ON THE SEMIGROUP INJECTIVE MONOID ENDOMORPHISMS OF THE MONOID $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ WITH A THREE ELEMENT FAMILY $\mathscr{F}^3$ OF INDUCTIVE NONEMPTY SUBSETS OF ω

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Abstract. We describe injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ with a three element family $\mathscr{F}^3$ of inductive nonempty subsets of ω. Also, we show that the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the multiplicative semigroup of positive integers.

1. Introduction, motivation and main definitions

We shall follow the terminology of [1, 2, 13]. By ω we denote the set of all non-negative integers and by $\mathbb{N}$ the set of all positive integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and any integer n we put $n + F = \{n+k \colon k \in F\}$ if $F \neq \varnothing$ and $n+\varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n+F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega \colon x \geqslant a\}$.

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i + 1 \in A$. Obvious, that $\varnothing$ is an inductive subset of ω.

Remark 1.1 ([5]).
1. By Lemma 6 from [4] nonempty subset $F \subseteq \omega$ is inductive in ω if and only $(-1 + F) \cap F = F$.
2. Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.
3. Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω.

For n arbitrary semigroup S any homomorphism $\alpha \colon S \to S$ is called an *endomorphism* of S. If the semigroup has the identity element 1_S then the endomorphism α of S such that $(1_S)\alpha = 1_S$ is said to be a *monoid endomorphism* of S. A bijective endomorphism of S is called an *automorphism*.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv} \colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [17].

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The *bicyclic monoid* $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_{\omega}=\omega\times\omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1)\cdot(i_2,j_2)=\left\{\begin{array}{ll}(i_1-j_1+i_2,j_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2), & \text{if } j_1\geqslant i_2.\end{array}\right. \tag{1}$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ to the semigroup $\boldsymbol{B}_{\omega}$ is isomorphic by the mapping $\mathfrak{h}\colon\mathscr{C}(p,q)\to\boldsymbol{B}_{\omega}$, $q^kp^l\mapsto(k,l)$ (see: [1, Section 1.12] or [15, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [4].

Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega}\times\mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2)=\left\{\begin{array}{ll}(i_1-j_1+i_2,j_2,(j_1-i_2+F_1)\cap F_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2,F_1\cap(i_2-j_1+F_2)), & \text{if } j_1\geqslant i_2.\end{array}\right. \tag{2}$$

In [4] is proved that if the family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I}=\{(i,j,\varnothing)\colon i,j\in\omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$. For any ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}}=\left\{\begin{array}{ll}(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F}\end{array}\right.$$

is defined in [4]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. In [4] it is proved that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, or it has the identity, are given. In particularly in [4] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω.

Group congruences on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [5]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [5], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are described. In [6] it is proved that an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is the indentity transformation if and only if ε has three distinct fixed points, which is equivalent to existence non-idempotent element $(i,j,[p))\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that $(i,j,[p))\varepsilon=(i,j,[p))$.

In [3,14] the algebraic structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω. The structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, for the family $\mathscr{F}_n$ which is generated by the initial interval $\{0,1,\ldots,n\}$ of ω, is studied in [8]. The semigroup of endomorphisms of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is described in [7,16].

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_{\omega}$ is the identity self-map of $\boldsymbol{B}_{\omega}$ [1], and hence the group $\mathbf{Aut}(\boldsymbol{B}_{\omega})$ of automorphisms of $\boldsymbol{B}_{\omega}$ is trivial. In [12] it is proved that the semigroups $\mathbf{End}(\boldsymbol{B}_{\omega})$ of the endomorphisms of the bicyclic semigroup $\boldsymbol{B}_{\omega}$ is isomorphic to the semidirect products $(\omega,+)\rtimes_{\varphi}(\omega,*)$, where $+$ and $*$ are the usual addition and the usual multiplication on the set of non-negative integers ω.

In the paper [9] injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω are studies. Here the authors describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, and show that Green's relations

$\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ coincide with the relation of equality. In [10, 11] the semigroup $\boldsymbol{End}^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is studied.

Later we assume that $\mathscr{F}^3$ is a family of inductive nonempty subsets of ω which consists of three sets. By Proposition 1 of [5] for any ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω-closed family $\mathscr{F}^*$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0) \in \mathscr{F}^*$ and the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ are isomorphic. Hence without loss of generality we may assume that the family $\mathscr{F}$ contains the set $[0)$, i.e., $\mathscr{F}^3 = \{[0), [1), [2)\}$. Later in the paper we denote $\mathscr{F}_{0,1} = \{[0), [1)\}$ and $\mathscr{F}_{1,2} = \{[1), [2)\}$ as subfamilies of $\mathscr{F}^3$.

In this paper we describe injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Also, we show that the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the multiplicative semigroup of positive integers.

2. On injective endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ which induce injective endomorphisms of its submonoid $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$

If $\mathscr{F}$ is an arbitrary ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ and $[s) \in \mathscr{F}$ for some $s \in \omega$ then

$$\boldsymbol{B}_{\omega}^{\{[s)\}} = \{(i, j, [s)) \colon i, j \in \omega\}$$

is a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ [5] and by Proposition 3 of [4] the semigroup $\boldsymbol{B}_{\omega}^{\{[s)\}}$ is isomorphic to the bicyclic semigroup.

Later we need the following theorem from [6].

Theorem 2.1 ([6, Theorem 2])**.** *Let $\mathscr{F}$ be an ω-closed family of inductive nonempty subsets of ω, which contains at least two sets. Then for an injective monois endomorphism ε of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ the following conditions are equivalent:*

- *(i) ε is the identity map;*
- *(ii) there exists a nonidempotent element $(i, j, [p)) \in \boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that $(i, j, [p))\varepsilon = (i, j, [p))$;*
- *(iii) the map ε has at leat three fixed points.*

Let $\mathscr{F}^2 = \{[0), [1)\}$. For an arbitrary positive integer k and any $p \in \{0, \ldots, k-1\}$ we define the transformation $\alpha_{k,p}$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ in the following way

$$\begin{aligned}
(i, j, [0))\alpha_{k,p} &= (ki, kj, [0)),\\
(i, j, [1))\alpha_{k,p} &= (p + ki, p + kj, [1)),
\end{aligned}$$

for all $i, j \in \omega$. Also, for an arbitrary positive integer $k \geqslant 2$ and any $p \in \{1, \ldots, k-1\}$ we define the transformation $\beta_{k,p}$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the following way

$$\begin{aligned}
(i, j, [0))\beta_{k,p} &= (ki, kj, [0)),\\
(i, j, [1))\beta_{k,p} &= (p + ki, p + kj, [0)),
\end{aligned}$$

for all $i, j \in \omega$.

The following theorem is proved in [9].

Theorem 2.2 ([9, Theorem 1])**.** *Let $\mathscr{F}^2 = \{[0), [1)\}$ and ε be an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$. Then either there exist a positive integer k and $p \in \{0, \ldots, k-1\}$ such that $\varepsilon = \alpha_{k,p}$ or there exist a positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$ such that $\varepsilon = \beta_{k,p}$.*

Example 2.3. Let $\mathscr{F}^3 = \{[0), [1), [2)\}$. Fix an arbitrary positive integer k. We define the transformation $\alpha_{[k]}$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ in the following way

$$(i, j, [p))\alpha_{[k]} = \begin{cases} (ki, kj, [p)), & \text{if } p \in \{0, 1\};\\ (k(i+1) - 1, k(j+1) - 1, [2)), & \text{if } p = 2, \end{cases}$$

for all $i, j \in \omega$. It is obvious that $\alpha_{[k]}$ is an injective transformation of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$.

Lemma 2.4. *For an arbitrary positive integer k the transformation $\alpha_{[k]}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$.*

Proof. It is obvious that in the case when $k=1$ the map $\alpha_{[k]}$ is the identity transformation of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, i.e., $\alpha_{[k]}$ is an automorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, and hence later without loss of generality we may assume that $k\geqslant 2$.

By Lemma 2 of [9] the restrictions of the map $\alpha_{[k]}$ onto the subsemigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}_{1,2}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ are injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}_{1,2}}$, respectively. Hence it is complete to show that the map $\alpha_{[k]}$ preserves the semigroup operation in the following two cases

$$(i_0,j_0,[0))\cdot(i_2,j_2,[2)) \qquad \text{and} \qquad (i_2,j_2,[2))\cdot(i_0,j_0,[0)).$$

Then we get that

$$\begin{aligned}
((i_0,j_0,[0))\cdot(i_2,j_2,[2)))\alpha_{[k]} &= \begin{cases}
(i_0-j_0+i_2,j_2,(j_0-i_2+[0))\cap[2))\alpha_{[k]}, & \text{if } j_0<i_2;\\
(i_0,j_2,[0)\cap[2))\alpha_{[k]}, & \text{if } j_0=i_2;\\
(i_0,j_0-i_2+j_2,[0)\cap(-1+[2)))\alpha_{[k]}, & \text{if } j_0=i_2+1;\\
(i_0,j_0-i_2+j_2,[0)\cap(i_2-j_0+[2)))\alpha_{[k]}, & \text{if } j_0\geqslant i_2+2
\end{cases}\\
&= \begin{cases}
(i_0-j_0+i_2,j_2,[2))\alpha_{[k]}, & \text{if } j_0<i_2;\\
(i_0,j_2,[2))\alpha_{[k]}, & \text{if } j_0=i_2;\\
(i_0,j_2+1,[1))\alpha_{[k]}, & \text{if } j_0=i_2+1;\\
(i_0,j_0-i_2+j_2,[0))\alpha_{[k]}, & \text{if } j_0\geqslant i_2+2
\end{cases}\\
&= \begin{cases}
(k(i_0-j_0+i_2+1)-1,k(j_2+1)-1,[2)), & \text{if } j_0<i_2;\\
(k(i_0+1)-1,k(j_2+1)-1,[2)), & \text{if } j_0=i_2;\\
(ki_0,k(j_2+1),[1))), & \text{if } j_0=i_2+1;\\
(ki_0,k(j_0-i_2+j_2),[0)), & \text{if } j_0\geqslant i_2+2,
\end{cases}
\end{aligned}$$

$$\begin{aligned}
&(i_0,j_0,[0))\alpha_{[k]}\cdot(i_2,j_2,[2))\alpha_{[k]} = (ki_0,kj_0,[0))\cdot(k(i_2+1)-1,k(j_2+1)-1,[2))\\
&= \begin{cases}
(ki_0-kj_0+k(i_2+1)-1,k(j_2+1)-1,(kj_0-(k(i_2+1)-1)+[0))\cap[2)), & \text{if } kj_0<k(i_2{+}1){-}1;\\
(ki_0,k(j_2+1)-1,[0)\cap[2)), & \text{if } kj_0=k(i_2{+}1){-}1;\\
(ki_0,kj_0-(k(i_2+1)-1)+k(j_2+1)-1,[0)\cap(k(i_2+1)-1-kj_0+[2))), & \text{if } kj_0>k(i_2{+}1){-}1
\end{cases}\\
&= \begin{cases}
(k(i_0-j_0+i_2+1)-1,k(j_2+1)-1,[2)), & \text{if } j_0<i_2{+}1{-}1/k;\\
(ki_0,k(j_2+1)-1,[2)), & \text{if } j_0=i_2{+}1{-}1/k;\\
(ki_0,k(j_0-i_2+j_2),[0)\cap(k(i_2+1)-1-kj_0+[2))), & \text{if } j_0>i_2{+}1{-}1/k;
\end{cases}\\
&= \begin{cases}
(k(i_0-j_0+i_2+1)-1,k(j_2+1)-1,[2)), & \text{if } j_0<i_2;\\
(k(i_0+1)-1,k(j_2+1)-1,[2)), & \text{if } j_0=i_2;\\
(ki_0,k(j_2+1),[1))), & \text{if } j_0=i_2+1;\\
(ki_0,k(j_0-i_2+j_2),[0)), & \text{if } j_0\geqslant i_2+2,
\end{cases}
\end{aligned}$$

because $k\geqslant 2$ and the equality $j_0=i_2+1-1/k$ is impossible; and

$$\begin{aligned}
((i_2,j_2,[2))\cdot(i_0,j_0,[0)))\alpha_{[k]} &= \begin{cases}
(i_2-j_2+i_0,j_0,(j_2-i_0+[2))\cap[0))\alpha_{[k]}, & \text{if } j_2<i_0;\\
(i_2,j_0,[2)\cap[0))\alpha_{[k]}, & \text{if } j_2=i_0;\\
(i_2,j_2-i_0+j_0,[2)\cap(i_0-j_2+[0)))\alpha_{[k]}, & \text{if } j_2>i_0
\end{cases}\\
&= \begin{cases}
(i_2-j_2+i_0,j_0,[0))\alpha_{[k]}, & \text{if } j_2+2\leqslant i_0;\\
(i_2+1,j_0,[1))\alpha_{[k]}, & \text{if } j_2+1=i_0;\\
(i_2,j_0,[2))\alpha_{[k]}, & \text{if } j_2=i_0;\\
(i_2,j_2-i_0+j_0,[2))\alpha_{[k]}, & \text{if } j_2>i_0
\end{cases}
\end{aligned}$$

$$= \begin{cases} (k(i_2-j_2+i_0), kj_0, [0)), & \text{if } j_2+2 \leqslant i_0; \\ (k(i_2+1), kj_0, [1)), & \text{if } j_2+1 = i_0; \\ (k(i_2+1)-1, k(j_0+1)-1, [2)), & \text{if } j_2 = i_0; \\ (k(i+1)-1_2, k(j_2-i_0+j_0+1)-1, [2)), & \text{if } j_2 > i_0, \end{cases}$$

$$(i_2, j_2, [2))\alpha_{[k]} \cdot (i_0, j_0, [0))\alpha_{[k]} = (k(i_2+1)-1, k(j_2+1)-1, [2)) \cdot (ki_0, kj_0, [0))$$
$$= \begin{cases} (k(i_2+1)-1-(k(j_2+1)-1)+ki_0, kj_0, (k(j_2+1)-1-ki_0+[2))\cap[0)), & \text{if } k(j_2{+}1){-}1 < ki_0; \\ (k(i_2+1)-1, kj_0, [2)\cap[0)), & \text{if } k(j_2{+}1){-}1 = ki_0; \\ (k(i_2+1)-1, k(j_2+1)-1-ki_0+kj_0, [2)\cap(ki_0-(k(j_2+1)-1)+[0))), & \text{if } k(j_2{+}1){-}1 > ki_0 \end{cases}$$
$$= \begin{cases} (k(i_2-j_2+i_0), kj_0, (k(j_2+1)-1-ki_0+[2))), & \text{if } j_2+1 < i_0+1/k; \\ (k(i_2+1)-1, kj_0, [2)), & \text{if } j_2+1 = i_0+1/k; \\ (k(i_2+1)-1, k(j_2-i_0+j_0+1)-1, [2)), & \text{if } j_2+1 > i_0+1/k \end{cases}$$
$$= \begin{cases} (k(i_2-j_2+i_0), kj_0, [0)), & \text{if } j_2+2 \leqslant i_0; \\ (k(i_2+1), kj_0, (k(j_2+1)-1-ki_0+[2))), & \text{if } j_2+1 = i_0; \\ (k(i_2+1)-1, k(j_0+1)-1, [2)), & \text{if } j_2 = i_0; \\ (k(i+1)-1_2, k(j_2-i_0+j_0+1)-1, [2)), & \text{if } j_2 > i_0, \end{cases}$$

because $k \geqslant 2$ and the equality $j_2+1 = i_0+1/k$ is impossible. This completes the proof of the lemma. $\square$

Proposition 2.5. *Let ε be an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that*

$$(0,0,[0))\varepsilon = (0,0,[0)), \qquad (0,0,[1))\varepsilon = (0,0,[1)), \qquad \textit{and} \qquad (0,0,[2))\varepsilon \in \boldsymbol{B}_{\omega}^{\{[2)\}}.$$

Then there exists a positive integer k such that $\varepsilon = \alpha_{[k]}$.

Proof. If $(0,0,[2))\varepsilon = (0,0,[2))$ then by Theorem 2.1 we get that ε is the identity map of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, and hence $\varepsilon = \alpha_{[k]}$ for $k=1$.

Later we assume that $(0,0,[2))\varepsilon \neq (0,0,[2))$. By Lemma 2 of [9] the restrictions of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$. The above arguments, the assumptions of the proposition, and Theorem 2.2 imply that there exists a positive integer k such that

$$(i,j,[0))\varepsilon = (ki,kj,[0)),$$
$$(i,j,[1))\varepsilon = (ki,kj,[1)),$$

for all $i,j \in \omega$. Again, by Lemma 2 of [9] the restrictions of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{1,2}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{1,2}}$. This, the above arguments, and Theorem 2.2 imply that there exists a positive integer $s \in \{1,\ldots,k-1\}$ such that

$$(i,j,[2))\varepsilon = (ki+s, kj+s, [1)),$$

for all $i,j \in \omega$.

We claim that $s = k-1$. Indeed, the semigroup operation of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ implies that

$$(1,1,[0))\cdot(0,0,[2)) = (1,1,[0)\cap(-1+[2))) = (1,1,[0)\cap([1))) = (1,1,[1)).$$

Since ε is an endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, we get that

$$\begin{aligned}
(k,k,[1)) &= (1,1,[1))\varepsilon = \\
&= ((1,1,[0))\cdot(0,0,[2)))\varepsilon = \\
&= (1,1,[0))\varepsilon\cdot(0,0,[2))\varepsilon = \\
&= (k,k,[0))\cdot(s,s,[2)) = \\
&= (k,k-s+s,[0)\cap(s-k+[2))) = \\
&= (k,k,[0)\cap[s-k+2)),
\end{aligned}$$

which implies that $\max\{0,s-k+2\}=1$. Then $s-k+2=1$, and hence $s=k-1$. $\square$

Proposition 2.6. *Let ε be an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. If $(0,0,[0))\varepsilon=(0,0,[0))$ and $(0,0,[1))\varepsilon=(0,0,[1))$, then $\varepsilon=\alpha_{[k]}$ for some positive integer k.*

Proof. Suppose that $(0,0,[2))\varepsilon\in\boldsymbol{B}_{\omega}^{\{[1)\}}$. Since $(0,0,[0))\varepsilon=(0,0,[0))$ and $(0,0,[1))\varepsilon=(0,0,[1))$, Theorem 2.2 implies that there exists a positive integer k such that $(i,j,[0))\varepsilon=(ki,kj,[0))$ and $(i,j,[1))\varepsilon=(ki,kj,[1))$ for all $i,j\in\omega$. Since $(0,0,[2))$ is an idempotent of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, Proposition 1.4.21(2) of [13] implies so is $(0,0,[2))\varepsilon$. By Lemma 2 of [4] there exists $s\in\omega$ such that $(0,0,[2))\varepsilon=(s,s,[1))$. The inequalities $(1,1,[1))\preccurlyeq(0,0,[2))\preccurlyeq(0,0,[1))$ and Proposition 1.4.21(6) of [13] imply that

$$\begin{aligned}
(k,k,[1)) = (1,1,[1))\varepsilon \preccurlyeq (0,0,[2))\varepsilon = (s,s,[1)) \preccurlyeq \\
\preccurlyeq (0,0,[1)) = (0,0,[1))\varepsilon.
\end{aligned}$$

Since the endomorphism ε is an injective map, Lemma 5 of [4] implies that $0<s<k$. The semigroup operation of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ implies that

$$(1,1,[0))\cdot(0,0,[2))=(1,1,[0)\cap(-1+[2)))=(1,1,[0)\cap([1)))=(1,1,[1)),$$

and hence we get that

$$\begin{aligned}
(k,k,[1)) &= (1,1,[1))\varepsilon = \\
&= ((1,1,[0))\cdot(0,0,[2)))\varepsilon = \\
&= (1,1,[0))\varepsilon\cdot(0,0,[2))\varepsilon = \\
&= (s,s,[1))\cdot(k,k,[0)) = \\
&= (s-s+k,k,(s-k+[1))\cap[0)) = \\
&= (k,k,[0)),
\end{aligned}$$

because $s<k$. The obtained contradiction implies that $(0,0,[2))\varepsilon\notin\boldsymbol{B}_{\omega}^{\{[1)\}}$.

Suppose that $(0,0,[2))\varepsilon\in\boldsymbol{B}_{\omega}^{\{[0)\}}$. Since $(0,0,[2))$ is an idempotent of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, Proposition 1.4.21(2) of [13] and Lemma 2 of [4] imply that there exists $t\in\omega$ such that $(0,0,[2))\varepsilon=(t,t,[0))$. The semigroup operation of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ implies that

$$(1,1,[0))\cdot(0,0,[2))=(1,1,[0)\cap(-1+[2)))=(1,1,[0)\cap([1)))=(1,1,[1)),$$

and by Theorem 2.2 we get that there exist a positive integer k such that $(i,j,[0))\varepsilon=(ki,kj,[0))$ and $(i,j,[1))\varepsilon=(ki,kj,[1))$ for all $i,j\in\omega$. Then we have that

$$\begin{aligned}
(k,k,[1)) &= (1,1,[1))\varepsilon = \\
&= ((1,1,[0))\cdot(0,0,[2)))\varepsilon = \\
&= (1,1,[0))\varepsilon\cdot(0,0,[2))\varepsilon = \\
&= (t,t,[0))\cdot(k,k,[0)) = \\
&= (\max\{t,k\},\max\{t,k\},[0))\in\boldsymbol{B}_{\omega}^{\{[0)\}},
\end{aligned}$$

a contradiction. Hence $(0,0,[2])\varepsilon \notin \boldsymbol{B}_{\omega}^{\{[0)\}}$.

The above argument imply that $(0,0,[2])\varepsilon \in \boldsymbol{B}_{\omega}^{\{[2)\}}$. Next we apply Proposition 2.5. $\square$

Proposition 2.7. *For an arbitrary injective monoid endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ there exist no a positive integer k and $p \in \{1,\ldots,k-1\}$ such that the restriction $\varepsilon{\restriction}_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}}$ of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ coincides with the endomorphism $\alpha_{k,p}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$.*

Proof. Suppose to the contrary that exist a positive integer k and $p \in \{1,\ldots,k-1\}$ such that $\varepsilon{\restriction}_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}} = \alpha_{k,p}$. Then we have that

$$(i,j,[0))\varepsilon = (ki,kj,[0)),$$
$$(i,j,[1))\varepsilon = (p+ki,p+kj,[1)),$$

for all $i,j \in \omega$.

Suppose that $(0,0,[2))\varepsilon \in \boldsymbol{B}_{\omega}^{\{[2)\}}$. By the choice of the integer p and by the description of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Lemma 5 of [4] or Proposition 3 in [5]) we get that there exists a positive integer t such that $(0,0,[2))\varepsilon = (t,t,[2))$. The semigroup operation of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ implies that

$$(1,1,[0))\cdot(0,0,[2)) = (1,1,[1)),$$

and hence we have that

$$(k,k,[0))\cdot(t,t,[2)) = (1,1,[0))\varepsilon\cdot(0,0,[2))\varepsilon = (1,1,[1))\varepsilon = (p+k,p+k,[1)).$$

The structure of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Proposition 3 in [5]) implies that

$$(1,1,[1)) \preccurlyeq (0,0,[2)) \preccurlyeq (0,0,[1)).$$

Hence by Proposition 1.4.21(6) of [13] we have that

$$(p+k,p+k,[1)) = (1,1,[1))\varepsilon \preccurlyeq (t,t,[2)) = (0,0,[2))\varepsilon \preccurlyeq (0,0,[1))\varepsilon = (p,p.[1)).$$

The above arguments and Lemma 5 of [4] imply that $p \leqslant t \leqslant k+p$. Then the equalities

$$(p+k,p+k,[1)) = (k,k,[0))\cdot(t,t,[2)) = \\ = \begin{cases} (t,t,[2)), & \text{if } k \leqslant t; \\ (k,k,[0)\cap(t-k+[2))), & \text{if } k > t \end{cases}$$

imply that $t-k=-1$ and $k=k+p$. The last equality contradicts the assumption.

Suppose that $(0,0,[2))\varepsilon \in \boldsymbol{B}_{\omega}^{\{[1)\}}$. Then by the choice of the integer p and by the structure of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Lemma 5 of [4] or Proposition 3 in [5]) we obtain that there exists a positive integer t such that $(0,0,[2))\varepsilon = (t,t,[1))$. Since

$$(1,1,[1)) \preccurlyeq (0,0,[2)) \preccurlyeq (0,0,[1)).$$

by Proposition 1.4.21(6) of [13] we have that

$$(p+k,p+k,[1)) = (1,1,[1))\varepsilon \preccurlyeq (t,t,[1)) = (0,0,[2))\varepsilon \preccurlyeq (0,0,[1))\varepsilon = (p,p.[1)).$$

The above arguments and Lemma 5 of [4] imply that $p \leqslant t \leqslant k+p$. These inequalities and the injectivity of the map ε imply that $p < t < k+p$. Then the equality

$$(1,1,[0))\cdot(0,0,[2)) = (1,1,[1)),$$

imply that

$$(p+k,p+k,[1)) = (k,k,[0))\cdot(t,t,[1)) = \\ = \begin{cases} (t,t,[1)), & \text{if } k \leqslant t; \\ (k,k,[0)), & \text{if } k > t, \end{cases}$$

and hence $t = k+p$, a contradiction.

Suppose that $(0,0,[2))\varepsilon \in \boldsymbol{B}_{\omega}^{\{[0)\}}$. Then by the choice of the integer p and the description of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Lemma 5 of [4] or Proposition 3 in [5]) we get that there exists a positive integer t such that $(0,0,[2))\varepsilon = (t,t,[0))$. Since

$$(1,1,[1)) \preccurlyeq (0,0,[2)) \preccurlyeq (0,0,[1)).$$

by Proposition 1.4.21(6) of [13] we have that

$$(p+k,p+k,[1)) = (1,1,[1))\varepsilon \preccurlyeq (t,t,[0)) = (0,0,[2))\varepsilon \preccurlyeq (0,0,[1))\varepsilon = (p,p.[1)).$$

The above arguments and Lemma 5 of [4] imply that $p \leqslant t \leqslant k+p$. Since

$$(1,1,[0)) \cdot (0,0,[2)) = (1,1,[1)),$$

we obtain that

$$(p+k,p+k,[1)) = (k,k,[0)) \cdot (t,t,[0)) = (\max\{k,t\},\max\{k,t\},[0)),$$

a contradiction.

The obtained contradictions imply the statement of the proposition. $\square$

Proposition 2.8. *For any an injective monoid endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ there exist no a positive integer $k \geqslant 2$ and $p \in \{1,\ldots,k-1\}$ such that the restriction $\varepsilon\!\downharpoonleft_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}}$ of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ coincides with the endomorphism $\beta_{k,p}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$.*

Proof. Suppose to the contrary that exist a positive integer k and $p \in \{1,\ldots,k-1\}$ such that $\varepsilon\!\downharpoonleft_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}} = \beta_{k,p}$. Then we have that

$$\begin{aligned}
(i,j,[0))\varepsilon &= (ki,kj,[0)),\\
(i,j,[1))\varepsilon &= (p+ki,p+kj,[0)),
\end{aligned}$$

for all $i,j \in \omega$.

Suppose that $(0,0,[2))\varepsilon \in \boldsymbol{B}_{\omega}^{\{[2)\}}$. Then by the choice of the integer p and the description of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Lemma 5 of [4] or Proposition 3 in [5]) we obtain that there exists a positive integer t such that $(0,0,[2))\varepsilon = (t,t,[2))$. Since

$$(1,1,[1)) \preccurlyeq (0,0,[2)) \preccurlyeq (0,0,[1)).$$

by Proposition 1.4.21(6) of [13] we have that

$$(p+k,p+k,[1)) = (1,1,[1))\varepsilon \preccurlyeq (t,t,[2)) = (0,0,[2))\varepsilon \preccurlyeq (0,0,[1))\varepsilon = (p,p.[1)).$$

The above arguments and Lemma 5 of [4] imply that $p \leqslant t \leqslant k+p$. The semigroup operation of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ implies that

$$(1,1,[0)) \cdot (0,0,[2)) = (1,1,[1)),$$

and hence we have that

$$(k,k,[0)) \cdot (t,t,[2)) = (1,1,[0))\varepsilon \cdot (0,0,[2))\varepsilon = (1,1,[1))\varepsilon = (p+k,p+k,[0)).$$

Then the equalities

$$\begin{aligned}
(p+k,p+k,[0)) &= (k,k,[0)) \cdot (t,t,[2)) =\\
&= \begin{cases} (t,t,[2)), & \text{if } k \leqslant t;\\ (k,k,[0) \cap (t-k+[2))), & \text{if } k > t \end{cases}
\end{aligned}$$

imply that $k = k+p$, and hence $p = 0$. A contradiction.

Suppose that $(0,0,[2))\varepsilon \in \boldsymbol{B}_{\omega}^{\{[1)\}}$. The choice of the integer p and the structure of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Lemma 5 of [4] or Proposition 3 in [5]) imply that there exists a positive integer t such that $(0,0,[2))\varepsilon = (t,t,[1))$. Similar as in the previous case we get that $p \leqslant t \leqslant k+p$. Then the equality

$$(1,1,[0)) \cdot (0,0,[2)) = (1,1,[1)),$$

implies that

$$(k,k,[0])\cdot(t,t,[1]) = (1,1,[0])\varepsilon\cdot(0,0,[2])\varepsilon = (1,1,[1])\varepsilon = (p+k,p+k,[0]),$$

and hence the equalities

$$\begin{aligned}(p+k,p+k,[0]) &= (k,k,[0])\cdot(t,t,[1]) = \\ &= \begin{cases}(t,t,[1]), & \text{if } k\leqslant t;\\ (k,k,[0]), & \text{if } k>t\end{cases}\end{aligned}$$

imply that $k=k+p$, and hence $p=0$. A contradiction.

Suppose that $(0,0,[2])\varepsilon\in\boldsymbol{B}_{\omega}^{\{[0]\}}$. The choice of the integer p and the structure of the natural partial order on $E(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ (see Lemma 5 of [4] or Proposition 3 in [5]) imply that there exists a positive integer t such that $(0,0,[2])\varepsilon=(t,t,[0])$. Similar as in the previous case we get that $p\leqslant t\leqslant k+p$. Then the equality

$$(1,1,[0])\cdot(0,0,[2]) = (1,1,[1]),$$

implies that

$$(k,k,[0])\cdot(t,t,[0]) = (1,1,[0])\varepsilon\cdot(0,0,[2])\varepsilon = (1,1,[1])\varepsilon = (p+k,p+k,[0]).$$

Then we have that

$$\begin{aligned}(p+k,p+k,[0]) &= (k,k,[0])\cdot(t,t,[0]) = \\ &= \begin{cases}(t,t,[0]), & \text{if } k\leqslant t;\\ (k,k,[0]), & \text{if } k>t.\end{cases}\end{aligned}$$

If $k=k+p$ then $p=0$, which contradicts the assumption of the proposition. If $t=p+k$ then

$$(1,1,[1])\varepsilon = (p+k,p+k,[0]) = (0,0,[2])\varepsilon,$$

which contradicts the injectivity of the map ε.

The obtained contradictions imply the statement of the proposition. $\square$

The following theorem summarises the main result of this section and it follows from Lemma 2.4 and Propositions 2.5–2.8.

Theorem 2.9. *Let $\mathscr{F}^3=\{[0],[1],[2]\}$ and ε be an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. If the restriction $\varepsilon{\restriction}_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}}$ of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$, then $\varepsilon=\alpha_{[k]}$ for some positive integer k.*

3. On injective endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ which do not induce injective endomorphisms of its submonoid $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$

Theorem 3.1. *Let $\mathscr{F}^3=\{[0],[1],[2]\}$. Then there exists no an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that the restriction $\varepsilon{\restriction}_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}}$ of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is not a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$.*

Proof. Suppose to the contrary that there exists an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that the restriction $\varepsilon{\restriction}_{\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}}$ of the map ε onto the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is not a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}$. By Proposition 3 of [4], for any $n=0,1,2$ the semigroup $\boldsymbol{B}_{\omega}^{\{[n)\}}$ is isomorphic to the bicyclic semigroup. By Proposition 4 of [5] we have that $(i,j,[0])\varepsilon\in\boldsymbol{B}_{\omega}^{\{[0)\}}$ for all $i,j\in\omega$, because ε is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Moreover, by Theorem 1 from [12] there exists a positive integer k such that $(i,j,[0])\varepsilon=(ki,kj,[0])$ for all $i,j\in\omega$. Again, Proposition 4 of [5] implies that for any $n\in\{1,2\}$ there exists $m_m\in\{0,1,2\}$ such that $(i,j,[n))\varepsilon\in\boldsymbol{B}_{\omega}^{\{[m_n)\}}$ for all $i,j\in\omega$. The above arguments and Theorem 2.2 imply that $(i,j,[1))\varepsilon\in\boldsymbol{B}_{\omega}^{\{[2)\}}$ for all $i,j\in\omega$.

We remark that the assumption that

$$(i,j,[2])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}_{0,1}}, \qquad \text{for all} \quad i,j\in\omega,$$

contradicts the equality

$$(1,1,[0])\cdot(0,0,[2])=(1,1,[1]).$$

Indeed, by Proposition 1.4.21(2) of [13], $(0,0,[2])\varepsilon$ is an idempotent of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. If $(0,0,[2])\varepsilon=(t,t,[0])$ for some $t\in\omega$ (see Lemma 2 in [4]), then we have that

$$\begin{aligned}
(1,1,[1])\varepsilon &= ((1,1,[0])\cdot(0,0,[2]))\varepsilon = \\
&= (1,1,[0])\varepsilon\cdot(0,0,[2])\varepsilon = \\
&= (k,k,[0])\cdot(t,t,[0]) = \\
&= (\max\{k,t\},\max\{k,t\},[0])\in \boldsymbol{B}_{\omega}^{\{[0]\}}.
\end{aligned}$$

This contradicts the condition that $(i,j,[1])\varepsilon\in\boldsymbol{B}_{\omega}^{\{[2]\}}$ for all $i,j\in\omega$. If $(0,0,[2])\varepsilon=(t,t,[1])$ for some $t\in\omega$ (see Lemma 2 in [4]), then we obtain that

$$\begin{aligned}
(1,1,[1])\varepsilon &= ((1,1,[0])\cdot(0,0,[2]))\varepsilon = \\
&= (1,1,[0])\varepsilon\cdot(0,0,[2])\varepsilon = \\
&= (k,k,[0])\cdot(t,t,[1]) = \\
&= \begin{cases} (t,t,[1]), & \text{if } t\geqslant k;\\ (k,k,[0]), & \text{if } t<k.\end{cases}
\end{aligned}$$

This contradicts the condition that $(i,j,[1])\varepsilon\in\boldsymbol{B}_{\omega}^{\{[2]\}}$ for all $i,j\in\omega$.

Suppose that $(0,0,[2])\varepsilon\in\boldsymbol{B}_{\omega}^{\{[2]\}}$. By Lemma 2 from [4] there exists $t\in\omega$ such that $(0,0,[2])\varepsilon=(t,t,[2])$. Since $(0,0,[2])\preccurlyeq(0,0,[1])$, Proposition 1.4.21(6) of [13] implies that $(0,0,[2])\varepsilon\preccurlyeq(0,0,[1])\varepsilon$, and hence $t\neq 0$.

Suppose that $(0,0,[1])\varepsilon=(p,p,[2])$ for some $p\in\omega$. Since

$$(1,0,[0])\cdot(0,0,[1])\cdot(0,1,[0])=((1,0,[1])\cdot(0,1,[0])=(1,1,[1]),$$

we have that

$$\begin{aligned}
(1,1,[1])\varepsilon &= ((1,0,[0])\cdot(0,0,[1])\cdot(0,1,[0]))\varepsilon = \\
&= (1,0,[0])\varepsilon\cdot(0,0,[1])\varepsilon\cdot(0,1,[0])\varepsilon = \\
&= (k,0,[0])\cdot(p,p,[2])\cdot(0,k,[0]) = \\
&= (k+p,p,[2])\cdot(0,k,[0]) = \\
&= (k+p,k+p,[2]).
\end{aligned}$$

Put $(0,1,[1])\varepsilon=(x,y,[2])$. By Proposition 1.4.21 from [13] and Lemma 4 of [4] we get that

$$\begin{aligned}
(1,0,[1])\varepsilon &= ((0,1,[1])^{-1})\varepsilon = \\
&= ((0,1,[1])\varepsilon)^{-1} = \\
&= (x,y,[2])^{-1} = \\
&= (y,x,[2]).
\end{aligned}$$

This implies that

$$\begin{aligned}
(p,p,[2]) = (0,0,[1])\varepsilon &= ((0,1,[1])\cdot(1,0,[1]))\varepsilon = \\
&= (0,1,[1])\varepsilon\cdot(1,0,[1])\varepsilon = \\
&= (x,y,[2])\cdot(y,x,[2]) = \\
&= (x,x,[2])
\end{aligned}$$

and

$$\begin{aligned}(k+p,k+p,[2]) &= (1,1,[1])\varepsilon = \\ &= ((1,0,[1])\cdot(0,1,[1]))\varepsilon = \\ &= (1,0,[1])\varepsilon\cdot(0,1,[1])\varepsilon = \\ &= (y,x,[2])\cdot(x,y,[2]) = \\ &= (y,y,[2]).\end{aligned}$$

Hence by the definition of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ we get that

$$(0,1,[1])\varepsilon = (p,k+p,[2]) \qquad \text{and} \qquad (1,0,[1])\varepsilon = (k+p,p,[2]).$$

Then for any $i,j\in\omega$ we have that

$$\begin{aligned}(i,j,[1])\varepsilon &= ((i,0,[1])\cdot(0,j,[1]))\varepsilon = \\ &= ((1,0,[1])^i\cdot(0,1,[1])^j)\varepsilon = \\ &= ((1,0,[1])\varepsilon)^i\cdot((0,1,[1])\varepsilon)^j = \\ &= (k+p,p,[2])^i\cdot(p,k+p,[2])^j = \\ &= (ki+p,p,[2])\cdot(p,kj+p,[2]) = \\ &= (ki+p,kj+p,[2]).\end{aligned}$$

Since $(1,1,[0])\preccurlyeq(0,0,[1])$ in $E\big(\boldsymbol{B}_{\omega}^{\mathscr{F}^3}\big)$, by Proposition 1.4.21(6) from [13] we have that

$$(k,k,[0]) = (1,1,[0])\varepsilon \preccurlyeq (0,0,[1])\varepsilon = (p,p,[2]).$$

Then Lemma 5 of [4] implies that $k\geqslant 2$. Also, the inequalities

$$(1,1,[1])\preccurlyeq(0,0,[2])\preccurlyeq(0,0,[1])$$

in $E\big(\boldsymbol{B}_{\omega}^{\mathscr{F}^3}\big)$ and Proposition 1.4.21(6) of [13] imply that

$$(k+p,k+p,[2]) = (1,1,[1])\varepsilon \preccurlyeq (0,0,[2])\varepsilon = (t,t,[2]) \preccurlyeq (0,0,[1])\varepsilon = (p,p,[2]).$$

By Lemma 5 of [4] we get that $p\leqslant t\leqslant k+p$. Since ε is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ we conclude that $p<t<k+p$.

The equality

$$(1,1,[0])\cdot(0,0,[2]) = (1,1,[1]).$$

implies that

$$\begin{aligned}(k+p,k+p,[2]) &= (1,1,[1])\varepsilon = \\ &= ((1,1,[0])\cdot(0,0,[2]))\varepsilon = \\ &= (1,1,[0])\varepsilon\cdot(0,0,[2])\varepsilon = \\ &= (k,k,[0])\cdot(t,t,[2]) = \\ &= \begin{cases}(t,t,[2]), & \text{if } k\leqslant t;\\ (k,k,[1]), & \text{if } k=t+1;\\ (k,k,[0]), & \text{if } k\geqslant t+2.\end{cases}\end{aligned}$$

Hence $k\leqslant t$ and $k+p=t$. The last equality implies that

$$(1,1,[1])\varepsilon = (k+p,k+p,[2]) = (0,0,[2])\varepsilon,$$

which contradicts the injectivity of the map ε.

The obtained contradictions imply the statement of the theorem. $\square$

4. On the monoid of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$

Theorems 2.9 and 3.1 imply the following theorem.

Theorem 4.1. *Let $\mathscr{F}^3=\{[0),[1),[2)\}$ and ε be an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Then $\varepsilon=\alpha_{[k]}$ for some positive integer k.*

By $(\mathbb{N},\cdot)$ we denote the multiplicative semigroup of positive integers.

Theorem 4.2. *Let $\mathscr{F}^3=\{[0),[1),[2)\}$. Then the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is isomorphic to $(\mathbb{N},\cdot)$.*

Proof. Fix arbitrary injective endomorphisms ε_1 and ε_2 of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Theorem 4.1 there exist positive integers k_1 and k_2 such that $\varepsilon_1=\alpha_{[k_1]}$ and $\varepsilon_2=\alpha_{[k_2]}$. Then we have that

$$((i,j,[0))\alpha_{[k_1]})\alpha_{[k_2]}=(k_1i,k_1j,[0))\alpha_{[k_2]}=(k_2k_1i,k_2k_1j,[0))=(i,j,[0))\alpha_{[k_1\cdot k_2]};$$
$$((i,j,[1))\alpha_{[k_1]})\alpha_{[k_2]}=(k_1i,k_1j,[1))\alpha_{[k_2]}=(k_2k_1i,k_2k_1j,[1))=(i,j,[1))\alpha_{[k_1\cdot k_2]};$$

and

$$\begin{aligned}((i,j,[2))\alpha_{[k_1]})\alpha_{[k_2]}&=(k_1(i+1)-1,k_1(j+1)-1,[2))\alpha_{[k_2]}=\\&=(k_2(k_1(i+1)-1+1)-1,k_2(k_1(j+1)-1+1)-1,[2))=\\&=(k_2k_1(i+1)-1,k_2k_1(j+1)-1,[2))=\\&=(i,j,[2))\alpha_{[k_1\cdot k_2]},\end{aligned}$$

for any $i,j\in\omega$. Hence we obtain that $\alpha_{[k_1]}\alpha_{[k_2]}=\alpha_{[k_1\cdot k_2]}$. It is obvious that the mapping $\mathfrak{i}\colon(\mathbb{N},\cdot)\to\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $k\mapsto\alpha_{[k]}$, is an injective homomorphism and by Theorem 4.1 it is surjective. $\square$

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