FAMILIES OF NUMERICAL SEMIGROUPS AND A SPECIAL CASE OF THE HUNEKE-WIEGAND CONJECTURE

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Abstract. The Huneke-Wiegand conjecture is a decades-long open question in commutative algebra. García-Sánchez and Leamer showed that a special case of this conjecture concerning numerical semigroup rings $\Bbbk[\Gamma]$ can be answered in the affirmative by locating certain arithmetic sequences within the numerical semigroup Γ. In this paper, we use their approach to prove the Huneke-Wiegand conjecture in the case where Γ is generated by a generalized arithmetic sequence and showcase how visualizations can be leveraged to find the requisite arithmetic sequences.

1. Introduction

Numerical semigroups, co-finite additive subsemigroups of the natural numbers, have long been studied for their relationship to important objects in commutative algebra. Given a numerical semigroup $\Gamma \subseteq \mathbb{Z}_{\geq 0}$ generated by $n_1, \ldots, n_k$, which we denote

$$\Gamma = \langle n_1, n_2, \ldots, n_k \rangle = \{z_1 n_1 + z_2 n_2 + \cdots + z_k n_k : z_i \in \mathbb{Z}_{\geq 0}\},$$

the semigroup algebra $\Bbbk[\Gamma] = \Bbbk[x^{n_1}, \ldots, x^{n_k}]$ over a field $\Bbbk$ is the subring of the polynomial ring $\Bbbk[x]$ for which every term x^n appearing with nonzero coefficient in an element of $\Bbbk[\Gamma]$ has $n \in \Gamma$. Understanding monomial ideals in this ring, which is inherently a problem in commutative algebra, can be attacked by studying the underlying semigroup Γ. The advantages of this approach are manifold, as numerical semigroups have a well-studied factorization theory [7, 9] and several computational packages [1, 6].

One specific open problem that has benefited explicitly from this relationship is the following special case of the Huneke-Wiegand conjecture [2, 4].

Conjecture 1.1. *If $R = \Bbbk[\Gamma]$ is the semigroup algebra of a symmetric numerical semigroup Γ, and M is a 2-generated ideal of R, viewed as a module over $\Bbbk[\Gamma]$, then the torsion submodule of $M \otimes_R \mathrm{Hom}_R(M, R)$ is non-trivial.*

In its general form, the Huneke-Wiegand conjecture [4], which has been open for 3 decades, concerns a one-dimensional Gorenstein domain R and a finitely generated R-module M that is not projective. If $R = \Bbbk[\Gamma]$ is a numerical semigroup algebra, then R is one-dimensional, and the Gorenstein hypothesis on R is equivalent to Γ being

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symmetric (that is, for every $n \notin \Gamma$, we have $F(S) - n \in \Gamma$, where $F(\Gamma) = \max(\mathbb{Z}_{\geq 0} \setminus \Gamma)$ is the *Frobenius number* of Γ).

In [2], García-Sánchez and Leamer showed that Conjecture 1.1 can be positively answered for a given numerical semigroup algebra $R = \Bbbk[\Gamma]$ if certain irreducible arithmetic sequences can be found inside Γ itself. Fix a positive $s \notin \Gamma$ and let

$$S_\Gamma^s = \{(n, \ell) : n, \ell \in \mathbb{Z}_{\geq 1} \text{ with } n, n+s, \ldots, n+\ell s \in \Gamma\}$$

encode the arithmetic sequences of step size s that are contained in Γ. Note that S_Γ^s is closed under component-wise addition since $(n_1, \ell_1), (n_2, \ell_2) \in S_\Gamma^s$ implies

$$n_1 + n_2, n_1 + n_2 + s, \ldots, n_1 + n_2 + (\ell_1 + \ell_2)s \in \Gamma.$$

The authors of [2] proved that Conjecture 1.1 holds if, for any numerical semigroup Γ and any positive $s \notin \Gamma$, some element $(n, 2) \in S_\Gamma^s$ is *irreducible*, meaning it cannot be written as a sum of other elements of S_Γ^s. This result has been leveraged to verify Conjecture 1.1 for some well-studied families of numerical semigroups, such as when Γ is a complete intersection [2] or when Γ is generated by an arithmetic sequence whose step size coincides with s [3].

In this paper, we utilize the results of [2] to prove Conjecture 1.1 whenever Γ is generated by a *generalized arithmetic sequence*, that is,

$$\Gamma = \langle a, ah + d, ah + 2d, \ldots, ah + kd \rangle$$

for some $a, h, d, k \in \mathbb{Z}_{\geq 1}$ with $\gcd(a, d) = 1$. This family of numerical semigroups, introduced in [8], are known for admitting concise characterizations of invariants that generally have high computational complexity in general (e.g., the Frobenius number).

Theorem 1.2. *If Γ is generated by a generalized arithmetic sequence, then the Huneke-Wiegand conjecture holds for any 2-generated monomial ideal in $\Bbbk[\Gamma]$.*

2. Visualizations for locating irreducible elements of S_Γ^s

Before giving the proof of Theorem 1.2, we demonstrate the utility of certain visuals that arose in obtaining this result. For a given numerical semigroup Γ, we may use the `Sage` [10] package `LeamerMonoid` [5] to compute, for each $s \notin \Gamma$, the set of all irreducible elements $(n, 2)$. A particularly helpful graphic emerges when we plot a point at (s, n) if $(n, 2) \in S_\Gamma^s$ is irreducible; Figures 1 and 2 each contain two examples. Thus, the semigroup algebra $\Bbbk[\Gamma]$ satisfies the Huneke-Wiegand conjecture for all 2-generated monomial ideals if, for each positive $s \notin \Gamma$, there exists at least one point (s, n) in that column.

Examining several of these graphs reveals a method of finding the requisite irreducible elements. For example, if $\Gamma = \langle n_1, n_2 \rangle$, such as in the left-hand graphic of Figure 1, then $(\mathsf{F}(\Gamma) + n_1 - s, 2) \in S_\Gamma^s$ is irreducible for each $s \notin \Gamma$; this is depicted with a diagonal red line defined by the equation $n = \mathsf{F}(\gamma) + n_1 - s$ that contains a point in every column.

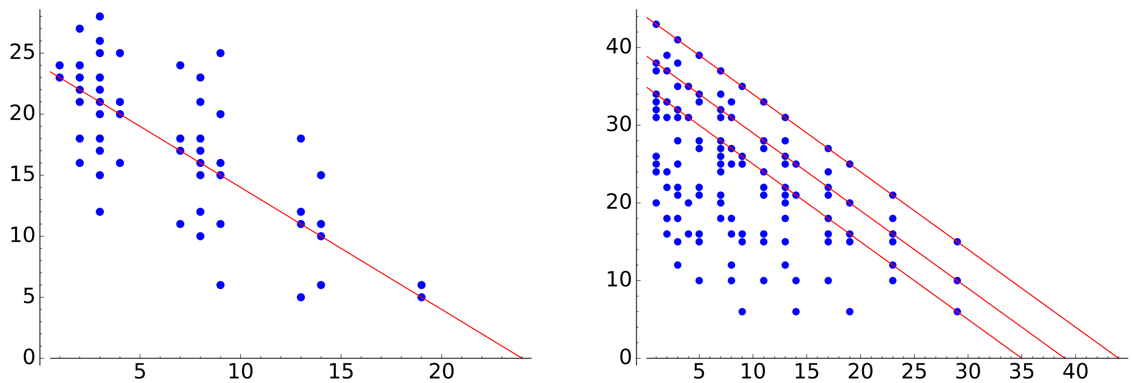

FIGURE 1. The points (s, n) for which $(n, 2)$ is irreducible in S^s_Γ, where $\Gamma = \langle 5, 6\rangle$ (left) and $\Gamma = \langle 6, 10, 15\rangle$ (right).

Lemma 2.1. *Fix a symmetric numerical semigroup* $\Gamma = \langle n_1, \ldots, n_k\rangle$ *with* $s \notin \Gamma$. *Fix a generator* n_j, *and let* $g = \gcd(\{n_1, \ldots, n_k\} \setminus \{n_j\})$. *We have* $(\mathsf{F}(\Gamma) - s + n_j, 2) \in S^s_\Gamma$, *and if this element is reducible in* S^s_Γ, *then* $g \mid s$.

Proof. Since Γ is symmetric, $\mathsf{F}(\Gamma) - s \in \Gamma$, and since

$$\mathsf{F}(\Gamma) + s + n_j > \mathsf{F}(\Gamma) + n_j > \mathsf{F}(\Gamma),$$

both $\mathsf{F}(\Gamma) + n_j$ and $\mathsf{F}(\Gamma) + s + n_j$ lie in Γ. This proves the first claim. For the second claim, suppose

$$(\mathsf{F}(\Gamma) - s + n_j, 2) = (y, 1) + (\mathsf{F}(\Gamma) - s + n_j - y, 1)$$

with $(y, 1), (\mathsf{F}(\Gamma) - s + n_j - y, 1) \in S^s_\Gamma$. Since $y, y + s \in \Gamma$, we conclude $y - n_j, y + s - n_j \notin \Gamma$ since Γ is symmetric. It must be that no expression for y or $y + s$ as a sum of generators involves the generator n_j. In particular, $y, y + s \in \langle n_1, \ldots, \hat{n}_j, \ldots, n_k\rangle$, meaning $g \mid y$ and $g \mid y + s$, from which we conclude $g \mid s$. □

Lemma 2.1 makes quick work of the case $\Gamma = \langle n_1, n_2\rangle$. Indeed, in addition to verifying $(\mathsf{F}(\Gamma) - s + n_1, 2) \in S^s_\Gamma$, Lemma 2.1 implies if this element were reducible, then $n_2 \mid s$, which is impossible since $s \notin \Gamma$.

For 3-generated numerical semigroups $\Gamma = \langle n_1, n_2, n_3\rangle$, one can see by inspection of the right-hand graphic of Figure 1 that, unlike the 2-generated case above, there is no single line that contains a point in every column. However, the 3 diagonal red lines depicted therein, each of which has the form $n = \mathsf{F}(\gamma) - s + n_j$ for some j, together contain at least one point in each column. These observations yield a relatively straightforward proof, included below, that Conjecture 1.1 holds in the case where Γ has at most 3 generators; however, note that this case also follows from [2] since any such numerical semigroup is complete intersection.

Proposition 2.2. *Given any symmetric numerical semigroup* $\Gamma = \langle n_1, n_2, n_3\rangle$ *and any* $s \notin \Gamma$*, the element* $(\mathsf{F}(\Gamma) - s + n_j, 2) \in S_\Gamma^s$ *is irreducible for some* j*.*

Proof. Let $g_j = \gcd(\{n_1, n_2, n_3\} \setminus \{n_j\})$ for each j. By Lemma 2.1, $(\mathsf{F}(\Gamma)-s+n_j, 2) \in S_\Gamma^s$ for each generator n_j, and in order to prove one of these is irreducible in S_Γ^s, it suffices to assume $\operatorname{lcm}(g_1, g_2, g_3) \mid s$.

Since Γ is symmetric, [9, Theorem 9.6] implies that, after rearranging n_1, n_2, n_3 as needed, $d = \gcd(n_1, n_2) > 1$ and

$$dn_3 = an_1 + bn_2 \tag{2.1}$$

for some $a, b \in \mathbb{Z}_{\geq 0}$. We claim $(\mathsf{F}(\Gamma) - s + n_1, 2) \in S_\Gamma^s$ is irreducible. Indeed, if this element were reducible, then it could be written as a sum

$$(\mathsf{F}((\Gamma) - s + n_1, 2) = (\mathsf{F}(\Gamma) - s + n_1 - z, 1) + (z, 1)$$

of atoms in S_Γ^s. In particular, $z, z+s \in \Gamma$ since $(z, 1) \in S_\Gamma^s$, whereas $z-n_1, z+s-n_1 \notin \Gamma$ since Γ is symmetric. As such, any expression of z and $z + s$ as a sum of generators must only involve n_2 and n_3, meaning

$$z = a_2n_2 + a_3n_3 \qquad \text{and} \qquad z + s = b_2n_2 + b_3n_3$$

for some $a_2, a_3, b_2, b_3 \in \mathbb{Z}_{\geq 0}$. Moreover, we must have $0 \leq a_3, b_3 < d$, as otherwise (2.1) would yield an expression involving n_1. Subtracting $z + s$ and z, we find

$$s = (z + s) - z = (b_2 - a_2)n_2 + (b_3 - a_3)n_3$$

and since $d \mid s$, $d \mid n_2$ and $d \nmid n_3$, we conclude $d \mid (b_3 - a_3)$. However, $|b_3 - a_3| < d$. Thus $b_3 - a_3 = 0$ and $s = (b_2 - a_2)n_2$, a contradiction. □

3. Numerical monoids generated by generalized arithmetic sequences

We now turn our attention back to numerical semigroups Γ generated by generalized arithmetic sequences; Figure 2 shows plots of irreducible elements of S_Γ^s for two such semigroups. Like the 3-generated case, multiple lines are required to find an irreducible element for each $s \in \mathbb{Z}_{\geq 1} \setminus \Gamma$. In particular, the combination of the line $n = \mathsf{F}(\Gamma) + d - s$ and the horizontal line $n = ah + d$ provide the requisite irreducible elements of S_Γ^s. These lines manifest in the form of Proposition 3.2, which we prove after a short lemma.

Lemma 3.1. *Suppose* $\Gamma = \langle a, ah + d, ah + 2d, \ldots, ah + kd\rangle$ *is a symmetric numerical semigroup with* $3 \leq k < a$ *and* $\gcd(a, d) = 1$*, and fix* $s \notin \Gamma$*. We have* $\mathsf{F}(\Gamma) - s + d \notin \Gamma$ *if and only if for some* $m \in \{0, \ldots, h - 1\}$*,*

$$s - d \equiv am \bmod (ah + kd).$$

Proof. First, suppose $s - d \equiv am \bmod (ah + kd)$ for some m as above. Fixing $l \in \mathbb{Z}$ such that $s - d = am + l(ah + kd)$ and noting that $l \geq 0$ since $s > 0$, we can write

$$\mathsf{F}(\Gamma) - s + d = \mathsf{F}(\Gamma) - (d + am + l(ah + kd)) + d = \mathsf{F}(\Gamma) - (am + l(ah + kd)).$$

Since Γ is symmetric and $am + l(ah + kd) \in \Gamma$, this implies $\mathsf{F}(\Gamma) - s + d \notin \Gamma$.

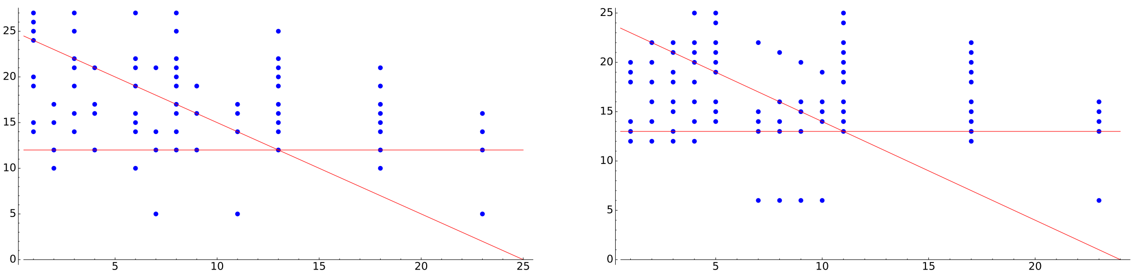

FIGURE 2. The points (s, n) for which $(n, 2)$ is irreducible in S^s_Γ, where $\Gamma = \langle 5, 12, 14, 16\rangle$ (left) and $\Gamma = \langle 6, 13, 14, 15, 16\rangle$ (right) are each generated by a generalized arithmetic sequence.

Conversely, suppose $\mathsf{F}(\Gamma) - s + d \notin \Gamma$. Since Γ is symmetric, $s - d \in \Gamma$, so suppose

$$s - d = z_0 a + z_1(ah + d) + \cdots + z_k(ah + kd).$$

We claim (i) $z_j = 0$ for each $0 < j < k$, and (ii) $0 \le z_0 \le h - 1$. Indeed, if $z_j > 0$ for $0 < j < k$, then

$$\begin{aligned} s = s - d + d &= z_0 a + \cdots + z_j(ah + jd) + \cdots + z_k(ah + kd) + d \\ &= z_0 a + \cdots + (z_j - 1)(ah + jd) + \cdots + z_k(ah + kd) + (ah + (j + 1)d), \end{aligned}$$

which is impossible since $s \notin \Gamma$, and if $z_0 \ge h$, then

$$\begin{aligned} s = s - d + d &= z_0 a + \cdots + z_k(ah + kd) + d \\ &= (z_0 - h)a + \cdots + z_k(ah + kd) + (ah + d), \end{aligned}$$

which is again impossible since $s \notin \Gamma$. Consequently, $s - d = z_0 a + z_k(ah + kd)$, thereby completing the proof with $m = z_0$. □

Proposition 3.2. *Suppose $\Gamma = \langle a, ah + d, ah + 2d, \ldots, ah + kd\rangle$ is a symmetric numerical semigroup with $3 \le k < a$ and $\gcd(a, d) = 1$, and fix $s \notin \Gamma$.*

(a) If $\mathsf{F}(\Gamma) - s + d \in \Gamma$, then $(\mathsf{F}(\Gamma) - s + d, 2)$ is irreducible in S^s_Γ.
(b) If $\mathsf{F}(\Gamma) - s + d \notin \Gamma$, then $(ah + d, 2)$ is irreducible in S^s_Γ.

So, the Huneke-Wiegand conjecture holds for any 2-generated monomial ideal in $\Bbbk[\Gamma]$.

Proof. For part (a), suppose $\mathsf{F}(\Gamma) - s + d \in \Gamma$. Since

$$\mathsf{F}(\Gamma) + s + d > \mathsf{F}(\Gamma) + d > \mathsf{F}(\Gamma),$$

we also have $\mathsf{F}(\Gamma) + d \in \Gamma$ and $\mathsf{F}(\Gamma) + d + s \in \Gamma$. This means $(\mathsf{F}(\Gamma) - s + d, 2) \in S^s_\Gamma$.

Now, by way of contradiction, suppose $(\mathsf{F}(\Gamma) - s + d, 2)$ is reducible, so that

$$(\mathsf{F}(\Gamma) - s + d, 2) = (\mathsf{F}(\Gamma) - s + d - n, 1) + (n, 1)$$

for some $(n, 1), (\mathsf{F}(\Gamma) - s + d - n, 1) \in S_\Gamma^s$. In particular, this means z and $\mathsf{F}(\Gamma) + d - n$ both lie in Γ, and since Γ is symmetric, $n - d \notin \Gamma$. We claim $n = z_0 a$ for some $z_0 \in \mathbb{Z}_{\geq 0}$. Indeed, if $n = z_0 a + \cdots + z_k(ah + kd)$ with $z_1 > 0$, then

$$n - d = (z_0 + h)a + (z_2 - 1)(ah + d) + \cdots + z_k(ah + kd),$$

and if $z_j > 0$ for some $j > 1$, then

$$n - d = z_0 a + \cdots + (z_{j-1} + 1)(ah + (j-1)d) + (z_j - 1)(ah + jd) + \cdots + z_k(ah + kd),$$

both of which are impossible since $n - d \notin \Gamma$. By similar reasoning, $n + s \in \Gamma$ and $n + s - d \notin \Gamma$, so $n + s = z_0' a$ for some $z_0' \in \mathbb{Z}_{\geq 0}$. This yields $s = (n+s) - n = (z_0' - z_0)a$, which is impossible since $s \notin \Gamma$. As such, we conclude $(\mathsf{F}(\Gamma) - s + d, 2)$ is irreducible in S_Γ^s, thereby proving part (a).

For part (b), suppose $\mathsf{F}(\Gamma) - s + d \notin \Gamma$. By Lemma 3.1, $s - d \equiv am \bmod (ah + kd)$ for some $m \in \{0, \ldots, h-1\}$, so let $l \in \mathbb{Z}$ with $s - d = am + l(ah + kd)$. Since $ah + d \in \Gamma$ and $k \geq 3$, we have

$$\begin{aligned} ah + d + s &= (ah + 2d) + am + l(ah + kd) \in \Gamma \text{ and} \\ ah + d + 2s &= (ah + 3d) + 2am + 2l(ah + kd) \in \Gamma, \end{aligned}$$

meaning $(ah + d, 2) \in S_\Gamma^s$. Lastly, suppose by way of contradiction that $(ah + d, 2)$ is reducible in S_Γ^s, so that

$$(ah + d, 2) = (ah + d - n, 1) + (n, 1)$$

for some $(n, 1), (ah + d - n, 1) \in S_\Gamma^s$. This means n and $ah + d - n$ are both in Γ, but since both are less than $ah + d$, there exists $z_0, z_0' \in \mathbb{Z}_{\geq 0}$ such that $n = z_0 a$ and $ah + d - n = z_0' a$. This implies $ah + d = (z_0 + z_0')a$, which is impossible since $\gcd(a, d) = 1$. This completes the proof. □

Proof of Theorem 1.2. If Γ has at most 3 generators, then Γ is complete intersection by [9, Corollary 10.5], so apply [2, Corollary 22]. Otherwise, apply Proposition 3.2. □

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