Gapless non-hydrodynamic modes in relativistic kinetic theory

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We rigorously prove, for the first time, that the non-hydrodynamic sector is gapless in any relativistic kinetic theory whose scattering cross-section decays to zero at large energies. In fact, if particles with very high energy (compared to the temperature) are free streaming, we can use them to build hot non-hydrodynamic waves, which live longer than any hydrodynamic wave. Since many standard cross-sections in quantum field theory vanish at high energies, the existence of these non-thermal long-lived waves is a rather general feature of relativistic systems.

Introduction - The existence of hydrodynamics usually relies on a "separation of timescales" assumption [1–6], which can be summarised as follows: Non-conserved quantities equilibrate (through collisions) over microscopic timescales, while conserved quantities equilibrate (through transport) over macroscopic timescales [7]. To appreciate the rationale of this assumption, consider the following example. Let $\rho(t, x^j)$ be the coarse-grained energy density of a gas of photons immersed in a medium. Suppose that interactions with the medium happen through elastic scattering, so ρ is a conserved density, and we can write a continuity equation $\partial_t \rho + \partial_j F^j = 0$, with F^j the energy flux [8]. If scatterings are frequent (compared to the evolution timescale of ρ), we can average the motion of photons over a few mean free times, and treat it as a random walk. Then, F^j is uniquely determined in terms of ρ by Fick's law, $F^j = -D\partial^j \rho$ [9], resulting in a hydrodynamic equation for the density: $\partial_t \rho = D\partial_j \partial^j \rho$. If, instead, the interactions are rare, we cannot construct a differential equation in terms of ρ alone, because F^j depends on infinitely many microscopic degrees of freedom (namely, the initial direction of propagation of each individual photon). Thus, a self-contained macroscopic description of the flow does not exist.

The tendency of gases and liquids to exhibit a scale separation at small gradients is often taken for granted [10–13]. The standard argument goes as follows: Let τ_{hy} be the decay timescale of a hydrodynamic wave of interest (e.g. a soundwave), and let $\tau_{\text{non-hy}}$ be the equilibration time of the longest-living collective excitation that is not described by hydrodynamics. If

$$\tau_{\text{non-hy}} \ll \tau_{\text{hy}} \,, \tag{1}$$

then all the "non-hydrodynamic excitations" decay faster than the hydrodynamic wave, which will dominate the late-time behavior. But since the damping factor of hydrodynamic waves due to viscosity scales like $\sim e^{-k^2 t}$, where k is the wave number, one can make $\tau_{\text{hy}} \sim k^{-2}$ arbitrarily large by taking $k \to 0$. Thus, at sufficiently small gradients (i.e. at sufficiently large lengthscales), τ_{hy} will outlive all non-hydrodynamic excitations, and (1) must hold.

Unfortunately, the above argument has a flaw [14, 15]. At infinitely large lengthscales, a fluid cell possesses infinitely many microscopic degrees of freedom. Thus, there may be an infinite sequence of collective excitations, whose equilibration times $\{\tau_n\}_{n\in\mathbb{N}}$ diverge at large n. Then, $\tau_{\text{non-hy}} = \infty$, and (1) never occurs. For example, suppose that $\tau_n = n$. Then, if we assign to each excitation an initial amplitude $1/n^2$, the total perturbation decays like

$$\sum_{n=1}^{\infty} \frac{e^{-t/n}}{n^2} \overset{\text{large } t}{\approx} \frac{1}{t} \,, \tag{2}$$

which survives longer than any soundwave (at non-zero viscosity). This possibility was extensively studied for non-relativistic gases [16–19]. There, it was shown that, if the collision cross-section decays to zero at high energies[1], then there is a continuous infinity of non-hydrodynamic excitations whose equilibration times are unbounded above. It was also shown that continuous superpositions of these excitations decay like $\sim e^{-t^b}$ (with $b<1$), outliving soundwaves. Whether realistic QFT interactions give rise to similar phenomena in relativistic gases is an open problem, and is known as the "poles or cut?" dilemma [21] (poles: $\tau_{\text{non-hy}} < \infty$; cut: $\tau_{\text{non-hy}} = \infty$). Several hints point towards "cut", coming from analytical estimates [22], qualitative models [23], and numerical experiments [21, 24]. However, a rigorous and general discussion is needed. Here, we prove that, in most cases, the correct answer is indeed "cut". Furthermore, we provide an intuitive argument for why this must be the case.

The metric signature $(-, +, +, +)$ is adopted throughout, and we work in natural units: $c = \hbar = k_B = 1$.

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[1] In the kinetic theory literature, interactions of this type are called "soft potentials" [18, 20].

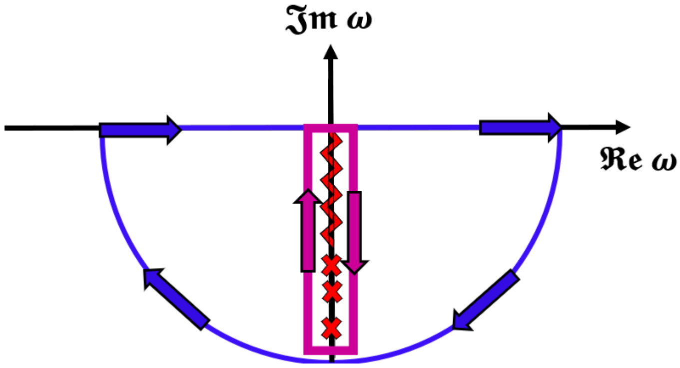

FIG. 1. To evaluate the integral $G(t)=(2\pi)^{-1}\int_{\mathbb{R}}G(\omega)e^{-i\omega t}d\omega$ for $t>0$, we close the path for negative $\mathfrak{Im}\,\omega$ (blue line), since $e^{-i\omega t}$ decays exponentially there. Then, we shrink the path around the singularities (purple line). In the limit in which the vertical lines approach the imaginary axis, we obtain (5), with $z(\nu)=(2\pi i)^{-1}\big[G(0^+\!-i\nu)-G(0^-\!-i\nu)\big]$.

Intuitive argument - Consider a relativistic gas of particles with mass m, density J^0, and temperature T. Suppose that one particle in this gas has energy much higher than the thermal energy, i.e. $p^0\gg\sqrt{m^2+T^2}\equiv E_{\rm th}$. The probability that this particle travels freely for a time interval t without interacting is $\mathcal{P}_s(t)=e^{-J^0\sigma t}$ [25, 26], where σ is the typical collision cross-section. Let us take, as a "realistic" (i.e. QFT-inspired) high-energy interaction, the cross-section of $\lambda\varphi^4$ scalar field theory, i.e. $\sigma\sim g/E_{\rm CM}^2$, where g is some constant, and $E_{\rm CM}$ is the energy in the center of momentum frame [27]. For collisions between our high-energy particle and the rest of the gas, we have $E_{\rm CM}^2\approx 4E_{\rm th}p^0$, giving [28]

$$\mathcal{P}_s(t)\approx\exp\left[-\frac{J^0gt}{4E_{\rm th}p^0}\right]. \tag{3}$$

Now, suppose that, instead of one high-energy particle, we have a population of such particles. If these are sufficiently diluted, we can treat each particle independently. Thus, working in the thermodynamic limit, imagine that there are N particles with energy $p^0(\gg E_{\rm th})$, plus other $N/2^3$ particles with energy $2p^0$, plus other $N/3^3$ particles with energy $3p^0$, plus other $N/4^3$ particles with energy $4p^0$..., and so on (up to infinity). Then, the total energy associated with hard particles that have traveled freely without interacting from time 0 to time t is a series, whose sum is finite:

$$\begin{split} & E_{\text{non-hy}}(0)=\pi^2Np^0/6\,, \\ & E_{\text{non-hy}}(t)=\sum_{n=1}^{\infty}\frac{Np^0}{n^2}\exp\left[-\frac{J^0gt}{4E_{\rm th}np^0}\right] \overset{\text{large } t}{\approx} \frac{4NE_{\rm th}(p^0)^2}{J^0gt}\,. \end{split} \tag{4}$$

Compare this situation with the photon gas in the introduction: $E_{\text{non-hy}}$ is the energy carried by particles that have traveled freely across the medium *without* relaxing to a hydrodynamic constitutive relation. No macroscopic law relates the large-scale flux of $E_{\text{non-hy}}$ to large-scale densities. The amount of this "non-hydrodynamic energy" falls like $1/t$, while hydrodynamic waves decay exponentially. Therefore, free-streaming hard particles dominate the late-time transport, and $\tau_{\text{non-hy}}=\infty$. Similar conclusions hold in any gas where $\sigma\to 0$ at large $E_{\rm CM}$.

In summary: If the high-energy tails of the kinetic distribution decay as power laws in p^0 (as in [29–31]), and the cross-section decays to zero at large energies (as in current models of QCD thermalization [23, 32–34]), high-energy particles form a non-hydrodynamic excitation that equilibrates slower than soundwaves, and $\tau_{\text{non-hy}}=\infty$.

The non-hydrodynamic sector is gapless - Let $G(t{-}t')$ be the retarded linear-response Green function of the shear stress component $\pi_{12}(t)$ induced by some external force $F(t')$, in the spatially homogenous limit. The singularities of its Fourier transform, $G(\omega)$, are the eigenfrequencies of the system, which (in kinetic theory) lay on the imaginary axis of the complex ω-plane [35, 36]. Thus, by deforming the integration path as in figure 1, we obtain [37]

$$G(t)=\Theta(t)\int_0^{+\infty}z(\nu)e^{-\nu t}d\nu\,, \tag{5}$$

where $z(\nu)$ is the spectral density of singularities at $\omega=-i\nu$. Comparing (3) with (5) in the limit $\nu\to 0$, we see that $z(\nu)$ is proportional to the density of hard particles per unit $(p^0)^{-1}\propto\nu$ that are disturbed by F. If the force F couples with particles at all energies, $z(\nu)$ is a continuous function (and not a sum of Dirac deltas) near $\nu{=}0$, meaning that $G(\omega)$ has a branch cut that touches the origin. Hence, the $\lambda\varphi^4$ gas possesses infinite non-hydrodynamic modes in a neighborhood of $\omega{=}0$, making its non-hydrodynamic sector gapless, in agreement with [21, 24].

In the rest of this Letter, we provide rigorous theorems supporting the above arguments, for general interactions.
Formulation of the problem - We consider a relativistic gas of classical particles of mass m (which may vanish), with kinetic distribution $f_p(x^\mu)$. The relativistic Boltzmann equation [38] has the form $p^\mu \partial_\mu f_p = \mathcal{C}[f_p]$, where $\mathcal{C}$ is some collision functional (it may be Boltzmann's collision integral or some approximation thereof [39]). Fixed a uniform equilibrium state $f_p^{\text{eq}} = e^{\alpha+\beta_\nu p^\nu}$ (α, β_ν constant), we make the decomposition $f_p = f_p^{\text{eq}}(1+\phi_p)$, and we linearise in ϕ_p. This results in the evolution equation $p^\mu \partial_\mu \phi_p = L\phi_p$, where L is a linear operator (whose domain of definition will be specified later). Fixed one reference frame, not necessarily at rest with respect to the gas, we restrict our attention to the homogenous problem, thereby dropping the term $p^j \partial_j \phi_p$. Hence, have

$$\partial_t \phi_p = (p^0)^{-1} L\phi_p \qquad \Longrightarrow \qquad \phi_p(t) = e^{(p^0)^{-1}Lt}\phi_p(0)\,. \tag{6}$$

Our goal in this Letter is to discuss the properties of the operator $(p^0)^{-1}L$, which generates the homogenous evolution. We stress the importance of keeping the factor $(p^0)^{-1}$, since the spectral properties of L may be very different from those of $(p^0)^{-1}L$. For example, if we take $L=-1$, whose spectrum is just $\{-1\}$, the spectrum of $(p^0)^{-1}L = (p^0)^{-1}$ is continuous, covering the interval $[-m^{-1}, 0]$.
Features of the collision operator - We require L to have the following properties (taking $\phi_p \in \mathbb{C}$ for later convenience):

$$\text{Equilibrium states:} \qquad L1 = Lp^\nu = 0\,, \tag{7}$$

$$\text{Dissipation:} \qquad \int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} \phi_p^* L\phi_p \leq 0\,, \tag{8}$$

$$\text{Onsager Symmetry:} \qquad \left[\int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} \psi_p^* L\phi_p\right]^* = \int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} \phi_p^* L\psi_p\,, \tag{9}$$

all of which are fulfilled by Boltzmann's binary collision integral [40] (see Supplementary Material for the proof). Equation (7) expresses the requirement that all equilibrium states $f_p = e^{\tilde\alpha+\tilde\beta_\nu p^\nu}$ (with $\tilde\alpha$ and $\tilde\beta_\nu$ constant) must be solutions of the Boltzmann equation. In fact, taking $\tilde\alpha = \alpha + a$ and $\tilde\beta_\nu = \beta_\nu + b_\nu$, and linearizing in a and b_ν, we obtain $\phi_p = a + b_\nu p^\nu$ which, plugged into the linearised Boltzmann equation, gives $L[a+b_\nu p^\nu] = p^\mu \partial_\mu (a+b_\nu p^\nu) = 0$. To derive (8), one expresses the entropy production in terms of the information current, $\varsigma = -\partial_\mu E^\mu \geq 0$, and imposes the second law of thermodynamics, $\varsigma \geq 0$. In kinetic theory, the information current E^μ is known [41], and we obtain

$$E^\mu = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu |\phi_p|^2\,, \qquad \varsigma = -\mathfrak{Re}\int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} \phi_p^* L\phi_p\,. \tag{10}$$

Equation (9) arises from microscopic time-reversal invariance [36, 42], and allows us to drop the real part in the formula for ς. Combining (7) and (9), one finds that the perturbed particle current and stress-energy tensor,

$$\delta J^\mu = \int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu \phi_p\,, \qquad \delta T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu p^\nu \phi_p\,, \tag{11}$$

automatically obey the conservation laws $\partial_\mu \delta J^\mu = \partial_\mu \delta T^{\mu\nu} = 0$.
A convenient Hilbert space - To establish rigorous results, we must fix a space of functions. We choose the (separable) Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, f_p^{\text{eq}})$, namely the space of (complex) square-integrable functions on $\mathbb{R}^3$ with respect to the measure $f_p^{\text{eq}} d^3p$. The associated inner product is

$$(\psi_p, \phi_p) = \int \frac{d^3p}{(2\pi)^3} f_p^{\text{eq}} \psi_p^* \phi_p\,, \tag{12}$$

with corresponding norm $||\phi_p|| = \sqrt{(\phi_p,\phi_p)} = \sqrt{2E^0}$. The space $\mathcal{H}$ is the set of homogeneous states with finite free energy density [43–45], and we use the free energy as a square norm[2] [20]. Clearly, any physically meaningful perturbation ϕ_p should belong to $\mathcal{H}$. Indeed, all the products $p^{\nu_1}p^{\nu_2}...p^{\nu_l}$ belong to $\mathcal{H}$, meaning that all the moments $\delta\rho^{0\nu_1\nu_2...\nu_l} = (p^{\nu_1}p^{\nu_2}...p^{\nu_l}, \phi_p)$ are well-defined inside $\mathcal{H}$, including $\delta J^0 = (1, \phi_p)$ and $\delta T^{0\nu} = (p^\nu, \phi_p)$.

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[2] Textbooks [38, 40] use a different inner product, replacing our measure $f_p^{\text{eq}} d^3p$ with the invariant measure $f_p^{\text{eq}} d^3p/p^0$. This convention presents a serious problem: If the initial state has finite norm in $L^2(\mathbb{R}^3, f_p^{\text{eq}}/p^0)$, there is no guarantee that such norm will remain finite at later times. By contrast, the second law forces $E^0(t)$ to be non-increasing. Thus, if $\phi_p(0) \in \mathcal{H}$, then $\phi_p(t) \in \mathcal{H}$ for all $t \geq 0$. Some readers may feel that, in relativity, the invariant measure d^3p/p^0 is "more natural". This feeling is an artifact of the homogenous limit. Thinking of the inhomogeneous case, it is clear that the norm depends on the Cauchy surface Σ upon which we define the state, and thus should *not* be Lorentz-invariant [46–48]. Indeed, using $2E^0$ as a squared norm is more natural, since it generalizes to $2\int_\Sigma E^\mu d\Sigma_\mu$ in the inhomogeneous case, and we can use Gauss' theorem [49] to link norms on different Cauchy surfaces: $||\phi_p||^2_{\Sigma_2} - ||\phi_p||^2_{\Sigma_1} = -\int 2\varsigma d^4x$.

The inner product (12) allows us to rewrite (8) and (9) as follows:

$$\varsigma = -\left(\phi_p, \frac{1}{p^0} L\phi_p\right) \geq 0\,, \qquad \left(\psi_p, \frac{1}{p^0} L\phi_p\right)^* = \left(\phi_p, \frac{1}{p^0} L\psi_p\right), \tag{13}$$

meaning that the operator $(p^0)^{-1}L$ is Hermitian negative-semidefinite. Furthermore, for $(p^0)^{-1}L$ to be physically meaningful, it must be densely defined in $\mathcal{H}$, since all functions in $\mathcal{H}$ are acceptable physical states[3]. The above facts allow us to apply the Friederichs extension (Theorem 5.1.13 of [51]), and promote $(p^0)^{-1}L$ to a self-adjoint operator.

A useful theorem - Let us decompose $\mathcal{H} = \mathcal{H}_{\text{hy}} \bigoplus \mathcal{H}_{\text{non-hy}}$, where $\mathcal{H}_{\text{hy}}$ is the space of states that are described by hydrodynamics, and $\mathcal{H}_{\text{non-hy}}$ is the space of non-hydrodynamic excitations. Since we are working in the homogeneous limit, all hydrodynamic "waves" take the form of global equilibria $\phi_p = a{+}b_\nu p^\nu$, so that $\mathcal{H}_{\text{hy}} = \text{span}\{1, p^\nu\}$. Conversely, the non-hydrodynamic excitations are those states that are "invisible" to hydrodynamics. Since hydrodynamics is only aware of the conserved densities, the non-hydrodynamic excitations are the states such that $\delta J^0 = \delta T^{0\nu} = 0$. Recalling that $\delta J^0 = (1, \phi_p)$ and $\delta T^{0\nu} = (p^\nu, \phi_p)$, we conclude that $\mathcal{H}_{\text{non-hy}} = \mathcal{H}_{\text{hy}}^{\perp}$. Therefore, $\mathcal{H}_{\text{non-hy}}$ is a Hilbert space in its own right. Furthermore, $\mathcal{H}_{\text{non-hy}}$ is an invariant space of $(p^0)^{-1}L$. This follows from the second equation of (13): Just take $\phi_p \in \mathcal{H}_{\text{non-hy}}$ and $\psi_p = a + b_\nu p^\nu$, so that, using (7),

$$\left(a{+}b_\nu p^\nu, \frac{1}{p^0} L\phi_p\right)^* = \left(\phi_p, \frac{1}{p^0} L[a{+}b_\nu p^\nu]\right) = 0 \qquad \Longrightarrow \qquad \frac{1}{p^0} L\phi_p \in \mathcal{H}_{\text{hy}}^{\perp} = \mathcal{H}_{\text{non-hy}}\,. \tag{14}$$

Thus, we can restrict the operator $(p^0)^{-1}L$ to a self-adjoint operator on the Hilbert subspace $\mathcal{H}_{\text{non-hy}}$ of pure deviations from local thermodynamic equilibrium. Then, we have the following result, whose proof is provided in the Supplementary Material.

Theorem 1. *Suppose that* 1 *and* p^ν *are the only collisional invariants belonging to* $\mathcal{H}$*, i.e.* $\text{Ker}(L) \equiv \mathcal{H}_{\text{hy}}$*. Then, the following facts are equivalent:*

(a) *We can construct non-hydrodynamic excitations that survive as long as we want, i.e. for any time* $t > 0$*,*

$$\sup_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{||\phi_p(t)||}{||\phi_p(0)||} = 1\,. \tag{15}$$

(b) *We can construct non-hydrodynamic states* ϕ_p *such that* $||(p^0)^{-1}L\phi_p||$ *is arbitrarily smaller than* $||\phi_p||$*, i.e.*

$$\inf_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{||(p^0)^{-1}L\phi_p||}{||\phi_p||} = 0\,. \tag{16}$$

(c) *We can construct non-hydrodynamic states that produce negligibly small entropy, i.e.*

$$\inf_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{\varsigma}{E^0} = 0\,. \tag{17}$$

(d) *Working within the space* $\mathcal{H}_{\text{non-hy}}$*, the frequency* $\omega = 0$ *is an accumulation point of the spectrum of* $(p^0)^{-1}L$*. Namely, for any* $\epsilon > 0$*, there is an element* $-i\omega$ *in the non-hydrodynamic spectrum of* $(p^0)^{-1}L$ *such that* $|\omega| < \epsilon$*.*

Statements (a,b,c,d) are four alternative (necessary and sufficient) criteria for $\tau_{\text{non-hy}}$ to be infinite. If any one of them is met, all four are met. Criterion (d) is the "cut" option considered in [21]. The implication (d)$\rightarrow$(a) is a rigorous formulation of the "dehydrodynamization" mechanism discussed in [23]. Note that, since (6) is linear, the states in (15) may be rescaled by an arbitrary constant factor. Thus, if (a) holds, there are long-lived non-hydrodynamic perturbations whose amplitude is (and will forever remain) much larger than an arbitrary preassigned soundwave.

The real novelty of Theorem 3 are criteria (b,c). In fact, the recent literature tries to assess (d) by direct evaluation of the spectrum of $(p^0)^{-1}L$ [21, 24]. This task requires numerical techniques that are by nature approximate, and sensitive to the details of L. Criteria (b) and (c) remarkably simplify the problem, since now one only needs to engineer a sequence $\{\phi_p^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{H}$ with unit norm, vanishing conserved densities, $(1, \phi_p^{(n)}){=}(p^\nu, \phi_p^{(n)}){=}0$, and such that either $||(p^0)^{-1}L\phi_p^{(n)}||$ or $\varsigma[\phi_p^{(n)}]$ converges to zero as $n{\rightarrow}\infty$. Below, we provide an explicit example.

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[3] For sufficiently regular cross-sections, Boltzmann's collision operator is well defined on C^∞ functions with compact support. These are dense in $\mathcal{H}$, making Boltzmann's operator densely defined, see e.g. [20, 22, 50] and references therein. In the following, we will adopt the usual physics convention of saying that "statement X holds in Hilbert space Y" whenever X holds in a *dense linear subset* of Y.

Application: Massless scalar particles - Let us apply Theorem 3 to a system of massless classical particles with $\lambda\varphi^4$ interaction (working in the equilibrium rest frame, so $\beta^\mu = \beta\delta^\mu_0$). In this model, the spectrum of the collision operator L is known analytically, and it can be shown that $\text{Ker}(L) = \mathcal{H}_{\text{hy}}$ [28]. We consider the sequence

$$\phi^{(n)}_p = L^{(5)}_n(\beta p^0)p^2p^3 \, , \tag{18}$$

where $L^{(2\ell+1)}_n$ are Laguerre polynomials. These states are eigenvectors of L, with eigenvalues

$$\chi_{n2} \propto -\frac{n+1}{n+3} \, . \tag{19}$$

Showing that $(1, \phi^{(n)}_p) = (p^\nu, \phi^{(n)}_p) = 0$ is straightforward. Furthermore, in the Supplementary Material we verify that

$$\frac{||(p^0)^{-1}L\phi^{(n)}_p||}{||\phi^{(n)}_p||} \leq \frac{\text{const}}{\sqrt{n+3}} \xrightarrow{n\to+\infty} 0. \tag{20}$$

Thus, criterion (b) is met. Applying Theorem 3, we conclude from (a) that the gas possesses non-hydrodynamic excitations that survive longer than soundwaves. We also conclude from (d) that the non-hydrodynamic sector is gapless, in agreement with equation (4), and with numerical studies [21, 24].

Main result - We are now ready to state the central theorem of this Letter.

In the full Boltzmann equation, the entropy production rate is [38]

$$\varsigma = \frac{1}{8}\int \frac{d^3p}{(2\pi)^3p^0}\frac{d^3p'}{(2\pi)^3p'^0}\frac{d^3q}{(2\pi)^3q^0}\frac{d^3q'}{(2\pi)^3q'^0}\, s\sigma(s,\theta)\, \delta^4(p+p'-q-q')\, f^{\text{eq}}_p f^{\text{eq}}_{p'}|\phi_p + \phi_{p'} - \phi_q - \phi_{q'}|^2 \, , \tag{21}$$

where $\sigma(s,\theta)$ is the differential cross-section, $s = E^2_{\text{CM}}$ is the Mandelstam variable, and θ is the scattering angle in the center of momentum frame. Now, suppose that there are two constants $A \geq 0$ and $0 < a \leq 1$ such that $\sigma(s,\theta) \leq A/s^a$. Then, if we consider the sequence of perturbations

$$\phi^{(n)}_p = e^{\frac{\beta}{2}(1-n^{-1})p^0}\frac{2p^2p^3}{(p^2)^2+(p^3)^2} \, , \tag{22}$$

which are orthogonal to 1 and p^ν, we find that (see Supplementary Material)

$$\frac{\varsigma[\phi^{(n)}_p]}{E^0[\phi^{(n)}_p]} \leq \frac{\text{const}}{n^a} \xrightarrow{n\to+\infty} 0 \, . \tag{23}$$

Hence, condition (c) of Theorem 3 is fulfilled. Considering that ς is linear in σ, we have the following rigorous result.

Theorem 2. *Consider a classical gas of relativistic particles, governed by Boltzmann's equation, with* $\text{Ker}(L)=\mathcal{H}_{hy}$. *Suppose that there are three constants $A \geq 0$, $B \geq 0$, and $0 < a < 1$ such that*

$$\sigma(s,\theta) \leq \frac{A}{s^a} + \frac{B}{s} \, , \tag{24}$$

for all θ and s. Then, the assumptions of Theorem 3 are met, and facts (a,b,c,d) *occur.*

This means that, if σ decays for $s \to +\infty$ like a power law s^{-a} (or faster), we can construct non-hydrodynamic excitations that survive as long as we want. The second term in (24) is there to remind us that $\sigma(s,\theta)$ is allowed to diverge at $s = 0$ (which occurs only for massless particles), but it should not grow faster than s^{-1} (otherwise our proof no longer works). We remark that Theorem 2 only provides sufficient conditions for (a,b,c,d) to hold. There may be other interactions that fulfill the conditions of Theorem 3.

Application: Screened gauge theories - Most scattering differential cross-sections in QED [52] and QCD [53] decay like $1/s$ at high energies, see e.g. the (ultrarelativistic) electron-electron and the gluon-gluon scatterings:

$$\sigma_{ee\to ee} \propto \frac{(3+\cos^2\theta)^2}{s\,\sin^4\theta} \, , \qquad \sigma_{gg\to gg} \propto \frac{1}{s}\left[3 - \frac{\sin^2\theta}{4} + \frac{4+12\cos^2\theta}{\sin^4\theta}\right] . \tag{25}$$

The factor $(\sin\theta)^{-4}$ makes the operator L small-angle divergent, reflecting the long-range nature of the interaction, which makes Boltzmann's assumption of local collisions invalid. However, if we correct the cross-sections (25) with medium-dependent effects (e.g., Debye screening [54]) the small-angle behavior is regularized [53, 55]. With certain regularizations, e.g. $(\sin\theta)^{-4} \to (\epsilon^2 + \sin^2\theta)^{-2}$ (with $\epsilon = \text{const}$), Theorem 2 applies.

Application: Yukawa theory - Ultrarelativistic fermions f interacting through either Yukawa coupling $\varphi\bar{f}f$ or $\varphi\bar{f}\gamma^5 f$ have scattering cross-section $\sigma_{ff\to ff} = \text{const}/s$ [56, 57] (the dependence on the angle disappears at high energies). Theorem 2 applies.

Final Remarks - Interactions that decay to zero at high energy are the most common in relativistic physics. Nearly all "textbook" (two-body) cross-sections decay like s^{-1}, as one would expect from naive dimensional analysis [27]. Indeed, $\sigma \propto s^{-1}$ is the universal scaling law of those theories that "forget" their reference mass scales in the ultrarelativistic limit. In those theories that "remember" some mass scale M at high energy, unitarity requires that $\sigma \leq \pi M^{-2}\ln^2(s)$ [58], which allows for mild growth of σ with s [59]. Hence, some diluted gases with a non-hydrodynamic gap are likely to exist. However, they seem to be the exception, not the rule.

Let us provide two arguments suggesting that the non-hydrodynamic sector of QCD plasmas is probably gapless.

- All relaxation-type models of QCD matter adopt an energy-dependent relaxation time $\tau \propto (p^0)^a$, with $0 < a < 1$ [23, 32, 34]. If this approximation is valid, at least qualitatively, we can use it to learn about the scaling law of the cross sections. In fact, for a high-energy particle interacting with a thermal bath, $s \propto p^0$, and $\tau \propto \sigma^{-1}$. We conclude that $\sigma \propto 1/s^a$, consistently with (24). Hence, in a corresponding "full-fledged" Boltzmann equation, Theorem 2 applies, and the non-hydrodynamic sector is gapless.

- According to [60, 61], particles with $p^0 \gg T$ traveling through a QCD plasma lose energy following the equation $\dot{p}^0 = -2C\sqrt{p^0}$, whose solution is $p^0(t) = \left[\sqrt{p^0(0)}-Ct\right]^2\Theta[\sqrt{p^0(0)}-Ct]$. Suppose that there is a diluted population of high-energy particles whose number (per unit energy) falls like $(p^0)^{-3}$ at $t = 0$. Then, at late times, the non-thermal energy carried by these particles decays slower than exponentially,

$$E_{\text{non-th}}(t) \propto \int_{(Ct)^2}^{\infty} \frac{(\sqrt{p^0}-Ct)^2}{(p^0)^3}dp^0 = \frac{1}{6(Ct)^2}\,, \tag{26}$$

proving the absence of a gap.

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Gapless non-hydrodynamic modes in relativistic kinetic theory
Supplementary Material

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I. PROPERTIES OF THE LINEARISED COLLISION OPERATOR

Here, we verify that Boltzmann's collision integral fulfills all the general properties mentioned in the main text. The action of L on a generic function ϕ_p is

$$L\phi_p = \int \frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_{p'} (\phi_q+\phi_{q'}-\phi_p-\phi_{p'}) \, , \tag{S1}$$

where $W_{pp'\leftrightarrow qq'} = W_{p'p\leftrightarrow qq'} = W_{qq'\leftrightarrow pp'} \geq 0$ is the transition rate, which vanishes unless $p{+}p' = q{+}q'$, due to energy-momentum conservation. This implies that $W_{pp'\leftrightarrow qq'} \neq 0$ only if $f^{\rm eq}_p f^{\rm eq}_{p'} = f^{\rm eq}_q f^{\rm eq}_{q'}$.

The fact that $L1 = Lp^\nu = 0$ is evident. Thus, we only need to prove the properties "Dissipation" and "Onsager Symmetry". To do that, we multiply (S1) by $f^{\rm eq}_p \psi^*_p/p^0$, and integrate over all momenta:

$$\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p = -\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} \psi^*_p (\phi_p+\phi_{p'}-\phi_q-\phi_{q'}) \tag{S2}$$

Since the integration variables $(pp'qq')$ are dummy variables, we can rename them at will. If we perform the change of variables $(pp'qq') \to (p'pqq')$ and use the symmetry $W_{p'p\leftrightarrow qq'} = W_{pp'\leftrightarrow qq'}$, we obtain

$$\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p = -\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} \psi^*_{p'} (\phi_p+\phi_{p'}-\phi_q-\phi_{q'}) \, . \tag{S3}$$

If, instead, we rename the variables in (S2) as follows: $(pp'qq') \to (qq'pp')$, we invoke the symmetry $W_{pp'\leftrightarrow qq'} = W_{qq'\leftrightarrow pp'}$ and the condition that $f^{\rm eq}_p f^{\rm eq}_{p'} = f^{\rm eq}_q f^{\rm eq}_{q'}$ whenever the transition is allowed, we obtain

$$\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p = -\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} (-\psi^*_q) (\phi_p+\phi_{p'}-\phi_q-\phi_{q'}) \, . \tag{S4}$$

Finally, let's perform the change of variables $(pp'qq') \to (pp'q'q)$ in (S4), and use the symmetry $W_{pp'\leftrightarrow q'q} = W_{pp'\leftrightarrow qq'}$. The result is

$$\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p = -\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} (-\psi^*_{q'}) (\phi_p+\phi_{p'}-\phi_q-\phi_{q'}) \, . \tag{S5}$$

Adding up (S2), (S3), (S4), and (S5), we finally obtain

$$\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p = -\frac{1}{4}\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} (\psi^*_p+\psi^*_{p'}-\psi^*_q-\psi^*_{q'}) (\phi_p+\phi_{p'}-\phi_q-\phi_{q'}) \, . \tag{S6}$$

The properties of L are now manifest. In fact, setting $\psi = \phi$, we obtain

$$\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p = -\frac{1}{4}\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} |\phi_p+\phi_{p'}-\phi_q-\phi_{q'}|^2 \leq 0 \, , \tag{S7}$$

thereby proving "Dissipation". To verify "Onsager Symmetry", we can write its two sides explicitly:

$$\begin{aligned}
\left[\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \psi^*_p L\phi_p\right]^* &= -\frac{1}{4}\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} (\psi_p+\psi_{p'}-\psi_q-\psi_{q'}) (\phi^*_p+\phi^*_{p'}-\phi^*_q-\phi^*_{q'}) \, , \\
\int \frac{d^3p}{(2\pi)^3 p^0} f^{\rm eq}_p \phi^*_p L\psi_p &= -\frac{1}{4}\int \frac{d^3p}{(2\pi)^3 p^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3q}{(2\pi)^3 q^0}\frac{d^3q'}{(2\pi)^3 q'^0} W_{pp'\leftrightarrow qq'} f^{\rm eq}_p f^{\rm eq}_{p'} (\phi^*_p+\phi^*_{p'}-\phi^*_q-\phi^*_{q'}) (\psi_p+\psi_{p'}-\psi_q-\psi_{q'}) \, .
\end{aligned} \tag{S8}$$

Clearly, they coincide.

II. PROOF OF THEOREM 1

Theorem 3. *Suppose that* 1 *and* p^ν *are the only collisional invariants belonging to* $\mathcal{H}$, *i.e.* $\text{Ker}(L) \equiv \mathcal{H}_{\text{hy}}$. *Then, the following facts are equivalent:*

(a) *We can construct non-hydrodynamic excitations that survive as long as we want, i.e. for any time* $t > 0$,

$$\sup_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{||\phi_p(t)||}{||\phi_p(0)||} = 1\,. \tag{S9}$$

(b) *We can construct non-hydrodynamic states* ϕ_p *such that* $||(p^0)^{-1}L\phi_p||$ *is arbitrarily smaller than* $||\phi_p||$, *i.e.*

$$\inf_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{||(p^0)^{-1}L\phi_p||}{||\phi_p||} = 0\,. \tag{S10}$$

(c) *We can construct non-hydrodynamic states that produce negligibly small entropy, i.e.*

$$\inf_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{\varsigma}{E^0} = 0\,. \tag{S11}$$

(d) *Working within the space* $\mathcal{H}_{\text{non-hy}}$, *the frequency* $\omega = 0$ *is an accumulation point of the spectrum of* $(p^0)^{-1}L$. *Namely, for any* $\epsilon > 0$, *there is an element* $-i\omega$ *in the non-hydrodynamic spectrum of* $(p^0)^{-1}L$ *such that* $|\omega| < \epsilon$.

Proof. Since statements (a,b,c,d) are properties of the restriction of $(p^0)^{-1}L$ to $\mathcal{H}_{\text{non-hy}}$, we will work inside the Hilbert subspace $\mathcal{H}_{\text{non-hy}}$ across the whole proof. In the following, we will use only two ingredients. First, that $(p^0)^{-1}L$ is a self-adjoint operator on $\mathcal{H}_{\text{non-hy}}$. Second, that, since $\text{Ker}(L) \cap \mathcal{H}_{\text{non-hy}} = \{0\}$, and $\text{Ker}(1/p^0) = \{0\}$, the number 0 is not a proper eigenvalue of $(p^0)^{-1}L$ inside $\mathcal{H}_{\text{non-hy}}$. This said, let us first prove the chain (b)→(c)→(d)→(b).
(b)→(c): We invoke the Cauchy-Schwartz inequality:

$$\frac{\varsigma}{2E^0} = \frac{\left|\left(\phi_p, \dfrac{1}{p^0}L\phi_p\right)\right|}{(\phi_p, \phi_p)} \leq \frac{||\phi_p|| \left|\left|\dfrac{1}{p^0}L\phi_p\right|\right|}{||\phi_p||^2} = \frac{\left|\left|\dfrac{1}{p^0}L\phi_p\right|\right|}{||\phi_p||}\,. \tag{S12}$$

Therefore, if there is a sequence of states along which $||(p^0)^{-1}L\phi_p||/||\phi_p||$ converges to zero, also ς/E^0 will tend to zero along such a sequence. Hence, (S10) implies (S11).
(c)→(d): Equation (S11) is equivalent to

$$\sup_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{\left(\phi_p, \dfrac{1}{p^0}L\phi_p\right)}{(\phi_p, \phi_p)} = 0\,. \tag{S13}$$

Since $(p^0)^{-1}L$ is self-adjoint, we invoke Theorem 2.19 of [62, Chapter 2, §2.4], and we conclude that $\sup \text{Sp}[(p^0)^{-1}L]$=0, where "Sp" denotes the spectrum (within $\mathcal{H}_{\text{non-hy}}$). However, according to Proposition 5.2.13 of [51, Chapter 5, §5.2], the spectrum of a self-adjoint operator is non-empty and closed. Thus, 0 belongs to the spectrum, by Theorem 2.28 of [63, Chapter 2]. However, we know that 0 is not a proper eigenvalue in $\mathcal{H}_{\text{non-hy}}$, meaning that it does not belong to the point spectrum. Furthermore, 0 cannot belong to the residual spectrum, which is empty, by the spectral theorem, see Theorem 1 of [64, Lecture 18]. Therefore, it must belong to the continuous spectrum, by Point 6 of [64, Lecture 17]. On the other hand, all isolated points in the spectrum of a self-adjoint operator belong to the point spectrum [65, Chapter 5, §3.5]. Therefore, since 0 does not belong to the point spectrum (within $\mathcal{H}_{\text{non-hy}}$), it cannot be isolated, and it must be an accumulation point of the spectrum, proving (d).
(d)→(b): Suppose that 0 is an accumulation point of the spectrum. Then, 0 belongs to the spectrum, since the spectrum of a self-adjoint operator is closed (again, by Proposition 5.2.13 of [51, Chapter 5, §5.2]). On the other hand, all points in the spectrum of a self-adjoint operator are approximate eigenvalues, by Theorem 1 of [64, Lecture 18]. Thus, (b) holds, by definition of approximate eigenvalue, see Points 2 and 5 of [64, Lecture 17].

Now we only need to connect (a) to one element of (b,c,d).
(a)↔(d): Since $\phi_p(t) = e^{(p^0)^{-1}Lt}\phi_p(0)$ (where the exponential is defined through spectral calculus [66]), equation (S9) can be equivalently rewritten as follows:

$$\sup_{\phi_p \in \mathcal{H}_{\text{non-hy}}} \frac{||e^{(p^0)^{-1}Lt}\phi_p||}{||\phi_p||} = 1\,. \tag{S14}$$

This is, in turn, equivalent to saying that $e^{(p^0)^{-1}Lt}$ is a bounded operator with norm $||e^{(p^0)^{-1}Lt}||=1$. However, since $(p^0)^{-1}L$ is self-adjoint, also $e^{(p^0)^{-1}Lt}$ is self-adjoint, by Theorem 3 of [66, Chapter XI, §12]. Hence,

$$||e^{(p^0)^{-1}Lt}|| = \sup_{\phi_p\in\mathcal{H}_{\text{non-hy}}} \frac{(\phi_p, e^{(p^0)^{-1}Lt}\phi_p)}{(\phi_p,\phi_p)} = \sup_{\mathcal{H}_{\text{non-hy}}} \text{Sp}\big[e^{(p^0)^{-1}Lt}\big] \,, \tag{S15}$$

where we have used Lemma 2.14 and Theorem 2.19 of [62, Chapter 2]. Therefore, condition (a) is equivalent to the requirement that the supremum of the spectrum of $e^{(p^0)^{-1}Lt}$ be equal to 1. On the other hand, condition (d) is equivalent to the requirement that the supremum of the spectrum of $(p^0)^{-1}L$ be equal to 0. Indeed, the two are related. In fact, applying Lemma 3.12 of [62, Chapter 3, §3.2], and considering that $f(x)=e^{tx}$ is continuous, we have

$$\text{Sp}[e^{(p^0)^{-1}Lt}] = \overline{e^{t\,\text{Sp}[(p^0)^{-1}L]}} \,, \tag{S16}$$

which implies (recall that $t>0$)

$$\sup_{\mathcal{H}_{\text{non-hy}}} \text{Sp}\big[e^{(p^0)^{-1}Lt}\big] = \exp\left\{ t \sup_{\mathcal{H}_{\text{non-hy}}} \text{Sp}[(p^0)^{-1}L] \right\}. \tag{S17}$$

This shows that (a) [i.e. left-hand side of (S17) equals 1] and (d) [i.e. right-hand side of (S17) equals e^0] are equivalent. This completes our proof. □

Remark: Point (d) tells us that the restriction of $(p^0)^{-1}L$ to the non-hydrodynamic space $\mathcal{H}_{\text{non-hy}}$ possesses an infinite list of spectral points that accumulate at 0. One may ask whether these spectral points "survive" when we go back to $\mathcal{H}$. The answer is, indeed, affirmative. In fact, all the spectral points of a self-adjoint operator are approximate eigenvalues. This means that λ is a spectral point of the restriction of $(p^0)^{-1}L$ to $\mathcal{H}_{\text{non-hy}}$ if and only if there is a sequence of states $\phi_p^{(n)}\in\mathcal{H}_{\text{non-hy}}$ with norm 1 such that

$$\lim_{n\to\infty} ||\big[(p^0)^{-1}L-\lambda\big]\phi_p^{(n)}|| = 0 \,. \tag{S18}$$

Clearly, equation (S18) is still fulfilled by the same sequence $\phi_p^{(n)}$ when we regard $(p^0)^{-1}L$ as an operator on $\mathcal{H}$. Thus, λ is a spectral point of $(p^0)^{-1}L$ also in $\mathcal{H}$.

III. MASSLESS PARTICLES WITH $\lambda\varphi^4$ INTERACTIONS

Here, we prove that an ultrarelativistic gas with $\lambda\varphi^4$ interaction fulfills criterion (b) of Theorem 3.

In massless $\lambda\varphi^4$ kinetic theory (for classical particles), with coupling strength g, the states

$$\phi_p^{(n)} = L_n^{(5)}(\beta p^0)p^2p^3 \tag{S19}$$

are eigenvectors of L, i.e. $L[L_n^{(5)}(\beta p^0)p^2p^3] = \chi_{n2}\, L_n^{(5)}(\beta p^0)p^2p^3$. The corresponding eigenvalues are

$$\chi_{n2} = -\frac{g\mathcal{M}}{2}\frac{n+1}{n+3}\,, \tag{S20}$$

which are bounded by the inequality $|\chi_{n2}| \leq g\mathcal{M}/2$.

For completeness, let us first prove that $\phi_p^{(n)} \in \mathcal{H}_{\text{non-hy}}$. This is straightforward to show in spherical coordinates,

$$p^\nu = \begin{bmatrix} p \\ p\cos\theta \\ p\sin\theta\cos\varphi \\ p\sin\theta\sin\varphi \end{bmatrix}, \qquad d^3p = p^2dp\,\sin\theta d\theta\, d\varphi\,. \tag{S21}$$

In fact, in these coordinates, $p^2p^3 \propto \sin(2\varphi)$, and the inner products $(1,\phi_p^{(n)})$, $(1,\phi_p^{(n)})$, and $(1,\phi_p^{(n)})$ vanish, being proportional to $\int_0^{2\pi}\sin(2\varphi)d\varphi = 0$. Also the inner products (p^2,ϕ_p^n) and (p^3,ϕ_p^n) vanish, since

$$(\phi_p^{(n)}, p^2{+}ip^3) \propto \int_0^{2\pi} e^{i\varphi}\sin(2\varphi)d\varphi = 0\,. \tag{S22}$$

Therefore, all the functions $\phi_p^{(n)}$ are non-hydrodynamic. Now we need to study the behavior of the norm of $(p^0)^{-1}L\phi_p^{(n)}$ at large n. Since $||(p^0)^{-1}L\phi_p^{(n)}|| = ||\chi_{n2}(p^0)^{-1}\phi_p^{(n)}|| \leq \frac{1}{2}g\mathcal{M}||(p^0)^{-1}\phi_p^{(n)}||$, we can write

$$\frac{||(p^0)^{-1}L\phi_p^{(n)}||^2}{||\phi_p^{(n)}||^2} \leq \left(\frac{g\mathcal{M}}{2}\right)^2 \frac{||(p^0)^{-1}L_n^{(5)}(p^0/T)p^2p^3||^2}{||L_n^{(5)}(p^0/T)p^2p^3||^2} = \left(\frac{g\mathcal{M}}{2}\right)^2 \frac{\displaystyle\int \frac{d^3p}{(2\pi)^3(p^0)^2}e^{-p^0/T}\left[L_n^{(5)}(p^0/T)p^2p^3\right]^2}{\displaystyle\int \frac{d^3p}{(2\pi)^3}e^{-p^0/T}\left[L_n^{(5)}(p^0/T)p^2p^3\right]^2}\,. \tag{S23}$$

Expressing both integrals in spherical coordinates, and simplifying the common factors, we obtain

$$\frac{||(p^0)^{-1}L\phi_p^{(n)}||^2}{||\phi_p^{(n)}||^2} \leq \left(\frac{g\mathcal{M}}{2T}\right)^2 \frac{\displaystyle\int_0^{+\infty} x^4e^{-x}\left[L_n^{(5)}(x)\right]^2dx}{\displaystyle\int_0^{+\infty} x^6e^{-x}\left[L_n^{(5)}(x)\right]^2dx} = \frac{1}{10(n{+}3)}\left(\frac{g\mathcal{M}}{2T}\right)^2 \xrightarrow{n\to+\infty} 0, \tag{S24}$$

which is what we wanted to prove. To evaluate the integrals, we used the following general formula [67]:

$$\int_0^{+\infty} x^\nu e^{-x}\left[L_n^{(a)}(x)\right]^2dx = \binom{n{+}a}{n}\binom{n{+}a{-}\nu{-}1}{n}\Gamma(\nu{+}1)\cdot {}_3F_2(-n,\nu{+}1,\nu{-}a{+}1;a{+}1,\nu{-}a{-}n{+}1;1)\,, \tag{S25}$$

which implies, in our case,

$$\begin{aligned} \int_0^{+\infty} x^4e^{-x}\left[L_n^{(5)}(x)\right]^2dx &= \frac{(n+5)!}{n!}\,\frac{1}{5}\,, \\ \int_0^{+\infty} x^6e^{-x}\left[L_n^{(5)}(x)\right]^2dx &= \frac{(n+5)!}{n!}\,(2n+6)\,, \end{aligned} \tag{S26}$$

whose ratio is, indeed, $(10n{+}30)^{-1}$.

IV. BOUND ON THE ENTROPY PRODUCTION USED IN THE PROOF OF THEOREM 2

We have the following sequence of entropy production rates:

$$\varsigma_n = \frac{1}{8}\int \frac{d^3p}{(2\pi)^3p^0}\frac{d^3p'}{(2\pi)^3p'^0}\frac{d^3q}{(2\pi)^3q^0}\frac{d^3q'}{(2\pi)^3q'^0}\, s\sigma(s,\theta)\,\delta^4(p{+}p'{-}q{-}q')\, f^{\text{eq}}_p f^{\text{eq}}_{p'}|\phi^{(n)}_p+\phi^{(n)}_{p'}-\phi^{(n)}_q-\phi^{(n)}_{q'}|^2\,, \tag{S27}$$

with equilibrium distribution $f^{\text{eq}}_p = e^{\alpha-\beta p^0}$, differential cross section $\sigma(s,\theta)\leq A/s^a$ (where $A{\geq}0$ and $0{\leq}a{\leq}1$), and

$$\phi^{(n)}_p = e^{\frac{\beta}{2}(1-n^{-1})p^0}\,\frac{2p^2p^3}{(p^2)^2{+}(p^3)^2}\,. \tag{S28}$$

Applying the Cauchy-Schwartz inequality to the dot product between $(1,1,-1,-1)$ and $(\phi^{(n)}_p,\phi^{(n)}_{p'},\phi^{(n)}_q,\phi^{(n)}_{q'})$, we obtain the following upper bound:

$$|\phi^{(n)}_p+\phi^{(n)}_{p'}-\phi^{(n)}_q-\phi^{(n)}_{q'}|^2 \leq 4\left[|\phi^{(n)}_p|^2+|\phi^{(n)}_{p'}|^2+|\phi^{(n)}_q|^2+|\phi^{(n)}_{q'}|^2\right] \tag{S29}$$

Therefore, recalling that $\sigma(s,\theta)\leq A/s^a$, we have an upper bound on the entropy production rate,

$$\varsigma_n \leq \frac{Ae^{2\alpha}}{2(2\pi)^{12}}\int \frac{d^3p}{p^0}\frac{d^3p'}{p'^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\, s^{1-a}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0+p'^0)}\left[|\phi^{(n)}_p|^2+|\phi^{(n)}_{p'}|^2+|\phi^{(n)}_q|^2+|\phi^{(n)}_{q'}|^2\right], \tag{S30}$$

which we can decompose into four parts:

$$\begin{aligned}
\varsigma_n \leq{}& \frac{Ae^{2\alpha}}{2(2\pi)^{12}}\int \frac{d^3p}{p^0}\frac{d^3p'}{p'^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\, s^{1-a}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0+p'^0)}|\phi^{(n)}_p|^2\\
&+\frac{Ae^{2\alpha}}{2(2\pi)^{12}}\int \frac{d^3p}{p^0}\frac{d^3p'}{p'^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\, s^{1-a}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0+p'^0)}|\phi^{(n)}_{p'}|^2\\
&+\frac{Ae^{2\alpha}}{2(2\pi)^{12}}\int \frac{d^3p}{p^0}\frac{d^3p'}{p'^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\, s^{1-a}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0+p'^0)}|\phi^{(n)}_q|^2\\
&+\frac{Ae^{2\alpha}}{2(2\pi)^{12}}\int \frac{d^3p}{p^0}\frac{d^3p'}{p'^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\, s^{1-a}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0+p'^0)}|\phi^{(n)}_{q'}|^2\,.
\end{aligned} \tag{S31}$$

In the second line, we perform the change on integration variables $(pp'qq')\rightarrow(p'pqq')$, finding that the first two lines are identical. Analogously, we perform the change of variables $(pp'qq')\rightarrow(pp'q'q)$ in the fourth line, finding that the last two lines are identical. Finally, we perform the change of variables $(pp'qq')\rightarrow(qq'pp')$ in the third line, and use the constraint $p^\mu+p'^\mu=q^\mu+q'^\mu$ imposed by the Dirac delta to show that the third line is identical to the first line[4]. Thus, all the first lines are equal to each other, and we can write

$$\varsigma_n \leq \frac{2Ae^{2\alpha}}{(2\pi)^{12}}\int \frac{d^3p}{p^0}\frac{d^3p'}{p'^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\, s^{1-a}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0+p'^0)}|\phi^{(n)}_p|^2\,. \tag{S32}$$

Now, we bound the Mandelstam variable $s=-(p^\mu{+}p'^\mu)(p_\mu{+}p'_\mu)$ as follows:

$$s=-p^\mu p_\mu - p'^\mu p'_\mu - 2p^\mu p'_\mu = 2m^2+2(p^0p'^0-p^jp'_j)\leq 2(m^2+p^0p'^0)\leq 4p^0p'^0\,. \tag{S33}$$

Since a does not exceed 1, the quantity s^{1-a} is a non-decreasing function of s, and we can set $s^{1-a}\leq(4p^0p'^0)^{1-a}$. We can also bound $|2p^2p^3|$ with $(p^2)^2{+}(p^3)^2$ in (S28). Thus, we have

$$\varsigma_n \leq \frac{8Ae^{2\alpha}}{4^a(2\pi)^{12}}\int \frac{d^3p}{(p^0)^a}\frac{d^3p'}{(p'^0)^a}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\,\delta^4(p{+}p'{-}q{-}q')\,e^{-\beta(p^0/n+p'^0)}\,. \tag{S34}$$

Defined $P^\mu=p^\mu+p'^\mu$, let us evaluate the integral

$$K(P)=\int\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\,\delta^4(P{-}q{-}q')\,. \tag{S35}$$

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[4] Note that $s=-(p^\mu{+}p'^\mu)(p_\mu{+}p'_\mu)=-(q^\mu{+}q'^\mu)(q_\mu{+}q'_\mu)$.

Since the measures d^3q/q^0 and d^3q/q^0 are Lorentz-invariant, we can carry out the calculation in the center-of-momentum frame, where $P=(\sqrt{s},0,0,0)$. Then,

$$K(P)=\int \frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\,\delta(\sqrt{s}-q^0-q'^0)\delta^3(\mathbf{q}+\mathbf{q}')=\int \frac{d^3q}{(q^0)^2}\,\delta(\sqrt{s}-2q^0)=2\pi\sqrt{1-\frac{(2m)^2}{s}}\leq 2\pi\,. \tag{S36}$$

Thus, (S34) becomes

$$\begin{aligned}\varsigma_n &\leq \frac{8Ae^{2\alpha}}{4^a(2\pi)^{11}}\int \frac{d^3p}{(p^0)^a}\,e^{-\beta p^0/n}\int \frac{d^3p}{(p^0)^a}\,e^{-\beta p^0}\\ &=\frac{32Ae^{2\alpha}}{4^a(2\pi)^9}\int_m^{+\infty} E^{1-a}e^{-\beta E/n}\sqrt{E^2-m^2}\,dE\int_m^{+\infty} E'^{1-a}e^{-\beta E'}\sqrt{E'^2-m^2}\,dE'\,.\end{aligned} \tag{S37}$$

The integrals can be bounded above replacing m with 0, and we finally obtain

$$\varsigma_n \leq \frac{32Ae^{2\alpha}\Gamma(3-a)^2}{4^a(2\pi)^9\beta^{6-2a}}\,n^{3-a}\,. \tag{S38}$$

On the other hand, the information density is given by

$$E_n^0=\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}f_p^{\mathrm{eq}}|\phi_p^{(n)}|^2=\frac{e^{\alpha}}{(2\pi)^2}\int_0^{+\infty} dp\,p^2\,e^{-\frac{\beta}{n}\sqrt{m^2+p^2}}\,. \tag{S39}$$

Since $\sqrt{m^2+p^2}\leq m+p$, we can bound below the information density as follows:

$$E_n^0 \geq \frac{e^{\alpha-\frac{m\beta}{n}}}{(2\pi)^2}\int_0^{+\infty} dp\,p^2\,e^{-\frac{\beta}{n}p}\geq \frac{2e^{\alpha-m\beta}}{(2\pi)^2\beta^3}\,n^3\,. \tag{S40}$$

Taking the ratio between (S37) and (S40), we find that

$$\frac{\varsigma_n}{E_n^0}\leq \frac{Ae^{\alpha+m\beta}\Gamma(3-a)^2}{4^a2^3\pi^7\beta^{3-2a}}\,n^{-a} \tag{S41}$$