Best Simultaneous Approximation of Functions and a Generalized Minimax Theorem

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Abstract

Best simultaneous approximation (BSA) for finitely or infinitely many functions are considered under the uniform norm and other important norms. Characterization theorems for a BSA from a finite-dimensional subspace are obtained by a generalized minimax theorem. From the characterization theorem a strong unicity theorem is also deduced for a BSA.

1. Introduction

Let $\{f_a\}$ be a family of functions obtained in association with each element a in a set A. The purpose is to approximate these functions $\{f_a\}_{a\in A}$ simultaneously from a subspace H contained in a function space. In this scetion X is a compact Hausdorff space and $C(X)$ denotes the set of all real-valued continuous functions on X.

In [6] such an approximation problem was considered for real-valued functions $\{f_a\}_{a\in A}$ defined on X. The continuity of functions themselves is not supposed, but we assume uniform boundedness of the functions. For a specified subspace H of finite dimension in $C(X)$, we say that $f^* \in H$ is a *best simultaneous approximation* (*BSA*) for $\{f_a\}_{a\in A}$ from H, whenever f^* satisfies the inequality

$$\max_{a\in A,\, x\in X} |f_a(x) - f^*(x)| \le \max_{a\in A,\, x\in X} |f_a(x) - f(x)| \ \text{ for all } f \in H.$$

In [6] a characterization theorem for a BSA was deduced under the following conditions:

- both functions (of x) $\inf_{a\in A} f_a(x)$ and $\sup_{a\in A} f_a(x)$ belong to $C(X)$;
- for each $x \in X$, the infimum and supremum of $f_a(x)$ are, respectively, attained by some $f_a(x)$.

Moreover, if H is a Haar subspace, a strong unicity theorem for a BSA was obtained from the characterization theorem (see Section 3). When X is a finite closed interval, an alternation theorem for a BSA was also obtained that is similar to the ordinary one (see [1]).

In the next section we consider a BSA problem in a function space $C(X, Y)$ (the set of all continuous functions from X to Y), Y being a normed linear space over the real field $\mathbb{R}$ with norm $\|\cdot\|$. When a family of functions $\{f_a\}_{a\in A} \subset C(X, Y)$ and a finite-dimensional subspace $H \subset C(X, Y)$ are given, $f^* \in H$ is said to be a BSA to the functions $\{f_a\}_{a\in A}$ from H, if the inequality

$$\max_{a\in A,\, x\in X} \| f_a(x) - f^*(x) \| \le \max_{a\in A,\, x\in X} \| f_a(x) - f(x) \|$$

holds for all $f \in H$. In this setting we will deduce a characterization theorem of a BSA for $\{f_a\}_{a\in A}$ that corresponds to the one of [6].

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In Section 3, from this characterization theorem, a strong unicity theorem is derived in the function space $C(X)$.

In Section 4 we treat another BSA problem in L^p-approximation and obtain a characterization theorem of a BSA for finitely or infinitely many functions $\{f_a\} \subset L^p$. These characterization theorems are proved by means of a generalized minimax theorem ([4, Corollary 3.3]). For convenience sake we restate it as a lemma.

Lemma 1. *Let U be an n-dimensional, compact convex subset of a Hausdorff topological vector space, V a compact Hausdorff space, and let $J : U \times V \to \mathbb{R}$ be a jointly continuous function. An element $u^* \in U$ minimizes $\max_{v \in V} J(u, v)$ over U, if and only if there exist nonnegative numbers $\lambda_1, \ldots, \lambda_{n+1}$ with sum one, and $v_1^*, \ldots, v_{n+1}^* \in V$ such that*

$$\sum_{i=1}^{n+1} \mu_i J(u^*, v_i) \leq \sum_{i=1}^{n+1} \lambda_i J(u^*, v_i^*) \leq \sum_{i=1}^{n+1} \lambda_i J(u, v_i^*) \tag{1}$$

holds for all $u \in U$, $v_1, \ldots, v_{n+1} \in V$, and for all nonnegative numbers $\mu_1, \ldots, \mu_{n+1}$ with sum one.

As a useful remark we add that, ignoring all i such that $\lambda_i = 0$ and rearranging the suffix, (1) can be described as

$$\sum_{i=1}^{n+1} \mu_i J(u^*, v_i) \leq \sum_{i=1}^{k} \lambda_i J(u^*, v_i^*) \leq \sum_{i=1}^{k} \lambda_i J(u, v_i^*),$$

for some $k\,(1 \leq k \leq n+1)$ with $\sum_{i=1}^{k} \lambda_i = 1$ $(\lambda_i > 0)$.

2. Characterization theorem

For a compact Hausdorff space X and a normed linear space Y over the real field $\mathbb{R}$ with norm $\| \cdot \|$, we consider the set $C(X, Y)$ of all continuous functions from X to Y. A family of functions $\{f_a\}_{a \in A} \subset C(X, Y)$ and an n-dimensional subspace $H \subset C(X, Y)$ are given, where n is a positive integer. For $f \in C(X, Y)$ we define the uniform norm of f by

$$|||f||| = \max_{x \in X} \| f(x) \|,$$

and we endow the function space $C(X, Y)$ with this norm. Therefore, a BSA $f^* \in H$ is characterized by

$$\max_{a \in A} |||f_a - f^*||| \leq \max_{a \in A} |||f_a - f||| \quad \text{for all } f \in H.$$

We assume that A is a Hausdorff topological space and impose the two conditions:

(a) A is compact;

(b) the mapping $A \to C(X, Y)$ defined by $a \mapsto f_a$ is continuous.

Now let us introduce the following function

$$J(f, a, x) = \| f_a(x) - f(x) \|$$

defined on $H \times A \times X$. It is a jointly continuous function and convex in the argument f. Moreover, $A \times X$ is a compact set with respect to the product topology. Under this setting we have the following characterization theorem for a BSA.

Theorem 2. *An element $f^* \in H$ is a BSA to $\{f_a\}$ from H if and only if, for some positive integer $k\,(1 \le k \le n+1)$, there exist $a_1, \dots, a_k \in A$, $x_1, \dots, x_k \in X$ and positive numbers $\lambda_1, \dots, \lambda_k$, whose sum is one, such that*

(i) $\sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f^*(x_i) \parallel \le \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f(x_i) \parallel$ *for all* $f \in H$;

(ii) $\parallel f_{a_i}(x_i) - f^*(x_i) \parallel = |||f_{a_i} - f^*||| = \max_{a\in A} |||f_a - f^*|||$ *for all* $i\,(1 \le i \le k)$.

(Proof) Let f^* be a BSA. We define $U = \{f \in H : |||f - f^*||| \le 1\}$. Then U is a compact convex set of H, since H is finite-dimensional. First we consider the approximation problem over the set U in place of H. Then f^* is also a minimizer of $\max_{(a,x)\in A\times X} J(f,a,x)$ over U. Applying Lemma 1 and its remark to this situation, we see that, for some $k\,(1 \le k \le n+1)$, there exist $(a_1,x_1), \dots, (a_k,x_k) \in A \times X$, and positive numbers $\lambda_1, \dots, \lambda_k$ with $\sum_{i=1}^{k} \lambda_i = 1$ such that the following two inequalities hold:

$$\sum_{i=1}^{k} \lambda_i J(f^*, a_i, x_i) \le \sum_{i=1}^{k} \lambda_i J(f, a_i, x_i) \text{ for all } f \in U; \tag{2}$$

$$\sum_{i=1}^{n+1} \mu_i J(f^*, b_i, y_i) \le \sum_{i=1}^{k} \lambda_i J(f^*, a_i, x_i) \tag{3}$$

for all $b_1, \dots, b_{n+1} \in A$, $y_1, \dots, y_{n+1} \in X$ and all nonnegative numbers $\mu_1, \dots, \mu_{n+1}$ with sum one.

The right-hand side of (2) is a convex function of f and has a local minimum at $f^* \in U$. By a property of convex functions it follows that it has a global minimum at $f^* \in H$, which implies (i). Next in (3) putting $\mu_1 = 1$ while other $\mu_i = 0$, and $b_1 = a$ for any $a \in A$, we have $\parallel f_a(y) - f^*(y) \parallel \le \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f^*(x_i) \parallel$ for all $y \in X$, and hence for every $a \in A$

$$|||f_a - f^*||| \le \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f^*(x_i) \parallel .$$

This shows that $\max_{a\in A} |||f_a - f^*||| \le \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f^*(x_i) \parallel$. Using $\sum_{i=1}^{k} \lambda_i = 1$ $(\lambda_i > 0)$, we conclude that

$$\max_{a\in A} |||f_a - f^*||| \le \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f^*(x_i) \parallel \le \max_{a\in A} |||f_a - f^*|||,$$

which implies (ii).

Conversely, suppose that $f^* \in H$ satisfies conditions (i) and (ii) for a_i's in A, x_i's of X and positive numbers λ_i's such that $\sum_{i=1}^{k} \lambda_i = 1$. Then these conditions imply that, for any $f \in H$,

$$\max_{a\in A} |||f_a - f^*||| = \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f^*(x_i) \parallel \le \sum_{i=1}^{k} \lambda_i \parallel f_{a_i}(x_i) - f(x_i) \parallel \le \max_{a\in A} |||f_a - f|||,$$

showing that f^* becomes a BSA. This completes the proof.

Next we consider the case where A is a finite set, as discussed in [5]. Let $g_1, \dots, g_\ell \in C(X,Y)$ be given. In order to consider BSA to $\{g_j\}$, we introduce a compact set

$$A = \Big\{ a = (\alpha_1, \dots, \alpha_\ell) : \sum_{j=1}^{\ell} \alpha_j = 1,\ \alpha_j \ge 0\ (1 \le j \le \ell) \Big\}.$$

For each $a \in A$ we set $g_a = \sum_{j=1}^{\ell} \alpha_j g_j$. For $f \in H$, an n-dimensional subspace of $C(X,Y)$, we have, using the convexity of norm

$$\max_{1\le j\le \ell} |||g_j - f||| \le \max_{a\in A} |||g_a - f||| = \max_{a\in A} |||\sum_{1\le j\le \ell}^{\ell} \alpha_j(g_j - f)||| \le \max_{1\le j\le \ell} |||g_j - f|||.$$

Thus our approximation problem is reduced to simultaneously approximate $\{g_a\}$ $(a \in A)$ from H. Then as a special case of Theorem 2 follows the characterization theorem in [5].

3. Strong unicity theorem

Suppose that the norm of Y is defined by means of an inner product $\langle\ ,\ \rangle$ so that $\| y \|^2 = \langle y, y\rangle$ for $y \in Y$. The condition (i) of Theorem 2 is equivalent to the assertion; the function of a real variable t

$$\sum_{i=1}^{k} \lambda_i \| f_{a_i}(x_i) - f^*(x_i) + tf(x_i) \|$$

attains the minimum at $t = 0$ for every $f \in H$. By a simple calculation we obtain the next corollary, using the inner product.

Corollary 3. *Let Y be an inner product space. An element $f^* \in H$ is a BSA to $\{f_a\}$ from H if and only if, for some positive integer $k\,(1 \le k \le n+1)$, there exist $a_1, \dots, a_k \in A$, $x_1, \dots, x_k \in X$ and positive numbers $\lambda_1, \dots, \lambda_k$, whose sum is one, such that*

(i′) $\displaystyle\sum_{i=1}^{k} \lambda_i \langle f_{a_i}(x_i) - f^*(x_i), f(x_i)\rangle = 0$ *for all* $f \in H$;

(ii) $\| f_{a_i}(x_i) - f^*(x_i) \| = |||f_{a_i} - f^*||| = \max_{a\in A} |||f_a - f^*|||$ *for all* $i\,(1 \le i \le k)$.

In what follows we consider the case of $Y = \mathbb{R}$ and so we deal with the function space $C(X)$ as in Section 1. Referring to [2, p.91] or [4, Section 5], an n-dimensional subspace $H \subset C(X)$ is called a Haar subspace if, for n distinct elements $x_1, \dots, x_n \in X$ and for n arbitrary numbers $r_1, \dots, r_n \in \mathbb{R}$, there exists a unique $f \in H$ such that $f(x_k) = r_k$ $(1 \le k \le n)$.

A one-dimensional Haar subspace is obviously spanned by any function that does not vanish in X. A two-dimensional Haar subspace is spanned by every pair of functions $f,\ g \in C(X)$ satisfying $f(x)g(y) \ne f(y)g(x)$ whenever $x \ne y$, and so on (by linear algebra). Here the uniform norm of $C(X)$ is defined by

$$\| f \| = \max_{x\in X} |f(x)|$$

for $f \in C(X)$. Now we prove the following strong unicity theorem.

Theorem 4. *Suppose that the compact set X contains at least $n+1$ elements, where $n \ge 1$. Let a BSA $f^* \in H$ to $\{f_a\}_{a\in A}$ from an n-dimensional Haar subspace H satisfy* (i') *and* (ii) *of Corollary 3, for some integer $k\,(1 \le k \le n+1)$. If k elements $x_1, \dots, x_k \in X$ are all distinct and the common value of* (ii) *is not zero, then there exists a positive number γ such that*

$$\max_{a\in A} \| f_a - h \| \ge \max_{a\in A} \| f_a - f^* \| + \gamma \| f^* - h \| \quad \textit{for all } h \in H.$$

(Proof) First we show that $k = n+1$. If $k \le n$, there is a function $f \in H$ such that $f(x_i) = f_{a_i}(x_i) - f^*(x_i)$ for $i\,(1 \le i \le k)$, since H is a Haar subspace. Inserting this f into the equality (i') we have

$$\sum_{i=1}^{k} \lambda_i |f_{a_i}(x_i) - f^*(x_i)|^2 = |f_{a_i}(x_i) - f^*(x_i)|^2 = 0.$$

However, we assumed that $\delta = |f_{a_i}(x_i) - f^*(x_i)| = \max_{a\in A} \| f_a - f^* \|$ is not zero. Hence we must have $k = n+1$.

Letting h be an arbitrary element in H such that $\| h \|= 1$, condition (i') can be written as

$$\sum_{i=1}^{n+1} \lambda_i \sigma_i h(x_i) = 0, \quad \sigma_i = (f_{a_i}(x_i) - f^*(x_i))/\delta \quad (1 \le i \le n+1).$$

Since H is a Haar subspace and $\| h \|= 1$, it follows that $\max_{1\le i\le n+1} \sigma_i h(x_i) > 0$. If we set

$$\gamma = \min_{\|h\|=1} \max_{1\le i\le n+1} \sigma_i h(x_i),$$

then γ is positive, since the set $\{h \in H : \| h \|= 1\}$ is compact.

Let $f \neq f^*$ be any element in H and set $h = (f^* - f)/ \| f^* - f \|$. Then there exists at least one i satisfying

$$\sigma_i h(x_i) = \sigma_i (f^*(x_i) - f(x_i))/ \| f^* - f \| \ge \gamma,$$

and the required inequality follows using this i and $|\sigma_i| = 1$:

$$\begin{aligned}\max_{a\in A} \| f_a - f \| \ge \sigma_i(f_{a_i}(x_i) - f(x_i)) &= \sigma_i(f_{a_i}(x_i) - f^*(x_i)) + \sigma_i(f^*(x_i) - f(x_i)) \\ &\ge \max_{a\in A} \| f_a - f^* \| + \gamma \| f^* - f \| .\end{aligned}$$

4. BSA on L^p-spaces

Let (S, m) be a σ-finite positive measure space and $L^p(S, m)$ $(1 \le p < \infty)$ the set of all real-valued measurable functions f such that $|f|^p$ are integrable over S. For such p let q be the real number determined by $p^{-1} + q^{-1} = 1$ for $p > 1$, and $q = \infty$ for $p = 1$. We use the following notation (similarly for $\| g \|_q$ of $g \in L^q$),

$$\| f \|_p = \Big(\int_S |f|^p dm \Big)^{1/p},$$

and $\| f \|_\infty = \text{ess sup}_{x\in S} |f(x)|$.

We also assume that A is a compact set of a Hausdorff topological space and to each $a \in A$ there corresponds a function f_a which belongs to $L^p(S, m)$, and that the mapping $A \to L^p(S, m)$ so defined is continuous.

Let H be an n-dimensional subspace of $L^p(S, m)$, where $n \ge 1$. The problem is to approximate simultaneously the functions $\{f_a\}$ by elements of H. If $f^* \in H$ satisfies

$$\max_{a\in A} \| f_a - f^* \|_p \le \max_{a\in A} \| f_a - f \|_p \tag{4}$$

for all $f \in H$, we say that f^* is a BSA to the functions $\{f_a\}$ from H.

In order to formulate this problem in relation to Lemma 1, we need the following well-known facts. Property (a) is the Banach-Alaoglu theorem (see [3, p.68]) and property (b) is the duality pairing (see [7, p.115]).

(a) The dual space of $L^p(S,m)$ is $L^q(S,m)$ and $G = \{g \in L^q(S,m) : \| g \|_q \leq 1\}$ is a compact set in the weak*-topology $\sigma(L^q(S,m), L^p(S,m))$.

(b) For each $f \in L^p(S,m)$ we have $\| f \|_p = \max_{g \in G} \int_S (gf) dm$.

Therefore, (4) is equivalent to the following:

$$\max_{(a,g) \in A \times G} \int_S g(f_a - f^*) dm \leq \max_{(a,g) \in A \times G} \int_S g(f_a - f) dm \tag{5}$$

for all $f \in H$, and the problem is to find a function f^* satisfying (5).

It follows from (a) that $A \times G$ is a compact set in the product topology and it is easy to see that

$$J(f,a,g) = \int_S g(f_a - f) dm$$

is a jointly continuous function of the three variables $a \in A$, $g \in G$ and $f \in H$, using Hölder's inequality and the definition of the weak*-topology. Moreover, it is a convex function with respect to f. Hence we can again invoke Lemma 1 for the characterization of best approximations.

Theorem 5. *An element $f^* \in H$ is a BSA to $\{f_a\}$ from H if and only if, for some positive integer $k\,(1 \leq k \leq n+1)$, there exist $a_1, \ldots, a_k \in A$, $g_1, \ldots, g_k \in G$ and positive numbers $\lambda_1, \ldots, \lambda_k$ with sum one, satisfying the following two conditions*:

(i) $\displaystyle\int_S (\sum_{i=1}^k \lambda_i g_i) h \, dm = 0$ *for all* $h \in H$;

(ii) $\displaystyle\int_S g_i(f_{a_i} - f^*) dm = \| f_{a_i} - f^* \|_p = \max_{a \in A} \| f_a - f^* \|_p$ *for all* $i\,(1 \leq i \leq k)$.

(Proof) Let f^* be a BSA. Define the set $U = \{f \in H : \| f - f^* \|_p \leq 1\}$. Then U is compact and convex, for H is a finite-dimensional subspace of $L^p(S,m)$. The convexity follows from Minkowski's inequality (see [7, p.33]). It is obvious that f^* also minimizes $\max_{(a,g) \in A \times G} J(f,a,g)$ over the set U. It follows from Lemma 1, for some $k\,(1 \leq k \leq n+1)$, that there exist $a_1, \ldots, a_k \in A$, $g_1, \ldots, g_k \in G$, and numbers $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$ such that the following two inequalities hold:

$$\sum_{i=1}^k \int_S \lambda_i g_i (f_{a_i} - f^*) dm \leq \sum_{i=1}^k \int_S \lambda_i g_i (f_{a_i} - f) dm \tag{6}$$

for all $f \in U$; and

$$\sum_{i=1}^{n+1} \int_S \mu_i h_i (f_{b_i} - f^*) dm \leq \sum_{i=1}^k \int_S \lambda_i g_i (f_{a_i} - f^*) dm \tag{7}$$

for all $(b_1, h_1), \ldots, (b_{n+1}, h_{n+1}) \in A \times G$, and all nonnegative numbers $\mu_1, \ldots, \mu_{n+1}$ with sum one.

Inequality (6) implies

$$\sum_{i=1}^k \int_S \lambda_i g_i (f - f^*) dm \leq 0 \quad \text{for all } f \in U.$$

Let us put $f = f^* + th$, where $h \in H$ is arbitrary and $t > 0$ is so small that this f belongs to U. Then we get

$$\sum_{i=1}^k \int_S \lambda_i g_i h \, dm = \int_S (\sum_{i=1}^k \lambda_i g_i) h \, dm \leq 0 \quad \text{for all } h \in H,$$

which means condition (i), since the left-hand side of the inequality must be zero.

By setting $b_i = a_i$ and $\mu_i = \lambda_i$ for all i $(1 \le i \le k)$ in (7) and remarking property (b), we see that (7) implies

$$\sum_{i=1}^{k} \lambda_i \| f_{a_i} - f^* \|_p \le \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f^*) dm.$$

Since the reverse inequality always holds, we conclude

$$\sum_{i=1}^{k} \lambda_i \| f_{a_i} - f^* \|_p = \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f^*) dm \le \max_{a \in A} \| f_a - f^* \|_p .$$

Next in (7) putting $\mu_1 = 1$ (so $\mu_i = 0$ for $i \neq 1$) and $b_1 = a$ for any $a \in A$, we have for any $a \in A$

$$\max_{g \in G} \int_S g(f_a - f^*) dm = \| f_a - f^* \|_p \le \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f^*) dm,$$

hence

$$\max_{a \in A} \| f_a - f^* \|_p \le \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f^*) dm \le \max_{a \in A} \| f_a - f^* \|_p . \tag{8}$$

Then we conclude from (8) and $\lambda_i > 0$ that for all $i \, (1 \le i \le k)$

$$\| f_{a_i} - f^* \|_p = \int_S g_i (f_{a_i} - f^*) dm = \max_{a \in A} \| f_a - f^* \|_p ,$$

which is condition (ii).

Conversely, suppose that f^* satisfies conditions (i) and (ii). Let $f \in H$ be any element. We have by (i)

$$\sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f) dm - \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f^*) dm = \int_S \Big(\sum_{i=1}^{k} \lambda_i g_i \Big) (f^* - f) dm = 0. \tag{9}$$

On the other hand, using (ii),

$$\begin{aligned} \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f) dm - \sum_{i=1}^{k} \int_S \lambda_i g_i (f_{a_i} - f^*) dm &\le \max_{1 \le i \le k} \| f_{a_i} - f \|_p \\ - \max_{a \in A} \| f_a - f^* \|_p \le \max_{a \in A} \| f_a - f \|_p &- \max_{a \in A} \| f_a - f^* \|_p, \end{aligned}$$

where the condition $\sum_{i=1}^{k} \lambda_i = 1$ $(\lambda_i > 0)$ is used. Immediately we conclude that f^* is a BSA in view of (9), thereby completing the proof.

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