# On groups whose conjugacy class sizes are not divisible by each other

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*Abstract: Let $G$ be a finite group and $N(G)$ be the set of its conjugacy class sizes excluding 1. Let us define a directed graph $\Gamma(G)$, the set of vertices of this graph is $N(G)$ and the vertices $x$ and $y$ are connected by a directed edge from $x$ to $y$ if $x$ divides $y$ and $N(G)$ does not contain a number $z$ different from $x$ and $y$ such that $x$ divides $z$ and $z$ divides $y$. We will call the graph $\Gamma(G)$ the conjugate graph of the group $G$. In this work, we will study finite groups whose conjugate graph is a set of points.*

*Keywords: finite group, conjugacy classes, conjugate graph.*

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## Introduction

Over the years, considerable work has been done to establish relations between the structure of a finite group and its set of sizes of conjugacy classes.

We say that a group $G$ has conjugate rank $n$ (shortly rank or $rank(G)$) if $|N(G)| = n$. Noboru Ito laid the foundation for the study of $F$-groups in his famous paper [8]. A finite group $G$ is $F$-group if $x, y \in G \setminus Z(G)$ and $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$. An important subclass of $F$-groups is the class of rank 1 groups, i.e. $I$-group is a group whose set of conjugacy classes sizes is $\{1, n\}$. Ito proved that rank 1 groups are nilpotent, in particular $n$ is a prime power. Later, Kenta Ishikawa showed [9] that rank 1 groups are of class at most 3. In [10] rank 1 groups when $p \neq 2$ were described.

Johen Rebmann [11] proved a classification theorem describing $F$-groups. He determined their structure, up to $F$-groups which are central extensions of groups of prime-power order.

One more subclass of $F$-groups is the class of $CA$-groups. Finite group $G$ is a $CA$-group if all centralizers of noncentral elements are abelian. The $CA$-groups were investigated by Roland Schmidt [12]. He determined their structure up to $CA$-groups which are central extensions of groups of prime-power order.

Silvio Dolfi, Marcel Herzog and Enrico Jabara [2] studied $CH$ groups, consisting of finite groups in which noncentral commuting elements have centralizers of the same order. In particular, the following inclusion was proved

$$CA \subset CH \subset F.$$

Given $\Theta \subseteq \mathbb{N}, |\Theta| < \infty$, define the directed graph $\Gamma(\Theta)$, with the vertex set $\Theta$ and edges $\overrightarrow{ab}$ whenever $a$ divides $b$ and $\Theta$ does not contain a number $c$ such that $a$ divides $c$

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and $c$ divides $b$. In [6], the conjugate graph $\Gamma(G)$ was defined for a finite group $G$. Set $\Gamma(G) = \Gamma(N(G) \setminus \{1\})$. We will say that $G$ is an $SP$-group if $\Gamma(G)$ does not contain edges. Note that the definition of $SP$-groups differs significantly from the definition of $CA$- and $CH$-groups. In this case, only the arithmetic properties of the group are used.

The main goal of this manuscript is to describe groups with the $SP$ property. In particular, we will prove the inclusion $SP \subset CH$.

**Theorem 1.** *$SP \subset CH$.*

Using this result and classification of $CH$-groups we obtain a classification of $SP$-groups.

**Theorem 2.** *A group $G$ is $SP$-group if and only if it is of one of the following types.*

*(I) $G = T \times P$ where $T$ is abelian and $P$ is a $p$-group for some prime $p$, $rank(P) = 1$.*

*(II) $G/Z$ is a Frobenius group with Frobenius kernel $K/Z$ and Frobenius complement $L/Z$, where $K$ and $L$ are abelian.*

*(III) $G/Z$ is a Frobenius group with Frobenius kernel $K/Z$ and Frobenius complement $L/Z$, such that $K = PZ$, where $rank(P) = 1$ and $P$ is a normal Sylow $p$-subgroup of $G$ for some $p \in \pi(G)$, $Z(P) = Z \cap P$ and $L$ is abelian.*

*(IV) $G/Z \simeq PSL(2, p^n)$ or $PGL(2, p^n)$ and $G' \simeq SL(2, p^n)$, where $p$ is a prime and $p^n > 3$.*

*(V) $G/Z \simeq PSL(2, 9)$ or $PGL(2, 9)$ and $G'$ is isomorphic to the Schur cover of $PSL(2, 9)$.*

**Corollary 1.** *If $\Gamma(G)$ is an edgeless graph with two vertices, then $G/Z$ is a solvable Frobenius group.*

In [3], Dolfi and Jabara studied groups of rank 2 and, in particular, from their description one can also obtain Corollary 1.

**Corollary 2.** *The graph $\Gamma(G)$ of an $SP$-group $G$ contains at most 3 vertices.*

# 1 Preliminaries

Let $G$ be a group and take $x \in G$. We denote by $x^G$ the conjugacy class of $G$ containing $x$ and $C_H(x)$ is the centralizer of $x$ in the subgroup $H$. If $N$ is a subgroup of $G$, then $Ind(N, x) = |N|/|C_N(x)|$. Note that $Ind(G, x) = |x^G|$.

**Lemma 1** ([1, Lemma 1])**.** *If, for some prime $p$, every $p'$-element of a group $G$ has index prime to $p$, then the Sylow $p$-subgroup of $G$ is a direct factor of $G$.*

**Lemma 2** ([7, Lemma 1.4])**.** *Let $G$ be a finite group, $K \trianglelefteq G$ and $\overline{G} = G/K$. Take $x \in G$ and $\overline{x} = xK \in G/K$. Then the following conditions hold*

*(i) $|x^K|$ and $|\overline{x}^{\overline{G}}|$ divide $|x^G|$.*

*(ii) If $L$ and $M$ are consequent members of a composition series of $G$, $L < M$, $S = M/L$, $x \in M$ and $\widetilde{x} = xL$ is an image of $x$, then $|\widetilde{x}^S|$ divides $|x^G|$.*

*(iii) If $y \in G, xy = yx$, and $(|x|, |y|) = 1$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

*(iv) If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

*(v) $\overline{C_G(x)} \leq C_{\overline{G}}(\overline{x})$.*

The following two lemmas are simple exercises.

**Lemma 3.** *Let $P$ be a $p$-group. Then $P/Z(P)$ is not a cyclic group.*

**Lemma 4.** *Let $G$ be a finite group, $K \trianglelefteq G, x \in G$. Then $C_G(x)K/K \leq C_{G/K}(xK)$.*

**Definition 1.** *A finite group $G$ is called a $CH$-group if for every $x, y \in G \setminus Z(G), xy = yx$ implies that $|C_G(x)| = |C_G(y)|$.*

**Lemma 5.** *[3, Theorem 4.2] Let $G$ be a nonabelian group and write $Z = Z(G)$. Then $G$ is a $CH$-group if and only if it is of one of the following types.*

*(I) $G$ is nonabelian and has an abelian normal subgroup of prime index.*

*(II) $G/Z$ is a Frobenius group with Frobenius kernel $K/Z$ and Frobenius complement $L/Z$, where $K$ and $L$ are abelian.*

*(III) $G/Z$ is a Frobenius group with Frobenius kernel $K/Z$ and Frobenius complement $L/Z$, such that $K = PZ$, where $P$ is a normal Sylow $p$-subgroup of $G$ for some $p \in \pi(G)$, $P$ is a $CH$-group, $Z(P) = Z \cap P$ and $L \simeq HZ$, where $H$ is an abelian $p'$-subgroup of $G$.*

*(IV) $G/Z \simeq S_4$ and if $V/Z$ is the Klein four group in $G/Z$, then $V$ is nonabelian.*

*(V) $G \simeq P \times A$, where $P$ is a nonabelian $CH$-group of prime-power order and $A$ is abelian.*

*(VI) $G/Z \simeq PSL(2, p^n)$ or $PGL(2, p^n)$ and $G' \simeq SL(2, p^n)$, where $p$ is a prime and $p^n > 3$.*

*(VII) $G/Z \simeq PSL(2, 9)$ or $PGL(2, 9)$ and $G'$ is isomorphic to the Schur cover of $PSL(2, 9)$.*

**Definition 2.** *We will say that $G$ lies in the class $F(i)$, where $i \in \{I, II, .., VII\}$ and write $G \in CH(i)$ if $G$ is a $CH$-group and item $i$ from Lemma* **??** *is satisfied.*

**Definition 3.** *A finite group $G$ is called an $SP$-group if for every $x, y \in G \setminus Z(G)$, $|C_G(x)|$ divides $|C_G(y)|$ implies that $|C_G(x)| = |C_G(y)|$.*

**Remark 1.** *A set of integers greater than 1 is primitive if no element of the set divides another. The definition of $SP$-groups is equivalent to a requirement that $N(G)$ is a primitive set.*

**Lemma 6.** [5, Theorem 5.2.3] *Let $A$ be a $p'$-group of automorphisms of abelian $p$-group $P$. Then $P = C_P(A) \times [P, A]$.*

# 2 Proof of Theorem 1

Let $G \in SP$, $Z = Z(G)$, and $x, y \in G \setminus Z$ be commuting elements. Let us show that $|C_G(x)| = |C_G(y)|$. We have $x = x_1, x_2, ..x_n$, where $x_i$ are elements of primary and coprime order. Note that $C_G(x_ix_j) = C_G(x_i) \cap C_G(x_j)$. Thus, among the elements $x_1, .., x_n$ there is an element $z$ that does not lie in $Z(G)$. For any $a \in C_G(z)$ such that $(a, z) = 1$, we have $C_G(a) = C_G(z)$ or $C_G(a) = G$. Similarly, in $C_G(y)$ there is an element $t$ of primary order such that $C_G(t) = C_G(y)$ and for any $b \in C_G(t)$ such that $(t, b) = 1$, we have $C_G(b) = C_G(t)$ or $C_G(b) = G$. Since the elements $x$ and $y$ commute, it follows that the elements $z$ and

$t$ commute. If $t$ and $z$ have coprime order, then $C_G(tz) = C_G(t) \cap C_G(z)$ and therefore $C_G(t) = C_G(tz) = C_G(z)$ and the statement has been proven.

Let's assume that $z$ and $t$ are $p$-elements, where $p$ is a prime number. If in the centralizer of $z$ or $t$ there is a $p'$-element $h$ such that $C_G(h) \neq G$, then $C_G(h) = C_G(z)$ and $C_G(h) = C_G(th) = C_G(t)$. Therefore $C_G(z) = C_G(t)$. Thus, if $C_G(z)$ or $C_G(t)$ contains a non-central $p'$ element, then $C_G(z) = C_G(t)$. Suppose that $|C_G(z)|_{p'} = |C_G(t)|_{p'} = |G/Z|_{p'}$. Then $Ind(G, z) = Ind(G, t)p^n$, in particular $Ind(G, z)$ divides $Ind(G, t)$. Therefore $|C_G(z)| = |C_G(t)|$.

Thus, we have shown that $SP \subset CH$. An example of a $CH$-group that is not $SP$ is given in Remark 2.

# 3 Proof of Theorem 2

It follows from Theorem 1 that any $SP$-group is a $CH$-group. To prove Theorem 2, let us study which $CH$-groups are $SP$-groups.

We will say that $G$ lies in the class $CH(i)$, where $i \in \{I, II, .., VII\}$ and write $G \in CH(i)$ if $G$ is a $CH$ group and item $i$ from Lemma **??** is satisfied. Note that the classes $CH(i)$ have intersections, therefore the number $i$ may not be uniquely determined.

Let $G \in SP$, $Z = Z(G)$.

**Lemma 7.** *$G \in F(I)$ if and only if one of the statements is true*

1. *$G = T \times P$ where $P$ is a Sylow $p$-subgroup of $G$ and $rank(P) = 1$;*

2. *$P$ is abelian.*

*Proof.* We have that $G$ contains a normal abelian subgroup $A$ of index $p$. Let $H$ be a Hall $p'$-subgroup of $G$. Since $H \leq A$, then $H$ is normal in $G$ and is an abelian group.

Let $P \leq H$ be a Sylow $p$-subgroup of $G$. Assume that $P$ acts non-trivially on $H$. From Lemma 6 and the fact that $H$ is an Abelian $p'$-group it follows that there is $x \in P$ such that $H = C_H(x) \times Y$, where $Y > 1$. In particular, $Ind(H, x) > 1$ and $x \notin A$. We have $N_G(Y) \geq A\langle x\rangle$. Since $N_G(Y)/A = G/A$, we have $Y \trianglelefteq G$.

Let $y \in Y$. Since $Y$ is a normal subgroup of $G$ and $A \leq C_G(y)$, we have $Ind(G, y) = p$. Note that $Ind(G, x)_p = Ind(P, x)$.

Assume that $P$ is not abelian. Assume that $Ind(P, x) = 1$. Since $C_G(A) \geq A\langle x\rangle = P$, we have $A \leq Z(P)$. From the fact that $|P|/|A| = p$ and Lemma 3 it follows that $P$ is abelian; a contradiction. Therefore, $Ind(G, x)_p > 1$. Thus $Ind(G, y)$ divides $Ind(G, x)$; a contradiction. Therefore $P$ is abelian and Statement 1 of the Lemma holds.

Suppose that $P$ acts trivially on $H$. Therefore, for any $h \in H$ we have $Ind(G, h)_p = 1$. It follows from Lemma 1, that $P$ is a direct factor of the group $G$. For any $x \in P$ we have $Ind(G, x) = Ind(P, x)$. Thus, if $P$ is not a group of rank 1 then $P$ contains elements $x$ and $y$ such that $1 < Ind(G, x) < Ind(G, y)$ and $Ind(G, x)|Ind(G, y)$; a contradiction. Therefore $rank(P) = 1$.

Thus, if $G \in SP$, then one of the statements of the lemma holds.

Let us prove that if one of the statements of the lemma is satisfied, then $G \in SP$.

Let us assume that the statement 1 of the Lemma is satisfied. Then $N(G) = \{p\}$ and hence $G \in SP$.

Let us assume that the statement 2 of the Lemma is satisfied. Then $N(G) = \{p, k\}$, where $k$ is coprime with $p$. Therefore, $G \in SP$.

$\square$

**Remark 2.** *Note that the class $F(I)$ contains a $CA$-group that is not an $SP$-group. For example, we can take a $p$-group $P = A \rtimes B$, where $A$ is an elementary abelian group of order $p^p$ and $B$ is a group of order $p$ acting permutatively on $A$. Obviously, the centralizer of any non-central element of $G$ is abelian and $N(G) = \{p, p^{p-1}\}$. If $p > 2$, then $G \in CA \setminus SP$. This means that the class of $SP$-groups does not coincide with the classes of $CA$- and $CH$-groups.*

**Lemma 8.** *Groups from $F(II)$ are $SP$-groups.*

*Proof.* We have that $G/Z$ is a Frobenius group with Frobenius kernel $K/Z$ and Frobenius complement $L/Z$, where $K$ and $L$ are abelian. Let $\bar{\ } : G \to G/Z$ be a natural homomorphism. Take $x \in K$, such that $Ind(G, x) > 1$. The fact that $\overline{C_G(x)} \leq C_{\overline{G}}(\overline{x})$ and $C_{\overline{G}}(\overline{x}) = \overline{K}$ implies that $Ind(G, x) = |L/Z|$. Similarly, if $y \in L$ is such that $Ind(G, y) > 1$, then $Ind(G, y) = |\overline{K}|$. Since any element of $G$ lies in the subgroup conjugate to one of the subgroups $L$ or $K$, then the set of conjugacy class sizes of the group $G$ is $\{1, |\overline{K}|, |\overline{L}|\}$. The groups $\overline{K}$ and $\overline{L}$ have coprime order. Thus $G \in SP$.

$\square$

**Remark 3.** *Note that groups satisfying statement 2 of Lemma 7 are $F(II)$ groups.*

**Lemma 9.** *$G \in F(III)$ if and only if $rank(K) = 1$.*

*Proof.* We have that $G/Z$ is a Frobenius group with Frobenius kernel $K/Z$ and Frobenius complement $L/Z$, such that $K = PZ$, where $P$ is a Sylow $p$-subgroup of $G$ for some $p \in \pi(G)$, $P$ is $CH$-group, $Z(P) = Z \cap P$ and $L \simeq HZ$, where $H$ is an abelian $p'$-subgroup of $G$. In particular, $L$ is abelian.

Suppose that there are elements $x, y \in P$, such that $Ind(P, x) > Ind(P, y) > 1$. Since $P$ is a normal Sylow $p$-subgroup of $G$ it follows that $Ind(G, x)_p = Ind(P, x)$. Similar to Lemma 8, it can be shown that $Ind(G, x) = Ind(P, x)|L/Z|$ and $Ind(G, y) = Ind(P, y)|L/Z|$. Thus $Ind(G, x)|Ind(G, y)$; a contradiction. Thus $rank(P) = 1$.

$\square$

**Lemma 10.** *If $G \in F(IV)$, then $G$ is not an $SP$-group.*

*Proof.* We have $G/Z \simeq S_4$ and if $V/Z$ is the Klein four group in $G/Z$, then $V$ is nonabelian.

Let $\overline{G} = G/Z$, $h \in G$ be such that its image $\overline{h} \in \overline{G}$ is of order 4. Note that $C_{\overline{G}}(\overline{h}) = \langle h \rangle$. From the fact that $\overline{C_G(h)} \leq C_{\overline{G}}(\overline{h})$, it follows that $C_G(h) = Z\langle h \rangle$. Therefore, $Ind(G, h) = 6$.

Let $g \in G$ be a 2-element such that $\overline{g}$ is an element of order 2 and does not lie in any proper normal subgroup of $\overline{G}$. In the standard permutation representation of $S_4$, the element $g$ has the form $(1, 2)$. Note that $\overline{C} = C_{\overline{G}}(\overline{g}) = \langle \overline{g} \rangle \times \langle \overline{z} \rangle$, where $\overline{z}$ is the central element of some Sylow 2-subgroup of the group $\overline{G}$. In the standard permutation representation of $S_4$, the element $z$ has the form $(1, 2)(3, 4)$. The group $\overline{C}$ is the Klein four group and therefore $C$ is nonabelian. Note that $Z < C_G(g) \leq C$. Therefore $C_G(g) = Z\langle g \rangle$. So $Ind(G, g) = 12$. Thus, $Ind(G, h)$ divides $Ind(G, g)$; a contradiction.

$\square$

**Lemma 11.** *$G \in F(V)$ if and only if $rank(P) = 1$.*

*Proof.* The statement of the lemma follows from the fact that $N(G) = N(P)$.

□

**Remark 4.** *Note that the groups from Statement* 1 *of Lemma 7 are $F(V)$-groups.*

**Lemma 12.** *Groups from $F(VI)$ are $SP$-groups.*

*Proof.* We have $G/Z \simeq PSL(2, p^n)$ or $PGL(2, p^n)$ and $G' \simeq SL(2, p^n)$ where $p$ is a prime and $p^n > 3$. Put $q = p^n$. It's not difficult to check that $N(SL_2(q)) = \{(q^2-1)/2, q(q-1), q(q+1)\}$ and $N(GL_2(q)) = \{q(q-1), q^2-1, q(q+1)\}$.

Let $X \leq G$ be a subgroup of minimal order such that $XZ/Z = G/Z$, $S \leq X$ be such that $S \simeq SL_2(q)$. Since $G = XZ$, we see that $X \trianglelefteq G$ and $N(G) = N(X)$. We have $X = S \rtimes L$, where $L$ is an abelian group acting on $S$ as a group of diagonal automorphisms or $L$ is the trivial group. Thus $N(X) = N(SL_2(q))$ or $GL_2(q)$.

□

**Lemma 13.** *Groups from $F(VII)$ are $SP$-groups.*

*Proof.* We have $G/Z \simeq PSL(2, 9)$ or $PGL(2, 9)$ and $G'$ is isomorphic to the Schur cover of $PSL(2, 9)$. Let $X \leq G$ be a subgroup of minimal order such that $XZ/Z = G/Z$. Using [4] it's easy to get that $N(X) = N(G') = \{72, 90, 120\}$. Similarly as in Lemma 12 we obtain that $N(G) = N(S)$ and therefore $G \in SP$.

□

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