On the weak homotopy types of small finite spaces

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We show that a connected finite topological space with 12 or less points has a weak homotopy type of a wedge of spheres. In other words, we show that the order complex of a connected finite poset with 12 or less points has a homotopy type of a wedge of spheres.

1. Introduction

Given a topological space X, a finite space Y, that is, a topological space with finitely many points, is a *finite model* of X if it is weak homotopy equivalent to X, and it is called a *minimal finite model* of X if it is a finite model of minimal cardinality. McCord [7] showed that given a finite simplicial complex K, there exists a finite topological space $\mathcal{X}(K)$ and a weak homotopy equivalence $\mu_K\colon K \to \mathcal{X}(K)$, whence any compact CW-complex has a minimal finite model. Cianci and Ottina [3] gave a complete characterization of minimal finite models of the real projective plane, the torus, and the Klein bottle. In particular, minimal finite models of the real projective plane have 13 points.

In that paper, they showed that if X is a connected finite space with $|X| \leq 12$, then $\pi_1(X)$ is a free group and its integral homology is torsion free, where $|X|$ denotes the cardinality of X. In this article, we refine their study and show a stronger result:

Theorem 1.1. *If X is a connected finite space with $|X| \leq 12$, then X has a weak homotopy type of a wedge of spheres, where we consider one point space as a wedge of 0-copies of spheres.*

The proof is given by induction on the cardinality of the space.

Definition 1.2. Let X be a finite connected space. We say that X *splits into smaller spaces* if there exist finite spaces A_i with $|A_i| < |X|$ and non negative integers $n_i \geq 0$ such that

$$X \simeq_w \bigvee_i \mathbb{S}^{n_i} A_i$$

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where $\mathbb{S}$ denotes the Non-Hausdorff suspension (see Definition 2.19).

When X is not connected, we say that X splits into smaller spaces if each connected component does.

More generally, if $|A_i| < |B|$ for some finite space B, we say that X *splits into spaces smaller than* B. Note that, if X splits into spaces smaller than B, then so does $\mathbb{S}X$ (see Example 2.3 and Corollary 2.21).

We also note that the weak homotopy type of a wedge sum of connected finite T_0 spaces is independent of base points and is weak homotopy invariant (see Corollary 2.21).

Example 1.3. Since $\mathbb{S}^0 A = A$, $\mathbb{S}^n\emptyset \simeq_w S^{n-1}$, if X is weak homotopy equivalent to a wedge of some spheres and a space A with $|A| < |X|$;

$$X \simeq_w A \vee \left(\bigvee_i S^{n_i} \right),$$

then X splits into smaller spaces.

Finite T_0 topological spaces and finite posets are essentially the same objects, and Cianci-Ottina [3] observed that, in most cases, a small finite T_0 space X can be decomposed into the form $U_a \cup F_b \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$ where U_a is the set of elements smaller than or equal to $a \in X$, F_b is the set of elements greater than or equal to $b \in X$, $\mathrm{mxl}(X)$ is the set of maximal elements, and $\mathrm{mnl}(X)$ is the set of minimal elements. We show that this decomposition splits X into wedge sum of suspensions and that, if X is a connected finite T_0 space with $1 < |X| \leq 12$, then X splits into smaller spaces, and from which, Theorem 1.1 follows by induction.

The paper is organized as follows: In Section 2, we recall some basic results on simplicial complexes and finite spaces and fix notations. In Section 3, we reformulate the poset splitting of Cianci-Ottina and show our fundamental splitting results Proposition 3.12 and Corollaries 3.13 and 3.14. In most cases, Corollaries 3.13 and 3.14 give splittings of small finite spaces into smaller spaces. To handle exceptional cases, we study the weak homotopy types of posets of intervals in Section 4. We describe some very small finite spaces in Section 5. In Sections 6 to 8, we show that, if X is a connected finite T_0 space with $1 < |X| \leq 12$, then X splits into smaller spaces, and we give a proof of Theorem 1.1 in Section 9.

2. Preliminaries

In this section, we recall some basic results on simplicial complexes and finite spaces and fix notations. If two topological spaces X and Y are homeomorphic, we write $X \cong Y$, if they are homotopy equivalent, we write $X \simeq Y$, and if they are weak homotopy equivalent, we write $X \simeq_w Y$. We denote the double mapping cylinder of maps $X \xleftarrow{f} A \xrightarrow{g} Y$ by $M_{f,g}$. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $M_{f_0,g_0} \simeq M_{f_1,g_1}$.

2.1. Simplicial complexes

All simplicial complexes are finite in this section. We will not distinguish between a simplicial complex and its geometric realization. The following are very basic facts of homotopy theory of simplicial complexes:

- Weak homotopy equivalent simplicial complexes are homotopy equivalent.
- If a simplicial complex K is a union of two subcomplexes $K = K_1 \cup K_2$, then K/K_2 is homeomorphic to $K_1/(K_1 \cap K_2)$.
- If L is a subcomplex of K (more generally, if (K, L) is a CW pair), then K/L is homotopy equivalent to the double mapping cylinder of $* \longleftarrow L \xrightarrow{i} K$ where $i\colon L \to K$ is the inclusion.
- If $K = K_1 \amalg K_2$, $L_i \subset K_i$ and $L = L_1 \amalg L_2$, then $K/L \cong K_1/L_1 \vee K_2/L_2$ (If $L_i = \emptyset$, then $K_i/L_i = K_i^+$, which is K_i with disjoint base point added).

If L_1 and L_2 are subcomplexes of K such that $L_1 \cap L_2 = \emptyset$, we denote the space obtained by collapsing L_1 to a single point and L_2 to a single point by $K/(L_1, L_2)$. The unreduced suspension of K is denoted by SK, that is,

$$SK = \frac{K \times I}{(K \times \{0\}, K \times \{1\})}.$$

Example 2.1. 1. If the inclusion $L \to K$ is null homotopic, then

$$K/L \simeq K \vee SL.$$

In particular,

a) if $K \simeq *$, then $K/L \simeq SL$,

b) if $L \simeq *$, then $K/L \simeq K$.

2. If $K = K_1 \cup K_2$, $K_2 \simeq *$, and $K_1 \cap K_2 \to K_1$ is null homotopic, then

$$K \simeq K/K_2 \cong K_1/(K_1 \cap K_2) \simeq K_1 \vee S(K_1 \cap K_2).$$

3. If $K = K_1 \cup K_2$ and $K_1, K_2 \simeq *$, then

$$K \simeq K/K_2 \cong K_1/(K_1 \cap K_2) \simeq S(K_1 \cap K_2).$$

Example 2.2. For a vertex $v \in V(K)$, we denote the subcomplex spanned by $V(K) - \{v\}$ by $K \setminus \{v\}$. Since

$$\begin{aligned} \mathrm{st}(v) &= \{\sigma \in K \mid \sigma \cup \{v\} \in K\} \simeq *, \\ \mathrm{lk}(v) &= \{\sigma \in \mathrm{st}(v) \mid v \notin \sigma\}, \\ K \setminus \{v\} &= \{\sigma \in K \mid v \notin \sigma\}, \end{aligned}$$

$$
\begin{aligned}
K &= (K \setminus \{v\}) \cup \mathrm{st}(v), \\
\mathrm{lk}(v) &= (K \setminus \{v\}) \cap \mathrm{st}(v),
\end{aligned}
$$

if the inclusion $\mathrm{lk}(v) \to K \setminus \{v\}$ is null homotopic, we have

$$K \simeq (K \setminus \{v\}) \vee S(\mathrm{lk}(v)).$$

Example 2.3. If K is a disjoint union of subcomplexes $K = \coprod_{i=1}^{n} K_i$ ($K_i \neq \emptyset$), then SK is homotopy equivalent to the wedge sum of SK_i and $n-1$ copies of S^1:

$$SK \simeq \left(\bigvee_{i=1}^{n} SK_i \right) \vee \left(\bigvee_{n-1} S^1 \right).$$

In particular, SK is a wedge of spheres when each connected components of K is a wedge of spheres.

One can easily show the following (We give a proof in Appendix A.):

Lemma 2.4. *Let K be a connected simplicial complex, L, M be subcomplexes and $K = L \cup M$. If all the connected components of L and M are contractible; $L = \coprod_{i=1}^{l} L_i$, $M = \coprod_{i=1}^{m} M_i$, $L_i \simeq *$, and $M_i \simeq *$, then*

$$K \simeq \left(\bigvee_{\substack{i,j \\ L_i \cap M_j \neq \emptyset}} S\left(L_i \cap M_j\right) \right) \vee \left(\bigvee_{n} S^1 \right)$$

where

$$n = |\{(i,j) \mid L_i \cap M_j \neq \emptyset\}| - l - m + 1.$$

Finally, we note that, since any point of a (geometric realization of) simplicial complex is a nondegenerate base point, the homotopy type of a wedge sum of connected simplicial complexes is independent of base points and is homotopy invariant, that is, if K_1, K_2, L_1, and L_2 are connected simplicial complexes such that $K_i \simeq L_i$, then $K_1 \vee K_2 \simeq L_1 \vee L_2$.

2.2. Finite spaces

In this subsection, we collect some basic facts on finite spaces. See [6] and [2] for details.

Finite spaces Alexandroff [1] showed that finite topological spaces and finite preordered sets are essentially the same objects: Given a preordered set X, the set of all down sets of X gives a topology on X, and if X is finite, this correspondence gives a bijection between preorders on X and topologies on X. We call a finite topological space (= finite preordered set) a *finite space*. Moreover, a map between finite spaces are continuous if and only if it is monotone.

All the maps between finite spaces are continuous unless otherwise stated.

Proposition 2.5. *Let $f, g\colon X \to Y$ be maps between finite spaces. Then $f \simeq g$ if and only if there exists a sequence of maps $f_0, \ldots, f_n\colon X \to Y$ satisfying $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$.*

In particular, if $f \leq g$, then $f \simeq g$.

McCord showed that a finite space is T_0 if and only if it is a poset. If X is a finite preordered set, then the projection to its maximal quotient poset is a homotopy equivalence. Therefore, when considering homotopy types of finite spaces, one may consider only finite T_0 spaces.

Let X be a finite space and $a \in X$. We denote

$$\begin{aligned} U_a^X &= \{x \in X \mid x \leq a\}, & \widehat{U}_a^X &= \{x \in X \mid x < a\}, \\ F_a^X &= \{x \in X \mid x \geq a\}, & \widehat{F}_a^X &= \{x \in X \mid x > a\}, \\ C_a^X &= U_a \cup F_a, & \widehat{C}_a^X &= C_a - \{a\}, \\ \mathrm{mxl}(X) &= \{x \in X \mid x \text{ is maximal.}\}, & \mathrm{mnl}(X) &= \{x \in X \mid x \text{ is minimal.}\}. \end{aligned}$$

We often omit the superscript X and write U_a instead of U_a^X etc.

It is easy to see the following:

Lemma 2.6. *Let X be a finite T_0 space and $a, b \in X$. The following holds:*

1. $a \leq b \Leftrightarrow a \in U_b \Leftrightarrow b \in F_a \Leftrightarrow U_a \subset U_b \Leftrightarrow F_a \supset F_b \Leftrightarrow F_a \cap U_b \neq \emptyset$.
2. $U_a \cap U_b \simeq *$ *or* $|U_a \cap U_b| < |X|$.
3. $U_a \cup U_b$ *is connected* $\Leftrightarrow U_a \cap U_b \neq \emptyset$.
4. *Let $A \subset X$ be a subspace and $a \in A$.*
   a) $U_a^A = U_a^X \cap A$.
   b) $F_a^A = F_a^X \cap A$.
   c) $\mathrm{mxl}(X) \cap A \subset \mathrm{mxl}(A)$.
   d) $\mathrm{mnl}(X) \cap A \subset \mathrm{mnl}(A)$.

Homotopy types Stong [9] studied homotopy types of finite spaces.

Definition 2.7. Let X be a finite T_0 space and $x \in X$. We call x a *down beat point* if x covers one and only one element of X. In other words, x is a down beat point if $\widehat{U}_x$ has a maximum. Dually, x is an *up beat point* if $\widehat{F}_x$ has a minimum.

Proposition 2.8 (Stong [9])**.** *Let X be a finite T_0 space and $x \in X$ a beat point. Then $X \setminus \{x\}$ is a strong deformation retract of X.*

Actually, we can remove down beat points at once.

Lemma 2.9. *Let X be a finite T_0 space, $x \in X$ a down beat point and $x \neq y \in X$. If y is a down beat point of X, then y is a down beat point of $X \setminus \{x\}$.*

Proof. Since x is a down beat point of X, $\widehat{U}_x^X$ has the maximum . We put $\underline{x} = \max \widehat{U}_x^X$. We have $\widehat{U}_x^X = U_{\underline{x}}^X$.

Assume that y is a down beat point of X and put $\underline{y} = \max \widehat{U}_y^X$.

Since $\widehat{U}_y^{X \setminus \{x\}} = \widehat{U}_y^X - \{x\}$, if $x \notin U_y^X$, then $\underline{y} = \max \widehat{U}_y^{X \setminus \{x\}}$.

Consider the case when $x \in U_y^X$, that is, $x \leq y$. Since $x \neq y$, we have $x \in \widehat{U}_y^X$.

If $x \neq \underline{y} = \max \widehat{U}_y^X$, we have $\underline{y} = \max \left(\widehat{U}_y^X - \{x\} \right) = \max \widehat{U}_y^{X \setminus \{x\}}$.

If $x = \underline{y}$, we have $\widehat{U}_y^X = U_x^X$ whence

$$\widehat{U}_y^{X \setminus \{x\}} = \widehat{U}_y^X - \{x\} = U_x^X - \{x\} = \widehat{U}_x^X = U_{\underline{x}}^X$$

Therefore $\underline{x} = \max \widehat{U}_y^{X \setminus \{x\}}$. $\square$

Corollary 2.10. *Let X be a finite T_0 space, $A \subset X$ a subset and assume that all points in $X - A$ are down beat points. Then A is a strong deformation retract of X.*

Remark 2.11. Removing down beat points may affect up beat points. Therefore, removing up and down beat points at once could change the weak homotopy type. See Fig. 1.

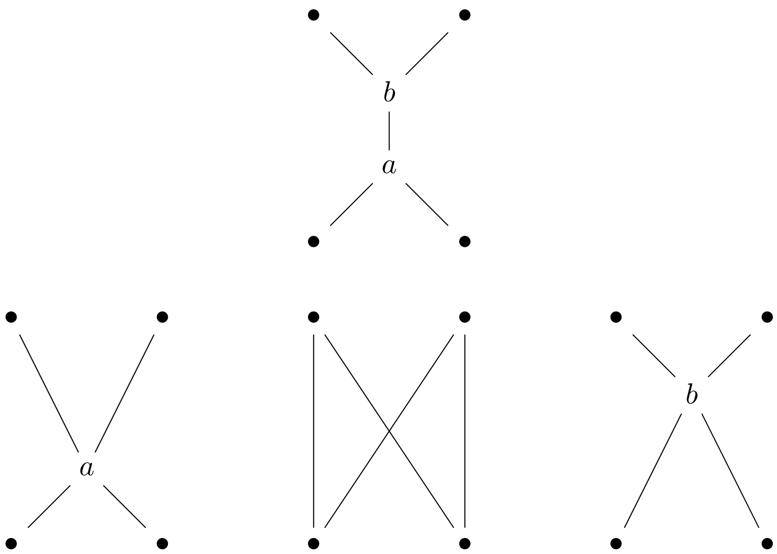


Figure 1: Removing up and down beat points.

Definition 2.12. A finite T_0 space is called a *minimal finite space* if it has no beat points. A *core* of a finite space is a strong deformation retract of the space which is a minimal finite space.

Theorem 2.13 (Stong [9])**.** *Any finite space has a core.*

Two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.

Observe the following:

Lemma 2.14. *Let X be a minimal finite space. If $b \in X$ is not a maximal element, then for all $a \in X$, $F_b - U_a \neq \emptyset$.*

Proof. We show that if there exists a point $a \in X$ such that $F_b - U_a = \emptyset$, then X has a beat point.

If $F_b - U_a = \emptyset$, then $F_b \subset U_a$. Since b is not maximal, there exists an element $b' \in X$ such that $b < b'$. Since $b' \in F_b \subset U_a$, we have $b < b' \leq a$, in particular, $b \neq a$. Hence $F_b - \{a\} \neq \emptyset$. Since $F_b - \{a\}$ is a nonempty finite set, there exists a maximal element $x \in \mathrm{mxl}(F_b - \{a\})$. We have $\widehat{F}_x \cap (F_b - \{a\}) = \emptyset$ and $\widehat{F}_x \subset F_b$, whence $\widehat{F}_x - \{a\} = \emptyset$, that is, $\widehat{F}_x \subset \{a\}$. Since $x \in F_b - \{a\} \subset U_a - \{a\}$, we have $x < a$. Therefore $\widehat{F}_x = \{a\}$ whence x is an up beat point. $\square$

Weak homotopy types Given a finite T_0 space X, we denote its *order complex*, that is, the simplicial complex of nonempty chains of X, by $\mathcal{K}(X)$. We denote the *face poset* of a simplicial complex K by $\mathcal{X}(K)$.

Theorem 2.15 (McCord [7])**.** *Let X be a finite T_0 space. The map $\mu_X\colon \mathcal{K}(X) \to X$ given by $\mu_X(\alpha) = \min(\mathrm{supp}(\alpha))$ is a weak homotopy equivalence.*

The map μ_X is natural with respect to X, that is, if $f\colon X \to Y$ is a map between finite spaces, then the following diagram is commutative:

$$\begin{tikzcd} \mathcal{K}(X) \arrow[r, "\mathcal{K}(f)"] \arrow[d, "\mu_X"', "\simeq_w"] & \mathcal{K}(Y) \arrow[d, "\simeq_w"', "\mu_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Corollary 2.16. *Let X, Y be finite T_0 spaces. Then, $X \simeq_w Y$ if and only if $\mathcal{K}(X) \simeq \mathcal{K}(Y)$.*

In particular, $X \simeq_w X^o$, where X^o is the opposite of X.

A finite T_0 space X is said to be *homotopically trivial* if $\mathcal{K}(X) \simeq *$, equivalently if $X \simeq_w *$.

Proposition 2.17. *Let $f, g\colon X \to Y$ be maps between finite T_0 spaces. If $f \simeq g$, then $\mathcal{K}(f) \simeq \mathcal{K}(g)$.*

When one considers the weak homotopy types of finite spaces, Quillen's Theorem A is quite a powerful tool.

Theorem 2.18 (Quillen [8])**.** *Let X, Y be finite T_0 spaces and $f\colon X \to Y$ a map.*

If $f^{-1}(U_y) \simeq_w *$ for every $y \in Y$, then f is a weak homotopy equivalence.

Dually, if $f^{-1}(F_y) \simeq_w *$ for every $y \in Y$, then f is a weak homotopy equivalence.

Definition 2.19. Let S^0 be the 0-dimensional sphere, that is, the discrete space with 2 points.

The ordinal sum $X * S^0$ of a finite T_0 space X and S^0 is called the *non-Hausdorff suspension* of X and is denoted by $\mathbb{S}X$. We define the n-fold non-Hausdorff suspension inductively by $\mathbb{S}^0 X = X$, $\mathbb{S}^n X = \mathbb{S}(\mathbb{S}^{n-1} X)$.

Proposition 2.20 (McCord [7], Barmak [2]). *Let X, Y be finite T_0 spaces. The following holds:*

$$\mathbb{S}X \simeq_w S\mathcal{K}(X),$$
$$X \vee Y \simeq_w \mathcal{K}(X) \vee \mathcal{K}(Y).$$

Corollary 2.21. *Let X, Y, X_1, X_2, Y_1, and Y_2 be finite T_0 spaces and K, K_1 and K_2 be simplicial complexes. The following holds:*

1. *If $X \simeq_w K$, then $\mathbb{S}X \simeq_w SK$.*
2. *If $X \simeq_w Y$, then $\mathbb{S}X \simeq_w \mathbb{S}Y$.*
3. *The weak homotopy type of a wedge sum of connected finite T_0 spaces is independent of basepoints.*
4. *If X_1 and X_2 are connected and $X_i \simeq_w K_i$, then $X_1 \vee X_2 \simeq_w K_1 \vee K_2$.*
5. *If X_1 and X_2 are connected and $X_i \simeq_w Y_i$, then $X_1 \vee X_2 \simeq_w Y_1 \vee Y_2$.*
6. *$\mathbb{S}(X \vee Y) \simeq_w \mathbb{S}X \vee \mathbb{S}Y$.*

Proof. Recall that the homotopy type of a wedge sum of connected simplicial complexes is independent of base points and is homotopy invariant.

1. If $X \simeq_w K$, then $\mathcal{K}(X) \simeq_w X \simeq_w K$ whence $\mathcal{K}(X) \simeq K$. Therefore $\mathbb{S}X \simeq_w S\mathcal{K}(K) \simeq SK$.
2. If $X \simeq_w Y$, then $S\mathcal{K}(X) \simeq S\mathcal{K}(Y)$ whence $\mathbb{S}X \simeq_w \mathbb{S}Y$.
3. This is because $X \vee Y \simeq_w \mathcal{K}(X) \vee \mathcal{K}(Y)$ and the homotopy type of the right hand side is independent of base points.
4. If $X_i \simeq_w K_i$, then $\mathcal{K}(X_i) \simeq K_i$. Therefore $X_1 \vee X_2 \simeq_w \mathcal{K}(X_1) \vee \mathcal{K}(X_2) \simeq K_1 \vee K_2$.
5. If $X_i \simeq_w Y_i$, then $\mathcal{K}(X_i) \simeq \mathcal{K}(Y_i) \simeq_w Y_i$. Therefore $X_1 \vee X_2 \simeq_w \mathcal{K}(X_1) \vee \mathcal{K}(X_2) \simeq_w Y_1 \vee Y_2$.
6. $$\begin{aligned}\mathbb{S}(X \vee Y) \simeq_w S(\mathcal{K}(X) \vee \mathcal{K}(Y)) &\simeq S\mathcal{K}(X) \vee S\mathcal{K}(Y) \\ &\simeq \mathcal{K}(\mathbb{S}X) \vee \mathcal{K}(\mathbb{S}Y) \simeq_w \mathbb{S}X \vee \mathbb{S}Y.\end{aligned}$$

□

The following generalizations of beat points are given by Barmak [2].

Definition 2.22. Let X be a finite T_0 space and $x \in X$. We call x a *down weak beat point* if $\widehat{U}_x$ is contractible. Dually, x is an *up weak beat point* if $\widehat{F}_x$ is contractible. We call x a *weak point* if it is either a down or up weak beat point.

As remarked in [2, 4.2.3], a point x is a weak point if and only if $\widehat{C}_x$ is contractible.

We call x a *γ-point* if $\widehat{C}_x$ is homotopically trivial.

Proposition 2.23 (Barmak [2])**.** *Let X be a finite T_0 space and $x \in X$ a γ-point. Then, the inclusion $i\colon X \setminus \{x\} \to X$ is a weak homotopy equivalence.*

A little bit more generally, the following holds:

Proposition 2.24. *Let X be a finite T_0 space. If the inclusion $\mathcal{K}\left(\hat{C}_x\right) \to \mathcal{K}\left(X - \{x\}\right)$ is null homotopic, then*

$$X \simeq_w (X - \{x\}) \vee \mathbb{S}\hat{C}_x$$

In particular, X splits into smaller spaces.

Proof. Since

$$\begin{aligned} \mathcal{K}(X) \setminus \{x\} &= \mathcal{K}(X - \{x\}), \\ \operatorname{lk}(x) &= \{\sigma \subset X - \{x\} \mid \sigma \neq \emptyset,\ \sigma \cup \{x\} \text{ is a chain}\} = \mathcal{K}\left(\hat{C}_x\right), \end{aligned}$$

the inclusion $\operatorname{lk}(x) \to \mathcal{K}(X) \setminus \{x\}$ is null homotopic by the assumption. Therefore

$$\begin{aligned} \mathcal{K}(X) &\simeq (\mathcal{K}(X) \setminus \{x\}) \vee S(\operatorname{lk}(x)) \\ &= \mathcal{K}(X - \{x\}) \vee S\mathcal{K}\left(\hat{C}_x\right) \end{aligned}$$

□

Corollary 2.25. *Let X be a finite T_0 space. If there exists a maximal element $a \in \operatorname{mxl}(X)$ such that $X - \{a\}$ is connected and $\hat{U}_a \simeq_w \bigvee_n S^0$, then*

$$X \simeq_w (X - \{a\}) \vee \bigvee_n S^1.$$

In particular, X splits into smaller spaces.

Proof. Since a is maximal, $\widehat{C}_a = \widehat{U}_a$, whence $\mathcal{K}\left(\widehat{C}_a\right) = \mathcal{K}\left(\widehat{U}_a\right) \simeq \bigvee_n S^0$. Since $\mathcal{K}(X - \{a\})$ is connected, the inclusion $\mathcal{K}\left(\widehat{C}_a\right) \to \mathcal{K}(X - \{a\})$ is null homotopic. □

3. Poset splitting

In this section, we reformulate the poset splitting of Cianci-Ottina and show our fundamental splitting results Proposition 3.12 and Corollaries 3.13 and 3.14.

Definition 3.1. Given a poset X, the poset

$$X' \coloneqq \mathcal{X}(\mathcal{K}(X))$$

is called the *barycentric subdivision* of X. X' is the set of nonempty finite chains of X ordered by the inclusion order.

X' is weak homotopy equivalent to X.

Clearly the following holds:

Lemma 3.2. *Let X be a poset and $A_i \subset X$.*

1. a) $\bigcap_i \mathcal{K}(A_i) = \mathcal{K}(\bigcap_i A_i)$.
   b) $\bigcup_i \mathcal{K}(A_i) \subset \mathcal{K}(\bigcup_i A_i)$.
   If every A_i is a down set or every A_i is an up set, then $\bigcup_i \mathcal{K}(A_i) = \mathcal{K}(\bigcup_i A_i)$.
2. a) *${A_i}'$ is a down set of X'.*
   b) $\bigcap_i {A_i}' = (\bigcap_i A_i)'$.
   c) $\bigcup_i {A_i}' \subset (\bigcup_i A_i)'$.
   If every A_i is a down set or every A_i is an up set, then $\bigcup_i {A_i}' = (\bigcup_i A_i)'$.

In general, the inclusion

$$\bigcup_i {A_i}' \subset \left(\bigcup_i A_i\right)'$$

is not a weak homotopy equivalence. To remedy this, Cianci-Ottina [3] used open stars and the second barycentric subdivisions.

Definition 3.3. 1. Let K be a simplicial complex and V its set of vertices.

a) For a vertex $v \in V$, the subset

$$\mathrm{ost}_K(v) \coloneqq \{\sigma \in K \mid v \in \sigma\}$$

of K is called the *open star* of v.

b) For a subset $A \subset V$,

$$\mathrm{ost}_K(A) \coloneqq \bigcup_{v\in A} \mathrm{ost}_K(v)$$

is called the *open star* of A.

Clearly, we have

$$\mathrm{ost}_K\left(\bigcup_i A_i\right) = \bigcup_i \mathrm{ost}_K(A_i),$$
$$\mathrm{ost}_K(A) = \{\sigma \in K \mid \sigma \cap A \neq \emptyset\}.$$

2. Let X be a poset and $A \subset X$ a subset. We define a subposet $\widetilde{A'} \subset X'$ by

$$\widetilde{A'} := \mathcal{X}\left(\mathrm{ost}_{\mathcal{K}(X)}(A)\right) = \{\sigma \in X' \mid \sigma \cap A \neq \emptyset\} \subset X'.$$

Remark 3.4. We have $\widetilde{A'} = X' - (X - A)'$.

Lemma 3.5. 1. *$\widetilde{A'}$ is an up set of X' and $A' \subset \widetilde{A'}$.*

2. *The inclusion $i\colon A' \to \widetilde{A'}$ and the map $r\colon \widetilde{A'} \to A'$ given by $r(\sigma) = \sigma \cap A$ are mutually inverse homotopy equivalences.*

3. *The map $\max\colon A' \to A$ is a weak homotopy equivalence.*

4. *If A is an up set, then $\max\sigma = \max(\sigma \cap A) \in A$ for any $\sigma \in \widetilde{A'}$, and $\max\colon \widetilde{A'} \to A$ is a weak homotopy equivalence.*

5. *If A is a down set, then $\min\sigma = \min(\sigma \cap A) \in A$ for any $\sigma \in \widetilde{A'}$, and $\min\colon \widetilde{A'} \to A^o$ is a weak homotopy equivalence.*

Proof. 1. If $\sigma \cap A \neq \emptyset$ and $\sigma \subset \tau$, then $\tau \cap A \neq \emptyset$.
If $\emptyset \neq \sigma \subset A$, then $\sigma \cap A \neq \emptyset$.

2. This is essentially shown in [4, II.9]. If $\sigma \in \widetilde{A'}$, then $\sigma \cap A \neq \emptyset$ hence $\sigma \cap A \in A'$. Clearly, the map r is monotone and $ri(\tau) = \tau$. We have $ir(\sigma) \leq \sigma$ for $\sigma \in \widetilde{A'}$. Therefore $ir \simeq 1_{\widetilde{A'}}$.

3. Well known. See [5] for example.

4. Suppose A is an up set and $\sigma \in \widetilde{A'}$. Since σ is a finite chain and $\emptyset \neq \sigma \cap A \subset \sigma$, we have $\max\sigma \geq \max(\sigma \cap A) \in A$, and since A is an up set, we have $\max\sigma \in A$. Therefore $\max\sigma \in \sigma \cap A$ and $\max\sigma \leq \max(\sigma \cap A)$.

   Since $\max \circ i = \max\colon A' \xrightarrow{i} \widetilde{A'} \xrightarrow{\max} A$, we see that $\max\colon \widetilde{A'} \to A$ is a weak homotopy equivalence.

5. This is the dual of part 4.

□

Corollary 3.6. *Let X be a poset, $A_i \subset X$ and $A = \bigcup_i A_i$.*
We have

$$\widetilde{A'} = \bigcup_i \widetilde{A_i'}, \qquad A \simeq_w \bigcup_i \mathcal{K}\left(\widetilde{A_i'}\right).$$

In particular, if $X = \bigcup_i A_i$, we have

$$X' = \bigcup_i \widetilde{A_i'}, \qquad X \simeq_w \bigcup_i \mathcal{K}\left(\widetilde{A_i'}\right).$$

Proof.

$$\begin{aligned}\bigcup_i \widetilde{A_i}' &= \bigcup_i \mathcal{X}\left(\mathrm{ost}_{\mathcal{K}(X)}(A_i)\right)\\ &= \mathcal{X}\left(\bigcup_i \mathrm{ost}_{\mathcal{K}(X)}(A_i)\right)\\ &= \mathcal{X}\left(\mathrm{ost}_{\mathcal{K}(X)}\left(\bigcup_i A_i\right)\right) = \mathcal{X}\left(\mathrm{ost}_{\mathcal{K}(X)}(A)\right) = \widetilde{A'}.\end{aligned}$$

Since $\widetilde{A_i}'$ is an up set, we have

$$A \simeq_w \widetilde{A'} \simeq_w \mathcal{K}\left(\widetilde{A'}\right) = \mathcal{K}\left(\bigcup_i \widetilde{A_i}'\right) = \bigcup_i \mathcal{K}\left(\widetilde{A_i}'\right).$$

□

Cianci-Ottina called the following the poset splitting technique: Let X be a finite T_0 space, A_1 and A_2 be subspaces of X. Then, $A_1 \cup A_2 \simeq_w \mathcal{K}\left(\widetilde{A_1}'\right) \cup \mathcal{K}\left(\widetilde{A_2}'\right)$ and $\mathcal{K}\left(\widetilde{A_i}'\right) \simeq_w A_i$, whence one may obtain the information of the weak homotopy type of $A_1 \cup A_2$ from those of A_1 and A_2.

Moreover, since

$$\mathcal{K}\left(\widetilde{A_1}'\right) \cap \mathcal{K}\left(\widetilde{A_2}'\right) = \mathcal{K}\left(\widetilde{A_1}' \cap \widetilde{A_2}'\right) \simeq_w \widetilde{A_1}' \cap \widetilde{A_2}',$$

the information on the weak homotopy type of $\widetilde{A_1}' \cap \widetilde{A_2}'$ and the inclusion $\widetilde{A_1}' \cap \widetilde{A_2}' \to \widetilde{A_i}'$ would give us more information.

Example 3.7. Let $X = A \cup B$ be a finite T_0 space. If A and B are homotopically trivial, then we have

$$X \simeq_w \mathbb{S}(\widetilde{A'} \cap \widetilde{B'})$$

Proof. Since A and B are homotopically trivial, $\mathcal{K}\left(\widetilde{A'}\right) \simeq \mathcal{K}(A) \simeq *$, $\mathcal{K}\left(\widetilde{B'}\right) \simeq *$. Therefore

$$\begin{aligned}X &\simeq_w \mathcal{K}\left(\widetilde{A'}\right) \cup \mathcal{K}\left(\widetilde{B'}\right)\\ &\simeq S\left(\mathcal{K}\left(\widetilde{A'}\right) \cap \mathcal{K}\left(\widetilde{B'}\right)\right)\\ &= S\left(\mathcal{K}\left(\widetilde{A'} \cap \widetilde{B'}\right)\right)\\ &\simeq_w \mathbb{S}(\widetilde{A'} \cap \widetilde{B'}).\end{aligned}$$

□

Lemma 3.8. *Let X be a poset and A and B nonempty subsets. Then, $\widetilde{A'} \cap \widetilde{B'} \neq \emptyset$ if and only if there exist an element in A and an element in B which are comparable.*

Proof. If $\widetilde{A'} \cap \widetilde{B'} \neq \emptyset$, pick an element $\sigma \in \widetilde{A'} \cap \widetilde{B'}$. Since $\sigma \in \widetilde{A'}$, we have $\sigma \cap A \neq \emptyset$, that is, there exists an element $a \in A$ such that $a \in \sigma$. Similarly, there exists an element $b \in B$ such that $b \in \sigma$. Since σ is a chain, a and b are comparable.

If $a \in A$ and $b \in B$ are comparable, then $\{a, b\}$ is a chain, $\{a, b\} \cap A \neq \emptyset$ and $\{a, b\} \cap B \neq \emptyset$. Hence $\{a, b\} \in \widetilde{A'} \cap \widetilde{B'}$. □

Definition 3.9. Let X be a poset and $A, B \subset X$.

We say that A and B are *comparable* if there exist an element in A and an element in B which are comparable.

Otherwise, that is, if any element of A and any element of B are incomparable, we say that A and B are *incomparable*.

Lemma 3.10. *Let $A \subset X$ be a down set and $b \in X$. Then A and $\{b\}$ are comparable if and only if $A \cap U_b \neq \emptyset$.*

Proof. If $A \cap U_b \neq \emptyset$, then clearly A and $\{b\}$ are comparable.

If A and $\{b\}$ are comparable, there exists $a \in A$ such that a and b are comparable. If $a \leq b$, then $a \in A \cap U_b$. If $b \leq a$, since A is a down set, $b \in A$ and so $b \in A \cap U_b$. In both cases, $A \cap U_b \neq \emptyset$. □

Lemma 3.11. *Let X be a poset.*

1. *Let $A \subset X$ be a down set and $b \in X$.*

   There exists a homotopy equivalence $q\colon \widetilde{A'} \cap \widetilde{\{b\}'} \xrightarrow{\simeq} (A \cap U_b)'$ which makes the following diagram homotopy commutative:

$$\begin{tikzcd}
\widetilde{A'} \cap \widetilde{\{b\}'} \arrow[r, "q", "\simeq"'] \arrow[d, phantom, "\bigcap" description] & (A \cap U_b)' \arrow[r, "\max", "\simeq_w"'] \arrow[d, phantom, "\bigcap" description] & A \cap U_b \arrow[d, phantom, "\bigcap" description] \\
\widetilde{A'} \arrow[r, "\simeq", "r"'] & A' \arrow[r, "\simeq_w", "\max"'] & A
\end{tikzcd}$$

2. *Let $a \in X$ and $B \subset X$ an up set.*

   There exists a homotopy equivalence $q\colon \widetilde{\{a\}'} \cap \widetilde{B'} \xrightarrow{\simeq} (F_a \cap B)'$ which makes the following diagram homotopy commutative:

$$\begin{tikzcd}
\widetilde{\{a\}'} \cap \widetilde{B'} \arrow[r, "q", "\simeq"'] \arrow[d, phantom, "\bigcap" description] & (F_a \cap B)' \arrow[r, "\max", "\simeq_w"'] \arrow[d, phantom, "\bigcap" description] & F_a \cap B \arrow[d, phantom, "\bigcap" description] \\
\widetilde{B'} \arrow[r, "\simeq", "r"'] & B' \arrow[r, "\simeq_w", "\max"'] & B
\end{tikzcd}$$

Proof. We show part 1. Part 2 is the dual.

Let A be a down set and $\sigma \in \widetilde{A'} \cap \widetilde{\{b\}'}$. As noted in Lemma 3.5, $\min \sigma \in A$ and since $b \in \sigma$, $\min \sigma \leq b$. Hence $\min \sigma \in A \cap U_b$ and so $\sigma \cap A \cap U_b \neq \emptyset$. We define a map $q\colon \widetilde{A'} \cap \widetilde{\{b\}'} \to (A \cap U_b)'$ by $q(\sigma) = \sigma \cap A \cap U_b$.

For $\tau \in (A \cap U_b)'$, clearly we have $\tau \cup \{b\} \in \widetilde{A'} \cap \widetilde{\{b\}'}$. We define a map $i_b \colon (A \cap U_b)' \to \widetilde{A'} \cap \widetilde{\{b\}'}$ by $i_b(\tau) = \tau \cup \{b\}$.

Clearly, q and i_b are order preserving.

For $\sigma \in \widetilde{A'} \cap \widetilde{\{b\}'}$, because we have $b \in \sigma$ and $\sigma \cap A \cap U_b \subset \sigma$,

$$i_b q(\sigma) = (\sigma \cap A \cap U_b) \cup \{b\} \leq \sigma.$$

On the other hand, for $\tau \in (A \cap U_b)'$, since $\tau \subset A \cap U_b$, we see that

$$q i_b(\tau) = (\tau \cup \{b\}) \cap (A \cap U_b) \geq \tau.$$

Therefore, q and i_b are mutually inverse homotopy equivalences.

For $\sigma \in \widetilde{A'} \cap \widetilde{\{b\}'}$, we have

$$q(\sigma) = \sigma \cap A \cap U_b \leq \sigma \cap A = r(\sigma)$$

whence the diagram is homotopy commutative. □

The next proposition and Corollaries 3.13 and 3.14 are our poset splitting results.

Proposition 3.12. *Let $X = A \cup B$ be a connected finite T_0 space. Assume that $A = \coprod_{i=1}^{l} A_i$, $B = \coprod_{i=1}^{m} B_i$, $A_i \simeq_w *$, $B_i \simeq_w *$, and if $i \neq j$, A_i and A_j, B_i and B_j are incomparable. Then*

$$X \simeq_w \left(\bigvee_{\substack{i,j \\ A_i \text{ and } B_j \text{ are comparable}}} \mathbb{S}\left(\widetilde{A_i'} \cap \widetilde{B_j'}\right) \right) \vee \left(\bigvee S^1\right)$$

and $\mathbb{S}\left(\widetilde{A_i'} \cap \widetilde{B_j'}\right) \simeq_w A_i \cup B_j$.

Proof. Since $X = A \cup B$, we have $X \simeq_w \mathcal{K}\left(\widetilde{A'}\right) \cup \mathcal{K}\left(\widetilde{B'}\right)$.

We set

$$K = \mathcal{K}\left(\widetilde{A'}\right) \cup \mathcal{K}\left(\widetilde{B'}\right),$$
$$L = \mathcal{K}\left(\widetilde{A'}\right), \qquad M = \mathcal{K}\left(\widetilde{B'}\right),$$
$$L_i = \mathcal{K}\left(\widetilde{A_i'}\right), \qquad M_i = \mathcal{K}\left(\widetilde{B_i'}\right).$$

If $i \neq j$, A_i and A_j are incomparable whence $\widetilde{A_i'} \cap \widetilde{A_j'} = \emptyset$ by Lemma 3.8. Therefore $\mathcal{K}\left(\widetilde{A_i'}\right) \cap \mathcal{K}\left(\widetilde{A_j'}\right) = \emptyset$ and

$$L = \mathcal{K}\left(\widetilde{A'}\right) = \bigcup_i \mathcal{K}\left(\widetilde{A_i'}\right) = \coprod_i \mathcal{K}\left(\widetilde{A_i'}\right) = \coprod_i L_i.$$

Similarly, we see that $M = \coprod_i M_i$. By the assumption, we have $L_i \simeq *$, $M_i \simeq *$.

Therefore by Lemma 2.4, K is homotopy equivalent to the wedge sum of $S(L_i \cap M_j)$ for those $L_i \cap M_j \neq \emptyset$ and some copies of S^1. We see that

$$\begin{aligned} L_i \cap M_j \neq \emptyset &\Leftrightarrow \mathcal{K}\left(\widetilde{A_i}'\right) \cap \mathcal{K}\left(\widetilde{B_j}'\right) \neq \emptyset \\ &\Leftrightarrow \widetilde{A_i}' \cap \widetilde{B_j}' \neq \emptyset \\ &\Leftrightarrow A_i \text{ and } B_j \text{ are comparable.} \end{aligned}$$

Finally, by Example 3.7, $S(L_i \cap M_j) \simeq_w \mathbb{S}\left(\widetilde{A_i}' \cap \widetilde{B_j}'\right) \simeq_w A_i \cup B_j$. ☐

A crucial observation of Cianci-Ottina [3] is that many small finite spaces can be decomposed into the form $U_a \cup F_b \cup \operatorname{mxl}(X) \cup \operatorname{mnl}(X)$. We show that this decomposition splits X into wedge sum of suspensions.

Corollary 3.13. *Let X be a connected finite T_0 space and assume that there exist $a, b \in X$ such that*

$$X = U_a \cup F_b \cup \operatorname{mxl}(X) \cup \operatorname{mnl}(X).$$

We put

$$\begin{aligned} A &= \operatorname{mnl}(X) - U_a - \{b\}, & A_{\sim b} &= \{x \in A \mid F_b \cap F_x \neq \emptyset\}, \\ B &= \operatorname{mxl}(X) - F_b - \{a\}, & B_{\sim a} &= \{x \in B \mid U_a \cap U_x \neq \emptyset\}. \end{aligned}$$

If U_a and F_b are comparable, then

$$X \simeq_w \mathbb{S}(\widetilde{U_a}' \cap \widetilde{F_b}') \vee \left(\bigvee_{x \in B_{\sim a}} \mathbb{S}(U_a \cap U_x)\right) \vee \left(\bigvee_{x \in A_{\sim b}} \mathbb{S}(F_b \cap F_x)\right) \vee \left(\bigvee_k S^1\right)$$

for some $k \geq 0$, and $\mathbb{S}(\widetilde{U_a}' \cap \widetilde{F_b}') \simeq_w U_a \cup F_b$.

If U_a and F_b are incomparable,

$$X \simeq_w \left(\bigvee_{x \in B_{\sim a}} \mathbb{S}(U_a \cap U_x)\right) \vee \left(\bigvee_{x \in A_{\sim b}} \mathbb{S}(F_b \cap F_x)\right) \vee \left(\bigvee_k S^1\right)$$

for some $k \geq 0$.

In particular, if U_a and F_b are incomparable or $U_a \cup F_b \subsetneq X$ or $U_a \cup F_b$ splits into smaller spaces, then X splits into smaller spaces.

Proof. Apply Proposition 3.12 to $U_a \cup A = U_a \amalg \coprod_{x \in A}\{x\}$ and $F_b \cup B = F_b \amalg \coprod_{x \in B}\{x\}$.

For the reader's convenience, we record the details.

Clearly, $X = U_a \cup A \cup F_b \cup B$.

Since $A \subset \operatorname{mnl}(X)$, different elements in A are incomparable. Let $x \in A$. Since x is a minimal element and $x \notin U_a$, $U_a \cap U_x = U_a \cap \{x\} = \emptyset$. Hence by Lemma 3.10, $\{x\}$ and U_a are incomparable. Moreover, U_a is contractible.

Similarly, different elements of B are incomparable, F_b and elements of B are incomparable and $F_b \simeq *$.

Therefore, we can apply Proposition 3.12.

By Lemma 3.10, U_a and $x \in B$ is comparable if and only if $U_a \cap U_x \neq \emptyset$, namely, $x \in B_{\sim a}$. In this case, we have $\widetilde{U_a}' \cap \widetilde{\{x\}}' \simeq_w U_a \cap U_x$ by Lemma 3.11 and either $U_a \cap U_x \simeq *$ or $|U_a \cap U_x| < |X|$ by Lemma 2.6.

Similarly, $x \in A$ and F_b are comparable if and only if $x \in A_{\sim b}$ and in this case, $\widetilde{F_b}' \cap \widetilde{\{x\}}' \simeq_w F_b \cap F_x$ and either $F_b \cap F_x \simeq *$ or $|F_b \cap F_x| < |X|$.

If $x \in A$ and $y \in B$ are comparable, then $x \leq y$ and $\{x, y\} = \min\left(\widetilde{\{x\}}' \cap \widetilde{\{y\}}'\right)$, hence $\widetilde{\{x\}}' \cap \widetilde{\{y\}}' \simeq *$. □

Corollary 3.14. *Let X be a connected finite T_0 space. If there exists $a \in X$ such that*

$$X = U_a \cup \operatorname{mxl}(X) \cup \operatorname{mnl}(X)$$

then

$$X \simeq_w \left(\bigvee_{\substack{b \in \operatorname{mxl}(X) \\ U_a \cap U_b \neq \emptyset}} \mathbb{S}(U_a \cap U_b) \right) \vee \left(\bigvee_k S^1 \right)$$

for some $k \geq 0$.

In particular, X splits into smaller spaces.

4. Weak homotopy types of posets of intervals

By Corollary 3.13, if $X = U_a \cup F_b \cup \operatorname{mxl}(X) \cup \operatorname{mnl}(X)$ and $U_a \cup F_b \subsetneq X$, then X splits into smaller spaces. However, in the case where $X = U_a \cup F_b$, we need to study the weak homotopy type of $U_a \cup F_b \simeq_w \mathbb{S}(\widetilde{U_a}' \cap \widetilde{F_b}')$. In this section, we analyse the weak homotopy type of $\widetilde{U_a}' \cap \widetilde{F_b}'$ using the poset of intervals.

Definition 4.1. Let X be a poset and $A, B \subset X$. Define a subposet $I(A, B)$ of $X^o \times X$ by

$$I^X(A, B) := \{(a, b) \in A \times B \mid a \leq b\} \subset X^o \times X.$$

We often omit the superscript X and write $I(A, B)$ instead of $I^X(A, B)$.

The poset $I(A, B)$ is the set of closed intervals in X whose end points are in A and B, ordered by inclusion.

Note that $I^X(A, B) \cong I^{X^o}(B, A)$ as posets.

Lemma 4.2. *Let X be a poset and $A, B \subset X$.*

1. *If A is a down set or B is an up set, then there exists a weak homotopy equivalence $e \colon \widetilde{A}' \cap \widetilde{B}' \xrightarrow{\simeq_w} I(A, B)$.*

   If A is a down set, then the following left diagram is commutative, and if B is an up set, then the following right diagram is commutative, where the maps p_A and

p_B *are projections.*

$$\begin{tikzcd}
\widetilde{A'} \arrow[d, "\min"', "\simeq_w"] & \supset & \widetilde{A'} \cap \widetilde{B'} \arrow[d, "\simeq_w"', "e"] & \widetilde{A'} \cap \widetilde{B'} \arrow[d, "\simeq_w"', "e"] & \subset & \widetilde{B'} \arrow[d, "\simeq_w"', "\max"] \\
A^o & & I(A,B) \arrow[ll, "p_A"'] & I(A,B) \arrow[rr, "p_B"'] & & B
\end{tikzcd}$$

2. *If both A and B are up sets, the projection p_B gives a weak homotopy equivalence $p_B \colon I(A,B) \to A \cap B$ and the following diagram is commutative.*

$$\widetilde{A'} \cap \widetilde{B'} \simeq_w I(A,B) \simeq_w A \cap B$$

$$\begin{tikzcd}
\widetilde{A'} \cap \widetilde{B'} \arrow[rr, "e"] \arrow[dr, "\max"'] & & I(A,B) \arrow[dl, "p_B"] \\
& A \cap B &
\end{tikzcd}$$

3. *If both A and B are down sets, the projection p_A gives a weak homotopy equivalence $p_A \colon I(A,B) \to (A \cap B)^o$ and the following diagram is commutative.*

$$\widetilde{A'} \cap \widetilde{B'} \simeq_w I(A,B) \simeq_w A \cap B$$

$$\begin{tikzcd}
\widetilde{A'} \cap \widetilde{B'} \arrow[rr, "e"] \arrow[dr, "\min"'] & & I(A,B) \arrow[dl, "p_A"] \\
& (A \cap B)^o &
\end{tikzcd}$$

Proof. 1. We consider the case where A is a down set.

Let $\sigma \in \widetilde{A'} \cap \widetilde{B'}$. As we noted in Lemma 3.5, $\min \sigma \in A$. Since $\sigma \cap B \neq \emptyset$ and $\sigma \cap B$ is a finite chain, there exists the maximum element $\max(\sigma \cap B) \in B$. We define a map $e \colon \widetilde{A'} \cap \widetilde{B'} \to I(A,B)$ by $e(\sigma) = (\min \sigma, \max(\sigma \cap B))$. We have $p_A e(\sigma) = \min \sigma$, that is, the left diagram is commutative.

If $\sigma \subset \tau$, then

$$\min \tau \leq \min \sigma \leq \max(\sigma \cap B) \leq \max(\tau \cap B),$$

hence e is monotone.

Suppose $(a,b) \in I(A,B)$, namely, $a \in A$, $b \in B$, and $a \leq b$.

Clearly, $\{a,b\} \in \widetilde{A'} \cap \widetilde{B'}$ and $e(\{a,b\}) = (a,b)$, hence $e^{-1}\left(U_{(a,b)}\right) \neq \emptyset$.

If $\sigma \in \widetilde{A'} \cap \widetilde{B'}$ and $e(\sigma) \leq (a,b)$, that is,

$$a \leq \min \sigma \leq \max(\sigma \cap B) \leq b,$$

then we easily see that

$$\sigma \cup \{a\},\ (\sigma \cap B) \cup \{a\},\ (\sigma \cap B) \cup \{a,b\},\ \{a,b\} \in e^{-1}\left(U_{(a,b)}\right),$$

and

$$\sigma \leq \sigma \cup \{a\} \geq (\sigma \cap B) \cup \{a\} \leq (\sigma \cap B) \cup \{a,b\} \geq \{a,b\},$$

and hence $e^{-1}\left(U_{(a,b)}\right) \simeq *$.

Therefore, by Quillen's Theorem A, e is a weak homotopy equivalence.

2. Since both A and B are up sets, $\max \sigma \in A \cap B$ for any element $\sigma \in \widetilde{A'} \cap \widetilde{B'}$. Hence we obtain a map $\max\colon \widetilde{A'} \cap \widetilde{B'} \to A \cap B$

   If $(a,b) \in I(A,B)$, then $a \leq b$, and since A is an up set, $b \in A \cap B$, namely, $p_B((a,b)) \in A \cap B$. Hence the projection gives a map $p_B\colon I(A,B) \to A \cap B$.

   In this (B being an up set) case, the map $e\colon \widetilde{A'} \cap \widetilde{B'} \to I(A,B)$ defined by $e(\sigma) = (\min(\sigma \cap A), \max \sigma)$ gives a weak homotopy equivalence, and we have $p_B e = \max$.

   Suppose $b \in A \cap B$.

   Since $(b,b) \in I(A,B)$ and $p_B((b,b)) = b$, $p_B^{-1}(U_b) \neq \emptyset$.

   If $(a,b') \in I(A,B)$ and $p_B((a,b')) \leq b$, then $a \leq b' \leq b$ and $(a,b') \leq (a,b) \geq (b,b)$. Hence $p_B^{-1}(U_b) \simeq *$.

   Therefore p_B is a weak homotopy equivalence.

3. This is the dual of part 2.

□

Corollary 4.3. *Let $X = A \cup B$ be a finite T_0 space. If A and B are homotopically trivial down sets, then we have*

$$X \simeq_w \mathbb{S}(A \cap B)$$

Proof.

$$X \simeq_w \mathbb{S}(\widetilde{A'} \cap \widetilde{B'}) \simeq_w \mathbb{S}(A \cap B)$$

Of course, one can directly show this using Quillen's Theorem A. □

Remark 4.4. It is well known that $U_a \cup U_b$ is *homotopy equivalent* to $\mathbb{S}(U_a \cap U_b)$. However, in general, even if both A and B are *contractible* down sets, $A \cup B$ and $\mathbb{S}(A \cap B)$ may not be homotopy equivalent. For example, consider the space S^1_3 in Example 5.2. $S^1_3 = (U_{a_0} \cup U_{a_1}) \cup U_{a_2}$, and $U_{a_0} \cup U_{a_1}$ and U_{a_2} are contractible. Since $(U_{a_0} \cup U_{a_1}) \cap U_{a_2} = \{b_0, b_1\}$, $\mathbb{S}\left((U_{a_0} \cup U_{a_1}) \cap U_{a_2}\right)$ is homeomorphic to S^1_2. Both S^1_3 and S^1_2 are minimal and they are not homeomorphic, so they are not homotopy equivalent.

Corollary 4.5. *Let X be a finite T_0 space and $a, b \in X$. We have*

$$U_a \cup F_b \simeq_w \mathbb{S}(I(U_a, F_b)).$$

Proof.

$$U_a \cup F_b \simeq_w \mathbb{S}(\widetilde{U_a'} \cap \widetilde{F_b'}) \simeq_w \mathbb{S}(I(U_a, F_b)).$$

□

We consider the height of the interval.

Definition 4.6. Let X be a finite T_0 space.

The *length* $l\,(c)$ of a chain c of X is one less than the cardinality of c: $l\,(c) := |c| - 1$. The number

$$\mathrm{h}(X) := \max\{l\,(c) \mid c \text{ is a chain of } X\}$$

is called the *height* of X.

If $\mathrm{h}(X) = 1$, then $\mathcal{K}\,(X)$ is one dimensional simplicial complex, whence each connected component of X is weak homotopy equivalent to a wedge of circles and $\mathbb{S}X$ is weak homotopy equivalent to a wedge of spheres of dimension at most 2.

Lemma 4.7. *Let X be a finite T_0 space and $A, B \subset X$.*

1. $\mathrm{h}(I(A, B)) \leq \mathrm{h}(A \cup B)$*. If $A \cap B = \emptyset$, then* $\mathrm{h}(I(A, B)) < \mathrm{h}(A \cup B)$*.*

2. *If both A and B are up sets or down sets, then* $\mathrm{h}(A \cup B) = \max\{\mathrm{h}(A), \mathrm{h}(B)\}$*.*

Proof. 1. Suppose $(a_0, b_0) < (a_1, b_1) < \cdots < (a_k, b_k)$ is a chain of $I(A, B)$ of length k. We have

$$a_k \leq \cdots \leq a_1 \leq a_0 \leq b_0 \leq \cdots \leq b_k$$

in $A \cup B$ and for each $1 \leq i \leq k$, $a_i < a_{i-1}$ or $b_{i-1} < b_i$. Therefore the length of the chain $\{a_k, \ldots, a_0, \ldots, b_k\}$ of $A \cup B$ is greater than or equal to k. If $a_0 < b_0$, then it is greater than or equal to $k + 1$. Hence, $\mathrm{h}(I(A, B)) \leq \mathrm{h}(A \cup B)$ and if $A \cap B = \emptyset$, then $\mathrm{h}(I(A, B)) < \mathrm{h}(A \cup B)$.

2. Clearly we have $\max\{\mathrm{h}(A), \mathrm{h}(B)\} \leq \mathrm{h}(A \cup B)$.

   Assume that both A and B are up sets. Let c be a nonempty chain of $A \cup B$. If $\min(c) \in A$, then $c \subset A$ for A is an up set, so $l\,(c) \leq \mathrm{h}(A)$. Similarly, if $\min(c) \in B$, then $l\,(c) \leq \mathrm{h}(B)$. Therefore, $\mathrm{h}(A \cup B) \leq \max\{\mathrm{h}(A), \mathrm{h}(B)\}$.

□

Definition 4.8. Let X be a finite T_0 space. We set

$$\mathcal{B} = X - \mathrm{mxl}(X) - \mathrm{mnl}(X).$$

Corollary 4.9. *If $\mathcal{B}$ is an antichain, then* $\mathrm{h}\,(I(A, B)) \leq 2$ *for $A, B \subset X$. If $A \cap B = \emptyset$, then* $\mathrm{h}\,(I(A, B)) \leq 1$*.*

Proposition 4.10. *Let X be a finite T_0 space such that $\mathcal{B}$ is an antichain and $a_0, b_0 \in X$.*

Then, each connected component of $I(U_{a_0}, F_{b_0})$ has the weak homotopy type of a wedge of spheres of dimension at most 2. Hence, $U_{a_0} \cup F_{b_0}$ has the weak homotopy type of a wedge of spheres of dimension at most 3.

In particular, if X is of the form $X = U_{a_0} \cup F_{b_0} \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$ and $\mathcal{B}$ is an antichain, then X splits into smaller spaces.

Proof. If $a_0 \not\geq b_0$, namely, if $U_{a_0} \cap F_{b_0} = \emptyset$, then $\mathrm{h}(I(U_{a_0}, F_{b_0})) \leq 1$ and each connected component is weak homotopy equivalent to a wedge of one dimensional spheres.

If $a_0 = b_0$, then $I(U_{a_0}, F_{b_0}) = U^o_{a_0} \times F_{b_0} \simeq *$.

Assume that $a_0 > b_0$.

We see that $(a_0, a_0),\ (b_0, b_0),\ (b_0, a_0) \in I(U_{a_0}, F_{b_0})$ because $a_0,\ b_0 \in U_{a_0} \cap F_{b_0}$.

We denote $\mathrm{mxl}(I(U_{a_0}, F_{b_0}))$ by mxl and $\mathrm{mnl}(I(U_{a_0}, F_{b_0}))$ by mnl.

We show that if $(a, b) \in I(U_{a_0}, F_{b_0}) - (\mathrm{mxl} \cup \mathrm{mnl})$, then either $(a, b) \in U_{(b_0,a_0)}$ or (a, b) is a down beat point of $I(U_{a_0}, F_{b_0})$.

Since $(a, b) \notin \mathrm{mnl}$, $a \neq b$, whence $a < b$. Since $(a, b) \notin \mathrm{mxl}$, $a \in \mathcal{B}$ or $b \in \mathcal{B}$. Because $\mathcal{B}$ is an antichain, we see that either $a \in \mathcal{B}$ and $b \in \mathrm{mxl}$, or $a \in \mathrm{mnl}$ and $b \in \mathcal{B}$.

Suppose $a \in \mathcal{B}$ and $b \in \mathrm{mxl}$. Since $\mathcal{B}$ is an antichain, $a \prec b$. Therefore, elements in $I(X, X)$ smaller than (a, b) are only (a, a) and (b, b). Since $(a, b) \notin \mathrm{mnl}$, at least one of them belongs to $I(U_{a_0}, F_{b_0})$. If both $(a, a),\ (b, b) \in I(U_{a_0}, F_{b_0})$, then $a \in F_{b_0}$ and $b \in U_{a_0}$, that is, $b_0 \leq a$ and $b \leq a_0$, hence $(a, b) \leq (b_0, a_0)$. Otherwise, (a, b) is a down beat point.

The case where $a \in \mathrm{mnl}$ and $b \in \mathcal{B}$ is similar.

Therefore, by removing these down beat points, we have

$$I(U_{a_0}, F_{b_0}) \simeq U_{(b_0,a_0)} \cup \mathrm{mxl} \cup \mathrm{mnl}\,.$$

By Corollary 3.14, the connected component of the right hand side containing $U_{(b_0,a_0)}$ is weak homotopy equivalent to a wedge of some copies of S^1 and $\mathbb{S}\left(U_{(b_0,a_0)} \cap U_{(a,b)}\right)$ for some (a, b). If $U_{(b_0,a_0)} \cap U_{(a,b)} = U_{(b_0,a_0)}$, then $\mathbb{S}\left(U_{(b_0,a_0)} \cap U_{(a,b)}\right)$ is contractible. If $U_{(b_0,a_0)} \cap U_{(a,b)} \subsetneq U_{(b_0,a_0)}$, then $U_{(b_0,a_0)} \cap U_{(a,b)} \subset \widehat{U}_{(b_0,a_0)}$ and

$$\mathrm{h}\left(U_{(b_0,a_0)} \cap U_{(a,b)}\right) \leq \mathrm{h}(\widehat{U}_{(b_0,a_0)}) < \mathrm{h}(U_{(b_0,a_0)}) \leq \mathrm{h}(I(U_{a_0}, F_{b_0})) \leq 2,$$

therefore $\mathcal{K}\left(U_{(b_0,a_0)} \cap U_{(a,b)}\right)$ is at most 1 dimensional and hence $\mathbb{S}\left(U_{(b_0,a_0)} \cap U_{(a,b)}\right)$ is weak homotopy equivalent to a wedge of some copies of S^2 ad S^1.

The other connected components has height at most 1. ☐

We need more general results.

Lemma 4.11. *Let X be a poset and $A, B \subset X$.*

1. a) *If $a_0 \in \mathrm{mxl}(A)$ and $b_0 \in \mathrm{mnl}(B \cap F_{a_0})$, then $(a_0, b_0) \in I(A, B)$ is a minimal element of $I(A, B)$.*

b) *If $a_0 \in A$ is an up beat point of A and $b_0 \in \operatorname{mnl}(B \cap F_{a_0})$, then $(a_0, b_0) \in I(A, B)$ is a minimal element or a down beat point of $I(A, B)$.*

2. a) *If $b_0 \in \operatorname{mnl}(B)$ and $a \in \operatorname{mxl}(A \cap U_{b_0})$, then $(a_0, b_0) \in I(A, B)$ is a minimal element of $I(A, B)$.*

   b) *If $b_0 \in B$ is a down beat point of B and $a \in \operatorname{mxl}(A \cap U_{b_0})$, then $(a_0, b_0) \in I(A, B)$ is a minimal element or a down beat point of $I(A, B)$.*

Proof. We show part 1. Part 2 is the dual.

a) Suppose $a_0 \in \operatorname{mxl}(A)$ and $b_0 \in \operatorname{mnl}(B \cap F_{a_0})$. If $(a, b) \in I(A, B)$ and $(a, b) \leq (a_0, b_0)$, then $a \in A$, $b \in B$ and $a_0 \leq a \leq b \leq b_0$. Since $a_0 \in \operatorname{mxl}(A)$, we have $a_0 = a$. Since $b_0 \in \operatorname{mnl}(B \cap F_{a_0})$, we have $b = b_0$. Therefore, $(a, b) = (a_0, b_0)$ and (a_0, b_0) is minimal.

b) Suppose $a_0 \in A$ is an up beat point of A and $b_0 \in \operatorname{mnl}(B \cap F_{a_0})$. We put $\hat{a}_0 = \min \widehat{F}^A_{a_0} = \min \left(A \cap \widehat{F}_{a_0} \right)$.

   Assume that (a_0, b_0) is not minimal in $I(A, B)$. We show that $(\hat{a}_0, b_0) = \max \widehat{U}_{(a_0, b_0)}$. If $(a, b) \in \widehat{U}_{(a_0, b_0)}$, namely, if $(a, b) \in I(A, B)$ and $(a, b) < (a_0, b_0)$, then $a_0 \leq a \leq b \leq b_0$ and $a_0 < a$ or $b < b_0$. Since $b \in B \cap F_{a_0}$ and $b_0 \in \operatorname{mnl}(B \cap F_{a_0})$, we have $b = b_0$. Therefore $a_0 < a$ and so $\hat{a}_0 \leq a$. Hence $a_0 < \hat{a}_0 \leq a \leq b = b_0$ and we have $(a, b) \leq (\hat{a}_0, b_0) < (a_0, b_0)$. Therefore, $(\hat{a}_0, b_0) = \max \widehat{U}_{(a_0, b_0)}$ and (a_0, b_0) is a down beat point.

$\square$

Lemma 4.12. *Let X be a finite T_0 space, $a_0 \in X - \operatorname{mnl}(X)$, $b_0 \in X - \operatorname{mxl}(X)$, and $a_0 \not\leq b_0$. We put*

$$A_0 = (U_{a_0} - \operatorname{mnl}(X)) - U_{b_0}, \qquad A_1 = (U_{a_0} - \operatorname{mnl}(X)) \cap U_{b_0},$$
$$B_0 = (F_{b_0} - \operatorname{mxl}(X)) - F_{a_0}, \qquad B_1 = (F_{b_0} - \operatorname{mxl}(X)) \cap F_{a_0}.$$

Suppose the following holds:

1. a) *All the elements of $A_0 \setminus \{a_0\}$ are up beat points of U_{a_0}.*

   b) *All the elements of $B_0 \setminus \{b_0\}$ are down beat points of F_{b_0}.*

2. $I(A_0, B_0) = \emptyset$.

Moreover, when $A_1 \neq \emptyset$ or $B_1 \neq \emptyset$, we also assume the following:

3. a) *When $A_1 \neq \emptyset$, there exists $\max A_1$. We put $a_1 = \max A_1$.*

   b) *When $B_1 \neq \emptyset$, there exists $\min B_1$. We put $b_1 = \min B_1$.*

Then we have

$$I(U_{a_0}, F_{b_0}) \simeq F_{(a_1,b_0)} \cup F_{(a_0,b_1)} \cup \mathrm{mxl}(I(U_{a_0}, F_{b_0})) \cup \mathrm{mnl}(I(U_{a_0}, F_{b_0}))$$

where we consider $F_{(a_1,b_0)} = \emptyset$ *when* $A_1 = \emptyset$ *and* $F_{(a_0,b_1)} = \emptyset$ *when* $B_1 = \emptyset$.

Moreover, the connected component containing $F_{(a_1,b_0)} \cup F_{(a_0,b_1)}$ *is weak homotopy equivalent to*

$$\left(\bigvee_{\substack{(a,b)\in \mathrm{mnl}(I(U_{a_0},F_{b_0})) \\ \left(F_{(a_1,b_0)}\cup F_{(a_0,b_1)}\right)\cap F_{(a,b)}\neq\emptyset}} \mathbb{S}\left(\left(F_{(a_1,b_0)} \cup F_{(a_0,b_1)}\right) \cap F_{(a,b)}\right) \right) \vee \left(\bigvee S^1\right).$$

Proof. We put

$$A = U_{a_0}, \qquad B = F_{b_0},$$
$$A_m = A \cap \mathrm{mnl}(X), \qquad B_m = B \cap \mathrm{mxl}(X).$$

Since

$$\begin{aligned} A &= (A - \mathrm{mnl}(X)) \cup (A \cap \mathrm{mnl}(X)) \\ &= A_0 \cup A_1 \cup A_m, \\ B &= (B - \mathrm{mxl}(X)) \cup (B \cap \mathrm{mxl}(X)) \\ &= B_0 \cup B_1 \cup B_m \end{aligned}$$

and, by the assumption, $I(A_0, B_0) = \emptyset$, we have

$$\begin{aligned} I(A,B) = I(A_1,B) &\cup I(A,B_1) \\ &\cup I(A_0,B_m) \cup I(A_m,B_0) \\ &\cup I(A_m,B_m). \end{aligned}$$

We show that $I(A_1, B) \subset F_{(a_1,b_0)}$.

We suppose $A_1 \neq \emptyset$. Since

$$A_1 = \mathcal{B} \cap U_{a_0} \cap U_{b_0}, \quad a_1 = \max A_1, \quad B = F_{b_0},$$

if $a \in A_1$ and $b \in B$, then we have $a \leq a_1 \leq b_0 \leq b$, henceforth $I(A_1, B) = A_1^o \times B$ and $(a_1, b_0) = \min I(A_1, B)$. Therefore $I(A_1, B) \subset F_{(a_1,b_0)}$.

Similarly or dually, we see that $I(A, B_1) \subset F_{(a_0,b_1)}$.

We show that

$$I(A_0, B_m) \subset \mathrm{mnl}(I(A,B)) \cup \widehat{F}_{(a_0,b_1)} \cup \{ \text{ down beat points of } I(A,B) \ \}.$$

Suppose $(a,b) \in I(A_0, B_m)$, that is, $a \in A_0$, $b \in B_m$ and $a \leq b$. By the assumption 1 (a), a is either the maximum element, namely, a_0 or an up beat point of $A = U_{a_0}$. Hence, by Lemma 4.11, if $b \in \mathrm{mnl}(B \cap F_a)$, then (a,b) is minimal or a down beat point of $I(A,B)$.

If $b \notin \mathrm{mnl}(B \cap F_a)$, then there exists an element $b' \in B$ such that $a \leq b' < b$. Since $b' \notin \mathrm{mxl}(X)$, we have $b' \in B - \mathrm{mxl}(X) = B_0 \cup B_1$, but since $a \in A_0$, $a \leq b'$, and $I(A_0, B_0) = \emptyset$, we see that $b' \notin B_0$ and so $b' \in B_1$, whence $b_1 = \min B_1 \leq b'$. Therefore, $a \leq a_0 \leq b_1 \leq b' < b$ and $(a, b) > (a, b') \geq (a_0, b_1)$.

Similarly, we see that $I(A_m, B_0) \subset \mathrm{mnl}(I(A, B)) \cup \widehat{F}_{(a_1,b_0)} \cup \{$ down beat points $\}$ and clearly we have $I(A_m, B_m) = \mathrm{mxl}(I(A, B))$.

Therefore, by removing down beat points, we have

$$I(U_{a_0}, F_{b_0}) \simeq F_{(a_1,b_0)} \cup F_{(a_0,b_1)} \cup \mathrm{mxl}(I(U_{a_0}, F_{b_0})) \cup \mathrm{mnl}(I(U_{a_0}, F_{b_0})).$$

Since $F_{(a_1,b_0)} \cap F_{(a_0,b_1)} = F_{(a_1,b_1)}$, we have $F_{(a_1,b_0)} \cup F_{(a_0,b_1)} \simeq \mathbb{S}(F_{(a_1,b_1)}) \simeq *$. Note that $F_{(a_1,b_0)} \cup F_{(a_0,b_1)}$ is an up set. By applying Proposition 3.12 and Lemma 3.11 to $F_{(a_1,b_0)} \cup F_{(a_0,b_1)} \cup \mathrm{mxl}(I(U_{a_0}, F_{b_0}))$ and $\mathrm{mnl}(I(U_{a_0}, F_{b_0}))$, we see that the connected component of the right hand side containing $F_{(a_1,b_0)} \cup F_{(a_0,b_1)}$ is weak homotopy equivalent to

$$\left(\bigvee_{\substack{(a,b) \in \mathrm{mnl}(I(U_{a_0}, F_{b_0})) \\ \left(F_{(a_1,b_0)} \cup F_{(a_0,b_1)}\right) \cap F_{(a,b)} \neq \emptyset}} \mathbb{S}\left(\left(F_{(a_1,b_0)} \cup F_{(a_0,b_1)} \right) \cap F_{(a,b)} \right) \right) \vee \left(\bigvee S^1 \right).$$

□

Remark 4.13. 1. If $a_0 \in \mathrm{mnl}(X)$, then $U_{a_0} = \{a_0\}$ and we have

$$I(U_{a_0}, F_{b_0}) = I(\{a_0\}, F_{b_0}) \cong F_{a_0} \cap F_{b_0}.$$

Similarly, if $b_0 \in \mathrm{mxl}(X)$, then we have

$$I(U_{a_0}, F_{b_0}) = I(U_{a_0}, \{b_0\}) \cong (U_{a_0} \cap U_{b_0})^o .$$

2. If $a_0 \leq b_0$, then we have

$$I(U_{a_0}, F_{b_0}) = U^o_{a_0} \times F_{b_0} \simeq *.$$

3. If $a_0 \not\leq b_0$, then $a_0 \notin U_{b_0}$ and hence $a_0 \notin A_1$. Therefore $a_1 < a_0$. Similarly, $b_0 < b_1$.

4. If $I(A_0, B_0) = \emptyset$, then we see that $U_{a_0} \cap F_{b_0}$ is $\{a_0, b_0\}$ or empty.

Lemma 4.14. *We consider the same situation as in Lemma* 4.12.

For all $(a, b) \in I(U_{a_0}, F_{b_0})$*, we have*

$$\begin{aligned} F_{(a_1,b_0)} \cap F_{(a,b)} &= (U_a \cap U_{a_1})^o \times F_b, \\ F_{(a_0,b_1)} \cap F_{(a,b)} &= U^o_a \times (F_b \cap F_{b_1}). \end{aligned}$$

If $(a, b) \leq (a_1, b_1)$*, then* $\left(F_{(a_1,b_0)} \cup F_{(a_0,b_1)}\right) \cap F_{(a,b)}$ *is contractible.*

Proof. Note that we have $a_1 < a_0 \leq b_1$ and $a_1 \leq b_0 < b_1$. Suppose $(c,d) \in I(U_{a_0}, F_{b_0})$. Then

$$\begin{aligned}(c,d) \in F_{(a_1,b_0)} \cap F_{(a,b)} &\Leftrightarrow (a_1,b_0) \leq (c,d) \text{ and } (a,b) \leq (c,d) \\ &\Leftrightarrow c \leq a_1 \text{ and } b_0 \leq d \text{ and } c \leq a \text{ and } b \leq d \\ &\Leftrightarrow c \in U_{a_1} \cap U_a \text{ and } d \in F_b.\end{aligned}$$

On the other hand, if $(c,d) \in (U_a \cap U_{a_1})^o \times F_b$, then $c \in U_{a_0}$, $d \in F_{b_0}$, and $c \leq a_1 \leq b_0 \leq d$, and hence $(c,d) \in I(U_{a_0}, F_{b_0})$. Therefore, as we saw, $(c,d) \in F_{(a_1,b_0)} \cap F_{(a,b)}$.

If $(a,b) \leq (a_1,b_1)$, namely, if $a \geq a_1$ and $b \leq b_1$, then $a = a_1$ or $b = b_1$ because, if $a \neq a_1$, then $a > a_1 = \max A_1$ hence $a \in A_0$, and since $I(A_0, B_0) = \emptyset$, we have $b \notin B_0$ and $b = b_1$.

Since $a_1 \leq a \leq a_0$ and $b_0 \leq b \leq b_1$, we have $U_{a_1} \subset U_a \subset U_{a_0}$ and $F_{b_0} \supset F_b \supset F_{b_1}$. Therefore, we have

$$\begin{aligned}\left(F_{(a_1,b_0)} \cup F_{(a_0,b_1)}\right) \cap F_{(a,b)} &= ((U_a \cap U_{a_1})^o \times F_b) \cup (U_a^o \times (F_b \cap F_{b_1})) \\ &= (U_{a_1}^o \times F_b) \cup (U_a^o \times F_{b_1}) \\ &= \begin{cases} U_{a_1}^o \times F_b \simeq *, & a = a_1 \\ U_a^o \times F_{b_1} \simeq *, & b = b_1. \end{cases}\end{aligned}$$

□

Corollary 4.15. *We consider the same situation as in Lemma* 4.12.

If $A_1 = B_1 = \emptyset$, *then* $U_{a_0} \cup F_{b_0}$ *is weak homotopy equivalent to a wedge of spheres of dimension at most* 2.

In particular, if $F_{b_0} \cap F_{a_0} \subset \mathrm{mxl}(X)$ *and* $U_{a_0} \cap U_{b_0} \subset \mathrm{mnl}(X)$, *then this holds.*

Proof. If $A_1 = B_1 = \emptyset$, then $I(U_{a_0}, F_{b_0})$ is homotopy equivalent to $\mathrm{mxl}(I(U_{a_0}, F_{b_0})) \cup \mathrm{mnl}(I(U_{a_0}, F_{b_0}))$, whose height is at most 1. Therefore, $U_{a_0} \cup F_{b_0} \simeq_w \mathbb{S}(I(U_{a_0}, F_{b_0}))$ is weak homotopy equivalent to a wedge of spheres of dimension at most 2.

If $F_{b_0} \cap F_{a_0} \subset \mathrm{mxl}(X)$, then $B_1 = \emptyset$. If $U_{a_0} \cap U_{b_0} \subset \mathrm{mnl}(X)$, then $A_1 = \emptyset$. □

Corollary 4.16. *We consider the same situation as in Lemma* 4.12.

If $A_1 = \emptyset$, *then* $I(U_{a_0}, F_{b_0})$ *splits into spaces smaller than* F_{b_1}, *and hence, so does* $U_{a_0} \cup F_{b_0}$. *If* $B_1 = \emptyset$, *then* $I(U_{a_0}, F_{b_0})$ *and* $U_{a_0} \cup F_{b_0}$ *split into spaces smaller than* U_{a_1}.

Proof. We consider the case where $A_1 = \emptyset$.

In this case, $I(U_{a_0}, F_{b_0})$ is homotopy equivalent to $F_{(a_0,b_1)} \cup \mathrm{mxl} \cup \mathrm{mnl}$, and each connected component is weak homotopy equivalent to a wedge of some copies of S^1 and $\mathbb{S}(F_{(a_0,b_1)} \cap F_{(a,b)})$ for some $(a,b) \in I(U_{a_0}, F_{b_0})$. We have

$$\mathbb{S}\left(F_{(a_0,b_1)} \cap F_{(a,b)}\right) = \mathbb{S}\left(U_a^o \times (F_b \cap F_{b_1})\right) \simeq \mathbb{S}\left(F_b \cap F_{b_1}\right)$$

and $|F_b \cap F_{b_1}| < |F_{b_1}|$ or $F_b \cap F_{b_1} \simeq *$. □

Corollary 4.17. *Let X be a connected finite T_0 space. Suppose the following holds:*

1. *All the elements of $\mathcal{B} - \mathrm{mxl}(\mathcal{B})$ are up beat points of $\mathcal{B}$.*
2. *One of the connected components of $\mathcal{B}$ is a chain. Let $\mathcal{B}_0$ be a connected component which is a chain.*
3. *There exists a point $a_0 \in \mathrm{mxl}(X)$ such that $\mathcal{B}_0 - U_{a_0} \neq \emptyset$. We put $b_0 = \min(\mathcal{B}_0 - U_{a_0})$.*

Then, $U_{a_0} \cup F_{b_0}$ is weak homotopy equivalent to a wedge of spheres of dimension at most 2.

Proof. We use Lemma 4.12.

Since X is connected, $\mathrm{mxl}(X) \cap \mathrm{mnl}(X) = \emptyset$, and since $a_0 \in \mathrm{mxl}(X)$, we have $a_0 \notin \mathrm{mnl}(X)$. Since $b_0 \in \mathcal{B}$, we have $b_0 \notin \mathrm{mxl}(X)$. Clearly, $a_0 \not\leq b_0$.

Since $a_0 \in \mathrm{mxl}(X)$, we have $B_1 = (F_{b_0} - \mathrm{mxl}(X)) \cap F_{a_0} = \emptyset$. Since $b_0 \in \mathcal{B}$, we have

$$\begin{aligned}(U_{a_0} - \mathrm{mnl}(X)) \cap U_{b_0} &\subset U_{a_0} \cap U_{b_0} \cap \mathcal{B} \\ &\subset U_{a_0} \cap \mathcal{B}_0 \\ &\subset (U_{a_0} - \mathrm{mnl}(X)) \cap U_{b_0}\end{aligned}$$

and since $b_0 = \min(\mathcal{B}_0 - U_{a_0}) \in \mathcal{B}_0$, $\mathcal{B} - \mathcal{B}_0$ and $\mathcal{B}_0$ are incomparable, and $\mathcal{B}_0 - U_{a_0}$ is an up set of $\mathcal{B}_0$, we have

$$\begin{aligned}F_{b_0} \cap \mathcal{B} = F_{b_0} \cap \mathcal{B}_0 &= \mathcal{B}_0 - U_{a_0}, \\ F_{b_0} &\subset (\mathcal{B}_0 - U_{a_0}) \cup \mathrm{mxl}(X).\end{aligned}$$

Therefore, we have

$$\begin{aligned}A_0 &= (U_{a_0} - \mathrm{mnl}(X)) - U_{b_0}, \\ A_1 &= (U_{a_0} - \mathrm{mnl}(X)) \cap U_{b_0} = \mathcal{B}_0 \cap U_{a_0}, \\ B_0 &= F_{b_0} - \mathrm{mxl}(X) = \mathcal{B}_0 - U_{a_0}, \\ B_1 &= \emptyset.\end{aligned}$$

We show that points of $A_0 - \{a_0\}$ are up beat points of U_{a_0}. Suppose $x \in A_0 - \{a_0\}$. Since $A_0 - \{a_0\} \subset \mathcal{B}$, x is either maximal or up beat point of $\mathcal{B}$ by the assumption. Note that $\hat{F}_x = \left(\hat{F}_x \cap \mathcal{B}\right) \cup \left(\hat{F}_x \cap \mathrm{mxl}(X)\right)$.

If x is a maximal element of $\mathcal{B}$, then $\hat{F}_x \subset \mathrm{mxl}(X)$ and

$$a_0 \in \hat{F}_x \cap U_{a_0} \subset \mathrm{mxl}(X) \cap U_{a_0} = \{a_0\},$$

and hence $\hat{F}_x \cap U_{a_0} = \{a_0\}$. Therefore, x is an up beat point of U_{a_0}.

If x is an up beat point of $\mathcal{B}$, then we put $\hat{x} = \min(\hat{F}_x \cap \mathcal{B})$. We have

$$\hat{F}_x \cap U_{a_0} = \hat{F}_x \cap (\mathcal{B} \cup \mathrm{mxl}(X)) \cap U_{a_0}$$

$$= \left(\hat{F}_x \cap \mathcal{B} \cap U_{a_0}\right) \cup \{a_0\}.$$

If $\hat{x} \notin U_{a_0}$, then $\left(\hat{F}_x \cap \mathcal{B}\right) \cap U_{a_0} = \emptyset$ and $\hat{F}_x \cap U_{a_0} = \{a_0\}$. If $\hat{x} \in U_{a_0}$, then $\hat{x} = \min(\hat{F}_x \cap U_{a_0})$. In any case, x is an up beat point of U_{a_0}.

Because $B_0 = F_{b_0} - \mathrm{mxl}(X) = \mathcal{B}_0 - U_{a_0}$ is a chain and $F_{b_0} \subset (\mathcal{B}_0 - U_{a_0}) \cup \mathrm{mxl}(X)$, every element of $B_0 - \{b_0\}$ is a down beat point of F_{b_0}.

We show that $I(A_0, B_0) = \emptyset$. Since

$$(A_0 - \{a_0\}) \cap \mathcal{B}_0 = (U_{a_0} \cap \mathcal{B} - U_{b_0}) \cap \mathcal{B}_0 = (U_{a_0} \cap \mathcal{B}_0) - U_{b_0} = \emptyset,$$

we see that $A_0 - \{a_0\} \subset \mathcal{B} - \mathcal{B}_0$, and since $a_0 \in \mathrm{mxl}(X)$ and $a_0 \notin B_0$, we have

$$I(A_0, B_0) = I(A_0 - \{a_0\}, B_0) \subset I(\mathcal{B} - \mathcal{B}_0, \mathcal{B}_0) = \emptyset.$$

Finally, if $A_1 \neq \emptyset$, then $A_1 = \mathcal{B}_0 \cap U_{a_0}$ is a nonempty finite chain and so there exists $\max A_1$.

Therefore, the assumption of Lemma 4.12 holds and $B_1 = \emptyset$.

If $A_1 = \emptyset$, then by Corollary 4.15, $U_{a_0} \cup F_{b_0}$ is weak homotopy equivalent to a wedge of spheres of dimension at most 2.

If $A_1 \neq \emptyset$, then by Corollary 4.16, $I(U_{a_0}, F_{b_0})$ splits into some copies of S^1 and $\mathbb{S}(U_a \cap U_{a_1})$ for some a. Since $U_{a_1} \subset \mathcal{B}_0 \cup \mathrm{mnl}(X)$ and $\mathcal{B}_0$ is a chain, we see that $U_a \cap U_{a_1}$ is homotopy equivalent to a discrete space. Therefore $U_{a_0} \cup F_{b_0}$ is weak homotopy equivalent to a wedge of spheres of dimension at most 2. $\square$

5. Some small finite spaces

Definition 5.1. We denote the finite space of Fig. 2 by S^1_n, that is, the underlying set of S^1_n is the $2n$-element set $S^1_n = \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ and the order is given by $b_i < a_i$ and $b_i < a_{i+1}$, where we consider $a_n = a_0$. Clearly, $S^1_n \simeq_w S^1$.

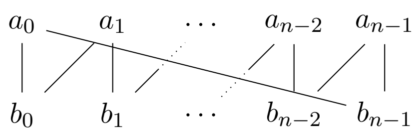
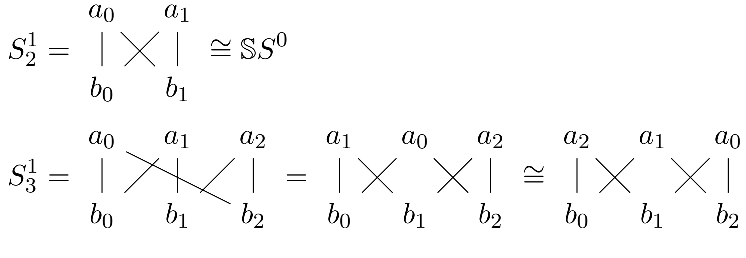


Figure 2: S^1_n

Example 5.2.



It is straightforward to see the following:

Lemma 5.3. *Any connected* $(3,3)$*-bipartite graph with no degree* 2 *vertex is isomorphic to one of the graphs in Fig.* 3*.*

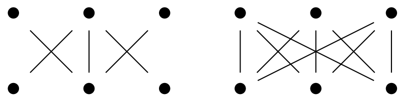


Figure 3: $(3,3)$-bipartite graphs with no degree 2 vertex.

Lemma 5.4. *Any* $(4,4)$*-bipartite graph whose all the vertices have degree* 2 *is isomorphic to* S^1_4 *or* $S^1_2 \amalg S^1_2$*.*

We list up connected finite T_0 spaces of cardinality 4 or less.

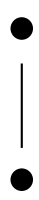
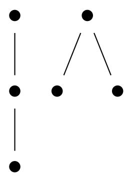
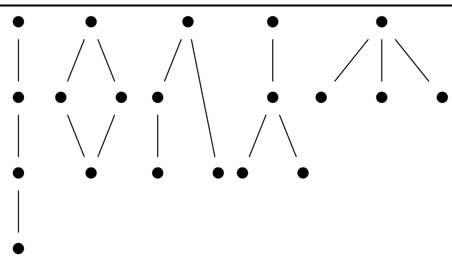
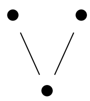
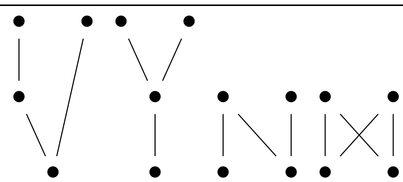
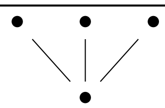
$\lvert\mathrm{mxl}(X)\rvert$ \ $\lvert X\rvert$	1	2	3	4
1				
2				
3				

Table 1: Connected finite T_0 spaces of cardinality at most 4.

6. $\lvert\mathrm{mxl}(X)\rvert \leq 3$

In this section, we assume that $|X| > 1$ and X is a connected minimal finite space, that is, X is a connected finite T_0 space without beat points. In this case, $\mathrm{mxl}(X) \cap \mathrm{mnl}(X) = \emptyset$ and $|X| \geq 4$.

Following Cianci-Ottina [3], we use the following notations:

Definition 6.1. We put

$$\begin{aligned}
\mathcal{B} &= X - \mathrm{mxl}(X) - \mathrm{mnl}(X), & l &= |\mathcal{B}|\,, \\
m &= |\mathrm{mxl}(X)|\,, & n &= |\mathrm{mnl}(X)|\,, \\
m' &= |\mathrm{mxl}(\mathcal{B})|\,, & n' &= |\mathrm{mnl}(\mathcal{B})|\,,
\end{aligned}$$

and for $x \in X$ and $a \in \mathrm{mxl}(X)$, we put

$$\begin{aligned}
\alpha_x &= |\mathrm{mxl}(F_x)| = |\mathrm{mxl}(X) \cap F_x|\,, \\
\beta_x &= |\mathrm{mnl}(U_x)| = |\mathrm{mnl}(X) \cap U_x|\,, \\
\gamma_a &= |U_a \cap \mathrm{mxl}(\mathcal{B})|
\end{aligned}$$

Since X does not have beat points, $\alpha_x \geq 2$ if $x \notin \mathrm{mxl}(X)$ and $\beta_x \geq 2$ if $x \notin \mathrm{mnl}(X)$.

Note that

$$|I(\mathrm{mxl}(\mathcal{B}), \mathrm{mxl}(X))| = \sum_{b \in \mathrm{mxl}\,\mathcal{B}} \alpha_b = \sum_{a \in \mathrm{mxl}(X)} \gamma_a$$

because

$$\begin{aligned}
I(\mathrm{mxl}(\mathcal{B}), \mathrm{mxl}(X)) &= \{(b, a) \in \mathrm{mxl}(\mathcal{B}) \times \mathrm{mxl}(X) \mid b \leq a\} \\
&= \bigcup_{b \in \mathrm{mxl}(\mathcal{B})} p_1^{-1}(b) = \bigcup_{b \in \mathrm{mxl}(\mathcal{B})} \{b\} \times (\mathrm{mxl}(X) \cap F_b) \\
&= \bigcup_{a \in \mathrm{mxl}(X)} p_2^{-1}(a) = \bigcup_{a \in \mathrm{mxl}(X)} (U_a \cap \mathrm{mxl}(\mathcal{B})) \times \{a\}.
\end{aligned}$$

We study the weak homotopy type of X of $m \leq 3$.

Lemma 6.2. *If $m \leq 2$, then X splits into smaller spaces.*

Proof. If $m = 1$, then X has the maximum and is contractible.
If $m = 2$ and $\mathrm{mxl}(X) = \{a, b\}$, then $X = U_a \cup U_b \simeq_w \mathbb{S}(U_a \cap U_b)$ and $U_a \cap U_b \simeq *$ or $|U_a \cap U_b| < |X|$. □

Lemma 6.3. *If $m = 3$ and $m' = 2$, then X splits into smaller spaces.*

Proof. Since $\sum_{b \in \mathrm{mxl}(\mathcal{B})} \alpha_b \geq 2m' = 4 > 3 = 1 \cdot m$, there exists $a \in \mathrm{mxl}(X)$ such that $\gamma_a > 1$, namely, $\gamma_a = 2 = m'$. Therefore, we have

$$X = U_a \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$$

and by Corollary 3.14, X splits into smaller spaces. □

Lemma 6.4. *If $m = 3$ and there exist two points $a_0, a_2 \in \mathrm{mxl}(X)$ such that $U_{a_0} \cap U_{a_2}$ is homotopically trivial, then X splits into smaller spaces.*

Proof. Suppose $\mathrm{mxl}(X) = \{a_0, a_1, a_2\}$.

Since $U_{a_0} \cup U_{a_2}$ is a down set, by Corollary 4.3, we see that

$$\begin{aligned} U_{a_0} \cup U_{a_2} &\simeq_w \mathbb{S}\left(U_{a_0} \cap U_{a_2}\right) \simeq_w * \\ X &= U_{a_1} \cup \left(U_{a_0} \cup U_{a_2}\right) \\ &\simeq_w \mathbb{S}\left(U_{a_1} \cap \left(U_{a_0} \cup U_{a_2}\right)\right) \end{aligned}$$

and $U_{a_1} \cap \left(U_{a_0} \cup U_{a_2}\right) \subsetneq U_{a_1}$. ☐

Remark 6.5. In fact, we can show that $U_{a_0} \cup U_{a_2} = X - \{a_1\}$, $U_{a_1} \cap \left(U_{a_0} \cup U_{a_2}\right) = \hat{U}_{a_1}$ and $X \simeq_w \mathbb{S}(\hat{U}_{a_1})$.

More generally, if X is a connected minimal finite space and $a_0 \in \mathrm{mxl}(X)$, then

$$\bigcup_{a \in \mathrm{mxl}(X) - \{a_0\}} U_a = X - \{a_0\}.$$

Actually, since $a_0 \notin U_a$, we have

$$\bigcup_{a \in \mathrm{mxl}(X) - \{a_0\}} U_a \subset X - \{a_0\}.$$

On the other hand, we have $X - \{a_0\} = (\mathrm{mxl}(X) - \{a_0\}) \cup \mathcal{B} \cup \mathrm{mnl}(X)$, and clearly, $\mathrm{mxl}(X) - \{a_0\}$ is contained in the left hand side.

If $b \in \mathrm{mxl}(\mathcal{B})$, then $\left|\hat{F}_b\right| \geq 2$ because b is not a beat point. Hence, there exists $a \in \mathrm{mxl}(X) - \{a_0\}$ such that $b < a$. Therefore, $\mathcal{B}$ is contained in the left hand side.

If $c \in \mathrm{mnl}(X)$, then $\hat{F}_c \neq \emptyset$ since X is connected. If $\hat{F}_c \cap \mathcal{B} \neq \emptyset$, then c is contained in the left hand side. If $\hat{F}_c \cap \mathcal{B} = \emptyset$, then $\hat{F}_c \subset \mathrm{mxl}(X)$, and since $\left|\hat{F}_c\right| \geq 2$, c is contained in the left hand side.

Lemma 6.6. *Suppose* $\mathrm{mxl}(X) = \{a_0, a_1, a_2\}$*. If* $U_{a_0} \cap U_{a_1}$ *is connected and* $\hat{U}_{a_2}$ *is weak homotopy equivalent to a simplicial complex of dimension at most* 1*, then* X *splits into smaller spaces.*

Proof. Note that $\widehat{C}_{a_2} = \widehat{U}_{a_2}$ and $X - \{a_2\} = U_{a_0} \cup U_{a_1}$.

Since $U_{a_0} \cap U_{a_1}$ is connected, $\mathcal{K}\left(U_{a_0} \cup U_{a_1}\right) \simeq S\mathcal{K}\left(U_{a_0} \cap U_{a_1}\right)$ is simply connected. Hence the inclusion

$$\mathcal{K}\left(\hat{U}_{a_2}\right) = \mathcal{K}\left(\hat{C}_{a_2}\right) \to \mathcal{K}\left(X - \{a_2\}\right) = \mathcal{K}\left(U_{a_0} \cup U_{a_1}\right)$$

is null homotopic. Therefore, by Proposition 2.24, we have

$$X \simeq_w (X - \{a_2\}) \vee \mathbb{S}(\hat{U}_{a_2}).$$

☐

Lemma 6.7. *If* $m = m' = 3$*, then* X *splits into smaller spaces, or* $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ *is isomorphic to* S^1_3*.*

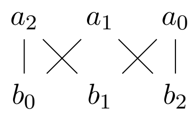


Figure 4: S_3^1

Proof. Since $|\mathrm{mxl}(\mathcal{B})| = m' = 3$, if there exists an element $a \in \mathrm{mxl}(X)$ such that $\gamma_a = |U_a \cap \mathrm{mxl}(\mathcal{B})| = 3$, then $X = U_a \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$ and X splits into smaller spaces by Corollary 3.14.

Assume that $\gamma_a \leq 2$ for all $a \in \mathrm{mxl}(X)$. Recall that $\alpha_b = |F_b \cap \mathrm{mxl}(X)| \geq 2$ for all $b \in \mathrm{mxl}(\mathcal{B})$. Since

$$2 \cdot 3 \geq \sum_{a \in \mathrm{mxl}(X)} \gamma_a = \sum_{b \in \mathrm{mxl}(\mathcal{B})} \alpha_b \geq 2 \cdot 3,$$

we see that $\gamma_a = 2$ for all $a \in \mathrm{mxl}(X)$ and $\alpha_b = 2$ for all $b \in \mathrm{mxl}(\mathcal{B})$. Now, it is straightforward to see that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is isomorphic to S_3^1.

$\square$

Lemma 6.8. *If $m = m' = 3$ and there exists $b \in \mathcal{B}$ such that $\left|\hat{F}_b^{\mathcal{B}}\right| = \left|\hat{F}_b \cap \mathcal{B}\right| = 2$, then X splits into smaller spaces.*

Proof. By Lemma 6.7, we may assume that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B}) = S_3^1$ of Fig. 4.

We show that $\hat{F}_b^{\mathcal{B}} \subset \mathrm{mxl}(\mathcal{B})$. Since $\hat{F}_b^{\mathcal{B}} \neq \emptyset$, we have $b \notin \mathrm{mxl}(\mathcal{B})$ and $\hat{F}_b^{\mathcal{B}} \cap \mathrm{mxl}(\mathcal{B}) \neq \emptyset$.

Suppose that $\hat{F}_b^{\mathcal{B}} \not\subset \mathrm{mxl}(\mathcal{B})$. Then we have $\left|\hat{F}_b^{\mathcal{B}} \cap \mathrm{mxl}(\mathcal{B})\right| = 1$. We may assume that $b_0 \in F_b$ and $b_1, b_2 \notin F_b$. Suppose $\hat{F}_b^{\mathcal{B}} = \{b', b_0\}$. Since $b < b' \in \mathcal{B}$ and $b' \notin \mathrm{mxl}(\mathcal{B})$, we have $b < b' < b_0$. Since $b' \not< b_1, b_2$ and b' is not a beat point, we have $b' < a_0$. Therefore, we have $\hat{F}_b = \{b', b_0, a_0, a_1, a_2\}$ and $b' = \min \hat{F}_b$. Then b is a beat point, which contradicts to the assumption.

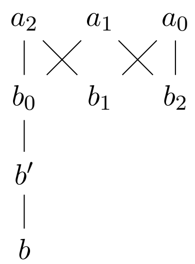


Figure 5: $\widehat{F}_b \not\subset \mathrm{mxl}(B)$.

Therefore, $\hat{F}_b^{\mathcal{B}} \subset \mathrm{mxl}(\mathcal{B})$.

We show that b is a weak beat point.

We may suppose $\hat{F}_b^{\mathcal{B}} = \{b_0, b_2\}$. Then, since $\mathrm{mxl}(X) = \{a_0, a_1, a_2\} \subset F_b$, we have

$$\hat{F}_b = (\hat{F}_b \cap \mathcal{B}) \cup \mathrm{mxl}(X)$$

$$= \{a_0, a_1, a_2, b_0, b_2\}$$

which is contractible, and so b is a weak beat point.

Therefore $X \simeq_w X - \{b\}$ hence X splits into smaller spaces. $\square$

Lemma 6.9. *If $m = 3$ and $m' = 4$, then X splits into smaller spaces, or* $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ *is isomorphic to one of spaces of Fig. 6.*

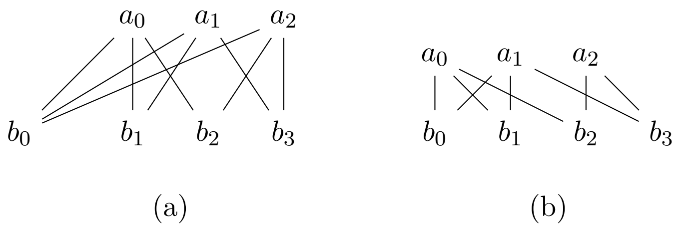


Figure 6: $m = 3$ and $m' = 4$.

Proof. This can be shown similarly to Lemma 6.7. If there exists an element $a \in \mathrm{mxl}(X)$ such that $\gamma_a = 4$, then $X = U_a \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$ and X splits into smaller spaces. If $\gamma_a \leq 3$ for all $a \in \mathrm{mxl}(X)$, then, since

$$3 \cdot 3 \geq \sum_{a \in \mathrm{mxl}(X)} \gamma_a = \sum_{b \in \mathrm{mxl}(\mathcal{B})} \alpha_b \geq 2 \cdot 4,$$

we see that $\gamma_a = 2$ or 3 for all $a \in \mathrm{mxl}(X)$ and $\gamma_a = 2$ for at most one of them, and that $\alpha_b = 2$ or 3 for all $b \in \mathrm{mxl}(\mathcal{B})$ and $\alpha_b = 3$ for at most one of them. If there exists an element $a \in \mathrm{mxl}(X)$ such that $\gamma_a = 2$, then $\alpha_b = 2$ for all $b \in \mathrm{mxl}(\mathcal{B})$ and we easily see that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is isomorphic to the space (b) of Fig. 6. Otherwise, $\gamma_a = 3$ for all a and there exists one element $b \in \mathrm{mxl}(\mathcal{B})$ such that $\alpha_b = 3$, and we see that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is isomorphic to the space (a) of Fig. 6. $\square$

7. $|\mathcal{B}| \leq 5$

We assume that X is a connected minimal finite space of $|X| > 1$ in this section, too, and we study the weak homotopy type of X of $l = |\mathcal{B}| \leq 5$.

Lemma 7.1. *If $l = 0$, namely, $X = \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$, then X is weak homotopy equivalent to a wedge of S^1's.*

Proof. In this case, $\mathcal{K}(X)$ is a connected 1-dimensional simplicial complex. $\square$

Lemma 7.2. *If $m' = 1$, namely, if there exists $\max \mathcal{B}$, then X splits into smaller spaces. In particular, if $l = 1$, then X splits into smaller spaces.*

Proof. Let $b = \max \mathcal{B}$. Clearly, we have $X = U_b \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$, and the result follows from Corollary 3.14. $\square$

Lemma 7.3. *If there exists an element $b \in \mathcal{B}$ such that $\mathcal{B} - U_b$ is a chain, then X splits into smaller spaces.*

In particular, if $l = 2$, then X splits into smaller spaces.

Proof. If $\mathcal{B} - U_b = \emptyset$, then $b = \max \mathcal{B}$ and the result follows from Lemma 7.2.

Assume that $\mathcal{B} - U_b \neq \emptyset$ and put

$$C = \mathcal{B} - U_b, \quad c_0 = \max C, \quad c_1 = \min C, \quad U = \bigcup_{a \in F_b} U_a.$$

By the assumption, C is a nonempty finite chain.

If $C - U = \emptyset$, namely, if $C \subset U$, then $c_0 \in U$, and hence there exists an element $a \in F_b$ such that $c_0 \in U_a$. Since $c_0 = \max C$, we have $C \subset U_a$, and since $a \in F_b$, we have $U_b \subset U_a$. Therefore, $\mathcal{B} \subset U_b \cup C \subset U_a$ and

$$X = U_a \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

Hence, X splits into smaller spaces.

If $C \cap U = \emptyset$, then $c_1 \notin U$ and so $F_{c_1} \cap F_b = \emptyset$. Hence $(U_b \cup F_{c_1}) \cap \hat{F}_b = \emptyset$, and since $b \in \mathcal{B}$, $\hat{F}_b \neq \emptyset$, and so $U_b \cup F_{c_1} \subsetneqq X$. Since $\mathcal{B} \subset U_b \cup C \subset U_b \cup F_{c_1}$, we have

$$X = U_b \cup F_{c_1} \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

Therefore, X splits into smaller spaces by Corollary 3.13.

Suppose $C - U \neq \emptyset$ and $C \cap U \neq \emptyset$, and put $d_1 = \min(C - U)$ and $d_0 = \max(C \cap U)$. Since $d_0 \in U$, there exists an element $a \in F_b$ such that $d_0 \in U_a$. Since $d_0 = \max(C \cap U)$, we have $C \cap U \subset U_a$, since $a \in F_b$, we have $U_b \subset U_a$, and since $d_1 = \min(C - U)$, we have $C - U \subset F_{d_1}$. Therefore, $\mathcal{B} \subset U_b \cup C \subset U_a \cup F_{d_1}$ and we see that

$$X = U_a \cup F_{d_1} \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

Since $d_1 \notin U$, we have $F_{d_1} \cap F_b = \emptyset$ and so $F_b - U_a \subset F_b \subset F_{d_1}^c$. Therefore, we have

$$F_b - U_a \subset U_a^c \cap F_{d_1}^c = (U_a \cup F_{d_1})^c.$$

Since X is a minimal finite space and b is not a maximal element, by Lemma 2.14, $F_b - U_a \neq \emptyset$ and hence $U_a \cup F_{d_1} \subsetneq X$. Therefore, by Corollary 3.13, X splits into smaller spaces. ☐

Corollary 7.4. *If $l \leq 5$ and $m' \leq 2$, then X splits into smaller spaces.*

Proof. If $m' < 2$, the result follows from Lemmas 7.1 and 7.2.

Suppose $m' = 2$ and $\mathrm{mxl}(\mathcal{B}) = \{b_1, b_2\}$. Since $|\mathcal{B}| = l \leq 5$, if $|U_{b_1} \cap \mathcal{B}| \geq 3$, then $\mathcal{B} - U_{b_1}$ is a chain, and if $|U_{b_1} \cap \mathcal{B}| \leq 2$, then $\mathcal{B} - U_{b_2}$ is a chain. The result follows from Lemma 7.3. ☐

Lemma 7.5. *Assume that all the connected components of $\mathcal{B}$ are chains.*

If $m' \leq 3$ and $m \leq 5$, or $m' \leq 5$ and $m \leq 3$, then X splits into smaller spaces.

Proof. The case $m \leq 2$ follows from Lemma 6.2, the case $m' = 0$ follows from Lemma 7.1, the case $m' = 1$ follows from Lemma 7.2, and the case $m' = 2$ follows from Lemma 7.3. Hence, we may assume $m' \geq 3$ and $m \geq 3$, that is, we may assume $m' = 3$ and $3 \leq m \leq 5$, or $3 \leq m' \leq 5$ and $m = 3$. In these cases, since

$$\left(\sum_{b \in \mathrm{mxl}(\mathcal{B})} \alpha_b \right) - (m'-2)m \geq 2m' - (m'-2)m = 4 - (m'-2)(m-2) > 0,$$

there exists an element $a \in \mathrm{mxl}(X)$ such that $\gamma_a = |U_a \cap \mathrm{mxl}(\mathcal{B})| > m' - 2$.

If $\gamma_a = m'$, then $\mathrm{mxl}(\mathcal{B}) \subset U_a$ and

$$X = U_a \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X),$$

therefore X splits into smaller spaces.

When $\gamma_a = m' - 1$, suppose $\mathrm{mxl}(\mathcal{B}) - U_a = \{b\}$ and let $\mathcal{B}_0$ be the connected component of $\mathcal{B}$ containing b, then $\mathcal{B}_0 - U_a$ is a nonempty chain. Put $b_0 = \min(\mathcal{B}_0 - U_a)$, then we have

$$X = U_a \cup F_{b_0} \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$$

and by Corollary 4.17, $U_a \cup F_{b_0}$ is weak homotopy equivalent to a wedge of spheres, hence X splits into smaller spaces by Corollary 3.13. □

Corollary 7.6. *If $l = 3$ and one of the following holds, then X splits into smaller spaces.*

1. $m' \leq 2$.

2. $m \leq 5$.

Proof. Part 1 follows from Corollary 7.4.

Consider part 2. Note that $|\mathcal{B}| = l = 3$.

If $\mathcal{B}$ is connected, then $\mathcal{B}$ has maximum or minimum. If $\max \mathcal{B}$ exists, then the result follows from Lemma 7.2. If $\min \mathcal{B}$ exists, then the opposite X^o splits into smaller spaces, and so does X since $X \simeq_w X^o$.

If $\mathcal{B}$ is not connected, then all the connected components of $\mathcal{B}$ are chains and the result follows from Lemma 7.5. □

Corollary 7.7. *If $|X| \leq 13$ and $l \leq 3$, then X splits into smaller spaces.*

Proof. By considering the opposite if necessary, we may assume $m \leq n$. We also may assume $m \geq 3$ by Lemma 6.2. If $l \leq 2$, then the result follows from Lemmas 7.1 to 7.3. If $3 \leq m \leq n$ and $l = 3$, then we have

$$13 \geq |X| = l + m + n \geq l + 2m = 3 + 2m.$$

Therefore $m \leq 5$ and the result follows from Corollary 7.6. □

Corollary 7.8. *If $l = 4$ and one of the following holds, then X splits into smaller spaces. In particular, if $l = 4$ and $m \leq 3$, then X splits into smaller spaces.*

1. $m' \leq 2$.

2. $m' = 3$ *and* $m \leq 5$.

3. $m' = 4$ *and* $m \leq 3$.

Proof. Part 1 follows from Corollary 7.4. Part 3 follows from Lemma 7.5 since $\mathcal{B}$ is an antichain if $l = m'$.

If $l = 4$ and $m' = 3$, then $\mathcal{B}$ is isomorphic to one of spaces of Fig. 7.

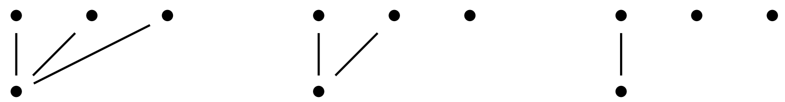


Figure 7: $\mathcal{B}$ for $l = 4$ and $m' = 3$.

Since $n' \leq 2$ in the first two cases, by taking the opposite, we see that X splits into smaller spaces. In the last case, every connected component of $\mathcal{B}$ is a chain and the result follows from Lemma 7.5. $\square$

We consider the case $l = 5$.

Lemma 7.9. *If* $l = 5$, $m' = 3$ *and* $m = 3$, *then* X *splits into smaller spaces.*

Proof. Note that, if $n' = |\mathrm{mnl}(\mathcal{B})| \leq 2$, by taking the opposite, we see that X splits into smaller spaces by Corollary 7.4.

If $\mathcal{B}$ is connected, then $\mathrm{mxl}(\mathcal{B}) \cap \mathrm{mnl}(\mathcal{B}) = \emptyset$. Since $|\mathrm{mxl}(\mathcal{B})| = m' = 3$, $|\mathrm{mnl}(\mathcal{B})| \leq 2$ and X splits into smaller spaces.

Suppose $\mathcal{B}$ is not connected. By Lemma 6.7, we may assume that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is isomorphic to S^1_3. Since $|\mathrm{mxl}(\mathcal{B})| = 3$, the number of connected components is at most 3.

(a) The case where $\mathcal{B}$ has 3 connected components.

Since the cardinalities of each component are

$$5 = 1 + 1 + 3 = 1 + 2 + 2,$$

there exists a component which consists of a single element, say $\{b_0\}$. Since $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B}) \cong S^1_3$, there exists an element $a_0 \in \mathrm{mxl}(X)$ such that $b_0 \notin U_{a_0}$. Then we have

$$X = U_{a_0} \cup F_{b_0} \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

Since all the elements of $\mathcal{B} - \mathrm{mxl}(\mathcal{B})$ are up beat points of $\mathcal{B}$, X splits into smaller spaces by Corollary 4.17.

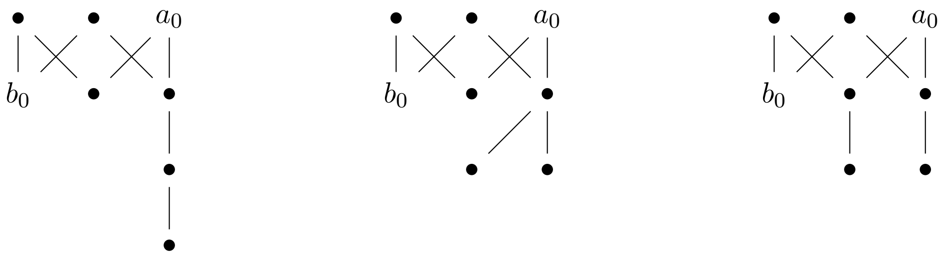


Figure 8: $\mathcal{B}$ with 3 components for $l = 5$, $m = m' = 3$.

(b) The case where $\mathcal{B}$ has 2 connected components.

The cardinalities of each component are

$$5 = 1 + 4 = 2 + 3.$$

In the latter case, $|\mathrm{mnl}(\mathcal{B})| = 2$ and the result follows. Consider the former case. Let $\mathcal{B}_0 = \{b_0\}$ be the component with a single element, $\mathcal{B}_1$ the component with $|\mathcal{B}_1| = 4$, and $\mathrm{mxl}(\mathcal{B}_1) = \{b_1, b_2\}$. Since $\mathcal{B}_1$ is connected and $|\mathrm{mxl}(\mathcal{B}_1)| = 2$, we see that $|\mathrm{mnl}(\mathcal{B}_1)| \leq 2$. If $|\mathrm{mnl}(\mathcal{B}_1)| = 1$, then $|\mathrm{mnl}(\mathcal{B})| = 2$ and the result follows. Suppose $|\mathrm{mnl}(\mathcal{B}_1)| = 2$. Since $\mathcal{B}_1$ is connected, there exists an element $b \in \mathrm{mnl}(\mathcal{B}_1)$ such that $b_1, b_2 \in F_b$ and, since $b_0 \notin F_b$, $\hat{F}_b^{\mathcal{B}} = \{b_1, b_2\}$. The result follows from Lemma 6.8.

$\square$

Lemma 7.10. *If $l = 5$, $m' = 4$ and $m = 3$, then X splits into smaller spaces.*

Proof. In this case, $\mathcal{B}$ is isomorphic to one of spaces of Fig. 9.

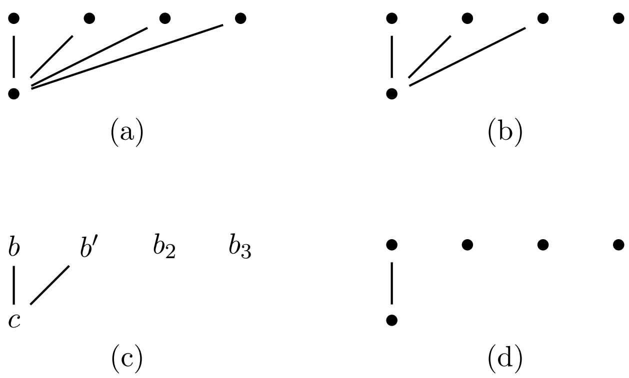


Figure 9: $\mathcal{B}$ for $l = 5$ and $m' = 4$.

In the cases (a) and (b), since $|\mathrm{mnl}\,\mathcal{B}| \leq 2$, X^o splits into smaller spaces by Corollary 7.4 and so does X. The case (d) follows from Lemma 7.5.

Consider the case (c). Since X is minimal and $m = |\mathrm{mxl}(X)| = 3$, we have $F_{b_2} \cap F_{b_3} \cap \mathrm{mxl}(X) \neq \emptyset$ and hence, there exists an element $a_0 \in \mathrm{mxl}(X)$ such that $b_2, b_3 \in U_{a_0}$. See Fig. 10.

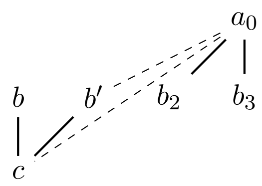


Figure 10: The case (c).

We may assume that $\gamma_{a_0} = |U_{a_0} \cap \mathrm{mxl}(\mathcal{B})| = 2$ or 3 (see Lemma 6.9).

If $\gamma_{a_0} = 2$, then we have

$$X = U_{a_0} \cup F_c \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

We put

$$\begin{aligned} A_0 &= (U_{a_0} - \mathrm{mnl}(X)) - U_c = \{a_0, b_2, b_3\}, & A_1 &= (U_{a_0} - \mathrm{mnl}(X)) \cap U_c = \emptyset \text{ or } \{c\}, \\ B_0 &= (F_c - \mathrm{mxl}(X)) - F_{a_0} = \{b, b', c\}, & B_1 &= (F_c - \mathrm{mxl}(X)) \cap F_{a_0} = \emptyset, \end{aligned}$$

then b_2 and b_3 are up beat points of U_{a_0}, b and b' are down beat points of F_c, and $I(A_0, B_0) = \emptyset$. If $A_1 \neq \emptyset$, then $c = \max A_1$. Therefore X splits into smaller spaces by Corollaries 4.15 and 4.16.

If $\gamma_{a_0} = 3$, we may assume that $b \notin U_{a_0}$ and $b' \in U_{a_0}$, and we have

$$X = U_{a_0} \cup F_b \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

We put

$$\begin{aligned} A_0 &= (U_{a_0} - \mathrm{mnl}(X)) - U_b = \{a_0, b', b_2, b_3\}, & A_1 &= (U_{a_0} - \mathrm{mnl}(X)) \cap U_b = \{c\}, \\ B_0 &= (F_b - \mathrm{mxl}(X)) - F_{a_0} = \{b\}, & B_1 &= (F_b - \mathrm{mxl}(X)) \cap F_{a_0} = \emptyset, \end{aligned}$$

then b', b_2, and b_3 are up beat points of U_{a_0}, $B_0 \backslash \{b\} = \emptyset$, $I(A_0, B_0) = \emptyset$, and $c = \max A_1$. Therefore X splits into smaller spaces by Corollary 4.16. $\square$

Corollary 7.11. *If $l = 5$ and one of the following holds, then X splits into smaller spaces.*

1. $m' \leq 2$
2. $m \leq 3$

Proof. Part 1 follows from Corollary 7.4. For part 2, we may assume $m = 3$ and $3 \leq m' \leq 5$. The case $m' = 3$ follows from Lemma 7.9, $m' = 4$ from Lemma 7.10, and $m' = 5$ from Lemma 7.5. $\square$

Corollary 7.12. *If $|X| \leq 11$, then X splits into smaller spaces.*

Proof. We may assume that $3 \leq m \leq n$. In this case, we have

$$11 \geq |X| = l + m + n \geq l + 6,$$

and hence $l \leq 5$.

By Corollary 7.7, we may assume that $l \geq 4$. Hence we have

$$11 \geq |X| = l + m + n \geq l + 2m \geq 4 + 2m$$

and so $m \leq 3$. The result follows from Corollaries 7.8 and 7.11. $\square$

8. $|X| = 12$

In this section, we assume that X is a connected minimal finite space of $|X| = 12$ and show that X splits into smaller spaces. We need laborious case by case analysis.

Proposition 8.1. *If X is a connected minimal finite space with $|X| = 12$, then X splits into smaller spaces.*

Proof. We may assume that $3 \leq m \leq n$. In this case, we have

$$12 = |X| = l + m + n \geq \begin{cases} l + 6 \\ l + 2m \end{cases}$$

and hence

$$\begin{aligned} l &\leq 6, \\ m &\leq (12 - l)/2. \end{aligned}$$

The case $l \leq 3$ follows from Corollary 7.7.

The case $l = 4$. In this case, we have $m \leq 4$. If $m \leq 3$ or $m = 4$ and $m' \leq 3$, then X splits into smaller spaces by Corollary 7.8. The remaining case is where $m = 4$ and $m' = 4$, that is, $l = m = n = 4$ and $\mathcal{B}$ is an antichain. We show this case in Lemma 8.2.

If $l = 5$, then $m \leq 3$ and X splits into smaller spaces by Corollary 7.11.

If $l = 6$, then $m \leq 3$ and hence $m = n = 3$. We show this case in Corollary 8.8. $\square$

Lemma 8.2. *If $l = m = n = 4$ and $\mathcal{B}$ is an antichain, then X splits into smaller spaces.*

Proof. Note that $\mathrm{mxl}(\mathcal{B}) = \mathcal{B}$. If there exists a maximal element $a \in \mathrm{mxl}(X)$ such that $\gamma_a = |U_a \cap \mathrm{mxl}(\mathcal{B})| \geq 3$, then there exists an element $b \in \mathcal{B}$ such that

$$X = U_a \cup F_b \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X).$$

Therefore, X splits into smaller spaces by Proposition 4.10.

Suppose $\gamma_a \leq 2$ for all $a \in \mathrm{mxl}(X)$.

Since X does not have any beat point, $\alpha_b = |\mathrm{mxl}(X) \cap F_b| \geq 2$ for all $b \in \mathcal{B}$. Since

$$2 \cdot 4 \geq \sum_{a \in \mathrm{mxl}(X)} \gamma_a = \sum_{b \in \mathcal{B}} \alpha_b \geq 2 \cdot 4,$$

we have $\alpha_b = 2$ for all $b \in \mathcal{B}$ and $\gamma_a = 2$ for all $a \in \mathrm{mxl}(X)$. Therefore, $\mathrm{mxl}(X) \cup \mathcal{B}$ is isomorphic to either $S^1_2 \amalg S^1_2$ or S^1_4 by Lemma 5.4. Similarly, or by considering the opposite, we may assume that $\mathcal{B} \cup \mathrm{mnl}(X)$ is isomorphic to $S^1_2 \amalg S^1_2$ or S^1_4.

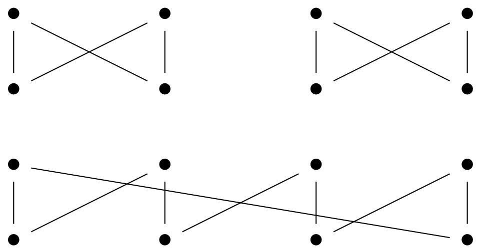


Figure 11: $S^1_2 \amalg S^1_2$ and S^1_4

Consider the case where at least one of $\mathrm{mxl}(X) \cup \mathcal{B}$ and $\mathcal{B} \cup \mathrm{mnl}(X)$ is isomorphic to S^1_4. We may assume that $\mathcal{B} \cup \mathrm{mnl}(X)$ is isomorphic to S^1_4. In this case, we easily see that, for any $a \in \mathrm{mxl}(X)$, $\hat{U}_a$ is contractible or homotopy equivalent to S^0, and $X - \{a\}$ is connected. Therefore, X splits into smaller spaces by Corollary 2.25.

If both $\mathrm{mxl}(X) \cup \mathcal{B}$ and $\mathcal{B} \cup \mathrm{mnl}(X)$ are isomorphic to $S^1_2 \amalg S^1_2$, then we see that X is isomorphic to the space of Fig. 12 because X is connected. We see that, for any $a \in \mathrm{mxl}(X)$, $\hat{U}_a$ is homotopy equivalent to S^0 and $X - \{a\}$ is connected. Therefore, X splits into smaller spaces by Corollary 2.25. (In fact, one can easily see that $X \simeq_w \bigvee_5 S^1$.)

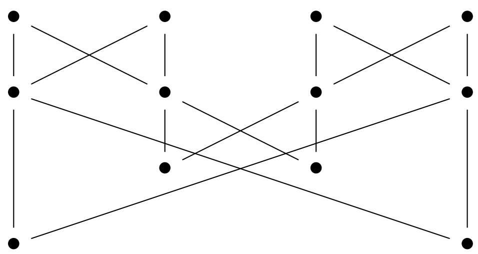


Figure 12: $S^1_2 \amalg S^1_2$ and $S^1_2 \amalg S^1_2$

$\square$

Lemma 8.3. *If $l = 6$, $m = n = 3$ and $m' = 3$, then X splits into smaller spaces.*

Proof. If $n' = |\mathrm{mnl}(\mathcal{B})| \leq 2$, then the opposite X^o splits into smaller spaces by Lemmas 6.3 and 7.2 and so does X. By Lemma 6.7, we may assume that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is identified with S^1_3 in Fig. 4, and so $\mathrm{mxl}(X) = \{a_0, a_1, a_2\}$ and $\mathrm{mxl}(\mathcal{B}) = \{b_0, b_1, b_2\}$.

Since $|\mathrm{mxl}(\mathcal{B})| = 3$, the number of connected components of $\mathcal{B}$ is at most 3.

(a) The case where $\mathcal{B}$ has 3 connected components.

The cardinalities of each component are

$$6 = 1 + 1 + 4 = 1 + 2 + 3 = 2 + 2 + 2.$$

The case $2 + 2 + 2$ follows from Lemma 7.5 and the case $1 + 2 + 3$ follows from Corollary 4.17 as in the case (a) of the proof of Lemma 7.9.

Consider the case $1 + 1 + 4$. Let $\mathcal{B}_4$ be the connected component with $|\mathcal{B}_4| = 4$. Since $\mathcal{B}_4$ is connected, we have $|\mathrm{mnl}(\mathcal{B}_4)| \leq 3$. If $|\mathrm{mnl}(\mathcal{B}_4)| = 2$ or 3, then the result follows from Corollary 4.17 similarly to the case $1 + 2 + 3$ (see Table 1).

Suppose $|\mathrm{mnl}(\mathcal{B}_4)| = 1$. We may assume that $b_1 \in \mathcal{B}_4$. Let $\mathcal{B}_4 = \{b_1, b_3, b_4, b_5\}$ and $b_4 = \min \mathcal{B}_4$. In this case, we have $\mathrm{mnl}(\mathcal{B}) = \{b_0, b_2, b_4\}$ and hence $n = n' = 3$, therefore we may assume that $\mathrm{mnl}(\mathcal{B}) \cup \mathrm{mnl}(X)$ is isomorphic to the one in Fig. 13 (a) by Lemma 6.7.

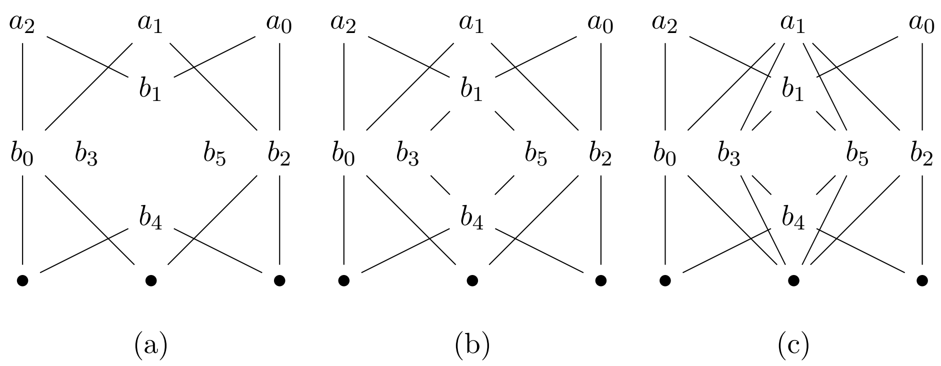


Figure 13: $1 + 1 + 4$

Since $\mathcal{B}_4$ is connected and has maximum and minimum, $\mathcal{B}_4$ is either a chain or isomorphic to the one in Fig. 13 (b), but since X does not have any beat point, $\mathcal{B}_4$ is not a chain.

Now, since $\{b_1, a_0, a_2\} \subset \hat{F}_{b_3} \subset \{b_1, a_0, a_1, a_2\}$ and b_3 is not a beat point, we have $b_3 \prec a_1$, and proceeding similarly, we see that X is isomorphic to the space Fig. 13 (c). Since $U_{a_0} \cap U_{a_2} = U_{b_1} \simeq *$, X splits into smaller spaces by Lemma 6.4. In fact, we easily see that

$$\hat{U}_{a_1} \simeq_w S^1 \vee S^1 \vee S^1,$$
$$X \simeq_w \mathbb{S}(\hat{U}_{a_1}) \simeq_w S^2 \vee S^2 \vee S^2.$$

(b) The case where $\mathcal{B}$ has 2 connected components.

The cardinalities of each component are

$$6 = 1 + 5 = 2 + 4 = 3 + 3.$$

In the case $3+3$, there exists an element $b \in \mathcal{B}$ such that $\left|\hat{F}_b^{\mathcal{B}}\right| = 2$, and the result follows from Lemma 6.8.

In the case $2+4$, let $\mathcal{B}_4$ be the connected component with $|\mathcal{B}_4| = 4$. Since $|\mathrm{mxl}(\mathcal{B}_4)| = 2$, we have $|\mathrm{mnl}(\mathcal{B}_4)| \leq 2$. If $|\mathrm{mnl}(\mathcal{B}_4)| = 1$, then $|\mathrm{mnl}(\mathcal{B})| = 2$ and hence the result follows. If $|\mathrm{mnl}(\mathcal{B}_4)| = 2$, then we see that there exists an element $b \in \mathcal{B}$ such that $\left|\hat{F}_b^{\mathcal{B}}\right| = 2$ (see Table 1), and the result follows.

Consider the case $1+5$, and let $\mathcal{B}_5$ be the connected component with $|\mathcal{B}_5| = 5$. Since $|\mathrm{mxl}(\mathcal{B}_5)| = 2$, we have $|\mathrm{mnl}(\mathcal{B}_5)| \leq 3$. If $|\mathrm{mnl}(\mathcal{B}_5)| = 1$, then $|\mathrm{mnl}(\mathcal{B})| = 2$, and if $|\mathrm{mnl}(\mathcal{B}_5)| = 3$, then there exists an element $b \in \mathcal{B}$ such that $\left|\hat{F}_b^{\mathcal{B}}\right| = 2$, and hence the result follows.

Suppose $|\mathrm{mnl}(\mathcal{B}_5)| = 2$. We may assume that $b_0 \notin \mathcal{B}_5$. Let $\mathcal{B}_5 = \{b_1, b_2, c, d_1, d_2\}$ and $\mathrm{mnl}(\mathcal{B}_5) = \{d_1, d_2\}$. We have $\mathrm{mnl}(\mathcal{B}) = \{b_0, d_1, d_2\}$ and hence $n = n' = 3$, therefore we may assume that $\mathrm{mnl}(\mathcal{B}) \cup \mathrm{mnl}(X)$ is isomorphic to the one in Fig. 14 (a).

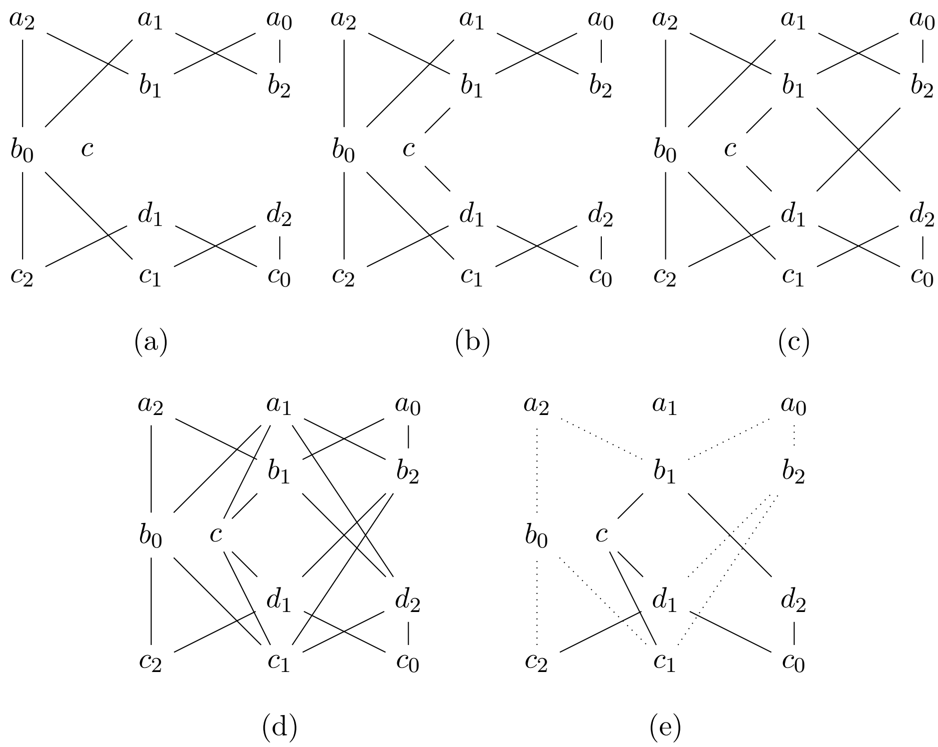


Figure 14: 1+5

Since $c \notin \mathrm{mnl}(\mathcal{B}_5) \cup \mathrm{mxl}(\mathcal{B}_5)$ and $\hat{F}_c^{\mathcal{B}} \subset \{b_1, b_2\}$, we have $0 < \left|\hat{F}_c^{\mathcal{B}}\right| \leq 2$. We may assume that $\left|\hat{F}_c^{\mathcal{B}}\right| = 1$ and, by considering the opposite, $\left|\hat{U}_c^{\mathcal{B}}\right| = 1$. We may assume that $\hat{F}_c^{\mathcal{B}} = \{b_1\}$ and $\hat{U}_c^{\mathcal{B}} = \{d_1\}$.

If $d_2 < b_2$, then, since $\mathcal{B}_5$ is connected, we have $d_2 < b_1$ or $d_1 < b_2$, and hence $\left|\hat{F}^{\mathcal{B}}_{d_2}\right| = 2$ or $\left|\hat{U}^{\mathcal{B}}_{b_2}\right| = 2$, therefore X splits into smaller spaces.

Suppose $d_2 \not< b_2$. Since $\mathcal{B}_5$ is connected, we have $d_2 < b_1$ and $d_1 < b_2$. Since c, b_2, and d_2 are not beat points, we have $c < a_1$, $c > c_1$, $b_2 > c_1$ and $d_2 < a_1$. Therefore X is isomorphic to the space of Fig. 14 (d). Since $U_{a_0} \cap U_{a_2} = U_{b_1} \simeq *$, X splits into smaller spaces by Lemma 6.4. In fact, we easily see that

$$\hat{U}_{a_1} \simeq_w S^1 \vee S^1 \vee S^1,$$
$$X \simeq_w \mathbb{S}(\hat{U}_{a_1}) \simeq_w S^2 \vee S^2 \vee S^2.$$

(c) The case where $\mathcal{B}$ is connected. We have $|\mathrm{mnl}(\mathcal{B})| \leq 3$ and we may assume that $|\mathrm{mnl}(\mathcal{B})| = 3$, that is, $m = m' = n = n' = 3$. In this case, we may assume that $\mathrm{mxl}(X) \cup \mathrm{mxl}\,\mathcal{B}$ and $\mathrm{mnl}(X) \cup \mathrm{mnl}(\mathcal{B})$ are isomorphic to S^1_3 as in Fig. 15 (a). We may also assume that $\left|\hat{F}^{\mathcal{B}}_{b}\right| \neq 2$ and $\left|\hat{U}^{\mathcal{B}}_{b}\right| \neq 2$ for all $b \in \mathcal{B}$, therefore $\mathcal{B}$ is isomorphic to one of graphs in Lemma 5.3.

If there exists an element $b \in \mathcal{B}$ such that $\left|\hat{F}^{\mathcal{B}}_{b}\right| = 1$, say, $\left|\hat{F}^{\mathcal{B}}_{d_0}\right| = 1$, then $\mathcal{B}$ is isomorphic to the one in Fig. 15 (b). Since d_0 is not a beat point and $d_0 \not< b_0, b_2$, we see that $d_0 \prec a_1$. Proceeding similarly, we see that X is isomorphic to the space of Fig. 15 (c). Since $U_{a_0} \cap U_{a_2} = U_{b_1} \simeq *$ (see Fig. 15 (d)), X splits into smaller spaces by Lemma 6.4 (In fact, $X \simeq_w S^2 \vee S^2 \vee S^2$).

If $\left|\hat{F}^{\mathcal{B}}_{b}\right| = 3$ for all $b \in B$, then X is isomorphic to the space of Fig. 15 (e). Since $U_{a_0} \cap U_{a_2} = U_{b_1} \simeq *$ (see Fig. 15 (f)), X splits into smaller spaces by Lemma 6.4 (In fact, $X \simeq_w S^3$).

□

Lemma 8.4. *If $l = 6$, $m = n = 3$, and $m' = 4$, then X splits into smaller spaces.*

Proof. If $n' = |\mathrm{mnl}(\mathcal{B})| \leq 3$, then the opposite X^o splits into smaller spaces by Lemmas 6.3, 7.2 and 8.3 and so does X.

By Lemma 6.9, we may assume that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is (a) or (b) of Fig. 16.

Since $|\mathrm{mxl}(\mathcal{B})| = 4$, the number of connected components of $\mathcal{B}$ is at most 4.

(a) The case where $\mathcal{B}$ has 4 connected components.

Note that each component has maximum since $|\mathrm{mxl}(\mathcal{B})| = 4$. The cardinalities of each component are

$$6 = 1 + 1 + 1 + 3 = 1 + 1 + 2 + 2.$$

The case $1 + 1 + 2 + 2$ follows from Lemma 7.5.

In the case $1 + 1 + 1 + 3$, let b be the maximum element of the component with 3 elements. Note that $b \leq a_0$ or $b \leq a_1$. If $b \leq a_0$, then $\mathrm{mxl}(\mathcal{B}) - U_{a_0} = \{b_3\}$, and we see that $X = U_{a_0} \cup F_{b_3} \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$ and the assumptions of Corollary 4.17 hold, therefore the result holds. The case $b \leq a_1$ is similar.

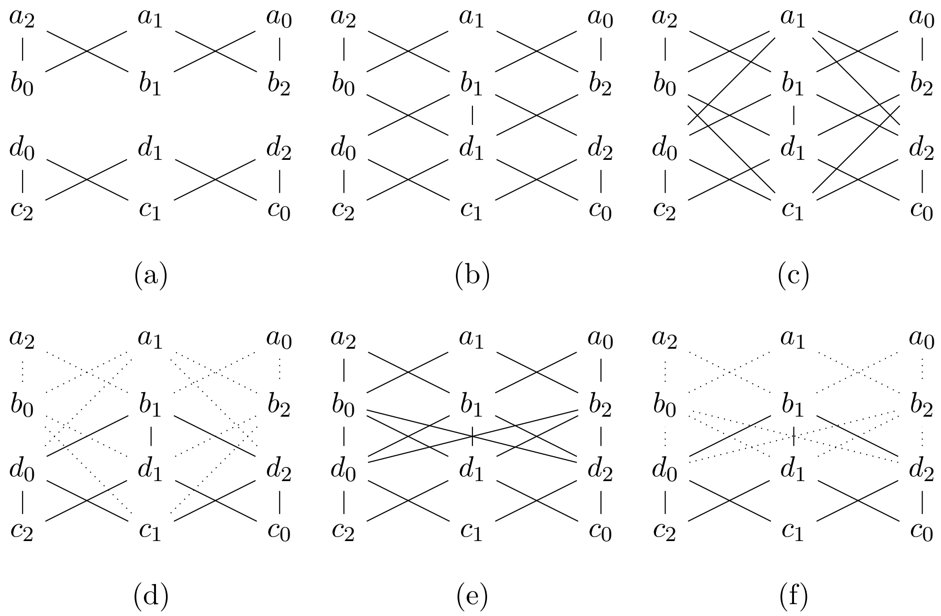


Figure 15: $l = 6$, $m = m' = n = n' = 3$, and $\mathcal{B}$ is connected.

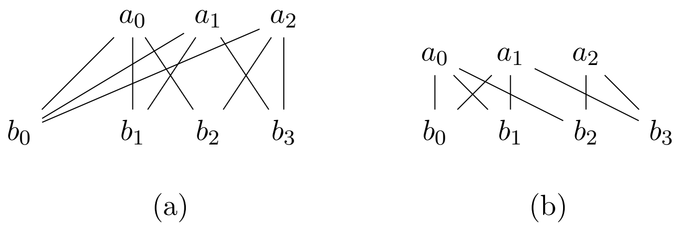


Figure 16: $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ for $m = 3, m' = 4$.

(b) The case where $\mathcal{B}$ has 3 connected components.

The possible cardinalities of each component are

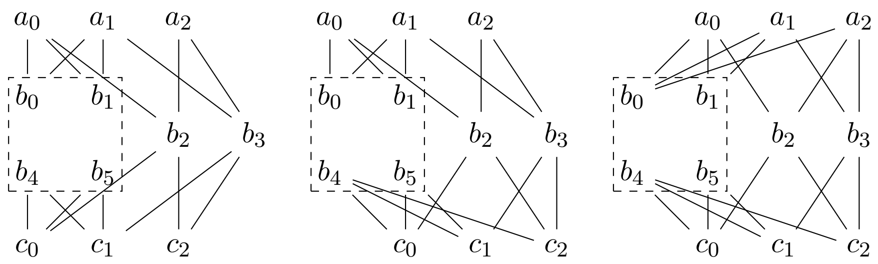
$$6 = 1 + 1 + 4 = 1 + 2 + 3 = 2 + 2 + 2,$$

but, since $|\mathrm{mxl}(\mathcal{B})| = 4$, the last case does not occur.

In the case $1 + 2 + 3$, we see that $n' = 3$ and the result follows.

In the case $1 + 1 + 4$, we see that $n' = |\mathrm{mnl}(\mathcal{B})| = 3$ or 4. We have to consider the case $n' = 4$. Let $\mathcal{B}_4$ be the connected component with $|\mathcal{B}_4| = 4$. We have $|\mathrm{mxl}(\mathcal{B}_4)| = 2$, and in the case $n' = 4$, we have $|\mathrm{mnl}(\mathcal{B}_4)| = 2$. We put $\mathrm{mnl}(\mathcal{B}_4) = \{b_4, b_5\}$. Since $\mathcal{B}_4$ is connected, there exists an element $b \in \mathrm{mnl}(\mathcal{B}_4)$ such that $\widehat{F}_b^{\mathcal{B}_4} = \mathcal{B}_4 \cap \widehat{F}_b = \mathrm{mxl}(\mathcal{B}_4)$ (see Table 1). In the case where $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is (b) of Fig. 16, if $\mathrm{mxl}(\mathcal{B}_4) \neq \{b_0, b_1\}$, then we see that $\widehat{F}_b \simeq *$, that is, b is a weak beat point. Hence $X \simeq_w X - \{b\}$ and the result follows. In the case where $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ is (a) of Fig. 16, if $b_0 \notin \mathrm{mxl}(\mathcal{B}_4)$, then we see that b is a weak beat point and the result follows. By the symmetry, we may consider the case where $\mathrm{mxl}(\mathcal{B}_4) = \{b_0, b_1\}$.

Suppose $\mathrm{mxl}(\mathcal{B}_4) = \{b_0, b_1\}$. By considering the opposite, we may suppose that $\mathrm{mxl}(X) \cup \mathrm{mxl}(\mathcal{B})$ and $\mathrm{mnl}(X) \cup \mathrm{mnl}(\mathcal{B})$ are those of the spaces in Fig. 17.



(a) (b) (c)

Figure 17: $1+1+4$, $n' = 4$.

We use Lemma 6.6.

Note that, in any case, we have $U_{a_0} = X - \{a_1, a_2, b_3\}$, $U_{a_1} = X - \{a_0, a_2, b_2\}$, and $U_{a_0} \cap U_{a_1} = X - \mathrm{mxl}(X) - \{b_2, b_3\} = \mathcal{B}_4 \cup \mathrm{mnl}(X)$, and in the cases (a) and (b), $\{b_2, b_3\} \cup \mathrm{mnl}(X) \subset \widehat{U}_{a_2} \subset \{b_2, b_3, b_4, b_5\} \cup \mathrm{mnl}(\mathcal{B})$ and so $\mathrm{h}\left(\widehat{U}_{a_2}\right) = 1$, therefore $\widehat{U}_{a_2}$ is weak homotopy equivalent to a simplicial complex of dimension at most 1.

Consider the case (a). If $c_2 < b_0$ or $c_2 < b_1$, then $U_{a_0} \cap U_{a_1}$ is connected, and hence X splits into smaller spaces by Lemma 6.6. Otherwise, we see that $\widehat{F}_{c_2} = \{b_2, b_3\} \cup \mathrm{mxl}(X) \simeq *$, namely, c_2 is a weak beat point, and the result follows.

In the case (b), $U_{a_0} \cap U_{a_1}$ is connected and the result follows.

In the case (c), $U_{a_0} \cap U_{a_1}$ is connected.

Note that $U_{b_2} \cup U_{b_3} \simeq *$ and $\{b_0, b_2, b_3\} \cup \mathrm{mnl}(X) \subset \widehat{U}a_2 \subset \{b_0, b_2, b_3, b_4, b_5\} \cup \mathrm{mnl}(X)$.

If $b_5 \not< b_0$, then $b_4 < b_0$ and we see that b_0 is a down beat point, which contradicts the minimality of X, therefore, $b_5 < b_0$.

If $\{b_4, b_5\} \subset U_{b_0}$, then we have $\widehat{U}a_2 = U_{b_0} \cup U_{b_2} \cup U_{b_3}$ and $U_{b_0} \cap (U_{b_2} \cup U_{b_3}) = \mathrm{mnl}(X)$. Therefore, $\widehat{U}a_2 \simeq_w \mathbb{S}\,\mathrm{mnl}(X) \simeq_w S^1 \vee S^1$.

If $b_4 \not< b_0$, then we have $b_4 < b_1$ and $b_5 < b_0, b_1$. Since b_4 is not an up beat point, we have $b_4 < a_2$, and since b_0 is not a down beat point, we have $c_2 < b_0$. We see that b_5 is an up beat point of $\widehat{U}a_2$, and hence $\widehat{U}a_2$ is homotopy equivalent to a space of height 1. See Fig. 18.

Therefore, X splits into smaller spaces by Lemma 6.6.

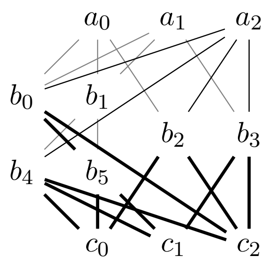


Figure 18: $\widehat{U}_{a_2}$.

(c) The case where $\mathcal{B}$ has 2 connected components. The cardinalities of each component are
$$6 = 1 + 5 = 2 + 4 = 3 + 3$$
and we see that $n' \leq 3$.

(d) The case where $\mathcal{B}$ is connected. In this case, we see that $n' \leq 2$.

$\square$

Lemma 8.5. *If $l = 6$, $m = n = 3$, and $m' = 5$, then X splits into smaller spaces.*

Proof. If $\mathcal{B}$ has 5 connected components, then X splits into smaller spaces by Lemma 7.5. Otherwise, we see that $n' \leq 4$ and hence X^o splits into smaller spaces by Lemmas 6.3, 7.2, 8.3 and 8.4, and so does X. $\square$

Lemma 8.6. *If $l = 6$, $m = n = 3$, and $\mathcal{B}$ is an antichain, then X splits into smaller spaces.*

Proof. We rely on a result of Cianci-Ottina [3]. Let
$$\mathcal{R} = \{(a, b, x) \in \mathrm{mxl}(X) \times \mathrm{mnl}(X) \times \mathcal{B} \mid a \not\geq x \text{ and } b \not\leq x\}$$
and $r = |\mathcal{R}|$. Note that $r = \sum_{x \in \mathcal{B}} (m - \alpha_x)(n - \beta_x)$, and, for $(a, b) \in \mathrm{mxl}(X) \times \mathrm{mnl}(X)$ and $x \in X$, $(a, b, x) \in \mathcal{R}$ if and only if $x \notin U_a \cup F_b \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$. By considering the projection $\mathcal{R} \to \mathrm{mxl}(X) \times \mathrm{mnl}(X)$, one sees that if $r < mn$, then there exist elements $a \in \mathrm{mxl}(X)$ and $b \in \mathrm{mnl}(X)$ such that $X = U_a \cup F_b \cup \mathrm{mxl}(X) \cup \mathrm{mnl}(X)$.

In our case, since $2 \leq \alpha_x, \beta_x \leq 3$ for all $x \in \mathcal{B}$, we see that
$$r = \sum_{x \in \mathcal{B}} (3 - \alpha_x)(3 - \beta_x) \leq \sum_{x \in \mathcal{B}} (3 - 2)(3 - 2) = |\mathcal{B}| = 6 < 3 \cdot 3 = mn.$$
Therefore, X splits into smaller spaces by Proposition 4.10. $\square$

Remark 8.7. As one sees, if $l \leq 8$, $m = n = 3$, and $\mathcal{B}$ is an antichain, then X splits into smaller spaces.

Corollary 8.8. *If $l = 6$ and $m = n = 3$, then X splits into smaller spaces.*

Proof. We have $1 \leq m' \leq 6$. The case $m' = 1$ follows from Lemma 7.2 and the case $m' = 2$ follows from Lemma 6.3. $\square$

9. Proof of the main theorem

Theorem 9.1. *If X is a connected finite space with $|X| \leq 12$ or $|X| = 13$ and $l \leq 3$, then X has a weak homotopy type of a wedge of spheres.*

Proof. Induction on $|X|$.

If $|X| = 1$, then X is a wedge of 0-copies of spheres.

Suppose $|X| > 1$ and the result holds for spaces whose cardinalities are smaller than $|X|$.

If X is not T_0, then X is homotopy equivalent to its maximal quotient poset, whose cardinality is smaller than X. If X is T_0 and has a beat point b, then $X \simeq X - \{b\}$. If X is minimal, then, by Corollaries 7.7 and 7.12 and Proposition 8.1, X splits into smaller spaces. Since each wedge summand is a wedge of spheres by the induction hypothesis, so does X. □

A. A proof of Lemma 2.4

We give a proof of Lemma 2.4 by induction on $l + m$.

If $l = m = 1$, then we have

$$\begin{aligned} K &= L_1 \cup M_1 \\ &\simeq S(L_1 \cap M_1) \end{aligned}$$

and, since K is connected, $L_1 \cap M_1 \neq \emptyset$. Therefore $n = 1 - 1 - 1 + 1 = 0$ and the result holds in this case.

Suppose $l + m > 2$. We may assume $l > 1$. Put $L' = \coprod_{i>1} L_i$. Let $K_1, \ldots, K_k$ be the connected components of $L' \cup M$. Since L_i and M_j are connected, we see that there exist decompositions

$$\{2, \ldots, l\} = \coprod_{i=1}^{k} I_i \qquad \{1, \ldots, m\} = \coprod_{i=1}^{k} J_i$$

and

$$K_i = \bigcup_{s \in I_i} L_s \cup \bigcup_{t \in J_i} M_t$$

We put

$$\begin{aligned} L^i &= \bigcup_{s \in I_i} L_s, \\ M^i &= \bigcup_{t \in J_i} M_t, \\ n_1^i &= |\{j \in J_i \mid L_1 \cap M_j \neq \emptyset\}| - 1. \end{aligned}$$

Note that, if $i \neq j$, then $L_s \cap M_t = \emptyset$ for all $s \in I_i$ and $t \in J_j$ because $K_i \cap K_j = \emptyset$.

Since $K_i = L^i \cup M^i$ and $|I_i| + |J_i| \leq l - 1 + m < l + m$, by the induction hypothesis, we have

$$K_i \simeq \left(\bigvee_{\substack{s \in I_i \\ t \in J_i \\ L_s \cap M_t \neq \emptyset}} S(L_s \cap M_t) \right) \vee \left(\bigvee_{n_i} S^1 \right),$$
$$n_i = |\{(s,t) \in I_i \times J_i \mid L_s \cap M_t \neq \emptyset\}| - |I_i| - |J_i| + 1.$$

We have

$$\begin{aligned} K = L_1 \cup L' \cup M &= L_1 \cup \left(\coprod K_i \right), \\ L_1 \cap \left(\coprod K_i \right) &= \coprod (L_1 \cap K_i), \\ L_1 \cap K_i = L_1 \cap \left(L^i \cup M^i \right) &= L_1 \cap M^i = \coprod_{\substack{j \in J_i \\ L_1 \cap M_j \neq \emptyset}} (L_1 \cap M_j), \end{aligned}$$

and, since K is connected, $L_1 \cap K_i \neq \emptyset$. Since $L_1 \simeq *$ and the inclusion $\coprod (L_1 \cap M_j) \to K_i$ is null homotopic because $L_1 \cap M_j \to M_j \subset K_i$ is null homotopic and K_i is connected, we have

$$\begin{aligned} K &\simeq K/L_1 \\ &\cong \frac{\coprod K_i}{\coprod (L_1 \cap K_i)} \\ &= \bigvee_i \frac{K_i}{L_1 \cap K_i} \\ &= \bigvee_i \frac{K_i}{\coprod_{\substack{j \in J_i \\ L_1 \cap M_j \neq \emptyset}} (L_1 \cap M_j)} \\ &\simeq \bigvee_i \left(K_i \vee S \left(\coprod_{\substack{j \in J_i \\ L_1 \cap M_j \neq \emptyset}} (L_1 \cap M_j) \right) \right) \end{aligned}$$

We have

$$\begin{aligned} \bigvee_i S \left(\coprod_{\substack{j \in J_i \\ L_1 \cap M_j \neq \emptyset}} (L_1 \cap M_j) \right) &\simeq \bigvee_i \left(\bigvee_{\substack{j \in J_i \\ L_1 \cap M_j \neq \emptyset}} S(L_1 \cap M_j) \vee \bigvee_{n_1^i} S^1 \right) \\ &= \left(\bigvee_{\substack{j \\ L_1 \cap M_j \neq \emptyset}} S(L_1 \cap M_j) \right) \vee \left(\bigvee_{\sum n_1^i} S^1 \right), \end{aligned}$$

$$
\begin{aligned}
\bigvee_i K_i &\simeq \bigvee_i \left(\left(\bigvee_{\substack{s\in I_i \\ t\in J_i \\ L_s\cap M_t\neq\emptyset}} S(L_s\cap M_t) \right) \vee \left(\bigvee_{n_i} S^1 \right) \right) \\
&= \left(\bigvee_{\substack{i>1,j \\ L_i\cap M_j\neq\emptyset}} S(L_i\cap M_j) \right) \vee \left(\bigvee_{\sum n_i} S^1 \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum n_1^i &= \sum_{i=1}^{k} \left(|\{j\in J_i \mid L_1\cap M_j\neq\emptyset\}| - 1 \right) \\
&= |\{j\in J \mid L_1\cap M_j\neq\emptyset\}| - k \\
\sum n_i &= \sum_{i=1}^{k} \left(|\{(s,t)\in I_i\times J_i \mid L_s\cap M_t\neq\emptyset\}| - |I_i| - |J_i| + 1 \right) \\
&= |\{(i,j) \mid i>1,\ L_i\cap M_j\neq\emptyset\}| - (l-1) - m + k \\
\sum n_i + \sum n_1^i &= |\{(i,j) \mid L_i\cap M_j\neq\emptyset\}| - l + 1 - m = n.
\end{aligned}
$$

Therefore we have

$$
K \simeq \left(\bigvee_{\substack{i,j \\ L_i\cap M_j\neq\emptyset}} S\left(L_i\cap M_j\right) \right) \vee \left(\bigvee_{n} S^1 \right)
$$

as desired. $\square$

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