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SOME PROPERTIES OF RELATIVE ROTA–BAXTER OPERATORS ON GROUPS

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Abstract. We find connection between relative Rota–Baxter operators and usual Rota–Baxter operators. We prove that any relative Rota–Baxter operator on a group H with respect to (G, Ψ) defines a Rota–Baxter operator on the semi-direct product $H \rtimes_\Psi G$. On the other side, we give condition under which a Rota–Baxter operator on the semi-direct product $H \rtimes_\Psi G$ defines a relative Rota–Baxter operator on H with respect to (G, Ψ). We introduce homomorphic post-groups and find their connection with λ-homomorphic skew left braces. Further, we construct post-group on arbitrary group and a family post-groups which depends on integer parameter on any two-step nilpotent group. We find all verbal solutions of the quantum Yang-Baxter equation on two-step nilpotent group.

Keywords: Group, nilpotent group, semi-direct product, skew brace, Rota–Baxter operator, relative Rota–Baxter operator, Yang–Baxter equation.
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Contents

1. Introduction

Rota–Baxter operators on groups were introduced in 2021 by L. Guo, H. Lang, Y. Sheng [4], a group with a Rota–Baxter operator is called a Rota–Baxter group (RB-group). Properties of RB-groups are actively studied in [5]. Connection between RB-groups, the Yang–Baxter equation and skew braces was found in [6]. The concept of

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braces was introduced by Rump [12] in 2007 in connection with non-degenerate involutive set theoretic solutions of the quantum Yang–Baxter equation. The concept of skew braces was introduced by Guarnieri and Vendramin [3] in 2017 in connection with non-involutive non-degenerate set theoretic solutions of the quantum Yang–Baxter equation. In [10] the Rota–Baxter operator was defined in cocommutative Hopf algebras. Important example of such algebras give group rings $\Bbbk[G]$ of group G over a field $\Bbbk$.

In [9] relative Rota–Baxter operators (RRB-operators) and relative Rota–Baxter groups (RRB-groups) were defined. In this case, the relative Rota–Baxter operator depends not only on the group, but also on the group of its automorphisms. In the case when this group of automorphisms is a group of inner automorphisms, we obtain the Rota–Baxter operator. Some properties of relative Rota–Baxter operators are studied in [11]. Relative Rota–Baxter operators on an arbitrary Hopf algebra were defined in [7]. In [8] Rota–Baxter and averaging operators on racks and rack algebras were introduced.

In the present paper we prove some properties of relative Rota–Baxter operators on groups. In particular, we are studying connections between relative Rota–Baxter operators and usual Rota–Baxter operators.

The paper is structured as follows. The next section is devoted to review some facts about skew braces, λ-maps, post-groups, Rota–Baxter operators, relative Rot–Baxter operators, and some connections between them.

In Section 3 we find connection between relative Rota–Baxter operators and usual Rota–Baxter operators. We prove (see Proposition 3.1) that any relative Rota–Baxter operator on a group H with respect to (G, Ψ) defines a Rota–Baxter operator on the semi-direct product $H \rtimes_\Psi G$. On the another side, Theorem 3.2 gives condition under which a Rota–Baxter operator on the semi-direct product $H \rtimes_\Psi G$ defines a relative Rota–Baxter operator on H with respect to (G, Ψ). An example shows that the construction of this theorem can map different Rota–Baxter operators to the same relative Rota–Baxter operator. Further, we introduce relative Rota–Baxter operator of weight -1 and find (see Proposition 3.6) connection between relative Rota–Baxter operators of weight 1 and -1. Corollary 3.7 gives a way to construct new RRB-operators from known RRB-operators.

It is known that if we have a RB-group or a RRB-group (G, B), then we can construct a skew left brace $(G, \cdot, \circ_B)$. In Section 4 we give example (Example 4.3) which shows that using RRB-operator $B\colon \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$, it is possible to construct a non-trivial left brace $(\mathbb{Z}_4, +, \circ_B)$, where $(\mathbb{Z}_4, +) \cong \mathbb{Z}_4$ and $(\mathbb{Z}_4, \circ_B) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. On the other side, any Rota-Baxter left brace on $\mathbb{Z}_4$ is trivial. But Theorem 4.4 shows that we can not take a prime number $p > 2$ instead 2. In this case any RRB-operator $B\colon \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ is a homomorphism and any relative Rota–Baxter left brace on $\mathbb{Z}_{p^2}$ is trivial.

In Section 5 we introduce homomorphic post-groups and find (see Proposition 5.2) their connection with λ-homomorphic skew left braces. Further, we construct post-group on arbitrary group and a family of post-groups on any two-step nilpotent group which depends on an integer parameter. By these post-groups we can construct skew left braces. We show that new operation in these skew left braces defines a two-step nilpotent group, which can be non isomorphic to the initial two-step nilpotent group. Moreover, it will be proved (in Proposition 5.5) that any skew left brace constructed above is a λ-homomorphic skew left brace.

As we have already mentioned above, any skew left braces gives a non-involutive non-degenerate set theoretic solutions of the quantum Yang-Baxter equation (YBE) [3]. In Theorem 5.13 we find all verbal solutions of the quantum Yang-Baxter equation on two-step nilpotent group.

2. Preliminary results

Let $(G, \cdot)$ and $(G, \circ)$ be groups on a same set G. The triple $(G, \cdot, \circ)$ is said to be a *skew left brace* if

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c),$$

where a^{-1} is the inverse of a with respect to the operation $\cdot$. To get a skew right brace we can take another axiom,

$$(b \cdot c) \circ a = (b \circ a) \cdot a^{-1} \cdot (c \circ a).$$

A skew left brace which is also a skew right brace is said to be a skew two-sided brace. If $(G, \cdot, \circ)$ is a skew left brace, then the map $\lambda\colon (G, \circ) \to \mathrm{Aut}(G, \cdot)$, where

$$\lambda_a(b) = a^{-1} \cdot (a \circ b),$$

is called the λ-*map* of $(G, \cdot, \circ)$. Here we denoted $\lambda_a = \lambda(a)$ the image of the element $a \in G$.

A map of skew left braces $\varphi\colon G \to H$ is called a homomorphism of skew left braces if $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ and $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$.

The next definition was introduced in [1].

Definition 2.1. Let $(G, \cdot)$ be a group and $\triangleright\colon G \times G \to G$ be a binary operation on G. The algebraic system $(G, \cdot, \triangleright)$ is called a *post-group*, if the operator $L^{\triangleright}_a\colon G \to G$, where $L^{\triangleright}_a(b) = a \triangleright b$, is an automorphism of the group $(G, \cdot)$, and

$$a \triangleright (b \triangleright c) = \big(a \cdot (a \triangleright b)\big) \triangleright c.$$

It is easy to see that the axiom of post-group is equivalent to the statement that for any $a, b \in G$ the following holds:

$$L^{\triangleright}_a L^{\triangleright}_b = L^{\triangleright}_{a L^{\triangleright}_a(b)}.$$

Also remark that the condition that $L^{\triangleright}_a$ is an automorphism of G means that

$$a \triangleright (b \cdot c) = (a \triangleright b) \cdot (a \triangleright c)$$

for any $a, b, c \in G$, which is left distributivity of the operation $\triangleright$. Connection between skew left braces and post-groups gives the following theorem.

Theorem 2.2 [1]**.** *1) For any skew left brace $(G, \cdot, \circ)$ let $a \triangleright b := \lambda_a(b)$. Then $(G, \cdot, \triangleright)$ is a post-group.*

2) For any post-group $(G, \cdot, \triangleright)$ let $a \circ b = a \cdot (a \triangleright b)$. Then $(G, \cdot, \circ)$ is a skew left brace.

In [4] were defined Rota–Baxter operators of weights ± 1 on a group.

Definition 2.3 [4]**.** Let G be a group.

(i) A map $B\colon G \to G$ is called a *Rota–Baxter operator of weight 1* if
$$B(g)B(h) = B\left(gB(g)hB(g)^{-1}\right), \ g,h \in G;$$
(ii) a map $C\colon G \to G$ is called a *Rota–Baxter operator of weight* -1 if
$$C(g)C(h) = C\left(C(g)hC(g)^{-1}g\right), \ g,h \in G.$$

A group endowed with an RB-operator is called a *Rota–Baxter group (RB-group)*.

When we say on a Rota–Baxter operator we say on a Rota–Baxter operator of weight 1.

For two Rota–Baxter groups (G, B_G) and (H, B_H) a group homomorphism $\varphi\colon G \to H$ is called a *homomorphism of Rota–Baxter groups*, if the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "B_G"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"'] \\ H \arrow[r, "B_H"] & H \end{tikzcd}$$

It has been shown in [6], that for a Rota–Baxter group (G, B) and the binary operation $\circ_B$, defined as
$$a \circ_B b = aB(a)bB(a)^{-1}, \tag{2.1}$$
the triple $(G, \cdot, \circ_B)$ is a skew left brace, which is called by RB skew left brace.

Proposition 2.4. The map from the category of Rota–Baxter groups to the category of skew left braces, given by equation (2.1), that acts trivially on homomorphisms, is a functor.

Proof. Let G and H be RotaBaxter groups, and $\varphi\colon G \to H$ a homomorphism of Rota–Baxter groups. For any $a, b \in G$ we have
$$\varphi(a \circ_G b) = \varphi(aB_G(a)bB_G(a)^{-1}) = \varphi(a)\varphi B_G(a)\varphi(b)\big(\varphi B_G(a)\big)^{-1} =$$
$$= \varphi(a)B_H\big(\varphi(a)\big)\varphi(b)B_H\big(\varphi(a)\big)^{-1} = \varphi(a) \circ_H \varphi(b).$$
□

The next definition can be found in [9].

Definition 2.5. Let G and H be groups, and $\Psi\colon G \to \operatorname{Aut} H$ is an action of G on H. A map $B\colon H \to G$ is called a *relative Rota–Baxter operator* (RRB-operator) on H with respect to (G, Ψ) if
$$B(h)B(k) = B\big(h\Psi_{B(h)}(k)\big), \ \ h,k \in H.$$
The quadruple (H, G, Ψ, B) is called a *relative Rota–Baxter group*.

Example 2.6. If $H = G$ and $\Psi\colon G \to \operatorname{Inn} G$, $\Psi_g = \Psi(g)\colon x \mapsto gxg^{-1}$, $x \in G$, then
$$B(h)B(k) = B\big(hB(h)kB(h)^{-1}\big), \ \ h,k \in H,$$
is the usual Rota–Baxter operator on G, and (G, B) is the Rota–Baxter group.

As in the case of Rota–Baxter groups, the operation

$$h \circ_B k = h\,\Psi_{B(h)}(k), \quad h, k \in G,$$

is a group operation on G (see [9, Proposition 3.5]).

The next question comes.

3. Rota–Baxter and relative Rota–Baxter operators

3.1. **Relative Rota–Baxter operators and semi-direct products.** Recall that a semi-direct product $H \rtimes_\Psi G$ of groups G and H under the action $\Psi\colon G \to \operatorname{Aut} H$ is the set of pairs

$$H \times G = \{(h, a) \mid h \in H, a \in G\}$$

with multiplication

$$(h, a)(k, b) = (h\Psi_a(k), ab),\ \ h, k \in H, a, b \in G.$$

The following proposition shows that any RRB-operator defines RB-operator on a semi-direct product.

Proposition 3.1. Let (H, G, Ψ, B) be a relative Rota–Baxter group. Then the operator

$$B'\colon H \rtimes_\Psi G \to H \rtimes_\Psi G,\ \ B'((h, a)) = (e, a^{-1}B(h)),\ \ h \in H, a \in G,$$

is a Rota–Baxter operator on the semi-direct product $H \rtimes_\Psi G$.

Proof. We need to check the equality

$$B'(u)\,B'(v) = B'\left(u\,B'(u)\,v\,B'(u)^{-1}\right),\ \ u, v \in H \rtimes_\Psi G.$$

If $u = (h, a)$, $v = (k, b)$, $h, k \in H$, $a, b \in G$, then the left hand side,

$$B'(u)\,B'(v) = B'((h, a))\,B'((k, b)) = (e, a^{-1}\,B(h))\,(e, b^{-1}\,B(k)) =$$
$$= \left(\Psi_{a^{-1}\,B(h)}(e), a^{-1}\,B(h)\,b^{-1}\,B(k)\right) = \left(e, a^{-1}\,B(h)\,b^{-1}\,B(k)\right).$$

The right hand side,

$$B'\left(u\,B'(u)\,v\,B'(u)^{-1}\right) = B'\left((h, a)\,B'((h, a))\,(k, b)\,B'((h, a))^{-1}\right) =$$
$$= B'\left((h\,\Psi_a(e), B(h))\,(k\Psi_b(e), b\,B(h)^{-1}\,a)\right) = B'\left((h\,\Psi_{B(h)}(k\Psi_b(e)),\ B(h)\,b\,B(h)^{-1}\,a)\right)$$
$$= B'\left((h\,\Psi_{B(h)}(k),\ B(h)\,b\,B(h)^{-1}\,a)\right) = \left(e,\ a^{-1}B(h)\,b^{-1}\,B(h)^{-1}\,B(h\,\Psi_{B(h)}(k))\right).$$

Comparing the left hand side and right hand side, we get

$$a^{-1}\,B(h)\,b^{-1}\,B(k) = a^{-1}B(h)\,b^{-1}\,B(h)^{-1}\,B(h\,\Psi_{B(h)}(k)).$$

Hence,

$$B(h)\,B(k) = B(h\,\Psi_{B(h)}(k)),\ \ h, k \in H.$$

Since B is a RRB-operator on H, this equality holds. $\square$

We will now present a construction that allows one to build relative Rota-Baxter operators using Rota–Baxter operators on semi-direct products.

Let $H \rtimes_\Psi G$ be a semi-direct product of groups H and G with respect to some left action Ψ of G on H. Let $B\colon H \rtimes_\Psi G \to H \rtimes_\Psi G$ be a Rota–Baxter operator. Consider the projections $\pi_H\colon H \rtimes_\Psi G \to H$ and $\pi_G\colon H \rtimes_\Psi G \to G$, as well as the restriction

$$B|_H\colon H \to H \rtimes_\Psi G$$

of B to H. Note that π_H is not necessarily a group homomorphism and the image of $B|_H$ does not necessary lie in H.

The next theorem gives possibilities to construct RRB-operators, using RB-operators on semi-direct products. Constructions of RB-operators on semi-direct and, in particular, on direct products of groups can be found in [6].

Theorem 3.2. *Let $B\colon H \rtimes_\Psi G \to H \rtimes_\Psi G$ be a Rota–Baxter operator. If the image of the map $\pi_H B|_H$ lies in the center of H, then the composition*

$$\pi_G B|_H\colon H \to G$$

is a relative Rota–Baxter operator with respect to (G, Ψ).

Proof. Let $h, k \in H$. Since B is a Rota-Baxter operator, we have

$$B(h)B(k) = B\big(hB(h)kB(h)^{-1}\big).$$

Now, express B as a product of $\pi_H B$ and $\pi_G B$. We have

$$B(h)B(k) = \pi_H B(h)\pi_G B(h)\pi_H B(k)\pi_G B(k) = \pi_H B(h)\pi_H B(k)^{\left(\pi_G B(h)\right)^{-1}} \cdot \pi_G B(h)\pi_G B(k) \tag{3.1}$$

and

$$B(hB(h)kB(h)^{-1}) = \pi_H B\big(hB(h)kB(h)^{-1}\big)\cdot \pi_G B\big(h\pi_H B(h)\pi_G B(h)k\pi_G B(h)^{-1}\pi_H B(h)^{-1}\big). \tag{3.2}$$

Since a semi-direct product of groups is a direct product of sets, $a = b$ if and only if $\pi_H(a) = \pi_H(b)$ and $\pi_G(a) = \pi_G(b)$. By applying this reasoning to expressions (3.1) and (3.2), we obtain:

$$\pi_G B(h)\pi_G B(k) = \pi_G B\big(h\pi_H B(h)\pi_G B(h)k\pi_G B(h)^{-1}\pi_H B(h)^{-1}\big).$$

Note that $\pi_H B(h) \in Z(H)$, and that $\pi_G B(h)k\pi_G B(h)^{-1} = \Psi_{\pi_G B(h)}(k) \in H$. We can now simplify:

$$\pi_G B\big(h\pi_H B(h)\pi_G B(h)k\pi_G B(h)^{-1}\pi_H B(h)^{-1}\big) = \pi_G B\big(h\Psi_{\pi_G B(h)}(k)\big)$$

and obtain

$$\pi_G B(h)\pi_G B(k) = \pi_G B\big(h\Psi_{\pi_G B(h)}(k)\big),$$

which shows that $\pi_G B|_H$ is indeed a relative Rota–Baxter operator. □

We will now provide an example that shows that a Rota–Baxter operator on $H \rtimes_\Psi G$ does not necessarily commute with the projection $\pi_G\colon H \rtimes_\Psi G \to G$, and thus, does not necessarily induce a Rota–Baxter operator on G.

Example 3.3. For the group $S_3 = A_3 \rtimes \langle s_1 \rangle$ consider the Rota–Baxter operator $B\colon S_3 \to S_3$, defined as

$$B(1) = 1, \qquad B(s_1) = s_1 s_2, \qquad B(s_2) = 1,$$

$$B(s_1 s_2) = s_2 s_1, \quad B(s_2 s_1) = s_1 s_2, \quad B(s_1 s_2 s_1) = s_2 s_1.$$

Note that $\pi_{\langle s_1 \rangle}\big(B(s_1)\big) = \pi_{\langle s_1 \rangle}(s_1 s_2) = 1$ and $B\big(\pi_{\langle s_1 \rangle}(s_1)\big) = B(s_1) = s_1 s_2$, which means that $\pi_{\langle s_1 \rangle} B \neq B \pi_{\langle s_1 \rangle}$.

We will now provide an example, that shows, the the construction of Theorem 3.2 can map different Rota–Baxter operators to the same relative Rota–Baxter operator.

Example 3.4. Let G and H be groups, and H be abelian. Let Ψ be an action of G on H. Define Rota–Baxter operators B_{-1} and B_e on $H \rtimes G$ by the following way: $B_{-1}(x) = x^{-1}$, $B_e(x) = e$. In general, the Rota–Baxter groups, defined by these operators, are not isomorphic, which can be checked by applying Proposition 2.4. Note that the image of $\pi_H B_{-1}|_H$ lies in the center of H, because $H = Z(H)$, and the image of $\pi_H B_e|_H$ is trivial. At the same time,

$$\pi_G B_{-1}(h) = \pi_G(h^{-1}) = e = \pi_G B_e(h),$$

which means that the relative Rota–Baxter operators, obtained by applying Theorem 3.2 to operators B_{-1} and B_e, are equal.

By analogy with RB-operators of weight -1 (see Definition 2.3) we introduce RRB-operators of weight -1.

Definition 3.5. Let G and H be groups and $\Psi\colon G \to \operatorname{Aut} H$ be an action of G on H. A map $C\colon H \to G$ is called a *relative Rota–Baxter operator* of weight -1 with respect to (G, Ψ), if

$$C(h) \cdot C(k) = C\big(\Psi_{C(h)}(k) \cdot h\big), \;\; h, k \in H.$$

Proposition 3.6. Let G and H be groups, $\Psi\colon G \to \operatorname{Aut} H$ be an action, and $B\colon H \to G$ be a relative Rota–Baxter operator. Then

1) the map C, defined as $C(h) = B(h^{-1})$, $h \in H$ is a relative Rota–Baxter operator of weight -1.

2) If $\varphi \in \operatorname{Aut} H$, $\psi \in \operatorname{Aut} G$, and for any $g \in G$ the following equality holds:

$$\varphi^{-1} \Psi_g \varphi = \Psi_{\psi(g)},$$

then the composition $\psi B \varphi$ is a relative Rota–Baxter operator.

Proof. 1) We can check directly: for any $h, k \in H$ we have

$$C(h)C(k) = B(h^{-1})B(k^{-1}) = B\big(h^{-1}\Psi_{B(h^{-1})}(k^{-1})\big) = \big((\Psi_{C(h)}(k)h)^{-1}\big) = C\big(\Psi_C(h)(k)h\big).$$

2) For any $h, k \in H$ we have

$$\psi B\varphi(h) \cdot \psi B\varphi(k) = \psi B\big(\varphi(h)\Psi_{B\varphi(h)}(\varphi(k))\big) = \psi B\varphi\big(h\varphi^{-1}\Psi_{B\varphi(h)}(\varphi(k))\big) = \psi B\varphi\big(h\Psi_{\psi B\varphi(h)}(k)\big).$$

$\square$

Corollary 3.7. Let (H, G, Ψ, B) be a relative Rota–Baxter group and $\varphi \in Z\big(\operatorname{Aut} H\big)$. Then $(H, G, \Psi, B\varphi)$ is a relative Rota–Baxter group.

Proof. For any $g \in G$ the automorphism $\Psi_g \in \operatorname{Aut} H$ commutes with φ, and we have $\varphi^{-1}\Psi\varphi = \Psi_g$. Therefore, $\psi B\varphi$ is a relative Rota–Baxter operator, where ψ is the identity automorphism. □

4. Skew braces from RB- and RRB-operators

As we know (see Section 2), if $(G, \cdot)$ is a group, $B\colon G \to G$ is a RB-operator, then $(G, \cdot, \circ_B)$ is a skew left brace, which is called a *Rota-Baxter skew left brace*, where

$$a \circ_B b = aB(a)bB(a)^{-1},\ a, b \in G.$$

The following lemma is evident.

Lemma 4.1. If $(G, \cdot)$ is an abelian group, then

1) any RB-operator on G is an endomorphism,

2) Any Rota–Baxter skew left brace $(G, \cdot, \circ_B)$ is trivial which means $a \circ_B b = a \cdot b$ for any $a, b \in G$.

Theorem 4.2 [1]**.** *Let $B\colon H \to G$ be a relative Rota–Baxter operator with respect to (G, Ψ). Put*

$$h \triangleright k = \Psi_{B(h)}(k).$$

for any $h, k \in H$. Then $(H, \cdot, \triangleright)$ is a post-group.

From Theorem 4.2 and Theorem 2.2 (see, also [11, Proposition 3.5]) it follows that if we define a new operation $\circ_B\colon H \to H$,

$$h \circ_B k = h\Psi_{B(h)}(k),\ h, k \in H,$$

using a relative Rota–Baxter operator $B\colon H \to G$ with respect to (G, Ψ), then $(H, \cdot, \circ_B)$ is a skew left brace. The following example compares construction of RB skew left braces and RRB skew left braces.

Example 4.3. Let $H = \mathbb{Z}_4$ be a cyclic group of order 4. Then, by Lemma 4.1(1) there are following RB-operators on $\mathbb{Z}_4$:

1) $B_0(h) = 0$ for any $h \in H$;

2) B_{-1}, which acts by the rules $0 \mapsto 0$, $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$;

3) B_2, which acts by the rules $0 \mapsto 0$, $1 \mapsto 2$, $2 \mapsto 0$, $3 \mapsto 2$.

By Lemma 4.1(2), on $\mathbb{Z}_4$ there exists only trivial RB skew left brace.

Now, let us construct relative Rota–Baxter skew left braces on $H = \mathbb{Z}_4$ with respect to $(G = \mathbb{Z}_2 \times \mathbb{Z}_2, \Psi)$. Note that $\operatorname{Aut} \mathbb{Z}_4 = \{\varepsilon, -\varepsilon\}$, where $\varepsilon = \mathrm{id}$ and $-\varepsilon(g) = -g$ for any $g \in \mathbb{Z}_4$. Let $\Psi\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined by the following way:

$$\Psi(0,0) = \varepsilon;\ \Psi(1,0) = -\varepsilon;\ \Psi(0,1) = \varepsilon;\ \Psi(1,1) = -\varepsilon.$$

Define $B\colon \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ as follows:

$$B(0) = (0,0),\ B(1) = (1,0),\ B(2) = (0,1),\ B(3) = (1,1).$$

One can check that B is a relative Rota–Baxter operator with respect to (G, Ψ) and by applying Theorems 4.2 and 2.2 to the operator B, we get a skew left brace $(\mathbb{Z}_4, +, \circ_B)$, where $(\mathbb{Z}_4, +) \cong \mathbb{Z}_4$ and $(\mathbb{Z}_4, \circ_B) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Hence, using RRB-operators we can construct more skew braces, than using only RB-operators.

It is interesting to generalize this example, by taking $H = \mathbb{Z}_{p^2}$, $G = \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime number. The following theorem shows that for $p > 2$ the set of skew left braces which can be defined on H using RRB-operators of the form $B\colon \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ is the same as using RB-operators $B'\colon \mathbb{Z}_{p^2} \to \mathbb{Z}_{p^2}$.

Theorem 4.4. *Let p be a prime number and $\Psi\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \operatorname{Aut} \mathbb{Z}_{p^2}$ be a group homomorphism. For any $b_1, b_p \in \mathbb{Z}_p \times \mathbb{Z}_p$, there is no more than one relative Rota-Baxter operator $B\colon \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ such that $B(1) = b_1$ and $B(p) = b_p$.*

Moreover, if $p > 2$, then any relative Rota-Baxter operator $B\colon \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ is a homomorphism.

Proof. Since $\operatorname{Aut} \mathbb{Z}_{p^2}$ is a group of order $p(p-1)$, then $\operatorname{Aut} \mathbb{Z}_{p^2} \simeq Z_p \times A$, where A is an abelian group of order $p-1$. Let us consider an automorphism χ_k of a group $\mathbb{Z}_{p^2}$ defined as

$$\chi_k\colon 1 \mapsto kp + 1$$

then χ_k is an element of order p i.e.

$$(kp+1)^p = (kp)^p + C_p^1 (kp)^{p-1} + \ldots + C_p^{p-2}(kp)^2 + C_p^{p-1} kp + 1, \text{ where } C_m^l = \frac{m!}{l!(m-l)!}.$$

All elements of the sum above obviously divided by p^2 except 1. Thus χ_k is an element of order p. And there is only p such elements (we can take $k = 0, 1, \ldots, p-1$).

It follows that any action $\Psi\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \operatorname{Aut} \mathbb{Z}_{p^2}$ has the form

$$\Psi_{(n_1,n_2)} x = \big(p(k_1 n_1 + k_2 n_2) + 1\big) x,$$

where the numbers $k_1, k_2 \in \{0, 1, \ldots, p-1\}$ define the action.

Now fix the action Ψ and suppose that $B\colon \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ is a relative Rota–Baxter operator, where $B(x) = (x_1, x_2)$ for some $x_1, x_2 \in \mathbb{Z}_p$. Let $\cdot$ denote the scalar multiplication of vectors from $\mathbb{Z}_p^2$, i.e. $(a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2$. Also, define a function $t\colon \mathbb{Z}_{p^2} \to p\mathbb{Z}_{p^2}$ as

$$t(x) = pB(x) \cdot (k_1, k_2) = p(k_1 x_1 + k_2 x_2).$$

Note that $\Psi_{B(x)} y = \big(t(x) + 1\big) y$ and since B is a relative Rota–Baxter operator, we have

$$B(x) + B(y) = B\big(x + \Psi_B(x) y\big) = B\big(x + y + t(x) y\big).$$

Using the fact that $t(x)$ is divisible by p, we can write

$$B(x) + B(py) = B(x + py + pt(x)y) = B(x + py).$$

It follows that the restriction $B|_{p\mathbb{Z}_{p^2}}$ is a homomorphism.

We will now prove by induction over n that

$$nB(x) = B\left(\frac{\bigl(t(x)+1\bigr)^n - 1}{t(x)}x\right).$$

Indeed, for $n = 1$ we have $B(x) = B(x)$, and if the statement holds for $n-1$, then

$$B(x) + (n-1)B\bigl(x\bigr) = B\left(x + (t(x)+1)\frac{\bigl(t(x)+1\bigr)^{n-1} - 1}{t(x)}x\right) =$$
$$= B\left(x + \frac{\bigl(t(x)+1\bigr)^n - t(x) - 1}{t(x)}x\right) = B\left(x + \frac{\bigl(t(x)+1\bigr)^n - t(x) - 1}{t(x)}x\right) =$$
$$= B\left(\frac{\bigl(t(x)+1\bigr)^n - 1}{t(x)}x\right).$$

Now note that

$$\frac{\bigl(1+t(x)\bigr)^n - 1}{t(x)} = n + \sum_{i=2}^{n} C_n^i t(x)^{i-1}.$$

We now have

$$nB(x) = B(nx + ps(x,n)x) = B(nx) + B(ps(x,n)x).$$

Note that $B|_{p\mathbb{Z}_{p^2}}$ is a homomorphism of abelian groups, and we can by extension treat it as a homomorphism of $\mathbb{Z}_{p^2}$-modules. We can thus write

$$B(ps(x,n)x) = s(x,n)xB(p) \text{ and } B(nx) = nB(x) - s(x,n)xB(p).$$

By substituting $x = 1$, we obtain

$$B(n) = nB(1) - s(1,n)B(p). \tag{4.1}$$

Note that $s(1,n)$ can be calculated knowing only $B(1)$ and the action Ψ, so we have proven that for any given action, the values of $B(1)$ and $B(p)$ define a unique relative Rota–Baxter operator.

Now let $p > 2$. Note that $C_p^2 = \frac{p!}{2!(p-2)!}$ is divisible by p. Since $t(x)$ is also divisible by p, it follows that $\sum\limits_{i=2}^{p} C_p^i t(x)^{i-1}$ is divisible by p^2. On one hand, $pB(x) = 0$, and on the other hand,

$$pB(x) = B\Bigl(px + \sum_{i=2}^{p} C_p^i t(x)^{i-1}\Bigr) = B(px).$$

It follows that $B(px) = 0$ for any x, so $B|_{p\mathbb{Z}_{p^2}}$ is a zero homomorphism, and in turn,

$$nB(1) = B(n) - s(1,n)B(p) = B(n),$$

which means that B is a homomorphism. $\square$

Question 4.5. Let us define a Rota–Baxter operator (RB-operator) on a skew left brace as a map which is a Rota–Baxter operator on both groups of skew left brace. Find Rota–Baxter operators on skew left braces. If we are considering RB-operators on a left brace $(G, +, \circ)$, then on the group $(G, +)$ it is an endomorphism.

5. Skew left braces, nilpotent groups and the YBE

5.1. **λ-homomorphic skew left braces.** Consider a particular type of skew left braces, which was introduced in [2]. A skew left brace $(G, \cdot, \circ)$ is called λ-*homomorphic*, if $\lambda\colon (G, \cdot) \to \mathrm{Aut}(G, \cdot)$ is a group homomorphism. The main idea for the introduction of λ-homomorphic skew left braces is the following. If we take a group G with a generating set A, and define a map $\lambda\colon A \to \mathrm{Aut}(G)$, then we can extend it on all elements of G. Under some conditions this map $\lambda\colon G \to \mathrm{Aut}(G)$ is a λ-map of a skew left brace $(G, \cdot, \circ)$, where the second operation is defined by the rule

$$a \circ b = a \cdot \lambda_a(b),\ \ a, b \in G.$$

Class of λ-homomorphic skew left braces is not a big class, but it has a good description. More precisely, any λ-homomorphic skew left brace is metatrivial that means that it is an extension of one trivial skew left brace by another trivial skew left brace (see [2]).

We introduce the following definition.

Definition 5.1. A post-group $(G, \cdot, \triangleright)$ is said to be a *homomorphic post-group* if it satisfies the identity

$$(a \cdot b) \triangleright c = (a \triangleright c) \cdot (b \triangleright c)$$

for all $a, b, c \in G$.

Note that the condition $(a \cdot b) \triangleright c = (a \triangleright c) \cdot (b \triangleright c)$ is the right distributivity.

By applying Theorem 2.2 to λ-homomorphic skew left braces we obtain the following result.

Proposition 5.2. Let $(G, \cdot, \circ)$ be a λ-homomorphic skew left brace. Then the post-group $(G, \cdot, \triangleright)$ has the following properties:

1) $(G, \cdot, \triangleright)$ is a homomorphic post-group.

2) $[a, b, c] = a \triangleright c$, where $[a, b, c]$ is the associator:

$$[a, b, c] := \big(a \triangleright (b \triangleright c)\big) \cdot \big((a \triangleright b) \triangleright c\big)^{-1}.$$

Proof. 1) Follows from

$$(a \cdot b) \triangleright c = \lambda_{a \cdot b}(c) = \lambda_a(c) \cdot \lambda_b(c) = (a \triangleright c) \cdot (b \triangleright c).$$

2) We have

$$(a \triangleright c) \cdot \big((a \triangleright b) \triangleright c\big) = \big(a \cdot (a \triangleright b)\big) \triangleright c = a \triangleright (b \triangleright c),$$

where the first equality follows from the definition of a homomorphic post-group and the second one follows from the definition of post-group (see Definition 2.1).

□

We will now use Theorem 2.2 to construct a particular class of skew left braces on two-step nilpotent groups.

Proposition 5.3. 1) For a group $(G,\cdot)$ let $a \triangleright b = a^{-1}ba$. Then $(G,\cdot,\triangleright)$ is a post-group.

2) For a two-step nilpotent group $(G,\cdot)$ and $n \in \mathbb{Z}\setminus\{0\}$ let $a \triangleright b = a^{-n}ba^n$. Then $(G,\cdot,\triangleright)$ is a post-group.

Proof. Since conjugation by an element is always an automorphism of the group, we only have to show that $a \triangleright (b \triangleright c) = \big(a(a \triangleright b)\big) \triangleright c$.

1) If $a \triangleright b = a^{-1}ba$, then

$$a \triangleright (b \triangleright c) = a^{-1}b^{-1}cba,$$

and

$$\big(a(a \triangleright b)\big) \triangleright c = (ba) \triangleright c = a^{-1}b^{-1}cba.$$

2) If G is a two-step nilpotent group and $a \triangleright b = a^{-n}ba^n = b[b,a]^n$, then we have

$$a \triangleright (b \triangleright c) = a \triangleright \big(c[c,b]^n\big) = c[c,b]^n\big[c[c,b]^n, a\big] = c[c,b]^n[c,a]^n\big[[c,b],a\big]^n = c[c,b]^n[c,a]^n,$$

and

$$\big(a(a \triangleright b)\big) \triangleright c = \big(ab[b,a]^n\big) \triangleright c = c\big[c, ab[b,a]^n\big]^n = c[c,b]^n[c,a]^n\big[c,[b,a]^n\big]^n = c[c,b]^n[c,a]^n.$$

□

Let G be a two-step nilpotent group and $a \triangleright b = a^{-n}ba^n$ for some integer n. By Proposition 5.3, $(G,\cdot,\triangleright)$ is a post-group. By Theorem 2.2, $(G,\cdot,\circ)$ is a skew left brace, where

$$a \circ b = a \cdot (a \triangleright b) = aa^{-n}ba^n = ab[b,a]^n.$$

The following statement holds for the group $(G,\circ)$

Proposition 5.4. The group $(G,\circ)$ defined above is two-step nilpotent.

Proof. Note that the inverse element with respect to the operation $\circ$ is the same element as the inverse with respect to the operation $\cdot$. Indeed, $a^{-1} \circ a = a^{-1}a[a,a^{-1}]^n = e$. Denote by $[a,b]_\circ$ the commutator with respect to the operation $\circ$:

$$[a,b]_\circ = a^{-1} \circ b^{-1} \circ a \circ b = \big(a^{-1}b^{-1}[b^{-1},a^{-1}]^n\big) \circ \big(ab[b,a]^n\big) = \big(a^{-1}b^{-1}[b,a]^n\big) \circ \big(ab[b,a]^n\big) =$$
$$= a^{-1}b^{-1}[b,a]^n ab[b,a]^n\big[ab[b,a]^n, a^{-1}b^{-1}[b,a]^n\big]^n = [a,b][b,a]^{2n}\big[ba[a,b],(ba)^{-1}\big] = [b,a]^{2n-1}.$$

Now we can see that $\big[[a,b]_\circ, c\big]_\circ = \big[c,[b,a]^{2n-1}\big]^{2n-1} = \big[c,[b,a]\big]^{(2n-1)^2} = e$ for any $a,b,c \in G$, which means that the group $(G,\circ)$ is two-step nilpotent. □

It is easy to show that $(G,\circ)$ is not necessarily isomorphic to $(G,\cdot)$. Indeed, if $(G,\cdot)$ satisfies the relation $[a,b]^{2n-1} = e$ for any a and b, then $[a,b]_\circ = [a,b]^{1-2n} = e$, hence the group $(G,\circ)$ has to be abelian. With n not equal to 0 or 1, groups that satisfy the relation $[a,b]^{2n-1} = e$ do not have to be abelian.

Proposition 5.5. A skew left brace constructed above is a λ-homomorphic skew left brace.

Proof. We have to prove that the λ-map which corresponds to skew left brace $(G, \cdot, \circ)$ is a homomorphism $\lambda\colon (G, \cdot) \to \mathrm{Aut}(G, \cdot)$. By the formula after Proposition 5.3, the new product is

$$a \circ b = aa^{-n}ba^n.$$

Hence, $\lambda_a(b) = a^{-1} \cdot (a \circ b) = a^{-n}ba^n$ and we have

$$\lambda_a(\lambda_b(c)) = \lambda_a(b^{-n}cb^n) = a^{-n}b^{-n}cb^na^n.$$

On the other side,

$$\lambda_{ab}(c) = (ab)^{-n}c(ab)^n = a^{-n}b^{-n}[a, b]^{-n(n-1)/2}cb^na^n[a, b]^{n(n-1)/2} = a^{-n}b^{-n}cb^na^n.$$

Comparing with the previous formula, we see that $\lambda_a\lambda_b = \lambda_{ab}$ for any $a, b \in G$. It means that λ is a homomorphism. $\square$

5.2. **Verbal solutions of the Yang-Baxter equation.** Let X be a nonempty set and $S\colon X^2 \to X^2$. The map S is called a *solution of the Yang–Baxter equation* on X, if

$$S_1\, S_2\, S_1 = S_2\, S_1\, S_2,$$

where $S_1 = S \times \mathrm{Id}$, $S_2 = \mathrm{Id} \times S$.

The following theorem allows us to use skew left braces in order to obtain solutions of the Yang–Baxter equation.

Theorem 5.6 [3]. *Let $(G, \cdot, \circ)$ be a skew left brace. Then the map $S\colon G^2 \to G^2$, defined as*

$$S(a, b) = \Big(\lambda_a(b), \overline{\lambda_a(b)} \circ a \circ b\Big),$$

where $\overline{x}$ is the inverse of x with respect to the operation $\circ$, is a solution of the Yang-Baxter equation on the set G.

In the previous section we constructed some skew left braces on nilpotent groups. We will now proceed to use Theorem 2.2, and 5.6 to construct solutions to the Yang–Baxter equation on two-step nilpotent groups. We will now explore verbal solutions of the Yang–Baxter equation on two-step nilpotent groups.

Definition 5.7. For a group G, a map $\varphi\colon G^n \to G$ is called a *verbal map* if there is a group word $w = w(x_1, \dots, x_n)$ on n letters such that for any $g_1, \dots, g_n \in G$ we have $\varphi(g_1, \dots, g_n) = w(g_1, \dots, g_n)$.

For any group word w we will denote the verbal map obtained in this way by φ_w.

Definition 5.8. Let G be a group. A solution S of the Yang–Baxter equation on G is called a *verbal solution* if there are group words w_1 and w_2 such that $S = \varphi_{w_1} \times \varphi_{w_2}$.

Note that in a two-step nilpotent group G any verbal map φ has a nice standard form $\varphi(x, y) = x^ay^b[y, x]^m$, and that even though it needs not be a group homomorphism, it has a well-defined abelianization $\varphi^{Ab}(x,y) = x^a y^b$, and the following diagram commutes:

$$\begin{tikzcd} G^2 \arrow[r, "\varphi"] \arrow[d] & G \arrow[d] \\ (G^{Ab})^2 \arrow[r, "\varphi^{Ab}"] & G^{Ab} \end{tikzcd}$$

The following proposition is immediate from this:

Proposition 5.9. If $S = \varphi_{w_1} \times \varphi_{w_2}$ is a verbal solution of the Yang–Baxter equation on a two-step nilpotent group G, then $S^{Ab} = \varphi_{w_1}^{Ab} \times \varphi_{w_2}^{Ab}$ is a verbal solution of the Yang–Baxter equation on G^{Ab}.

Verbal maps $(G^{Ab})^2 \to (G^{Ab})^2$ can be represented as matrices with integer coefficients, and for 2×2 matrices we can fully describe which of them satisfy the Yang–Baxter equation:

Theorem 5.10. *Let M be a 2×2 matrix with coefficients in an integral domain R. The map from R^2 to R^2 defined by left multiplication by M is a solution of the Yang-Baxter equation on R if and only if M has at least one of the following forms:*

$$\begin{pmatrix} 1-bc & b \\ c & 0 \end{pmatrix};\ \begin{pmatrix} 0 & b \\ c & 1-bc \end{pmatrix};\ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix};\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We can write down the Yang–Baxter equation in the following form:

$$0 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} =$$
$$= \begin{pmatrix} a(a+bc-1) & abd & 0 \\ acd & ad(d-a) & -abd \\ 0 & -acd & -d(d+bc-1) \end{pmatrix}$$

and obtain the following system of algebraic equations:

$$\begin{aligned} &abd = 0; \\ &acd = 0; \\ &a(a+bc-1) = 0; \\ &d(d+bc-1) = 0; \\ &ad(d-a) = 0. \end{aligned}$$

Since the coefficients are taken from a ring with no zero divisors, the solution of the system can be decomposed into a union of solutions of four simpler systems of equations:

$$\begin{aligned}
&1)\ a = 0,\ d = 1 - bc;\\
&2)\ d = 0,\ a = 1 - bc;\\
&3)\ a = 0,\ d = 0;\\
&4)\ b = 0,\ c = 0,\ a(a-1) = 0,\ d(d-1) = 0.
\end{aligned}$$

Note that the solution of the system 4) is a union of 4 points, 3 of which are also solutions of 1), 2) or 3). With this in mind, 4) can be reduced to $a = 1, b = 1, c = 0, d = 0$, which completes the proof. □

We will now investigate verbal solutions of the Yang–Baxter equation on two-step nilpotent groups. We are interested in such pairs of group words $w_1(x, y) = x^a y^b [y, x]^m$, $w_2(x, y) = x^c y^d [y, x]^n$ that the map $S = \varphi_{w_1} \times \varphi_{w_2}$ is a solution of the Yang–Baxter equation on any two-step nilpotent group G. If w_1 and w_2 are such words, then the maps $(S \times \mathrm{Id})(\mathrm{Id} \times S)(S \times \mathrm{Id})$ and $(\mathrm{Id} \times S)(S \times \mathrm{Id})(\mathrm{Id} \times S)$ from F^3 to F^3 must coincide for any free two-step nilpotent group F.

Abelianization of a free two-step nilpotent group is a free abelian group, so S^{Ab} must be a solution of the Yang–Baxter equation on $\mathbb{Z}$, and as such, the matrix $M_S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must be of at least one of the forms listed in theorem 5.10. We will denote $S_1 = S \times \mathrm{Id}$, $S_2 = \mathrm{Id} \times S$ and write down the corresponding Yang–Baxter equation for each of these matrices with free parameters m and n.

Starting with $M_S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$S_1 S_2 S_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = S_1 S_2 \begin{pmatrix} x[y,x]^m \\ y[y,x]^n \\ z \end{pmatrix} = S_1 \begin{pmatrix} x[y,x]^m \\ y[y,x]^n[z,y]^m \\ z[z,y]^n \end{pmatrix} = \begin{pmatrix} x[y,x]^{2m} \\ y[y,x]^{2n}[z,y]^m \\ z[z,y]^n \end{pmatrix};$$

$$S_2 S_1 S_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = S_2 S_1 \begin{pmatrix} x \\ y[z,y]^m \\ z[z,y]^n \end{pmatrix} = S_2 \begin{pmatrix} x[y,x]^m \\ y[y,x]^n[z,y]^m \\ z[z,y]^n \end{pmatrix} = \begin{pmatrix} x[y,x]^m \\ y[y,x]^n[z,y]^{2m} \\ z[z,y]^{2n} \end{pmatrix}.$$

The Yang–Baxter equation here implies $n = 0$ and $m = 0$, so the only verbal solution corresponding to this matrix is

$$S(x, y) = (x, y).$$

Now assume $M_S = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. We have

$$S_1S_2S_1\begin{pmatrix}x\\y\\z\end{pmatrix}=S_1S_2\begin{pmatrix}y^b[y,x]^m\\x^c[y,x]^n\\z\end{pmatrix}=S_1\begin{pmatrix}y^b[y,x]^m\\z^b[z,x]^{cm}\\x^{c^2}[y,x]^{cn}[z,x]^{cn}\end{pmatrix}=\begin{pmatrix}z^{b^2}[z,x]^{bcm}[z,y]^{b^2m}\\y^{bc}[y,x]^{cm}[z,y]^{b^2n}\\x^{c^2}[y,x]^{cn}[z,x]^{cn}\end{pmatrix};$$
$$S_2S_1S_2\begin{pmatrix}x\\y\\z\end{pmatrix}=S_2S_1\begin{pmatrix}x\\z^b[z,y]^m\\y^c[z,y]^n\end{pmatrix}=S_1\begin{pmatrix}z^{b^2}[z,x]^{bm}[z,y]^{bm}\\x^c[z,x]^{bn}\\y^c[z,y]^n\end{pmatrix}=\begin{pmatrix}z^{b^2}[z,x]^{bm}[z,y]^{bm}\\y^{bc}[y,x]^{c^2m}[z,y]^{bn}\\x^{c^2}[y,x]^{c^2n}[z,x]^{bcn}\end{pmatrix}.$$

The Yang-Baxter equation in this case is equivalent to the following system of algebraic equations:

$$\begin{aligned}b(c-1)m&=0;\\b(b-1)m&=0;\\c(c-1)m&=0;\\b(b-1)n&=0;\\c(c-1)n&=0;\\c(b-1)n&=0.\end{aligned}$$

For the sake of uniformity, we will rename the free parameters to u and v. With that in mind, the set of solutions to the system of algebraic equations above and the corresponding verbal solutions S is as follows:

$$\begin{aligned}b=0,\ c=0:&\ \ S(x,y)=\big([y,x]^u,[y,x]^v\big);\\b=0,\ c=1,\ n=0:&\ \ S(x,y)=\big([y,x]^u,x\big);\\b=1,\ c=1:&\ \ S(x,y)=\big(y[y,x]^u,x[y,x]^v\big);\\b=1,\ c=0,\ m=0:&\ \ S(x,y)=\big(y,[y,x]^u\big);\\m=0,\ n=0:&\ \ S(x,y)=\big(y^u,x^v\big).\end{aligned}$$

Now assume $M_S=\begin{pmatrix}1-bc & b\\ c & 0\end{pmatrix}$. Note that in a two-step nilpotent group the following expression holds:

$$(xy)^k=x^ky^k[y,x]^{\frac12k(k-1)},$$

which can be proven by induction. Indeed, for $k=0$ the expression holds. If the expression holds for k, then

$$(xy)^{k+1}=x^ky^kxy[y,x]^{\frac12k(k-1)}=x^{k+1}y^{k+1}[y,x]^{\frac12k(k-1)+k}=x^{k+1}y^{k+1}[y,x]^{\frac12(k+1)(k+1-1)}$$

and

$$(xy)^{k-1}=x^ky^ky^{-1}x^{-1}[y,x]^{\frac12k(k-1)}=x^{k-1}y^{k-1}[y,x]^{\frac12k(k-1)-(k-1)}=x^{k-1}y^{k-1}[y,x]^{\frac12(k-1)(k-2)}.$$

Now, for the map S we have

$$
\begin{aligned}
S_1S_2S_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= S_1S_2\begin{pmatrix} x^{1-bc}y^b[y,x]^m \\ x^c[y,x]^n \\ z \end{pmatrix} = S_1\begin{pmatrix} x^{1-bc}y^b[y,x]^m \\ (x^c[y,x]^n)^{1-bc}z^b[z,x^c]^m \\ x^{c^2}[y,x]^{cn}[z,x^c]^n \end{pmatrix} = \\
&= S_1\begin{pmatrix} x^{1-bc}y^b[y,x]^m \\ x^{c(1-bc)}z^b[y,x]^{(1-bc)n}[z,x]^{cm} \\ x^{c^2}[y,x]^{cn}[z,x]^{cn} \end{pmatrix} = \\
&= \begin{pmatrix} (x^{1-bc}y^b[y,x]^m)^{1-bc}(x^{c(1-bc)}z^b[y,x]^{(1-bc)n}[z,x]^{cm})^b[x^{c(1-bc)}z^b, x^{1-bc}y^b]^m \\ (x^{1-bc}y^b[y,x]^m)^c[x^{c(1-bc)}z^b, x^{1-bc}y^b]^n \\ x^{c^2}[y,x]^{cn}[z,x]^{cn} \end{pmatrix} = \\
&= \begin{pmatrix} x^{(1-bc)}y^{b(1-bc)}z^{b^2}[y,x]^{(1-bc)m+\frac{1}{2}b^2c(1-bc)^2+b(1-bc)n-bc(1-bc)m}[z,x]^{bcm+\frac{1}{2}b^2c(1-bc)(b-1)+b(1-bc)m}[z,y]^{b^2m} \\ x^{c(1-bc)}y^{bc}[y,x]^{mc+\frac{1}{2}bc(1-bc)(c-1)-bc(1-bc)n}[z,x]^{b(1-bc)n}[z,y]^{b^2n} \\ x^{c^2}[y,x]^{cn}[z,x]^{cn} \end{pmatrix};
\end{aligned}
$$

$$
\begin{aligned}
S_2S_1S_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= S_2S_1\begin{pmatrix} x \\ y^{1-bc}z^b[z,y]^m \\ y^c[z,y]^n \end{pmatrix} = S_2\begin{pmatrix} x^{1-bc}(y^{1-bc}z^b[z,y]^m)^b[y^{1-bc}z^b, x]^m \\ x^c[y^{1-bc}z^b, x]^n \\ y^c[z,y]^n \end{pmatrix} = \\
&= S_2\begin{pmatrix} x^{(1-bc)}y^{b(1-bc)}z^{b^2}[y,x]^{(1-bc)m}[z,x]^{bm}[z,y]^{bm+\frac{1}{2}b^2(1-bc)(b-1)} \\ x^c[y,x]^{(1-bc)n}[z,x]^{bn} \\ y^c[z,y]^n \end{pmatrix} = \\
&= \begin{pmatrix} x^{(1-bc)}y^{b(1-bc)}z^{b^2}[y,x]^{(1-bc)m}[z,x]^{bm}[z,y]^{bm+\frac{1}{2}b^2(1-bc)(b-1)} \\ (x^c[y,x]^{(1-bc)n}[z,x]^{bn})^{1-bc}(y^c[z,y]^n)^b[y^c,x^c]^m \\ (x^c[y,x]^{(1-bc)n}[z,x]^{bn})^c[y^c,x^c]^n \end{pmatrix} = \\
&= \begin{pmatrix} x^{(1-bc)}y^{b(1-bc)}z^{b^2}[y,x]^{(1-bc)m}[z,x]^{bm}[z,y]^{bm+\frac{1}{2}b^2(1-bc)(b-1)} \\ x^{c(1-bc)}y^{(bc)}[y,x]^{(1-bc)^2n+c^2m}[z,x]^{b(1-bc)n}[z,y]^{bn} \\ x^{c^2}[y,x]^{c(1-bc)n+c^2n}[z,x]^{bcn} \end{pmatrix}.
\end{aligned}
$$

The Yang–Baxter equation in this case is equivalent to the following system of algebraic equations:

1) $(1-bc)m+\frac{1}{2}b^2c(1-bc)^2+b(1-bc)n-bc(1-bc)m-(1-bc)m=0;$

2) $bcm+\frac{1}{2}b^2c(1-bc)(b-1)+b(1-bc)m-bm=0;$

3) $b^2m-bm-\frac{1}{2}b^2(1-bc)(b-1)=0;$

4) $cm+\frac{1}{2}bc(1-bc)(c-1)-bc(1-bc)n=0;$

5) $b^2n-bn=0;$

6) $cn-c(1-bc)n-c^2n=0;$

7) $cn-bcn=0.$

Or, in an alternative form,

1) $b(1-bc)(\frac{1}{2}bc+n-cm)=0;$

2) $bc(b-1)(m+\frac{1}{2}b(1-bc))=0;$

3) $b(b-1)(m-\frac{1}{2}b(1-bc))=0;$

4) $\frac{1}{2}bc(1-bc)(c-1)-c(c-1)m-(1-bc)n=0;$

5) $b(b-1)n=0;$

6) $c^2(b-1)n=0;$

7) $c(b-1)n=0.$

Equations 5) – 7) all hold if and only if at least one of the following conditions is satisfied: either $b=c=0$, or $b=1$, or $n=0$. We will examine these three cases separately, making corresponding substitutions into equations 1)–4).

Case $b=c=0$. Equations 1)–3) hold automatically, and 4) is reduced to $n=0$. We have $1-bc=1$ and m a free parameter. This gives us the verbal solution

$$S(x,y)=\big(x[x,y]^u,1\big).$$

Case $b=1$. Equations 2) and 3) hold automatically, and we are left with the system

1) $(1-c)(\frac{1}{2}c+n-cm)=0;$

4) $(c-1)(-\frac{1}{2}c(c-1)+n-cm)$

If $c=1$, then the case is reduced to the previously examined case with the matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$.
If $n=cm-\frac{1}{2}c$ and $n=cm+\frac{1}{2}c(c-1)$, then we have $c(c-1)+c=0$, hence $c=0$ and

$n = 0$, and the case is reduced to a particular example of the case $n = 0$, which we will examine next.

Case $n = 0$. By substituting $n = 0$ into equations 1)–4) we get the system

$$\begin{aligned}
&1)\ bc(1-bc)(\frac{1}{2}b - m) = 0;\\
&2)\ bc(b-1)(m + \frac{1}{2}b(1-bc)) = 0;\\
&3)\ b(b-1)(m - \frac{1}{2}b(1-bc)) = 0;\\
&4)\ -c(c-1)(m - \frac{1}{2}b(1-bc)) = 0.
\end{aligned}$$

Equations 3) and 4) hold if and only if at least one of the following conditions holds: either $b, c \in \{0, 1\}$, or $m = \frac{1}{2}b(1-bc)$.

If $b = c = 0$ or $b = c = 1$, the case is reduced to one of the previously examined cases. If $b = c - 1 = 0$ or $c = b - 1 = 0$, then equations 1) and 2) hold automatically, m stays a free parameter, and we get two new verbal solutions:

$$\begin{aligned}
S(x, y) &= (x[y, x]^u, x);\\
S(x, y) &= (xy[y, x]^u, 1).
\end{aligned}$$

If $m = \frac{1}{2}b(1-bc)$, then the system is further reduced to

$$\begin{aligned}
&1)\ \frac{1}{2}b^3c^2(1-bc) = 0;\\
&2)\ b^2c(b-1)(1-bc) = 0,
\end{aligned}$$

which holds if and only if $b = 0$ or $c = 0$ or $1 - bc = 0$. If $1 - bc = 0$, then $m = 0$, and the case is reduced to a previously examined case. If $b = 0$, then $m = 0$, c is a free parameter, and we have the verbal solution

$$S(x, y) = (x, x^u).$$

Finally, if $c = 0$, then m is a free parameter, $b = 2m$, and we have the verbal solution

$$S(x, y) = (xy^{2u}[y, x]^u, 1).$$

As for the matrix $\begin{pmatrix} 0 & b \\ c & 1-bc \end{pmatrix}$, we will obtain the corresponding verbal solutions by using the symmetries of the Yang–Baxter equation.

Lemma 5.11. Let X be a set and $S\colon X^2 \to X^2$ is a solution of the Yang–Baxter equation on X. Then $S^\sigma = \sigma S \sigma$ is a solution of the Yang–Baxter equation on X, where $\sigma(x, y) = (y, x)$.

Proof. Define the map $\tau\colon X^3 \to X^3$ the following way: $\tau(x,y,z) = (z,y,x)$. We can assume that $S(x,y) = (f(x,y), g(x,y))$. Then $S^\sigma(x,y) = (g(y,x), f(y,x))$. Note that

$$\tau(S \times \mathrm{Id})\tau(x,y,z) = \tau(f(z), g(y), x) = (x, g(y), f(z)) = (\mathrm{Id} \times S^\sigma)(x,y,z).$$

Similarly, we have $\tau(\mathrm{Id} \times S)\tau = S^\sigma \times \mathrm{Id}$. Now,

$$(S^\sigma \times \mathrm{Id})(\mathrm{Id} \times S^\sigma)(S^\sigma \times \mathrm{Id}) = \tau(\mathrm{Id} \times S)(S \times \mathrm{Id})(\mathrm{Id} \times S)\tau;$$
$$(\mathrm{Id} \times S^\sigma)(S^\sigma \times \mathrm{Id})(\mathrm{Id} \times S^\sigma) = \tau(S \times \mathrm{Id})(\mathrm{Id} \times S)(S \times \mathrm{Id})\tau,$$

and since S is a solution of the Yang–Baxter equation, the right sides of these equalities coincide, and hence

$$(S^\sigma \times \mathrm{Id})(\mathrm{Id} \times S^\sigma)(S^\sigma \times \mathrm{Id}) = (\mathrm{Id} \times S^\sigma)(S^\sigma \times \mathrm{Id})(\mathrm{Id} \times S^\sigma).$$

□

Corollary 5.12. If $S(x,y) = \big(x^a y^b [y,x]^m, x^c y^d [y,x]^n\big)$ is a verbal solution of the Yang–Baxter equation on a 2-step nilpotent group, then $\bar{S}(x,y) = \big(x^d y^c [y,x]^{dc-n}, x^b y^a [y,x]^{ab-m}\big)$ is also a verbal solution.

Now, by combining all the solutions obtained and applying the symmetries, we can finally formulate the theorem.

Theorem 5.13. *If (w_1, w_2) is a pair of group words on two letters such that for any two-step nilpotent group G the induced map $S\colon G^2 \to G^2$, $S(x,y) = \big(w_1(x,y), w_2(x,y)\big)$ is a solution of the Yang–Baxter equation, then there are $u, v \in \mathbb{Z}$ such that $S(x,y)$ has one (or more) of the following forms:*

$$S(x,y) = (x,y);$$
$$S(x,y) = \big([y,x]^u, [y,x]^v\big);$$
$$S(x,y) = \big(y[y,x]^u, x[y,x]^v\big);$$
$$S(x,y) = \big(y^u, x^v\big);$$
$$S(x,y) = \big([y,x]^u, x\big); \qquad S(x,y) = \big(y, [y,x]^u\big);$$
$$S(x,y) = \big(x[y,x]^u, 1\big); \qquad S(x,y) = \big(1, y[y,x]^u\big);$$
$$S(x,y) = \big(x[y,x]^u, x\big); \qquad S(x,y) = \big(y, y[y,x]^u\big);$$
$$S(x,y) = \big(xy[y,x]^u, 1\big); \qquad S(x,y) = \big(1, xy[y,x]^u\big);$$
$$S(x,y) = (x, x^u); \qquad S(x,y) = (y^u, y);$$
$$S(x,y) = \big(xy^{2u}[y,x]^u, 1\big); \qquad S(x,y) = \big(1, x^{2u}y[y,x]^u\big).$$

Conversely, all of the maps above define verbal solutions of the Yang–Baxter equation on any two-step nilpotent group for any values of the parameters $u, v \in \mathbb{Z}$.

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