# On 3-colourability of $(bull, H)$-free graphs

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**Abstract**

The 3-colourability problem is a well-known NP-complete problem and it remains NP-complete for *bull*-free graphs, where *bull* is the graph consisting of $K_3$ with two pendant edges attached to two of its vertices. In this paper we study 3-colourability of $(bull, H)$-free graphs for several graphs $H$. We show that these graphs are 3-colourable or contain an induced odd wheel $W_{2p+1}$ for some $p \geq 2$ or a spindle graph $M_{3p+1}$ for some $p \geq 1$. Moreover, for all our results we can provide certifying algorithms that run in polynomial time.

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## 1 Introduction

We consider finite, simple, and undirected graphs. For terminology and notations not defined here, we refer to [1].

An *induced subgraph* of a graph $G$ is a graph on a vertex set $S \subseteq V(G)$ for which two vertices are adjacent if and only if they are adjacent in $G$. In particular, we say that the subgraph is *induced by* $S$. We also say that a graph $H$ is an *induced subgraph* of $G$ if $H$ is isomorphic to an induced subgraph of $G$.

Given a family $\mathcal{H}$ of graphs and a graph $G$, we say that $G$ is $\mathcal{H}$*-free* if $G$ contains no graph from $\mathcal{H}$ as an induced subgraph. In this context, the graphs of $\mathcal{H}$ are referred to as *forbidden induced subgraphs*.

A graph is $k$*-colourable* if each of its vertices can be coloured with one of $k$ colours so that adjacent vertices obtain distinct colours. The smallest integer $k$ such that a given graph $G$ is $k$-colourable is called the *chromatic number* of $G$, denoted by $\chi(G)$. Clearly, $\chi(G) \geq \omega(G)$ for every graph $G$, where $\omega(G)$ denotes the *clique number* of $G$, that is, the order of a maximum complete subgraph of $G$. Furthermore, a graph $G$ is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph $G'$ of $G$. For a subgraph $H$ and a vertex $v$, let $d_H(v) = |N(v) \cap V(H)|$.

The graph on five vertices $v_1$, $v_2$, $v_3$, $v_4$, $v_5$ and with the edges $v_1v_2$, $v_2v_3$, $v_3v_4$, $v_4v_5$, $v_2v_4$ is called the *bull*. Let $S_{i,j,k}$ be the graph consisting of three induced paths of lengths $i$, $j$ and $k$, with a common initial vertex. The graph $S_{1,1,1}$ is called *claw*, $S_{1,1,2}$ is called *chair* and $S_{1,2,2}$ is called $E$.

The 3-colourability problem is a well-known NP-complete problem and it remains NP-complete for *claw*-free and *bull*-free graphs. In the last two decades, a large number of results of colourings of graphs with forbidden subgraphs have been shown (cf. [2], [3], [4], [10], [12], [14], [15] and cf. [9], [11], [13] for three surveys).

Following [5], an algorithm is certifying, if it returns with each input a simple and easily verifiable certificate that the particular input is correct. For example, a certifying algorithm for the bipartite graph recognition would return either a 2-colouring of the input graph proving that it is bipartite, or an odd cycle proving that it is not bipartite. In this paper we study 3-colourability of $(bull, H)$-free graphs for several graphs $H$. For all of our results we will provide certifying algorithms that run in polynomial time.

Our research has been motivated by [4] and we use some definitions and notations from it. A graph $G$ of order $3p + 1$, $p \geq 1$ is called a *spindle graph* $M_{3p+1}$ if it contains a cycle $C$: $u_0u_1 \dots u_{3p}u_0$, where $\{u_{3i-2}, u_{3i-1}, u_{3i+1}, u_{3i+2}\} = N_G(u_{3i})$ and $\{u_{3i-3}, u_{3i}\} = N_G(u_{3i-1}) \cap N_G(u_{3i-2})$ for each $i \in [p]$, where $[p] := \{1, 2, \dots, p\}$.

Observe that $M_4 \cong K_4$ and $M_7$ is known as the Moser spindle.

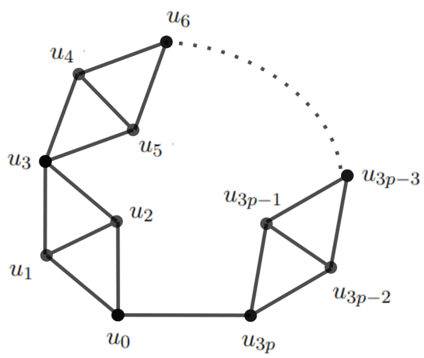



Figure 1: The spindle graph $M_{3p+1}$.

**Proposition 1** ([4])**.** *The graph $M_{3p+1}$ is not 3-colourable for every $p \geq 1$.*

Since the 3-colourability problem is NP-complete for claw-free graphs and $K_3$-free graphs (cf. [9]), it is also NP-complete for *bull*-free graphs. The following theorem in [4] has motivated our research.

**Theorem 2** ([4])**.** *Let $G$ be $(bull, claw)$-free graph. Then one of the following holds*

*(i) $G$ contains $W_5$ or*

*(ii) G contains a (not necessarily induced) spindle graph $M_{3i+1}$ for some $i \geq 1$ or*
*(iii) G is 3-colourable.*

The goal of this paper is to consider 3-colourability of $(bull, H)$-free graphs, where $H$ is a supergraph of the claw.

**Theorem 3.** *Let $G$ be a connected $(bull, chair)$-free graph. Then*
*(i) G contains an odd wheel or*
*(ii) G contains a (not necessarily induced) spindle graph $M_{3i+1}$ for some $i \geq 1$ or*
*(iii) G is 3-colourable.*

In fact Theorem 3 can be extended to the larger class of $(bull, E)$-free graphs. However, for this proof, we will show and make use of several additional graph properties.

**Theorem 4.** *Let $G$ be a connected $(bull, E)$-free graph. Then*
*(i) G contains an odd wheel or*
*(ii) G contains a (not necessarily induced) spindle graph $M_{3i+1}$ for some $i \geq 1$ or*
*(iii) G is 3-colourable.*

If we forbid in addition induced 5-cycles, then Theorem 4 can be extended as follows.

**Theorem 5.** *Let $G$ be a connected $(bull, C_5, H)$-free graph with $H \in \{S_{1,1,3}, S_{1,2,3}\}$. Then*
*(i) G contains an odd wheel or*
*(ii) G contains a (not necessarily induced) spindle graph $M_{3i+1}$ for some $i \geq 1$ or*
*(iii) G is 3-colourable.*

The 3-colourability problem has been also studied for $P_k$-free graphs for $k \geq 5$. Let $G_1$, $G_2$, $G_3$ be graphs on 7, 10 and 13 vertices, respectively (see Figure 2). In [5] the following theorem was shown.

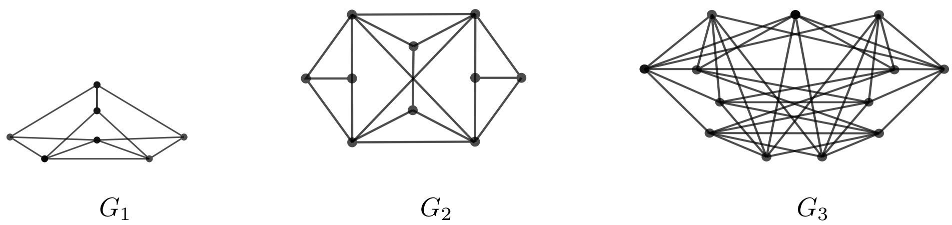



Figure 2: The graphs $G_1$, $G_2$ and $G_3$ from Theorem 6.

**Theorem 6** ([5])**.** *A $P_5$-free graph is 3-colourable if and only if it does not contain $K_4$, $W_5$, $M_7$, $G_1$, $G_2$ or $G_3$ as a subgraph.*

Note that $G_1$, $G_2$ and $G_3$ are not *bull*-free. This leads to the following corollary from Theorem 5.

**Corollary 7.** *Let $G$ be a $(bull, P_5)$-free graph. Then*

*(i) $G$ contains $W_5$ or*

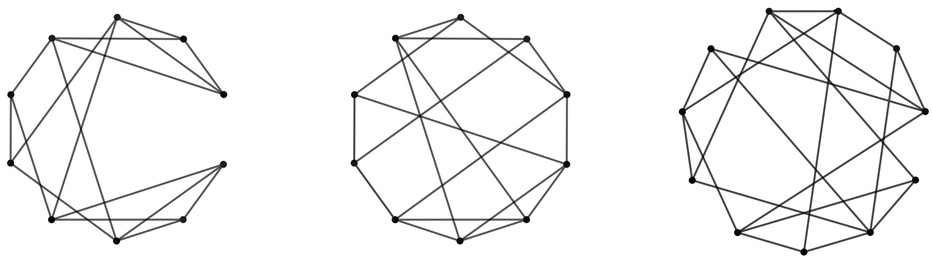
*(ii) $G$ contains a (not necessarily induced) spindle graph $M_4$ or $M_7$ or*

*(iii) $G$ is 3-colourable.*

Moreover, for $P_6$-free graphs we can recall that in [6] the following theorem was shown.

**Theorem 8** ([6])**.** *A $P_6$-free graph is 3-colourable if and only if it does not contain $F_1 \cong K_4$, $F_2 \cong W_5$, $F_3 \cong M_7$, $F_4$,..., $F_{24}$ as a subgraph, defined in [6].*

It is easy to check that $F_2 \cong W_5$, $F_3 \cong M_7$, $F_{12}$, $F_{15}$ and $F_{18}$ (see Figure 3) are the only $(K_4, bull)$-free graphs. Note that $F_{18}$ is well known as the Mycielski graph. This leads to another corollary from Theorem 5.



$F_{12}$ $F_{15}$ $F_{18}$

Figure 3: The graphs $F_{12}$, $F_{15}$ and $F_{18}$ from Theorem 8.

**Corollary 9.** *Let $G$ be a $(bull, P_6)$-free graph. Then*

*(i) $G$ contains $W_5$ or*

*(ii) $G$ contains a (not necessarily induced) spindle graph $M_4$ or $M_7$ or*

*(iii) $G$ contains $F_{12}$, $F_{15}$ or $F_{18}$ or*

*(iv) $G$ is 3-colourable.*

The organization of the paper is the following. In Section 2 we provide preliminary results and properties for *bull*-free graphs. Next, in Section 3 we prove Theorem 3 and in Section 4 we prove Theorem 4. Finally, in Section 5 we show that the proofs of Theorem 3 and Theorem 4 provide polynomial time certifying algorithms for 3-colourability in the class of $(bull, H)$-free graphs for $H \in \{S_{1,1,2}, S_{1,2,2}\}$.

## 2 Preliminary results

We recall that a *hole* in a graph $G$ is an induced cycle of length at least 4, and an *antihole* in $G$ is an induced subgraph whose complement is a cycle of length at least 4. A hole (antihole) is *odd* if it has an odd number of vertices. As the main tool for proving Theorems 3 and 4, we will use the well-known Strong Perfect Graph Theorem shown by Chudnovsky et al. [8].

**Theorem 10** (Chudnovsky et al. [8]). *A graph is perfect if and only if it contains neither an odd hole nor an odd antihole as an induced subgraph.*

In the following we will consider 3-colourability in subclasses of *bull*-free graphs. Here are some useful reductions:

- If $\Delta(G) \leq 3$, then $G$ is 3-colourable by Brook's Theorem.
- If $G$ has a vertex $w$ of degree at most 2, then $G$ is 3-colourable if and only if $G - w$ is 3-colourable. So we can reduce $G$ to $G - w$.
- If $G$ contains $K_4 = M_4$, then $G$ is not 3-colourable.
- If a graph $G$ contains an odd antihole $\overline{C_{2t+1}}$ with $t \geq 4$, then $G$ contains $K_4$. If $t = 3$, then $G$ contains the spindle graph $M_7$, and finally, if $t = 2$, we have an antihole $\overline{C_5}$, which is isomorphic to the hole $C_5$.
- If $G$ is not connected, then we can check 3-colourability for each component of $G$ seperately. Moreover, if $G$ has a cut-vertex $w$, let $G_1, G_2, \ldots, G_t$ be the components of $G - w$. Now we check whether each induced subgraph $G'_i = G[G_i \cup \{w\}]$ is 3-colourable. If all of $G'_1, \ldots, G'_t$ are 3-colourable, then we can combine their 3-colourings to obtain a 3-colouring of $G$.

These reductions show that we can restrict our 3-colourability test to the class of *bull*-free graphs that are 2-connected, $K_4$-free, and where $\delta(G) \geq 3$. Furthemore, we can assume without losing generality that the graph $G$ contains an odd hole $C_{2p+1}$.

### 2.1 Properties for *bull*-free graphs

Let $Q = v_1 v_2 \ldots v_p v_1$ be the smallest induced odd hole in the graph $G$ and $w \in V(G) \setminus Q$. We define $q(w)$ as the largest $i$ such that $w$ has $i$ consecutive neighbours on the cycle $Q$. Thus, there is $1 \leq j \leq p$ satisfying $\{v_j, v_{j+1}, \ldots, v_{j+i-1}\} \subset N_Q(w)$. All indices are taken modulo $p$.

We will prove some useful facts about this value.

**Fact 11.** *If $p > 5$, then $q(w) \in \{1, 3\}$. If $p = 5$, then $q(w) \in \{1, 3, 4\}$.*

*Proof.* Firstly, note that if $q(w) = 2$, then the set of vertices $\{v_{j-1}, v_j, v_{j+1}, v_{j+2}, w\}$ induces *bull*. If $4 \leq q(w) < p$ and $p \geq 7$, then the set of vertices $\{v_{j-1}, v_j, v_{j+1}, v_{j+3}, w\}$ induces *bull*. If $q(w) = p$, then the graph $G$ contains an odd wheel $W_p$. □

**Fact 12.** *If $q(w) = 3$, then $d_Q(w) = 3$.*

*Proof.* Suppose $q(w) = 3$ and there is $v_k \in N_Q(w)$ with $v_k \notin \{v_j, v_{j+1}, v_{j+2}\}$. Hence, we have $p \geq 7$. Then one of the sets $\{v_{j-1}, v_j, v_{j+1}, v_k, w\}$ or $\{v_{j+1}, v_{j+2}, v_{j+3}, v_k, w\}$ induces *bull*. □

**Fact 13.** *If $q(w) = 1$, then $d_Q(w) \in \{1, 2\}$. Moreover, if $q(w) = 1$ and $d_Q(w) = 2$, then there is $i$ such that $N_Q(w) = \{v_i, v_{i+2}\}$.*

*Proof.* Suppose $w$ has two neighbours $v_i, v_j$ in $Q$, satisfying $|i - j| \geq 3$, where $i > j$ . Then either the cycle $wv_iv_{i+1} \ldots v_j$ or the cycle $wv_jv_{j+1} \ldots v_i$ is odd. This cycle must contain an induced odd cycle $Q'$, which is shorter than $Q$. Since $w$ has no consecutive neighbours on $Q$, the cycle $Q'$ is not $K_3$. □

## 2.2 Properties for $(bull, E)$-free graphs

We can now define the following sets:
- $A_i = \{v \in V \setminus Q : N_Q(v) = \{v_i\}\}$.
- $B_i = \{v \in V \setminus Q : N_Q(v) = \{v_i, v_{i+2}\}\}$.
- $C_i = \{v \in V \setminus Q : N_Q(v) = \{v_i, v_{i+1}, v_{i+2}\}\}$.
- $D_i = \{v \in V \setminus Q : N_Q(v) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}\}$.

Let $A'_i = \{v \in A_i : \exists v' \in A_i, vv' \in E(G)\}$ and $B'_i = \{v \in B_i : \exists v' \in B_i \cup C_i, vv' \in E(G)\}$. Then let $A^*_i = A_i \setminus A'_i$ and $B^*_i = B_i \setminus B'_i$.

Let also $A = \bigcup_{i=1}^{p} A_i$, $B = \bigcup_{i=1}^{p} B_i$, $B' = \bigcup_{i=1}^{p} B'_i$, $C = \bigcup_{i=1}^{p} C_i$ and $D = \bigcup_{i=1}^{p} D_i$.

We want to prove that $V(G) = Q \cup A \cup B \cup C \cup D$. This is true due to the facts above and to the following lemma.

**Lemma 14.** *$Q$ is a dominating set in $G$.*

*Proof.* Suppose there exists a vertex $v \in V(G)$ for which $\mathrm{dist}(v, Q) = 2$. Hence, there exist $v' \notin Q$ and $v_i \in Q$ such that $v'v_i, vv' \in E(G)$. If $v'v_j \in E(G)$ for every $j$, then the set of vertices $\{v', v_1, \ldots, v_p\}$ induces odd wheel. If there exists $k \in [p]$ such that $v'v_k, v'v_{k-1} \in E(G)$ and $v'v_{k+1} \notin E(G)$, then the set $\{v, v', v_{k-1}, v_k, v_{k+1}\}$ induces *bull*.

If none of these two cases occur, then $q(v') = 1$. Since $Q$ is an odd cycle, there exists $j \in [p]$ such that $v'v_{j-1}, v'v_{j+1}, v'v_{j+2} \notin E(G)$ and $v'v_j \in E(G)$. Then the set of vertices $\{v, v', v_j, v_{j+1}, v_{j+2}, v_{j-1}\}$ induces $E$ (so it also induces *chair*). □

**Fact 15.** *Let $w \in B_i \cup C_i$ and $w' \in B_j \cup C_j$. If $|i - j| \geq 2$, then $ww' \notin E(G)$.*

*Proof.* We can assume without losing generality that $2 \leq j - i < p - (j - i)$. Suppose $ww' \in E(G)$. Let us consider possible cases.

If $j - i \geq 3$, then one of the sets $\{v_{i+2}, v_{i+3}, \ldots, v_j, w', w\}$ or $\{v_{j+2}, v_{j+3}, \ldots, v_i, w, w'\}$ induces smaller odd cycle in $G$.

If $j - i = 2$ and $p > 5$, then the set of vertices $\{v_i, w, v_j, w', v_{j+2}\}$ induces *bull*.

If $j - i = 2$, $p = 5$ and $w' \in B$, then the set of vertices $\{v_i, w, w', v_j, v_{j+1}\}$ induces *bull*. Analogously if $w \in B$.

If $j - i = 2$, $p = 5$ and $w, w' \in C$, then the graph $G$ contains the spindle graph $M_7$. □

**Fact 16.** *If $C_i \neq \emptyset$, then $C_{i+1}, C_{i-1} = \emptyset$.*

*Proof.* Suppose that $w \in C_i$ and $w' \in C_{i+1}$. If $ww' \in E(G)$, then the induced graph $G[\{w, v_{i+1}, v_{i+2}, w'\}]$ is complete. If $ww' \notin E(G)$, then the set of vertices $\{v_{i-1}, v_i, w, v_{i+1}, w'\}$ induces *bull*. $\square$

**Fact 17.** *If $w \in B_i \cup C_i$, $w' \in B_{i+1} \cup C_{i+1}$ and $ww' \in E(G)$, then $w \in B_i^*$ or $w' \in B_{i+1}^*$.*

*Proof.* Assume $ww' \in E(G)$. Let us consider possible cases.

If $w \in C_i$, then $w' \notin C_{i+1}$ by Fact 16.

If $w \in C_i$ and $w' \in B'_{i+1}$, then there exists $w'' \in B'_{i+1}$ such that $w'w'' \in E(G)$ and either the set of vertices $\{v_{i-1}, v_i, w, v_{i+1}, w''\}$ induces *bull* (if $ww'' \notin E(G)$) or the set $\{w, v_{i+1}, w', w''\}$ induces $K_4$ (otherwise).

If $w \in B'_i$ and $w' \in B'_{i+1}$, then there exist $w'' \in B'_i \cup B_i$ and $w''' \in B'_{i+1} \cup C_{i+1}$ such that $ww'', w'w''' \in E(G)$. Then the set of vertices $\{v_{i-1}, v_i, w', w, w''\}$ induces *bull* (if $w'w'' \notin E(G)$), or the set $\{v_{i+4}, v_{i+3}, w''', w', w\}$ induces *bull* (if $ww''' \notin E(G)$), or the set $\{v_{i+4}, v_{i+3}, w''', w', w''\}$ induces *bull* (if $w'w'' \in E(G)$, but $w''w''' \notin E(G)$), or the set $\{w, w', w'', w'''\}$ induces $K_4$ (otherwise). $\square$

## 3 Proof of Theorem 3

Note that if $G$ is a $(bull, chair)$-free graph, then the sets $A$ and $B$ are empty. It is true, because if $q(w) = 1$, then (since $Q$ is an odd cycle) there exists $i \in [p]$ such that $wv_{i-1}, wv_{i+1}, wv_{i+2} \notin E(G)$ and $wv_i \in E(G)$. Then the set of vertices $\{v_{i-1}, v_i, v_{i+1}, v_{i+2}, w\}$ induces *chair*.

Moreover, we can point out that for every $i \in [p]$ we have $|C_i| \leq 1$. It is true, because if we have two distinct vertices $w, w' \in C_i$, then either the graph $G[\{v_i, v_{i+1}, w, w'\}]$ is complete (if $ww' \in E(G)$) or the set of vertices $\{v_{i-2}, v_{i-1}, v_i, w, w'\}$ induces *chair* (if $ww' \notin E(G)$).

Consider the case with $p = 5$. We know that $V(G) = Q \cup C \cup D$. Our assumption is that $\delta(G) \geq 3$, so every vertex from $Q$ must have at least one neighbour in the set $C \cup D$. Since $p = 5$, the graph $G$ always contains $M_7$.

Now, we will describe the structure of the graph $G$ for $p > 5$. By Fact 11 we have $V(G) = C \cup Q$. Moreover, by Fact 16, $|N_Q(w) \cap N_Q(w')| \leq 1$ for every $w, w' \in C$. Then $|C| \leq \frac{p-1}{2}$. Let us recall that $C$ is an isolated set of vertices (by Fact 15).

This graph is either 3-colourable or contains the spindle graph. If $|C| = \frac{p-1}{2}$, then it is easy to see that $G$ is the spindle graph of order $\frac{3(p-1)}{2} + 1$. Otherwise, there either exists vertex $v \in Q$ such that $d_C(v) = 0$ or there exist two vertices $v_i, v_j \in Q$ such that $C_i, C_{j-2} \neq \emptyset, C_{i-2}, C_j = \emptyset$ and $j = i + 2k$, where $0 < k < \frac{p-1}{2}$. The first case is impossible due to our assumption that $\delta(G) > 2$. In second case we colour vertices $v_i, v_{i+1}, \ldots, v_j$ alternately with colours blue and red (starting with blue), and the rest of $Q$ alternately with colours red and green. Finally, we colour vertices from $C$ with remaining colours. Note that for every vertex $w \in C$ we have at least one free colour - the colouring method provides us that for every $i$ such that $C_i \neq \emptyset$, vertices $v_i$ and $v_{i+2}$ get the same colour. Therefore, the obtained colouring is proper.

# 4 Proof of Theorem 4

The proof of Theorem 4 will be split into two cases.

## 4.1 Case $p > 5$

Notice that in this case, if $w \in A_i$, then the set of vertices $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, w\}$ induces $E$. Then (because $D = \emptyset$ due to Fact 11), we have $V(G) = Q \cup B \cup C$.

Let us recall that due to Fact 15 edges $ww'$ non-incident to the cycle $Q$ can exist only for $w \in B_i \cup C_i$ and $w' \in B_i \cup C_i \cup B_{i+1} \cup C_{i+1}$.

To shorten our considerations, we will call an $ww'$ a "1-type edge" if $w, w' \in B_i \cup C_i$, and a "2-type edge" if $w \in B_i \cup C_i, w' \in B_{i+1} \cup C_{i+1}$. Of course, every edge non-incident to $Q$ is either 1-type or 2-type. Fact 17 tells us that if $ww'$ is an 2-type edge, then $w \in B^*$ or $w' \in B^*$. It is obvious, that no 1-type edge is incident to the set $B^*$. Therefore, the graph $G' = G[V(G) \setminus (B^*)]$ does not contain any 2-type edge and does contain all 1-type edges.

Note that if there exists a proper 3-colouring $c'$ of the graph $G'$, then there also exists a proper colouring $c$ of the graph $G$.

Assume that $c'$ is a proper 3-colouring of the graph $G'$. Let us precolour with $c'$ all vertices outside the set $B^*$. By Fact 15, every neighbour of non-precoloured vertex $w \in B_i^*$ is $v_i$, or $v_{i+2}$, or belongs to the set $C_{i-1} \cup B_{i-1} \cup C_{i+1} \cup B_{i+1}$. That means every neighbour of $w$ is also incident to the vertex $v_{i+1}$. Thus, the vertex $w$ can get the colour $c'(v_{i+1})$.

How can we decide whether a proper 3-colouring of $G'$ exists or not? We want to show that $c'$ exists if and only if there exists a proper colouring $c''$ of the cycle $Q$ satisfying the following property:

$$\forall i \in [p] : C_i \cup B_i' \neq \emptyset \Rightarrow c''(v_i) = c''(v_{i+2}). \tag{1}$$

Of course, if $C_i \cup B_i' \neq \emptyset$, then any proper colouring must assign the same colour to the vertices $v_i$ and $v_{i+2}$. The inverse implication is true due to the fact that the graph $G'$ does not contain any 2-type edge and to the following observation.

**Fact 18.** *The graph $G[B_i \cup C_i]$ is bipartite for any $i$.*

*Proof.* Suppose $G[B_i \cup C_i]$ contains an odd cycle. Then it contains the induced odd cycle $w_1 w_2 \ldots w_s$, where $s \geq 3$, and the set of vertices $\{v_i, w_1, w_2, \ldots, w_s w_1\}$ induces either $K_4$ or an odd wheel. □

Thus, having $c''$, we can construct $c'$ in a very simple way, assigning every vertex the first available colour.

To find the colouring $c''$ satisfying the property (1), we proceed according to the following algorithm:

1. Colour vertex $v_1$ with red.
2. Colour with red all those vertices whose colouring is enforced by the property (1).
3. If there occurres a colour conflict, stop. The graph $G$ contains the spindle graph.
4. Let $k$ be an index such that $v_{k-2}$ is non-coloured and $v_k$ is red. Colour $v_{k-1}$ with green.
5. Colour with green all those vertices whose colouring is enforced by the property (1).
6. If there there occurred colour conflict, stop. The graph $G$ contains the spindle graph.
7. Colour vertex $v_{k-2}$ with blue.
8. Colour with blue every non-coloured vertex $v_{k-2l}$, where $l \in \mathbb{N}$.
9. Colour with red all the remaining vertices.

Steps 8 and 9 are possible, since $Q$ is an odd cycle. Using this procedure we obtain a proper colouring of the cycle $Q$.

### 4.2 Case $p = 5$

Since $G$ is $(bull, E)$-free graph, and $C_5$ is a dominating cycle (by Lemma 14), it follows that $V(G) = Q \cup A \cup B \cup C \cup D$. Assuming $G$ does not contain $W_5$, $K_4$ and $M_7$, we will prove a number of properties of these subsets.

**Fact 19.** *The graphs $G[C_i \cup B_i]$ and $G[A_i]$ are bipartite for any $i$.*

*Proof.* The proof is analogous to the proof of Fact 18. □

**Fact 20.** *If $w \in A_i$ and $w' \in A_{i+1} \cup A_{i-1} \cup B_{i-1}$, then $ww' \in E(G)$.*

*Proof.* Suppose $w' \in A_{i+1} \cup B_{i-1}$ and $ww' \notin E(G)$. Then the set of vertices $\{w, v_i, v_{i+1}, v_{i+2}, v_{i+3}, w'\}$ induces $E$. Analogously for $A_{i-1}$. □

**Fact 21.** *If $w \in A_i$ and $w' \in A_{i+2} \cup A_{i+3} \cup B_i \cup B_{i+3} \cup C_i \cup C_{i+1} \cup C_{i+2} \cup C_{i+3}$, then $ww' \notin E(G)$.*

*Proof.* Suppose $ww' \in E(G)$.

If $w' \in A_{i+2}$, then the set of vertices $\{v_{i+3}, v_{i+4}, v_i, v_{i+1}, w, w'\}$ induces $E$. Analogously for $A_{i+3}$.

If $w' \in B_i \cup C_i$, then the set of vertices $\{v_{i-1}, v_i, w, w', v_{i+2}\}$ induces $bull$. Analogously for $B_{i+3} \cup C_{i+3}$.

If $w' \in C_{i+1}$, then the set of vertices $\{w, w', v_{i+2}, v_{i+3}, v_{i+4}\}$ induces $bull$. Analogously for $C_{i+2}$. □

**Fact 22.** *There are at most two $i, j$ such that $A'_i$ and $A'_j$ are nonempty. Moreover $|i-j| > 1$.*

*Proof.* Suppose $w, w' \in A'_i$ and $u, u' \in A'_{i+1}$, where $ww', uu' \in E(G)$. Then by Fact 20 the set of vertices $\{w, w', u, u'\}$ induces $K_4$. Thus $|j-i| > 1$. Since $Q$ consists of five vertices, the conclusion holds. $\square$

**Fact 23.** *If $w \in A_i$, $w' \in A_{i+1}$ and $w'' \in B_{i+2}$, then $ww'' \notin E(G)$ or $w'w'' \notin E(G)$.*

*Proof.* Suppose $ww'', w'w'' \in E(G)$. By Fact 20 $ww' \in E(G)$. Then the set of vertices $\{v_i, w, w', w'', v_{i+2}\}$ induces *bull*. $\square$

**Fact 24.** *If $w \in C_i$ and $w' \in A_{i+1} \cup B_{i-1} \cup B_{i+1}$, then $ww' \in E(G)$.*

*Proof.* Suppose $ww' \notin E(G)$. If $w' \in A_{i+1} \cup B_{i+1}$, then the set of vertices $\{v_{i-1}, v_i, v_{i+1}, w, w'\}$ induces *bull*. Analogously for $w' \in B_{i-1}$ $\square$

**Fact 25.** *If $w \in C_i$ and $w' \in A_i \cup A_{i+2} \cup A_{i+3} \cup A_{i+4}$, then $ww' \notin E(G)$.*

*Proof.* Suppose $ww' \in E(G)$.

If $w' \in A_i$, then the set of vertices $\{v_{i+3}, v_{i+2}, v_{i+1}, w, w'\}$ induces *bull*. Analogously for $w' \in A_{i+2}$

If $w' \in A_{i+3}$, then the set of vertices $\{v_{i-1}, v_i, v_{i+1}, w, w'\}$ induces *bull*. Analogously for $w' \in A_{i+4}$.

$\square$

**Fact 26.** *If $w, w' \in A'_i$, $ww' \in E(G)$ and there exists $w'' \in V(G) \setminus (A_i \cup D)$ such that $ww'' \in E(G)$, then $w'w'' \in E(G)$.*

*Proof.* Suppose $ww', ww'' \in E(G)$ and $w'w'' \notin E(G)$.

By Fact 21 $w'' \notin A_{i+2} \cup A_{i+3} \cup B_i \cup B_{i+3} \cup C_i \cup C_{i+1} \cup C_{i+2} \cup C_{i+3}$. Let us consider possible cases.

If $w'' \in A_{i+1} \cup B_{i+1}$, then the set of vertices $\{v_{i-1}, v_i, w, w', w''\}$ induces *bull*. Analogously for $A_{i-1}$ and $B_{i+2}$.

If $w'' \in B_{i-1}$, then by Fact 20 we have $w'w'' \in E(G)$.

If $w'' \in C_{i-1}$, then by Fact 24 the set of vertices $\{v_i, w, w', w''\}$ induces $K_4$.

$\square$

**Fact 27.** *There is only one $i$ such that $C_i \neq \emptyset$. Moreover, if $C_i \neq \emptyset$ then $A'_{i+1} \cup B'_{i+1} \cup B'_{i+2} \cup B'_{i+3} \cup B'_{i+4} = \emptyset$.*

*Proof.* Let $w \in C_i$. We consider possible cases.

If $w', w'' \in A'_{i+1}$ and $ww' \in E(G)$, then by Fact 24 the set of vertices $\{v_{i+1}, w, w', w''\}$ induces $K_4$.

If $w' \in C_{i+1}$, then either the set of vertices $\{v_{i-1}, v_i, w, v_{i+1}, w'\}$ induces *bull* (if $ww' \notin E(G)$) or the set of vertices $\{v_i, v_{i+1}, w, w'\}$ induces $K_4$ (otherwise). Analogously for $w' \in C_{i-1}$.

If $w' \in C_{i+2}$, then $G$ contains the spindle graph $M_7$. Analogously for $w' \in C_{i+3}$.

If $w', w'' \in B'_{i+1}$, then the set of vertices $\{v_{i-1}, v_i, w, v_{i+1}, w'\}$ induces *bull* (if $ww' \notin E(G)$), or the set of vertices $\{v_{i+4}, v_{i+3}, w'', w', w\}$ induces *bull* (if $ww' \in E(G)$ but $ww'' \notin E(G)$), or the set of vertices $\{v_i, w, w', w''\}$ induces $K_4$. Analogously for $w', w'' \in B_{i-1}$.

If $w', w'' \in B'_{i+2}$, then $G$ contains the spindle graph $M_7$. Analogously for $w', w'' \in B'_{i+3}$. □

**Fact 28.** *If $w, w' \in B'_i$, $ww' \in E(G)$ and there exists $w'' \in V(G) \setminus (B_i \cup C_i \cup D)$ such that $ww'' \in E(G)$, then $w'w'' \in E(G)$.*

*Proof.* By Facts 15, 21 we have $w'' \notin B_{i+2} \cup B_{i+3} \cup A_{i+2} \cup A_i$.

By Fact 27 we know that $C \setminus C_i = \emptyset$. Then $w'' \notin C$. If $w'' \in A_{i+1}$, then $w'w''$ exists by Fact 20.

Suppose now $w'' \in B_{i+1} \cup A_{i+3}$ and $ww'' \in E(G), w'w'' \notin E(G)$. Then the set of vertices $\{v_{i-1}, v_i, w', w, w''\}$ induces *bull*. Analogously if $w'' \in B_{i-1} \cup A_{i+4}$.

□

**Fact 29.** *Let $w \in D_i$. If $w' \in A_{i+1} \cup A_{i+2} \cup B_{i-1} \cup B_i \cup B_{i+1} \cup B_{i+2}$, then $ww' \in E(G)$.*

*Proof.* Suppose $ww' \notin E(G)$.

If $w' \in A_{i+1}$, then the set of vertices $\{v_{i-1}, v_i, w, v_{i+1}, w'\}$ induces *bull*. Analogously for $w' \in A_{i+2}$.

If $w' \in B_i \cup B_{i+1}$, then the set of vertices $\{w', v_i, v_{i+1}, w, v_{i+4}\}$ induces *bull*.

If $w' \in B_{i+2}$, then the set of vertices $\{v_i, w, v_{i+3}, v_{i+2}, w'\}$ induces *bull*. □

**Fact 30.** *Let $D \neq \emptyset$. Then $D = D_i^*$ for some $i$ and $A_i \cup A_{i+3} \cup A'_{i+1} \cup A'_{i+2} \cup B' \cup C = \emptyset$.*

*Proof.* Suppose $w \in D_i$. If $w' \in D \setminus D_i$, then $G$ contains the spindle graph $M_7$.

If $w' \in D_i$ and $ww' \in E(G)$, then the set of vertices $\{v_i, v_{i+1}, w, w'\}$ induces $K_4$.

If $w' \in A_i$, then either the set of vertices $\{w', w, v_i, v_{i+1}, v_{i+3}\}$ induces *bull* (if $ww' \notin E(G)$) or the set of vertices $\{v_{i-1}, v_i, w', w, v_{i+2}\}$ induces *bull* (otherwise). Analogously for $A_{i+3}$.

Suppose $w', w'' \in A'_{i+1}$ and $w'w'' \in E(G)$. By Fact 29 the set of vertices $\{w, w', w'', v_{i+1}\}$ induces $K_4$. Analogously for $A'_{i+2}$.

If $w' \in C_i$, then the set of vertices $\{v_{i+4}, v_{i+3}, w, v_{i+2}, w'\}$ induces *bull* if and $ww' \notin E(G)$ or the set of vertices $\{v_i, v_{i+1}, w, w'\}$ induces $K_4$ if $ww' \in E(G)$. Analogously for $w' \in C_{i+1}$.

If $w' \in C_{i+2} \cup C_{i+3} \cup C_{i+4}$, then the graph $G$ contains the spindle graph $M_7$.

Finally, suppose $w'w'' \in E(G)$.

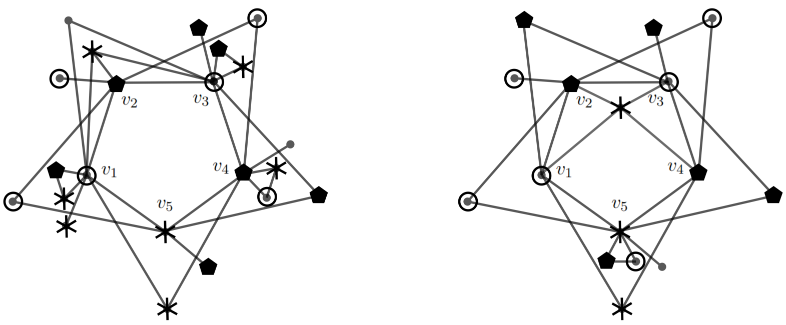
If $w', w'' \in B'_i$, then by Fact 29 the set of vertices $\{v_i, w, w', w''\}$ induces $K_4$. Analogously for $w', w'' \in B'_{i+1}$.

If $w', w'' \in B'_{i+2} \cup B'_{i+3} \cup B'_{i+4}$, then the graph $G$ contains the spindle graph $M_7$. $\square$

Due to all the facts above, we can distinguish following types of possible edges:

- Type 0: edges incident to the cycle $Q$.
- Type 1: edges $ww'$ such that $w, w' \in B_i \cup C_i$.
- Type 2: edges $ww'$ such that $w \in B_i \cup C_i$ and $w' \in B_{i+1} \cup C_{i+1}$.
- Type 3: edges $ww'$ such that $w \in A_i$ and $w' \in B_{i-1} \cup C_{i-1}$.
- Type 4: edges $ww'$ such that $w, w' \in A_i$.
- Type 5: edges $ww'$ such that $w \in A_i$ and $w' \in A_{i+1}$.
- Type 6: edges $ww'$ such that $w \in A_i$ and $w' \in B_{i+1} \cup B_{i+2}$.
- Type 7: edges non-incident to $Q$ and incident to the set $D$.

Let us recall that by Fact 20 edges of types 3 and 5 are obligatory, that is, if respective sets are nonempty, every edge between them exists.




(a) Case 1/2. (b) Case 3.

**Algorithm.**

Let us colour the graph $G$ as follows.

1. We colour with red vertices $v_1$ and $v_3$, with green vertices $v_2$ and $v_4$, and with blue vertex $v_3$.
2. For $i \neq 1$ and $w \in B^*_i$ we colour $c(w) = c(v_{i+1})$.
3. For $i \neq 4, 5$ and $w \in A^*_i$ we colour $c(w) = c(v_{i-1})$.
4. We colour $C_1$ with blue and $A'_1, A'_3, A'_4, B'_1$ with remaining colours.
5. If $D = \emptyset$, we colour $A^*_5$ with green. Then we colour with red every $w \in A^*_4$ adjacent to $A'_3 \cup B'_1$. The rest of $A_4$ we colour with blue. We colour with blue every $w \in B^*_1$ adjacent to $A^*_5$. The rest of $B^*_1$ we colour with green.

6. If $D \neq \emptyset$, we colour $D_1$ with blue. Then we colour $B_1^*$ with green and $A_5'$ with red and green. Every $w \in A_5^*$ adjacent to $B_1^*$ we colour with red. The rest of $A_5^*$ we colour with green.

**Proposition 31.** *The algorithm colours all vertices of $G$ and this colouring is proper.*

*Proof.* We want to prove that our algorithm colours all vertices of $G$ and this gives a proper colouring. Of course, it is easy to see that there will not occur any colour conflict on edges of types 0, 2 and 3. By Fact 19 there is no colour conflict also on edges of types 1 and 4 (we have two free colours, and the respective subgraphs are bipartite). The further proof will be split into three subcases.

*Case 1. $C \neq \emptyset$.*

Of course in this case $D = \emptyset$ (by Fact 30). Note that we can assume without loss of generality that $C = C_1$, $B' = B_1'$ (by Fact 27) and $A_5' = \emptyset$ (by Fact 22). Let us also recall that in this case $A_2' = \emptyset$ (by Fact 27).

We start with checking the possible colour conflicts on the edges of type 5. Of course, one of the sets $A_3', A_4'$ is empty (by Fact 22), so there is no colour conflict on the edges of type 5 between sets $A_3$ and $A_4$. Suppose $A_1'$ and $A_5^*$ are both nonempty. If $A_4' \neq \emptyset$, then the graph $G[\{v_4\} \cup A_4' \cup A_5^* \cup A_1' \cup \{v_1, v_2, v_3\} \cup C_1]$ contains $M_{10}$. If $A_3' \neq \emptyset$ and $A_4^* \neq \emptyset$ then the graph $G[A_5^* \cup A_1' \cup \{v_1, v_2, v_3\} \cup C_1 \cup A_3' \cup A_4]$ contains $M_{10}$. Thus, at least one of the sets $A_3'$, $A_4^*$ is empty. Hence, we can consider the symmetry of the graph such that it swaps $v_4$ and $v_5$. Then one of the sets $A_1'$ or $A_5^*$ is empty.

Now we want to exclude colour conflicts on the edges of type 6. If $A_1'$ is adjacent to $B_3^*$, then the graph $G[B_3^* \cup A_1' \cup \{v_1, v_2, v_3\} \cup C]$ contains $M_7$ (by Facts 24, 26). Analogously for $A_3'$ incident to $B_4^*$. If $A_4'$ is incident to $B_5^*$, then the graph $G[\{v_4\} \cup A_4' \cup B_5^* \cup \{v_2, v_3\} \cup C]$ contains $M_7$ (by Fact 24, 26). The colour conflict between $A_4'$ and $B_1^*$ may occur only if $B_1$ is blue. By definition of the colouring, $B_1$ is blue only if it is incident to $A_5^*$. By Fact 23 this is impossible.

If there is a colour conflict between $A_4^*$ and $B_5^*$, then there must be two adjacent vertices $w \in A_4^*$ and $w' \in B_5^*$, both coloured with red. But $w$ can be red only when it is adjacent to $B_1'$ (by Fact 24 the graph $G[\{v_1, v_2\} \cup C \cup B_1' \cup \{w, w'\}]$ contains $M_7$) or when $A_3'$ is nonempty (and the graph $G[w, w', v_2, v_3 \cup C \cup A_3']$ contains $M_7$).

The colour conflict between $A_4^*$ and $B_1^*$ can occur only if there are two vertices $w \in A_5^*$ and $w' \in B_1$, both coloured with blue. But if $w'$ is coloured with blue, by definition of the colouring, there is $w'' \in A_5^*$ such that $w'w'' \in E(G)$. It contradicts Fact 23, so this colour conflict is also impossible.

If $w \in A_5^*$ is adjacent to $w', w'' \in B_1'$, then $u \in A_4$ and $w'$ cannot be adjacent (otherwise the graph $G[\{w, w', w'', u\}]$ contains $K_4$ or $G[\{w, w', w'', v_1\} \cup A_4]$ contains $M_7$, by Fact 28). Thus, we can consider the symmetry of the graph such that it swaps $v_4$ and $v_5$. Note that this operation does not change our previous assumption. Indeed, before the reflection, if $A_4' \neq \emptyset$, then the graph $G[\{v_4\} \cup A_4' \cup A_5 \cup B_1' \cup \{v_3\}]$ contains $M_7$, and if $A_3' \neq \emptyset$ and $A_4^* \neq \emptyset$, then the graph $G[A_5^* \cup B_1' \cup \{v_3\} \cup A_3' \cup A_4^*]$ contains $M_7$.

By definition of the colouring, there is no colour conflict between $A_5^*$ and $B_1^*$.

*Case 2.* $C \cup D = \emptyset$

In case $C \cup D = \emptyset$ we can also assume $B' = \emptyset$ (because otherwise we can find another 5-cycle $Q'$ and a vertex incident to three consecutive vertices of $Q'$). By Fact 22 we can assume that $A'_2 \cup A'_5 = \emptyset$. Also, we can assume that there is no 6-type edges with vertices from $A'$ (again, by Fact 26, otherwise we could find there another 5-cycle so that we return to *Case 1*) and that if $A'$ is nonempty, then $A_5$ is empty (because for $A'_i \neq \emptyset$ we have one of sets $A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}$ empty by the same argument as before).

Now, let us check the possible colour conflicts. It is easy to see that if $A' = \emptyset$, our colouring is proper. Assume $A' \neq \emptyset$. Then, thanks to $A_5 = \emptyset$, there is no conflicts on edges of type 5. The only possible (given our assumptions) conflicts may occur on edges of type 6 with vertices from $A^*_4$. As in previous case, there is no conflict between $A^*_4$ and $B_1$. If $A^*_4$ is connected to $B^*_5$ and coloured with red, then by definition of our colouring $A'_3 \neq \emptyset$. Hence, we can find another 5-cycle so that we return to *Case 1*.

*Case 3.* $D \neq \emptyset$

We will see that there cannot occur any colour conflict on edges of type 7. Let us recall that by Fact 30 in this case we have $A_i \cup A_{i+3} \cup A'_{i+1} \cup A'_{i+2} \cup B' \cup C = \emptyset$. If $B_4$ is adjacent to $D = D_1$, then the graph $G$ contains $W_5$.

Hence colour conflicts can occur only on the edges of type 6 with vertices from $A_5$. If $A'_5$ is adjacent to $B^*_1$, then the graph $G[\{v_5\} \cup A'_5 \cup B^*_1 \cup \{v_3, v_4\} \cup D]$ contains $M_7$ by Fact 29. Analogously for $A'_5$ adjacent to $B^*_2$. Now suppose there are $w \in A^*_5$, $w' \in B^*_1$ and $w'' \in B^*_2$ such that $ww', ww'' \in E(G)$. Note that $w'w'' \in E(G)$ (because otherwise the set $\{v_5, v_1, w', u, w''\}$ induces *bull*, for any $u \in D$). Then the set of vertices $\{v_1, w', w, w'', v_4\}$ induces *bull*.

$\square$

### 4.3 Proof of Theorem 5

We can follow the proof of Theorem 4 with a few changes. Because there are no 5-cycles, we only have to consider the case $p > 5$. Notice that in this case, if $w \in A_i$, then the set of vertices $\{v_{i-3}, v_{i-2}, v_i, v_{i+1}, v_{i+2}, v_{i+3}, w\}$ induces $S_{1,2,3}$. Then (because $D = \emptyset$ due to Fact 11), we have $V(G) = Q \cup B \cup C$. Now, we can follow the proof of Theorem 4.

## 5 Certifying algorithms

**Theorem 32.** *There exists a polynomial time certifying algorithm for 3-colourability in the class of* $(bull, H)$*-free graphs for* $H \in \{S_{1,1,2}, S_{1,2,2}\}$*, and in the class of* $(bull, C_5, H)$*-free graphs with* $H \in \{S_{1,1,3}, S_{1,2,3}\}$*.*

*Proof.* An odd hole can be found in polynomial time $O(n^9)$ by an algorithm in [7]. So let $Q$ be this odd hole of length $p$. To check whether $G$ contains an odd wheel $W_{2t+1}$ for some $t \geq 1$, one can check for every vertex $w \in V(G)$ in polynomial time $O(|E|)$ whether $G[N(w)]$ is bipartite. If this is not the case, then $G$ contains an odd wheel with center vertex $w$. In the case $p > 5$, all structural investigations can be performed in polynomial time and the algorithm either finds a proper 3-colouring or detects a spindle graph $M_{3t+1}$ for some $t \geq 3$. In the proofs for the case $p = 5$, proper 3-colourings of $G$ or a subgraph from $\{M_4, M_7, M_{10}\}$ will be found in polynomial time.

Summarizing, all structural investigations and algorithms run in polynomial time and either find a proper 3-colouring of $G$ or detect an odd wheel or a spindle graph $M_{3i+1}$ for some $i \geq 1$. □

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