f(v), f(w)\}}$ at a single point q .²²

If j is a general position map $j: |G| \rightarrow \text{Int } \mathcal{H}$ satisfying these conditions, we call j a **generic R-approximation** of f .

Lemma 3 ([18, Propositions 2.9]). *Let $f: G \rightarrow H$ be a simplicial map between graphs, and let $i: |H| \hookrightarrow S$ be an embedding of H into a surface S with a metric. Then the following statements are equivalent:*

- (1) f is approximable by embeddings.
- (2) f is R -approximable by an embedding.

From this point onwards, we focus on stable maps between graphs. The definition provided below is a one-dimensional simplicial version of the definition of piecewise linear stable maps from [16, Appendix B].

Definition. Let $f: G \rightarrow H$ be a simplicial map between graphs.

We call a vertex $v \in G$ **regular** with respect to f if f is bijective on the star $\text{st}(v) = \bigcup_{e \in E(G), v \in e} e$.

The map f is called **stable** if, for any vertex $w \in V(H)$,

- (1) if $\deg(w) \neq 2$, all the vertices in $f^{-1}(w)$ are regular,
- (2) if $\deg(w) = 2$, $f^{-1}(w)$ contains at most one non-regular vertex.

If a simplicial map f is stable, it must also be non-degenerate. Indeed, if $f(e) = v$, where e is an edge and v is a vertex, the endpoints of e are non-regular with respect to f , implying that f is not stable.

Now we are ready to state the theorem.

Theorem 7. *Let $f: G \rightarrow J$ be a stable simplicial map from a graph G to a path graph J (that is, a triangulation of a segment). Let $i: |J| \hookrightarrow \mathbb{R}^2$ be an embedding. Then the following statements are equivalent:*

- (1) f is approximable by embeddings with respect to $i: |J| \rightarrow \mathbb{R}^2$.
- (2) $|f|$ lifts to an embedding.

Let us start by proving a simple auxiliary lemma.

Lemma 4. *Let S be a rectangular region $[C, D] \times [H, L]$ in $\mathbb{R}^2$. Let α and β denote the sides $\{C\} \times [H, L]$ and $\{D\} \times [H, L]$, respectively. Let γ be a simple curve connecting a point $a \in \alpha$ and a point $b \in \beta$, such that $\gamma \cap (\alpha \sqcup \beta) = \{a, b\}$. Assume Φ is a path-connected subset of S not intersecting γ , such that $\Phi \cap \alpha$ and $\Phi \cap \beta$ are finite sets of points.*

²²Thus, the points $x = j^{-1}(p)$ and $y = j^{-1}(q)$ decompose $|\{v, w\}|$ into three subsegments $|\{v, x\}|$, $|\{x, y\}|$, and $|\{y, w\}|$, where $j(|\{v, x\}|) \subset D_{f(v)}$, $j(|\{x, y\}|) \subset D_{\{f(v), f(w)\}}$, and $j(|\{y, w\}|) \subset D_{f(w)}$.

Then, for all $p \in (\Phi \cap \alpha) \sqcup (\Phi \cap \beta)$, either $p_y < s(p)_y$ or $p_y > s(p)_y$ holds simultaneously. Here $s(p) = a$ if $p \in \alpha$, $s(p) = b$ if $p \in \beta$, and q_y denotes the y -coordinate of a point q .

Proof. Suppose there exists a pair of points $p, q \in (\Phi \cap \alpha) \sqcup (\Phi \cap \beta)$ such that $p_y < s(p)_y$ and $q_y > s(q)_y$, or $p_y > s(p)_y$ and $q_y < s(q)_y$. Since Φ is path-connected and does not intersect γ , there must be a path ϕ in Φ (and thus in $S \setminus \gamma$) connecting p and q . However, according to [20, Theorem 8, Chapter 2], $S \setminus \gamma$ consists of two path-connected components with p and q lying in different components. This contradicts the existence of the path ϕ . $\square$

Proof of Theorem 7. First, let us number the vertices of J according to an arbitrary orientation on J and denote them by $v_0, \dots, v_N$.

Observe that any embedding $i: |J| \hookrightarrow \mathbb{R}^2$ induces the same ribbon graph $\mathcal{J}$ up to a piecewise linear homeomorphism.²³ Thus, it follows from Lemma 3 that the approximability of f by embeddings does not depend on the embedding i .

Therefore, for convenience, we can assume that i embeds $|J|$ into $\mathbb{R}^2$ with the Euclidean distance d_2 as a line segment with $i(v_k) = (2k + \frac{1}{2}, 0)$, and $\mathcal{J}$ is a ribbon $[0, 2N + 1] \times [-1, 1]$ in $\mathbb{R}^2$, where $D_{v_k} = [2k, 2k+1] \times [-1, 1]$ and $D_{\{v_k, v_{k+1}\}} = [2k+1, 2(k+1)] \times [-1, 1]$.²⁴

Now suppose that $|f|$ lifts to an embedding $\widetilde{|f|}: |G| \rightarrow |J| \times \mathbb{R}$. Let M be the maximum of $|h(p)|$, where $h(p) = \text{pr}_{\mathbb{R}} \widetilde{|f|}(p)$. For any $\varepsilon > 0$, let $F: |G| \rightarrow \mathbb{R}^2$ be the composition $S_\varepsilon \circ |f|$ where $S_\varepsilon(q, y) = (i(q)_x, \frac{\varepsilon y}{2M})$; here $i(q)_x$ stands for the x -coordinate of $i(q)$. Therefore, putting $q = f(p)$, we have

$$d_2(F(p), i(q)) = d_2\left(\left(i(q)_x, \frac{\varepsilon h(p)}{2M}\right), (i(q)_x, 0)\right) = \left|\frac{\varepsilon h(p)}{2M}\right| < \varepsilon.$$

Thus, f is approximable by embeddings.

Now let us prove the reverse implication. Suppose f is approximable by embeddings. By Lemma 3, there exists an R-approximation $j: |G| \hookrightarrow \text{Int } \mathcal{J}$. We are going to define orders on the sets $f^{-1}(v_k)$ and $f^{-1}(\{v_k, v_{k+1}\})$. It will be more convenient to start with the sets $f^{-1}(\{v_k, v_{k+1}\})$.

Consider the images of the edges from $f^{-1}(\{v_k, v_{k+1}\})$ in the rectangular regions $D_{\{v_k, v_{k+1}\}}$. It follows from the definition of R-approximation that $j(|e|) \cap D_{\{v_k, v_{k+1}\}}$ for an edge $e \in f^{-1}(\{v_k, v_{k+1}\})$ is a simple curve γ_e going from the left side of $D_{\{v_k, v_{k+1}\}}$ to its right side, and intersecting the boundary of $D_{\{v_k, v_{k+1}\}}$ only at the endpoints. Hence, by applying Lemma 4, for each pair of edges $e, g \in f^{-1}(\{v_k, v_{k+1}\})$, the endpoints of γ_e have both either larger or smaller y -coordinates than the endpoints of γ_g lying in the corresponding sides of the rectangle. Therefore, we can define a linear order by putting $e \prec_{k, k+1} g$ if both endpoints of γ_e have smaller y -coordinates than the corresponding endpoints of γ_g .

²³Given two distinct ribbon graphs $\mathcal{J}$ and $\mathcal{J}'$, we can take arbitrary homeomorphisms between arcs $D_v \cap D_e$ and $D'_v \cap D'_e$ and then extend them to a homeomorphism $\mathcal{J} \cong \mathcal{J}'$ that preserves the handle decompositions of $\mathcal{J}$ and $\mathcal{J}'$.

²⁴ $\mathcal{J}$ collapses onto $i(|J|)$, hence it is a regular neighborhood by [24, Corollary 3.30].

Now let us proceed to the sets $f^{-1}(v_k)$. The situation with them, although slightly more complex, is essentially the same. In what follows, we put $\gamma_w = j(|\text{st } w|) \cap D_{f(w)}$ for a vertex $w \in V(G)$.

First note that, since f is stable, all the vertices in $f^{-1}(v_0)$ and $f^{-1}(v_N)$ have degree one, and for any $w \in f^{-1}(v_0)$ the set γ_w is a simple curve connecting $j(w)$ with a point on the right side of ∂D_{v_0} , and this point is the only intersection of γ_w with the boundary of D_{v_0} . Therefore, we can order the vertices $\{w_i\}$ in $f^{-1}(v_0)$ with respect to the order of the y -coordinates of the points $\gamma_{w_i} \cap \partial D_{v_0}$, which gives us a linear order $\prec_0$ on $f^{-1}(v_0)$.

The same holds true for vertices from the set $f^{-1}(v_N)$, with the only difference being that the curves γ_w for $w \in f^{-1}(v_N)$ connect $j(w)$'s with points on the left side of D_{v_N} . As for $f^{-1}(v_0)$, we can order the vertices in $f^{-1}(v_N)$ according to the order of the y -coordinates of the points $\gamma_w \cap \partial D_{v_N}$.

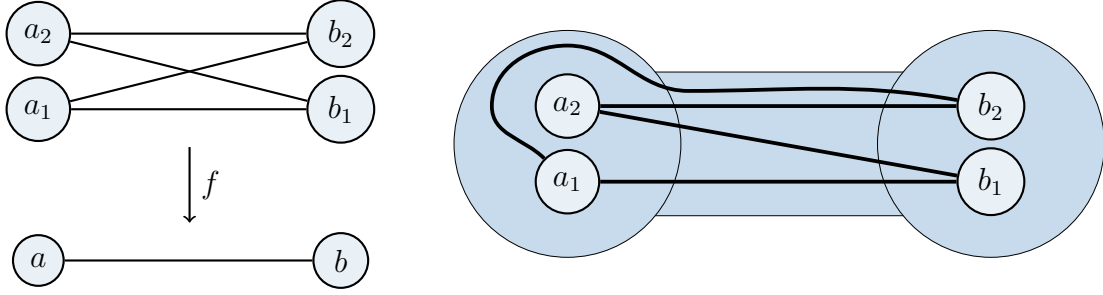
Now consider the set $f^{-1}(v_k)$, where $0 < k < N$. Since f is stable, for any pair of vertices $w, s \in f^{-1}(v_k)$, one of the sets γ_w and γ_s is a simple curve connecting a point on the left side of D_{v_k} and a point on its right side, and intersecting the boundary of D_{v_k} only at these two points. At the same time, the other set is clearly path-connected, and it intersects the boundary of D_{v_k} in a finite number of points, these points will be referred to as “endpoints”. Therefore, by applying [Lemma 4](#) we get that the y -coordinates of the endpoints of γ_w are all larger or all smaller than the y -coordinates of the endpoints of γ_s once we compare only the pairs of points lying on the same side of D_{v_k} . Hence we can define a linear order $\prec_k$ by letting $w \prec_k s$ if all the endpoints of γ_w lying on the left and right sides of D_{v_k} have the smaller y -coordinates than all the endpoint of γ_s lying on the same sides.

Finally, we are going to prove that $\{\prec_k, 0 \leq k \leq N\}$ forms an admissible collection of linear orders, and therefore, $|f|$ lifts to an embedding according to [Theorem 2](#). Indeed, take a pair of edges $e = \{w_e, s_e\}, g = \{w_g, s_g\}$ of G such that $f(w_e) = f(w_g) = v_k$ and $f(s_e) = f(s_g) = v_{k+1}$. Without loss of generality, assume $e \prec_{k,k+1} g$. This implies that the “left” endpoint of γ_e , lying on the left side of $D_{\{v_k, v_{k+1}\}}$, has a smaller y -coordinate than the “left” endpoint of γ_g . However, the “left” endpoints of γ_e and γ_g are endpoints of γ_{w_e} and γ_{w_g} , lying on the same side of D_{v_k} ; hence, $w_e \preceq_k w_g$. Repeating this reasoning for the “right” endpoints of γ_e and γ_g , we conclude that $s_e \preceq_{k+1} s_g$. Therefore, the orders $\prec_k$ form an admissible collection of linear orders. $\square$

The next example shows that the assumption that f is stable in [Theorem 7](#) is needed.

Example 4. Let f be a map $G \rightarrow J$, where G triangulates the circle S^1 , that is shown on the left-hand side of [Fig. 6](#). It is R-approximable by embeddings, see the right-hand side of [Fig. 6](#), therefore, it is approximable by embeddings. However, it does not lift to an embedding, since there is a 2-obstructor for f defined as the path $(a_1, a_2) \rightarrow (b_1, b_2) \rightarrow (a_2, a_1)$ in $G_f^{(2)}$.

Now we use some known results on the problem of existence of approximation by embeddings to obtain sufficient conditions for the existence of a lifting to an embedding in the case of stable maps from trees to segments.

FIGURE 6. $f: G \rightarrow I \hookrightarrow \mathbb{R}^2$ and its R-approximation.

Definition. Let $f: G \rightarrow H$ be a simplicial map between graphs, let $i: |H| \hookrightarrow S$ be an embedding of H into a surface S , and let $\mathcal{H}$ be a corresponding ribbon graph. We call a generic R-approximation $j: |G| \rightarrow \text{Int } \mathcal{H}$ a **$\mathbb{Z}_2$ -approximation** if for any pair of non-adjacent edges $e, g \in E(G)$ (i.e., $e \cap g = \emptyset$) the set $j(|e|) \cap j(|g|)$ is finite and consists of an even number of points.

f is called **$\mathbb{Z}_2$ -approximable** if there exists a $\mathbb{Z}_2$ -approximation of f .

The following theorem is proved in [9] (in general case) and in [27] (for the case when T is a trivalent tree):

Theorem 8 ([9, Corollary 3] and [27, Theorem 1.5]). *Let $f: T \rightarrow J$ be a simplicial map from a tree T to a path graph J , and assume that J is embedded into $\mathbb{R}^2$. If f is $\mathbb{Z}_2$ -approximable then f is approximable by embeddings.*

The variant of van Kampen obstruction for $\mathbb{Z}_2$ -approximability of f by embeddings described below is taken from [27] and [23].

Assume we have a simplicial map $f: G \rightarrow H$ between graphs G and H , and H is embedded into a surface S . Let $G^\square$ be the two-dimensional cubical complex $\bigcup \{e \times g \mid e, g \in E(G): e \cap g = \emptyset\}$. Clearly, the transposition $\tau \in S_2$ acts on $|G^\square|$ by $\tau(p \times q) = q \times p$. Let $\tilde{G}^\square$ be a two-dimensional cell complex $G^\square/S_2$, whose cells are images of the cubes of $G^\square$ under the quotient map. Therefore, 2-cells of $\tilde{G}^\square$ might be associated with unordered pairs of disjoint edges of G .

Take any generic R-approximation $j: |G| \rightarrow \mathcal{H}$ of f , where $\mathcal{H}$ stands for a ribbon graph corresponding to the embedding $|H| \hookrightarrow S$; by “generic” here we mean that j is injective on $|V(G)|$, and for any pair of edges $e, g \in E(G)$ the set $j(|e|) \cap j(|g|)$ consists of a finite number of points.

Let $G_f^\square \subset G^\square$ be the subcomplex $\bigcup \{e \times g \mid e, g \in E(G): e \cap g = \emptyset, f(e) \cap f(g) = \emptyset\}$. Since $|G_f^\square|$ is closed under the involution on $|G^\square|$, the induced involution on $|G_f^\square|$ is well-defined. Let $\tilde{G}_f^\square$ be $G_f^\square/S_2$.

Let us define a cellular 2-cochain $c_v \in C^2(\tilde{G}^\square; \mathbb{Z}_2)$ by putting $c_v(e \cdot g) = |j(|e|) \cap j(|g|)| \bmod 2$, where $e \cdot g$ is the image of $e \times g$ under the projection $G^\square \rightarrow \tilde{G}^\square$. Observe that c_v is in fact a relative cochain lying in $C^2(\tilde{G}^\square, \tilde{G}_f^\square; \mathbb{Z}_2)$. Indeed, for edges $e, g \in E(G)$ with $f(e) \cap f(g) = \emptyset$, the curves $j(|e|)$ and $j(|g|)$ lie in different discs

with respect to the chosen generic R-approximation and therefore do not intersect. Let $v_f = [c_v] \in H^2(|\tilde{G}^\square|, |\tilde{G}_f^\square|; \mathbb{Z}_2)$ be the corresponding cohomology class.

It can be shown that v_f is well-defined by following [25, Lemmas 3.2–3.5]. These lemmas consider the case of the classical van Kampen obstruction, but it is easy to adapt them to our case.²⁵

Theorem 9 ([27, Proposition 3.1]). *Suppose $f: G \rightarrow H$ is a simplicial map between graphs G and H , and H is embedded into an oriented surface S . Then $v_f = 0$ if and only if f is $\mathbb{Z}_2$ -approximable.*

Now let us introduce a van Kampen-like obstruction for the existence of a lifting. We start with a cochain-free definition of the corresponding cohomology class inspired by [3] and [14]. Next, we present an explicit cochain representing it, defined in a way similar to the cochain c_v defining v_f .

Recall that f induces the principal S_2 -bundle $p_2: |G_f^{(2)}| \rightarrow |\tilde{G}_f^{(2)}|$. Let $g: |\tilde{G}_f^{(2)}| \rightarrow \mathbb{R}P^\infty$ be a map classifying it.²⁶ Let w_1 be the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$. We define $w_f \in H^1(|\tilde{G}_f^{(2)}|, \mathbb{Z}_2)$ as the pullback g^*w_1 .²⁷

It follows from the definition that w_f is the first Stiefel–Whitney class of the one-dimensional real vector bundle associated with the covering p_2 .²⁸ Indeed, this vector bundle is a pullback under g of the canonical line bundle over $\mathbb{R}P^\infty$, and the latter bundle is associated with the universal bundle $S^\infty \rightarrow \mathbb{R}P^\infty$. Therefore, $w_f = 0$ if and only if the bundle associated with p_2 is orientable, that is, admits a non-vanishing section. Hence $w_f = 0$ if and only if p_2 is trivial.

Now let $\tilde{f}: |G| \rightarrow |H| \times \mathbb{R}$ be a generic lifting of f ; as before, the word “generic” means that $\tilde{f}$ is injective on $V(G)$, and for any pair of edges $e, g \in E(G)$, the set $\tilde{f}(|e|) \cap \tilde{f}(|g|)$ consists of a finite number of points. Let c_w be a cochain in $C^1(\tilde{G}_f^{(2)}; \mathbb{Z}_2)$ that takes the value $|\tilde{f}(|e|) \cap \tilde{f}(|g|)| \bmod 2$ on an edge $e \cdot g \in E(\tilde{G}_f^{(2)})$, where by $e \cdot g$ we denote the edge of $\tilde{G}_f^{(2)}$ corresponding to an unordered pair of edges e and g .

Lemma. $w_f = [c_w]$.

Proof. Let $F: |G| \rightarrow |H| \times \mathbb{R} \times \mathbb{R}$ be an embedding such that $\tilde{f} = \text{pr}_{|H| \times \mathbb{R}} \circ F$, and let $\rho: |\tilde{G}_f^{(2)}| \rightarrow \mathbb{R}P^1$ be a map that takes an unordered pair $\{p, q\}$ of points of $|G|$ to an equivalence class of lines parallel to a vector going from the point $\text{pr}_{\mathbb{R} \times \mathbb{R}}(F(q))$ to the point $\text{pr}_{\mathbb{R} \times \mathbb{R}}(F(p))$.

Clearly, we can choose F so that ρ is a local homeomorphism at each $\{p, q\} \in |\tilde{G}_f^{(2)}|$ where $\tilde{f}(p) = \tilde{f}(q)$.

²⁵Note that v_f coincides with the classical van Kampen obstruction for graphs when $f: G \rightarrow \{p\} \hookrightarrow \mathbb{R}^2$.

²⁶The reader may refer to [19], [17], or [11] for the needed information on universal bundles and classifying maps.

²⁷See [17, Theorem 7.1] for a computation of the ring $H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

²⁸See [11, Theorem 3.1] or [17, Theorem 7.1].

It can be seen that p_2 is the pullback of the universal S_2 -bundle under the composition $i \circ \rho$, where i is the inclusion $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$; indeed, it follows from the universal property of pullbacks that there is a morphism between principal S_2 -bundles $(i \circ \rho)^*(S^\infty \rightarrow \mathbb{R}P^\infty)$ and p_2 , which must be an isomorphism by [19, Proposition 2.1].

As the embedding $i: \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ induces an isomorphism $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong H^1(\mathbb{R}P^1; \mathbb{Z}_2)$,²⁹ $w_f = \rho^*w_1$, where w_1 is the generator of $H^1(\mathbb{R}P^1; \mathbb{Z}_2)$.

Note that $\rho^{-1}([0 \times \mathbb{R}]) = \left\{ \{p, q\} \mid \tilde{f}(p) = \tilde{f}(q) \right\} \subset |\tilde{G}_f^{(2)}|$. Let us consider a (simplicial) cycle C . Clearly, there is an Euler cycle going through the edges belonging to C .³⁰ Let ψ_C be a non-degenerate piecewise linear map $\psi_C: S^1 \rightarrow |\tilde{G}_f^{(2)}|$ realising it. According to the definition of c_w ,

$$c_w(C) = \left| \psi_C^{-1} \left(\left\{ \{p, q\} \mid \tilde{f}(p) = \tilde{f}(q) \right\} \right) \right| \bmod 2 = |(\rho \circ \psi_C)^{-1}([0 \times \mathbb{R}])| \bmod 2.$$

Since ρ is a local homeomorphism at the points $\{p, q\}$ where $\tilde{f}(p) = \tilde{f}(q)$, and ψ_C is a local homeomorphism at the preimages of the interior points of the edges of $\tilde{G}_f^{(2)}$, the last number is, in fact, the sum of the local degrees modulo 2 at the preimages of $[0 \times \mathbb{R}]$. Therefore, $[c_w]([C])$ is the degree modulo 2 of the map $\rho \circ \psi_C$.

On the other hand, $w_f([C]) = w_1([\rho \circ \psi_C(S^1)])$ is also equal to the degree modulo 2 of the map $\rho \circ \psi_C$. Thus, $w_f = [c_w]$.³¹ $\square$

Now, we will prove the following theorem:

Theorem 10. *Let $f: G \rightarrow J$ be a stable simplicial map from a graph to a path graph. Then $w_f = 0$ if and only if $v_f = 0$.*

Sketch of a geometrical proof. Take a small neighborhood of $|\tilde{G}_f^{(2)}|$ in $|\tilde{G}^\square|$ homeomorphic to a one-dimensional disc bundle over $|\tilde{G}_f^{(2)}|$. After taking suitable triangulations, we can observe that the total space of the bundle is the closure of the complement of $|\tilde{G}_f^\square|$ in $|\tilde{G}^\square|$ (see Fig. 7). Therefore, the cohomology class v_f belongs to the second cohomology group of the Thom space of this bundle. Furthermore, it can be shown that the Thom isomorphism maps w_f to v_f . $\square$

Cochain-level proof. Let $v_0, \dots, v_N$ be the vertices of J . By replacing the triangulations of G and J with their subdivisions, we can assume that if $f^{-1}(v_i)$ contains a non-regular vertex, then all the preimages of v_{i-1} and v_{i+1} are regular.

Let $\tilde{f}: |G| \rightarrow |J| \times \mathbb{R}$ be a generic lifting of f . By choosing S_ε as in the proof of Theorem 7, we obtain a generic approximation of f .

²⁹The induced map of (cellular) cochain complexes is essentially truncating the cochains of dimension two and higher. Since all differentials in both complexes are zero, we obtain the desired isomorphism.

³⁰Recall that an Euler cycle is a cycle that visits every edge exactly once. In our case, it visits each edge in C exactly once and does not include other edges.

³¹Note that $H^1(X; \mathbb{Z}_2) \cong \text{Hom}_{\mathbb{Z}_2}(H_1(X; \mathbb{Z}_2), \mathbb{Z}_2)$ by the universal coefficient theorem.

Note that the cellular cochains in $C^\bullet(\tilde{G}^\square, \tilde{G}_f^\square; \mathbb{Z}_2)$ and the simplicial cochains in $C^\bullet(\tilde{G}_f^{(2)}; \mathbb{Z}_2)$ could be interpreted as, respectively, cubical cochains in $C^\bullet(G^\square, G_f^\square; \mathbb{Z}_2)$ and simplicial cochains in $C^\bullet(G_f^{(2)}; \mathbb{Z}_2)$, equivariant (as functions) with respect to the involution. Thus, let $c_v \in C^2(G^\square, G_f^\square; \mathbb{Z}_2)$ be the equivariant cubical 2-cochain defining v_f and corresponding to the generic approximation, and let $c_w \in C^\bullet(G_f^{(2)}; \mathbb{Z}_2)$ be the equivariant simplicial 1-cochain defining w_f and corresponding to $\tilde{f}$.

Note that $G_f^{(2)}$ can be naturally embedded into $G^\square$ as the diagonals of the squares $(a, b) \times (c, d)$ where $f(a) = f(c)$ and $f(b) = f(d)$. It follows from the definition of c_w and c_v that c_v takes the value 1 on exactly those squares that contain a diagonal lying in $G_f^{(2)}$ on which c_w takes 1. The opposite is also true: c_w takes the value 1 on the diagonals of those squares on which c_v takes the value 1.

It can be observed that $|G_f^{(2)}|$ lies in the complement of $|G_f^\square|$ in $|G^\square|$. Furthermore, as a subcomplex of $G^\square$, $G_f^\square$ precisely consists of those squares and edges that do not intersect $|G_f^{(2)}|$.

Consider two regular vertices $a, b \in V(G)$ with the same image $f(a) = f(b) = v_i$ lying in the interior of J (i.e., $0 < i < N$). Let a be incident to edges e_1 and e_2 , and b be incident to edges g_1 and h_1 , where $f(g_1) = f(e_1)$ and $f(h_1) = f(e_2)$. The star of $a \times b$ in $G^\square$ consists of a large square divided into four smaller squares $\{e_1, e_2\} \times \{g_1, h_1\}$.³² Observe also that $G_f^{(2)}$ forms a diagonal of the larger square composed of diagonals of $e_1 \times g_1$ and $e_2 \times h_1$.

Now, let us consider the case where b is a non-regular vertex (the case where a is non-regular is symmetrical). We can assume that b is incident to edges $g_1, \dots, g_k, h_1, \dots, h_l$, and a is incident to edges e_1 and e_2 such that $f(g_1) = f(g_2) = \dots = f(g_k) = f(e_1)$ and $f(h_1) = f(h_2) = \dots = f(h_l) = f(e_2)$. As illustrated in Fig. 7, the star of $a \times b$ in $G^\square$ is a $(k + l)$ -page book, with the spine consisting of two segments, namely $e_1 \times b$ and $e_2 \times b$. Each page corresponds to an edge $v \in \{g_1, \dots, g_k, h_1, \dots, h_l\}$ that is incident to b and consists of two half-pages, namely $e_1 \times v$ and $e_2 \times v$. Regarding $G_f^{(2)}$ around $a \times b$, it consists of $k + l$ diagonals of half-pages, one in each page. Moreover, in the pages $\{e_1, e_2\} \times g_i$, those are diagonals of the half-pages $e_1 \times g_i$, while in the pages $\{e_1, e_2\} \times h_i$, those are diagonals of the half-pages $e_2 \times h_i$.

If $a, b \in f^{-1}(v_0)$ or $a, b \in f^{-1}(v_N)$, both a and b are regular degree one vertices. Therefore, the star of $a \times b$ in $G^\square$ is a single square formed by the edges incident to them, and $G_f^{(2)}$ is a diagonal of that square.

Hence, the closure of the complement of $G_f^\square$ in $G^\square$ can be seen as a collection of books connected with strips running along the graph $G_f^{(2)}$. Note that $G^\square$ does not contain any edges where both endpoints belong to $G_f^{(2)}$.

For each $a \times b \in V(G_f^{(2)})$, we choose a subset $E(a \times b)$ of edges incident with $a \times b$ in $G^\square$ as follows. When $a \times b$ is formed by a pair of regular degree one vertices, we simply select

³²The star of a vertex v in a cubical complex C is the union of cubes of C that have v as a vertex.

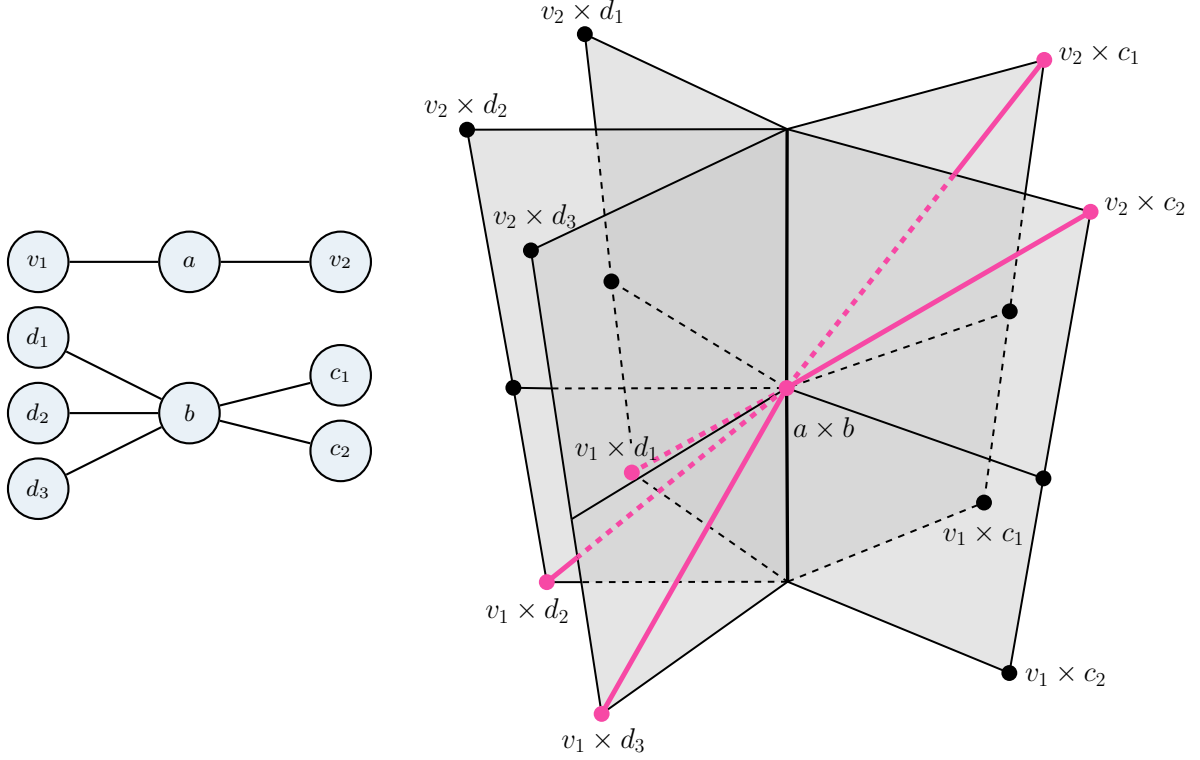


FIGURE 7. The neighborhood of $a \times b$ in $G^\square$. The graph $G_f^{(2)}$ is represented by the pink lines.

one of the two edges incident with $a \times b$, no matter which one, and put it in $E(a \times b)$. In the case where a is a regular degree two vertex, and b is a regular or non-regular vertex, we define $E(a \times b)$ as either $\{a \times h_1, \dots, a \times h_l, e_1 \times b\}$ or $\{a \times g_1, \dots, a \times g_k, e_2 \times b\}$, using the notation provided earlier. The selection of $E(b \times a)$ is done symmetrically with respect to the involution, meaning that $E(b \times a) = \tau(E(a \times b)) = \{y \times x \mid x \times y \in E(a \times b)\}$.

We refer to the squares of $G^\square$ that contain the edges of $G_f^{(2)}$ (on the diagonal) as full squares, while the squares that do not contain such edges are referred to as empty squares.

An important observation is that for any vertex $a \times b \in V(G_f^{(2)})$, the following holds for the squares of its star:

- (1) for a full square, exactly one of its sides lies in $E(a \times b)$,
- (2) for an empty square, either zero or two of its sides lie in $E(a \times b)$.

This property is evident for pairs of regular degree one vertices. For pairs of other types, it can be verified through direct calculations, considering the explicit description provided earlier of how $G_f^{(2)}$ intersects the star of $a \times b$.

Suppose $w_f = 0$, and, therefore, there exists an equivariant 0-cocycle $c'_w \in C^0(G_f^{(2)}; \mathbb{Z}_2)$ with $\delta c'_w = c_w$. Let us introduce an equivariant 1-cocycle $c'_v \in C^1(G^\square, G_f^\square; \mathbb{Z}_2)$.

For any edge $x \times y$ incident with $a \times b$ and belonging to $E(a \times b)$, we define $c'_v(x \times y) = c'_w(a \times b)$; for other edges, we set $c'_v(x \times y) = 0$. Since the sets $E(a \times b)$ are compatible with the involution, and given that c'_w is equivariant, it follows that c'_v is equivariant as well. Additionally, note that c'_v assigns zero values to edges not intersecting $G_f^{(2)}$. Therefore, c'_v is well-defined.

Based on the properties of the sets $E(a \times b)$ discussed above, we observe the following:

- (1) On empty squares, the coboundary $\delta c'_v$ evaluates to zero. This is because each empty square contains an even number of edges on which c'_v takes the same value.
- (2) On full squares, the value of $\delta c'_v$ is equal to the value of c_w on the diagonal. Indeed, a full square contains one edge from each $E(a \times b)$ and $E(a' \times b')$, where $a \times b$ and $a' \times b'$ are the endpoints of the diagonal belonging to $G_f^{(2)}$. Consequently, $\delta c'_v$ sums up the values of c'_w on $a \times b$ and $a' \times b'$. Since $\delta c'_w = c_w$, their sum is equal to the value of c_w on the diagonal.

Thus, we have $\delta c'_v = c_v$, implying that $v_f = 0$.

Conversely, assume that $v_f = 0$. Let $c'_v \in C^1(G^\square, G_f^\square; \mathbb{Z}_2)$ be an equivariant 1-cochain such that $\delta c'_v = c_v$. We define an equivariant cochain $c'_w \in C^0(G_f^{(2)}; \mathbb{Z}_2)$ on the vertices $a \times b$ of $G_f^{(2)}$ as follows:

- (1) If a and b are regular vertices of degree one, the value of c'_w on $a \times b$ is the sum of the values of c'_v on the two edges incident to $a \times b$.
- (2) Otherwise, it is the sum of the values of c'_v on the two “spine edges”, which are the edges forming the spine of the star of $a \times b$ (as we have discussed above, geometrically, this star is a book). In the case when both a and b are regular degree two vertices, it is important to carefully choose the “spine edges” for $a \times b$, so that our choice is compatible with the involution. This means that if $x \times y$ and $x' \times y'$ are the “spine edges” for $a \times b$, then the edges $y \times x$ and $y' \times x'$ must be the “spine edges” for $b \times a$.

As mentioned earlier, the values of c_v on squares coincide with the values of c_w on their diagonals. Thus, to establish $\delta c'_w = c_w$, it suffices to show that $c'_w(a \times b) + c'_w(a' \times b')$ equals the sum of the values of c'_v on the edges of the full square containing a diagonal $\{a \times b, a' \times b'\} \in E(G_f^{(2)})$.

We claim that for a vertex $a \times b$ of $G_f^{(2)}$ and a full square S in its star, the value of c'_w on $a \times b$ equals the sum of the values of c'_v on the two edges of S incident to $a \times b$. If a and b are regular vertices of degree one, this follows from the definition of c'_w .

Otherwise, S contains exactly one of the two spine edges of $a \times b$ as a side. Let us denote this edge by e_+ , and let the other side of S incident to $a \times b$ be denoted by g . Together with an empty square S' , S forms a full page in the book of $a \times b$. The other spine edge e_- and g are the sides of S' incident to $a \times b$. The other two sides of S' do not intersect $G_f^{(2)}$, thus, c'_v takes the value zero on them. Consequently, since $0 = c_v(S') = \delta c'_v(S') = c'_v(e_-) + c'_v(g)$, the cochain c'_v takes the same value on the edges

g and e_- . Therefore, $c'_w(a \times b) = c'_v(e_+) + c'_v(e_-) = c'_v(e_+) + c'_v(g)$, where e_+ and g are the sides of S incident to $a \times b$.

Therefore, for every full square S containing a diagonal $\{a \times b, a' \times b'\} \in G_f^{(2)}$, we conclude that $c'_w(a \times b) + c'_w(a' \times b')$ equals the sum of the values of c'_v on the sides of S . Thus, we have $\delta c'_w = c_w$ and $w_f = 0$. $\square$

Theorem 11. *Let $f: T \rightarrow J$ be a stable simplicial map from a tree to a path graph. Then the following statements are equivalent:*

- (1) *There are no 2-obstructors for f ;*
- (2) *$|f|$ lifts to an embedding.*

Proof. If f lifts to an embedding, the non-existence of 2-obstructors follows from [Theorem 1](#).

Now assume that there are no 2-obstructors for f . Therefore, it follows from [Lemma 1](#) that p_2 is trivial, and, thus, $w_f = 0$.

Applying [Theorem 10](#), we conclude that $v_f = 0$. Hence, [Theorem 9](#) says that f is $\mathbb{Z}_2$ -approximable with respect to an embedding $|J| \hookrightarrow \mathbb{R}^2$. Since T is a tree, by [Theorem 8](#), f is approximable by embeddings. Finally, f lifts to an embedding by applying [Theorem 7](#). $\square$

The proven result has the potential for various generalizations. For instance, we believe that the non-existence of 2-obstructors condition remains sufficient for the existence of a lifting even for an arbitrary, not necessarily stable, non-degenerate map from a graph to a segment.

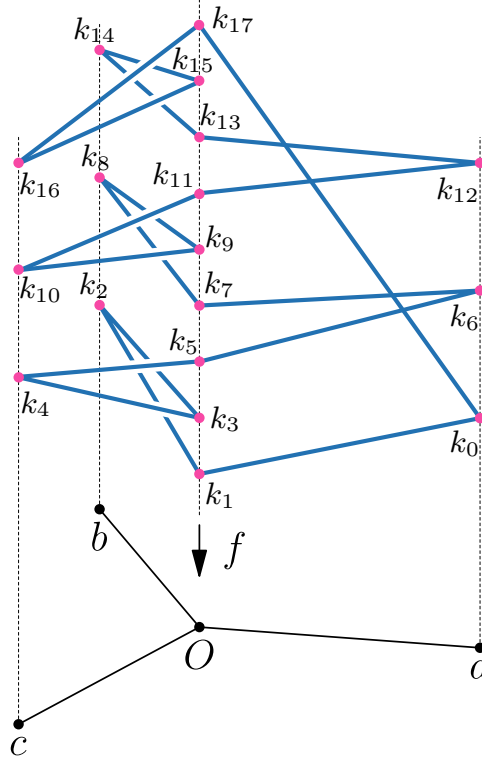
Furthermore, instead of considering maps to a segment, we can consider maps to trees. However, the following example shows that the stated 2-obstructor condition is not sufficient for maps from arbitrary graphs to trees, while the question of its sufficiency for maps from trees to trees remains open.

Example 5. Let T be a tripod with 4 vertices O , a , b , and c , where O is the central vertex. Let $|f|: S^1 \rightarrow |T|$ be a “walking around” $|T|$.

Specifically, let C_6 be a hexagon triangulating S^1 , whose six vertices v_i numbered in the counterclockwise order starting from zero, and let $g: C_6 \rightarrow T$ be simplicial map defined by

$$\begin{aligned} f(v_0) &= a, & f(v_1) &= O, \\ f(v_2) &= b, & f(v_3) &= O, \\ f(v_4) &= c, & f(v_5) &= O. \end{aligned}$$

Now, consider the composition $f: C_{18} \xrightarrow{w} C_6 \xrightarrow{g} T$ illustrated in [Fig. 8](#), where C_{18} represents an 18-gon. Here the first map is a simplicial analogue of the standard 3-winding $w: S^1 \rightarrow S^1$ defined as $z \mapsto z^3$ assuming that S^1 is embedded into the complex plane as the unit circle centered at zero. Explicitly, $w(k_i) = v_{(i \bmod 6)}$ where $k_0, \dots, k_{17}$ are the vertices of C_{18} numbered in the counterclockwise order.

FIGURE 8. The 3-winding map $f: C_{18} \rightarrow T$.

Note that f has a 3-obstructor because w does, and, therefore, it does not lift to an embedding.

We claim that f does not have any 2-obstructors. To demonstrate this, let us look at the graph $(C_{18})_f^{(2)}$.

In this graph, a vertex corresponding to a pair (k_i, k_j) of distinct vertices from $f^{-1}(O)$ has degree two when $w(k_i) = w(k_j)$, meaning that it is also a vertex of $(C_{18})_w^{(2)}$, and degree one otherwise.

Consider a vertex (k_t, k_s) of $(C_{18})_f^{(2)}$ where $f(k_t) = f(k_s) \neq O$. This is a degree-four vertex. Moreover, two of the four vertices adjacent to it have degree one, precisely those that do not lie in $(C_{18})_w^{(2)}$.

Consequently, by removing the vertices of degree one from $(C_{18})_f^{(2)}$, we get $(C_{18})_w^{(2)}$. Since the latter graph does not have a 2-obstructor path, the same holds true for the former graph.

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