On balanced homodyne measurement – simple proof of wave function collapse

E. Brüning[a] and S. Nagamachi[b]

[a]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, and and National Institute for Theoretical Physics (NITheP), KwaZulu-Natal, South Africa
[b]The University of Tokushima, Tokushima 770-8506, Japan

ARTICLE HISTORY
Compiled April 22, 2024

ABSTRACT
Using only elementary calculus we prove that the balanced homodyne detector measures the quadrature phase amplitude of a signal. More precisely, after the measurement of photon numbers l, the collapse of the composite state of a strong laser beam and a signal approximates the collapse of the signal to the eigen-state of quadrature phase amplitude with eigen-value r.

KEYWORDS
quantum teleportation, continuous variables, laser beam, homodyne detection; 81V80,81P45,47N50

Contents

1 Introduction 1

2 Holomorphic representation of CCR 8

3 Balanced homodyne measurement 12

4 Proof of the convergence of $|\alpha|\,{}_2\langle\alpha|\Pi^{[x|\alpha|]}|\alpha\rangle_2$ 16

5 Conclusion 27

1. Introduction

Optical homodyne detection is the extremely useful and flexible measuring method and has broad applications in optical communication, e.g., quantum key distribution [7, 8], quantum teleportation [2], quantum computation [9] and weak signal detection [4, 5]. In the balanced homodyne detection, the signal is mixed with a relative strong local oscillator on a half-beamsplitter, the two output modes are detected by a pair of detectors and the difference of two photo currents can be measured [1, 4, 5].

CONTACT E. Brüning. Email: bruninge@ukzn.ac.za

This article is a continuation of our articles [3, 6]. In order to explain how balanced homodyne detection arises and how it is used in the theory of quantum teleportation we recall the relevant parts of [3, 6]: The total system is composed of three Hilbert spaces $\mathcal{H}_j$ $(j = 0, 1, 2)$ which are isomorphic each other. c_l^* and c_j are the creation and the annihilation operators defined in $\mathcal{H}_j$ and $|n\rangle_j = \frac{1}{\sqrt{n!}} c_j^{*n}|0\rangle$ $(n = 0, 1, \ldots)$ constitutes an orthonormal basis of $\mathcal{H}_j$, where $|0\rangle$ is the vacuum vector. The system composed of $\mathcal{H}_1$ and $\mathcal{H}_2$ describes some localized part denoted part (A). The Hilbert space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ has a generalized basis

$$\left\{ \pi^{-1/2} \sum_{n=0}^{\infty} (D_0(\alpha)|n\rangle_0) \otimes |n\rangle_1; \alpha \in \mathbb{C} \right\}, \quad D_0(\alpha) = \mathrm{e}^{\alpha c_0^* - \bar{\alpha} c_0} \tag{1.1}$$

called a generalized Bell basis, where $D_j(\alpha)$ denotes the operator of translation by α on $\mathcal{H}_j$. The system $\mathcal{H}_3$ describes another localized part denoted by (B). Alice (a local observer at part (A)) performs the Bell-state measurements with respect to the generalized Bell basis on the state $|\Phi\rangle$ of the composite system of $|\psi\rangle_0 = \sum_{n=0}^{\infty} a_n |n\rangle_0$ (to be teleported) and a quantum channel $\sum_{n=0}^{\infty} q^n |n\rangle_1 \otimes |n\rangle_2$ (via which $|\psi\rangle_0$ is teleported to Bob (a local observer at part (B))), i.e., on

$$|\Phi\rangle = \pi^{1/2} |\psi\rangle_0 \otimes \sum_{n=0}^{\infty} q^n |n\rangle_1 \otimes |n\rangle_2 \in \mathcal{H}_0 \hat{\otimes} \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, \ |q| < 1.$$

The Bell-state measurement is done by sending the state $|\Phi\rangle$ through the half-beamsplitter $\mathrm{e}^{\mathrm{i} H_{hbs}^{01}}$ and by the measurements of the observables x_0 and p_1:

$$H_{hbs}^{01} = \mathrm{i}(\pi/4)(c_0^* c_1 - c_0 c_1^*), \quad x_0 = \frac{1}{\sqrt{2}}(c_0 + c_0^*), \quad p_1 = \frac{1}{\sqrt{2}\mathrm{i}}(c_1 - c_1^*). \tag{1.2}$$

Let $\{|x_-\rangle_0; x_- \in \mathbb{R}\}$ be the generalized basis which consists of generalized eigen-vectors $|x_-\rangle_0$ of x_0 with the eigen-value x_-, and $\{|p_+\rangle_1; p_+ \in \mathbb{R}\}$ the generalized basis consisting of generalized eigen-vectors $|p_+\rangle_1$ of p_1 with eigen-value p_+. Then we have (see Remark 2.7)

$$\begin{aligned} & \mathrm{e}^{-\mathrm{i} H_{hbs}^{01}} |x_-\rangle_0 \otimes |p_+\rangle_1 && = \pi^{-1/2} \textstyle\sum_{k=0}^{\infty} (D_0(x_- + \mathrm{i} p_+)|k\rangle_0) \otimes |k\rangle_1 \\ \Leftrightarrow \quad & {}_0\langle x_-| \otimes {}_1\langle p_+| \mathrm{e}^{\mathrm{i} H_{hbs}^{01}} && = \pi^{-1/2} \textstyle\sum_{k=0}^{\infty} {}_0\langle k| D_0(x_- + \mathrm{i} p_+)^* \otimes {}_1\langle k|. \end{aligned} \tag{1.3}$$

The measurement theory called projective measurement is based on the following postulates (see [12]):

Definition 1.1 (Projective measurement)**.**
Postulate 1) When one measures the observable $\Lambda = \sum_\lambda \lambda \Pi^\lambda$ on the state $|\psi\rangle$, the result one obtains is one of the eigenvalue λ. The probability for obtaining that particular eigenvalue λ is $\langle \psi | \Pi^\lambda | \psi \rangle / \langle \psi | \psi \rangle$, where Π^λ is the projection appearing in the spectral resolution $\Lambda = \sum_\lambda \lambda \Pi^\lambda$ of Λ.
Postulate 2) After the measurement the initial state $|\psi\rangle$ changes to $\Pi^\lambda |\psi\rangle$.

When Alice performs the projective measurement of the observable $x_0 \otimes p_1 \otimes I$ to the state $(\mathrm{e}^{\mathrm{i} H_{hbs}^{01}} \otimes I)|\Phi\rangle$ and has the result $(x_0, p_1) = (x_-, p_+)$, then the state $(\mathrm{e}^{\mathrm{i} H_{hbs}^{01}} \otimes I)|\Phi\rangle$

changes to

$$
\begin{aligned}
&(|x_-\rangle_0 \otimes |p_+\rangle_1 {}_0\langle x_-| \otimes {}_1\langle p_+| \otimes I)(\mathrm{e}^{\mathrm{i}H^{01}_{hbs}} \otimes I|\Phi\rangle) \\
&= |x_-\rangle_0 \otimes |p_+\rangle_1 \pi^{-1/2} \textstyle\sum_{k=0}^{\infty} {}_0\langle k|D_0(x_- + \mathrm{i}p_+)^* \otimes {}_1\langle k| \otimes I|\Phi\rangle.
\end{aligned}
$$

The essential part of (the equation describing) teleportation is

$$
\begin{aligned}
&\pi^{-1/2} \textstyle\sum_{k=0}^{\infty} {}_0\langle k|D_0(\alpha)^* \otimes {}_1\langle k| \otimes I|\Phi\rangle = \\
&\textstyle\sum_{k=0}^{\infty} {}_0\langle k|D_0(\alpha)^*|\psi\rangle_0 {}_1\langle k| \sum_{n=0}^{\infty} q^n |n\rangle_1 \otimes |n\rangle_2 \\
&= \textstyle\sum_{k=0}^{\infty} q^k {}_0\langle k|D_0(\alpha)^*|\psi\rangle_0 |k\rangle_2 \to D_2(\alpha)^*|\psi\rangle_2
\end{aligned}
\tag{1.4}
$$

as $q \to 1$, where $|\psi\rangle_2 = \sum_{n=0}^{\infty} a_n |n\rangle_2$. Note that $\mathcal{H}_j$ are isomorphic to each other by the correspondence $\mathcal{H}_i \ni |n\rangle_i \leftrightarrow |n\rangle_j \in \mathcal{H}_j$. This is the scheme of quantum teleportation of continuous variables (see [2, 3, 6]). The postulate 2 of projective measuement plays an important role for teleportation.

Remark 1.2. Sometimes the Bell-measurement is understood as the measurement of $\mathrm{e}^{-\mathrm{i}H^{01}_{hbs}}(x_0 \otimes p_1)\mathrm{e}^{\mathrm{i}H^{01}_{hbs}} \otimes I$ on the state $|\Phi\rangle$ instead of the measurement $x_0 \otimes p_1 \otimes I$ on the state $(\mathrm{e}^{\mathrm{i}H^{01}_{hbs}} \otimes I)|\Phi\rangle$.

Postulate 1 gives the same result for both cases, but the postulate 2 gives different results. In any case, $|\Phi\rangle$ and $(\mathrm{e}^{\mathrm{i}H^{01}_{hbs}} \otimes I)|\Phi\rangle$ change to

$$
\mathrm{e}^{-\mathrm{i}H^{01}_{hbs}}(|x_-\rangle_0 \otimes |p_+\rangle_1) \otimes D_0(\alpha)^*|\psi\rangle_2 \quad \text{and} \quad |x_-\rangle_0 \otimes |p_+\rangle_1 \otimes D_0(\alpha)^*|\psi\rangle_2
$$

respectively, that is, $|\psi\rangle_0$ is teleported to $D_2(\alpha)^*|\psi\rangle_2$.

For the measurement of x_0, the balanced homodyne detection is often used [1, 2, 4, 5].

For the balanced homodyne detection, we prepare one more creation c_3^* and annihilation c_3 operatores of a local oscillator and a half-beamsplitter

$$
\mathrm{e}^{\mathrm{i}H^{03}_{hbs}}, \ H^{03}_{hbs} = \mathrm{i}(\pi/4)(c_0^* c_3 - c_0 c_3^*).
$$

Let $|\alpha\rangle_3$ be the coherent state

$$
|\alpha\rangle_3 = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha c_3^*}|0\rangle, \ \alpha \in \mathbb{C},
$$

where $|0\rangle$ is the vacuum state. For the measurement of x_0 we choose $\alpha = |\alpha|$ and let N_j $(j = 0, 3)$ be the number operators defined by

$$
N_j = c_j^* c_j.
$$

Then $|n\rangle_j = \frac{1}{\sqrt{n!}} c_j^{*n}|0\rangle$ is an eigen-state of N_j with eigen-value n, and

$$
|l + j\rangle_3 \otimes |j\rangle_0 \ (l \geq 0), \quad |j\rangle_3 \otimes |j - l\rangle_0 \ (l \leq 0), \quad j = 0, 1, \ldots
$$

are eigen-vectors of $N_3 - N_0$ with eigen-value l.

The balanced homodyne detection is the measurement of $N_3 - N_0$ on the state $e^{\mathrm{i}H^{03}_{\mathrm{hbs}}}(|\phi\rangle_0 \otimes |\alpha\rangle_3)$, where $|\phi\rangle_0$ is the state defined by

$$|\phi\rangle_0 = \phi(c_0^*)|0\rangle$$

for an entire function $\phi(z)$ of z with ${}_0\langle\phi|\phi\rangle_0 < \infty$. The expectation value of this measurement is

$${}_0\langle\phi| \otimes {}_3\langle\alpha|e^{-\mathrm{i}H^{03}_{\mathrm{hbs}}}(N_3 - N_0)e^{\mathrm{i}H^{03}_{\mathrm{hbs}}}|\phi\rangle_0 \otimes |\alpha\rangle_3$$

which is the expectation value of

$$e^{-\mathrm{i}H^{03}_{\mathrm{hbs}}}(N_3 - N_0)e^{\mathrm{i}H^{03}_{\mathrm{hbs}}} = c_3^*c_0 + c_0^*c_3$$

on the state $|\phi\rangle_0 \otimes |\alpha\rangle_3$.

Let $\tilde{\Pi}^l$ and Π^l be projections which appear in the spectral resolutions of $N_3 - N_0$ and $c_3^*c_0 + c_0^*c_3$ respectively, i.e.,

$$N_3 - N_0 = \sum_{l=-\infty}^{\infty} l\tilde{\Pi}^l, \;\; c_3^*c_0 + c_0^*c_3 = \sum_{l=-\infty}^{\infty} le^{-\mathrm{i}H^{03}_{\mathrm{hbs}}}\tilde{\Pi}^l e^{\mathrm{i}H^{03}_{\mathrm{hbs}}} = \sum_{l=-\infty}^{\infty} l\Pi^l.$$

Remark 1.3. Under the projective measurement postulate, there are no differences of the measured value l and the probability for obtaining l, between the measurement of $N_3 - N_0$ on the state $e^{\mathrm{i}H^{03}_{\mathrm{hbs}}}(|\phi\rangle_0 \otimes |\alpha\rangle_3)$ and the measurement of $c_3^*c_0 + c_0^*c_3$ on the state $|\phi\rangle_0 \otimes |\alpha\rangle_3$, but the changed states are different, i.e., $\tilde{\Pi}^l e^{\mathrm{i}H^{03}_{\mathrm{hbs}}}(|\phi\rangle_0 \otimes |\alpha\rangle_3)$ and $e^{-\mathrm{i}H^{03}_{\mathrm{hbs}}}\tilde{\Pi}^l e^{\mathrm{i}H^{03}_{\mathrm{hbs}}}(|\phi\rangle_0 \otimes |\alpha\rangle_3)$ (see (Remark 1.2)).

Note that one can deduce this result from the latter, the result of projective measurement of the observable $c_3^*c_0+c_0^*c_3$. So, we study the measurement of the observable $c_3^*c_0 + c_0^*c_3$ on the state $|\phi\rangle_0 \otimes |\alpha\rangle_3$ under the framework of projective measurement.

The expectation value of the observable $c_3^*c_0 + c_0^*c_3$ on the state $|\phi\rangle_0 \otimes |\alpha\rangle_3$ is calculated as

$${}_0\langle\phi| \otimes {}_3\langle\alpha|(c_3^*c_0 + c_0^*c_3)|\phi\rangle_0 \otimes |\alpha\rangle_3 = {}_0\langle\phi|(\bar{\alpha}c_0 + \alpha c_0^*)|\phi\rangle_0.$$

The above formula is sometimes expressed as saying

$$\begin{array}{c}\text{“the local oscillator is strong, so that the operators } c_3 \text{ and } c_3^* \\ \text{can be replaced by complex numbers } \alpha \text{ and } \bar{\alpha} \text{ respectively”.}\end{array} \tag{1.5}$$

One might find the assumption (1.5) strange because of the following: Let $\alpha = |\alpha|$. Since the difference of the number operators haves eigen values in $\mathbb{Z}$, the statement (1.5) seems to cause the following contradiction:

$$\begin{array}{c}\text{the operator } c_3^*c_0 + c_0^*c_3 \text{ has eigen values in } \mathbb{Z}, \text{while the operator} \\ \bar{\alpha}c_0 + \alpha c_0^* = |\alpha|\,(c_0 + c_0^*) = \sqrt{2}\,|\alpha|\,x_0 \;\text{ has its spectrum in } \mathbb{R}.\end{array} \tag{1.6}$$

Since the balanced homodyne detection seems to detect the number of photons, the eigen-values of $c_3^*c_0 + c_0^*c_3$, it seems impossible to make a measurement of x_0 by the balanced homodyne detection.

But in [1] it is proven that the statement (1.5) is valid for the balanced homodyne measurement in the sense that the probability of the outcome $l = x|\alpha|$ for $c_3^* c_0 + c_0^* c_3$ and $|\phi\rangle_0 \otimes |\alpha\rangle_3$ approximates the probability of the outcome x for $(1/|\alpha|)(\bar{\alpha} c_0 + \alpha c_0^*)$ and $|\phi\rangle_0$ for large $|\alpha|$.

It is convenient to use $(1/|\alpha|)(c_3^* c_0 + c_0^* c_3)$ and $(1/|\alpha|)(\bar{\alpha} c_0 + \alpha c_0^*)$ instead of $c_3^* c_0 + c_0^* c_3$ and $\bar{\alpha} c_0 + \alpha c_0^*$ for large $|\alpha| \to \infty$.

Let $\alpha = |\alpha| \mathrm{e}^{\mathrm{i}\theta}$ and $|\theta; r\rangle$ be the generalized eigen-vector of $\mathrm{e}^{-\mathrm{i}\theta} c_0 + \mathrm{e}^{\mathrm{i}\theta} c_0^*$ with eigen-value r such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\theta; r\rangle_{00}\langle\theta; r| dr = I.$$

The spectral resolution of $(1/|\alpha|)(c_3^* c_0 + c_0^* c_3)$ is $\sum_{l=-\infty}^{\infty} (l/|\alpha|) \Pi^l$ and the projection onto the space spanned by the eigen-states whose eigen-values $l/|\alpha|$ are less than x is

$$\sum_{-\infty < l \le x|\alpha|} \Pi^l = \int_{-\infty}^{x|\alpha|} \Pi^{[s]} ds = \int_{-\infty}^{x} |\alpha| \, \Pi^{[r|\alpha|]} dr,$$

which corresponds to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} |\theta; r\rangle_{00}\langle\theta; r| dr,$$

where $[z]$ denotes the integer part of z.

In [1], Braunstein calculated the probability distribution $P(x)$ of the outcome $x = l/|\alpha|$ for $(c_3^* c_0 + c_0^* c_3)/|\alpha|$ and $|\beta\rangle_0 \otimes |\alpha\rangle_3$ by calculating the characteristic function $\chi(k) = ({}_0\langle\beta| \otimes {}_3\langle\alpha|) \exp\{\mathrm{i}k(c_3^* c_0 + c_0^* c_3)/|\alpha|\}(|\beta\rangle_0 \otimes |\alpha\rangle_3)$ and its Fourier transformation, and he showed that $P(x)$ approximates

$$\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\{x - (\bar{\alpha}\beta + \alpha\bar{\beta})/|\alpha|\}^2/2}, \; x = l/|\alpha| \tag{1.7}$$

well as $|\alpha| \to \infty$. It is easily seen that $\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\{x - (\bar{\alpha}\beta + \alpha\bar{\beta})/|\alpha|\}^2/2} = {}_0\langle\beta|\theta; x\rangle_{00}\langle\theta; x|\beta\rangle_0$ by using the expression (2.4), where $(\bar{\alpha} c_0 + \alpha c_0^*)/|\alpha| = \int_{-\infty}^{\infty} x |\theta; x\rangle_{00}\langle\theta; x| dx$ is the spectral resolution of $(\bar{\alpha} c_0 + \alpha c_0^*)/|\alpha|$ for $\alpha = \mathrm{e}^{\mathrm{i}\theta} |\alpha|$. Thus the balanced homodyne detection fulfill the postulate 1 for the projective measurement.

However, in [1] Braunstein did not study the postulate 2 of projective measurement, that is, does the collapse of $|\phi\rangle_0 \otimes |\alpha\rangle_3$ to an eigen-vector with eigen-value l approximate the collapse of $|\phi\rangle_0$ to an eigen-vector with eigen-value x? The postulate 2 of projective measurement plays a crucial role for the theory of quantum teleportation. We will prove the convergence of $|\alpha| \, {}_3\langle\alpha|\Pi^{[x|\alpha|]}|\alpha\rangle_3 \to |\theta; x\rangle_{00}\langle\theta; x|$ as $|\alpha| \to \infty$, and give the affirmative answer to this question. Our proof gives also the approximation of the probability distribution. That is, our proof is an alternative proof of that given in [1], and this proof is elementary and direct (not via characteristic function).

In order to study the change of the state, we want to show that

$$|\alpha| \, \Pi^l (|\phi\rangle_0 \otimes |\alpha\rangle_3) \to \frac{1}{\sqrt{2\pi}} |\theta; x\rangle_{00}\langle\theta; x|\phi\rangle_0 \otimes |\alpha\rangle_3 \tag{1.8}$$

as $|\alpha| \to \infty$ for $l = [x\,|\alpha|]$, where $[z]$ is the integer part of z.

In Section 2, we give a brief review of the holomorphic representation of creation c_j and annihilation c_j^* $(j = 0, 1, \ldots, n)$ operators, where the coherent states $|\alpha\rangle_2$ and $|\theta; x\rangle_{11}\langle\theta; x|$ are represented by

$$\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\bar{u}_2} \quad \text{and} \quad \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\overline{\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{v}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}\mathrm{e}^{-x^2/4}}$$

respecctively.

In Sections 3 and 4, we use only c_0 and c_3. So we change the subscripts $0, 3$ to $1, 2$, i.e., we replace c_0 and c_3 with c_1 and c_2 respectively.

In Section 3, we introduce a set of vectors

$$|m, n\rangle = \frac{1}{\sqrt{2^m 2^n m! n!}}(\bar{u}_1 + \bar{u}_2)^m(-\bar{u}_1 + \bar{u}_2)^n, \quad 0 \le m, n \in \mathbb{Z}$$

which constitutes an orthonormal basis of the Fock space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ of c_1 and c_2, and show that the orthogonal projection Π^l onto the eigen space of $c_2^* c_1 + c_1^* c_2$ with eigen value l is

$$\Pi^l = \sum_{j=0}^{\infty} |l + j, j\rangle\langle l + j, j| \quad (\text{for} \quad l \ge 0), \quad \Pi^l = \sum_{j=0}^{\infty} |j, j - l\rangle\langle j, j - l| \quad (\text{for} \quad l \le 0).$$

Moreover, we introduce generalized vectors

$$|\varphi, l\rangle = \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi}|l + j, j\rangle \quad (\text{for} \quad l \ge 0), \quad |\varphi, l\rangle = \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi}|l, j - l\rangle \quad (\text{for} \quad l \le 0)$$

and show

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi, l\rangle\langle\varphi, l| d\varphi = \Pi^l.$$

For $\alpha = \mathrm{e}^{\mathrm{i}\theta}\,|\alpha|$, consider

$$\begin{aligned}
{}_2\langle\alpha|\varphi, 0\rangle &= \mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2 + \bar{\alpha}^2)^j \\
&= \mathrm{e}^{-|\alpha|^2/2} f(\varphi, 0, \bar{u}_1, \mathrm{e}^{-\mathrm{i}\theta}\,|\alpha|) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2 + \mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)/2} \\
&= \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2}\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}(\varphi - 2\theta)}|\alpha|^2/2} = \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j(\varphi - 2\theta)}\frac{(|\alpha|^2/2)^j}{j!}
\end{aligned} \tag{1.9}$$

which follows from

$$|\varphi, 0\rangle = \sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}|j, j\rangle = \sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2 + \bar{u}_2^2)^j = \mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2 + \bar{u}_2^2)/2}.$$

and ${}_2\langle\alpha|f\rangle = \mathrm{e}^{-|\alpha|^2/2} f(\bar{\alpha})$.

If $\mathrm{e}^{\mathrm{i}\varphi} = \mathrm{e}^{2\mathrm{i}\theta}$ then

$$\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)/2} = \mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}.$$

It is interesting that $\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}$ is a generalized eigen-function of $\mathrm{e}^{-\mathrm{i}\theta}c_1 + \mathrm{e}^{\mathrm{i}\theta}c_1^*$ with eigen-value 0. For other φ,

$${}_2\langle\alpha|\varphi,0\rangle = \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2}\psi_P(t) = \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2}\mathrm{e}^{(\mathrm{e}^{\mathrm{i}t}-1)m} \to 0,\ t = \varphi - 2\theta$$

as $m = |\alpha|^2/2 \to \infty$, where $\psi_P(t)$ is the characteristic function of the Poisson distribution $P(j;m) = \mathrm{e}^{-m}\frac{m^j}{j!}$. This suggests the convergence

$$|\alpha|\,{}_2\langle\alpha|\Pi^0|\alpha\rangle_2 = \frac{|\alpha|}{2\pi}\int_{2\theta-\pi}^{2\theta+\pi} {}_2\langle\alpha|\varphi,0\rangle\langle\varphi,0|\alpha\rangle_2\,d\varphi \to \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\overline{\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}} \tag{1.10}$$

as $|\alpha| \to \infty$, since the Poisson distribution $P(j;m) = \mathrm{e}^{-m}\frac{m^j}{j!}$ approximates the Gaussian distribution $G(j;m,\sigma^2) = \frac{1}{\sqrt{2\pi m}}\mathrm{e}^{-(j-m)^2/2\sigma^2}$ ($\sigma^2 = m = |\alpha|^2/2$) well and

$$(\sqrt{2m}/\sqrt{2\pi})\overline{\psi_P(t)}\psi_P(t) \approx (\sqrt{2m}/\sqrt{2\pi})\overline{\psi_G(t)}\psi_G(t) = (\sqrt{2m}/\sqrt{2\pi})\mathrm{e}^{-mt^2} \to \delta(t)$$

as $m \to \infty$. In Corollary 4.6, the relation $(\sqrt{2m}/\sqrt{2\pi})\overline{\psi_P(t)}\psi_P(t) \to \delta(t)$ is proved without using the relation between $P(j;m)$ and $G(j;m,\sigma^2)$.

In Section 4, we will prove the convergence

$$|\alpha|\,{}_2\langle\alpha|\Pi^l|\alpha\rangle_2 \to \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/4}\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\overline{\mathrm{e}^{-x^2/4}\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{v}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}}, \tag{1.11}$$

as $|\alpha| \to \infty$ for fixed $x \in \mathbb{R}$ such that $l = [x|\alpha|]$, the main theorem, Theorem 4.17 which implies (1.8). For the proof we compare the formula

$${}_2\langle\alpha|\varphi,l\rangle = \mathrm{e}^{-|\alpha|^2/2}\left[\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{\sqrt{j!}}{\sqrt{2^l(l+j)!}}(\bar{u}_1+\bar{\alpha})^l\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j\right]$$

with (1.9). The terms $\frac{\sqrt{j!}}{\sqrt{2^l(l+j)!}}(\bar{u}_1+\bar{\alpha})^l$ will change ${}_2\langle\alpha|\varphi,0\rangle$ into ${}_2\langle\alpha|\varphi,l\rangle$. We find a useful formula (4.9):

$$\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j = \mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi,l\rangle.$$

$\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}}$ gives the term $\lim_{|\alpha|\to\infty}\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}} = \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}$, where we use the formula $\lim_{n\to\infty}(1+a/n)^n = \mathrm{e}^a$, and $\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}}$ will give the normalization term $\mathrm{e}^{-x^2/4}$. This can be seen by Chebyshev's inequality for Poisson distribution (see Lemma 4.8) and $\lim_{m\to\infty}\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}} = \mathrm{e}^{-x^2/4\mu}$ for $l = x\sqrt{2m}$ and $j = \mu m$ which follows from the Stirling's approximation $n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$ and the formula

$\lim_{n\to\infty}(1+a/n)^n = \mathrm{e}^a$. The Chebyshev's inequality shows that the summation $\sum_{0\le j<m-\lambda\sqrt{m}} + \sum_{m+\lambda\sqrt{m}<j}$ is small and $\sum_{j=l/2}^{\infty}$ can be replaced by $\sum_{j=m-\lambda\sqrt{m}}^{m+\lambda\sqrt{m}}$ for large λ. If $j \in [m-\lambda\sqrt{m}, m+\lambda\sqrt{m}]$, then $\mu = j/m \in [1-\lambda/\sqrt{m}, 1+\lambda/\sqrt{m}]$. So, $\mathrm{e}^{-x^2/4\mu}$ can be replaced by $\mathrm{e}^{-x^2/4}$ for large $m = |\alpha|^2/2$. In this way, we have the main proposition Proposition 4.16:

$$\mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi, l\rangle \approx \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-x^2/4}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(\bar{u}_1^2 + \mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)^j$$

$$\approx \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j(\varphi-2\theta)}\frac{|\alpha|^2/2}{j!}$$

for sufficiently large $l = x|\alpha|$. The above formula and Corollary 4.6 imply the formula (1.11): Theorem 4.17.

In Section 5, It is shown that the balanced homodyne measurement causes the quantum teleportation.

2. Holomorphic representation of CCR

Later we will use the holomorphic representation of the canonical commutation relation defined below (see [10]).

Definition 2.1. Consider the Hilbert space $L^2(\mathbb{C}^n, d\mu_n)$ with

$$u_j = x_j + \mathrm{i}y_j, \quad d\mu_n = \prod_{j=1}^{n}\mathrm{e}^{-\bar{u}_j u_j}\frac{d\bar{u}_j du_j}{2\pi\mathrm{i}} = \prod_{j=1}^{n}\mathrm{e}^{-(x_j^2+y_j^2)}\frac{dx_j dy_j}{\pi}, \quad \int_{\mathbb{C}^n} d\mu_n = 1$$

and its subspace $\mathcal{H}$ generated by holomorphic functions of $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ (anti-holomorphic functions of u). For $f, g \in \mathcal{H}$ define the inner product by

$$\langle f|g\rangle = \int_{\mathbb{C}^n}\overline{f(\bar{u})}g(\bar{u})d\mu_n = \int_{\mathbb{C}^n}\overline{f(\bar{u})}g(\bar{u})\prod_{j=1}^{n}\mathrm{e}^{-\bar{u}_j u_j}\frac{d\bar{u}_j du_j}{2\pi\mathrm{i}}.$$

Then the multiplication operator $c_j^* : f(\bar{u}) \to \bar{u}_j f(\bar{u})$ and the differentiation operator $c_j : f(\bar{u}) \to \partial f(\bar{u})/\partial\bar{u}_j$ ($\partial/\partial\bar{u}_j = (1/2)(\partial/\partial x_j + \mathrm{i}\partial/\partial y_j)$) are adjoint to each other and satisfy the canonical commutation relations (CCR)

$$[c_j, c_k] = [c_j^*, c_k^*] = 0, \quad [c_j, c_k^*] = \delta_{j,k}.$$

This representation of the commutation relations is called the holomorphic representation.

Remark 2.2. $\{\bar{u}_1^{k_1}\cdots\bar{u}_n^{k_n}/\sqrt{k_1!\cdots k_n!}\}_{k_1,\ldots k_n=0}^{\infty}$ is a complete orthonormal system of

$\mathcal{H}$. The following calculations show the orthonormality: If $k \leq l$ then we have

$$\int_{\mathbb{C}^n} \overline{\bar{u}_j^k} \bar{u}_j^l d\mu_n = \int_{\mathbb{C}^n} u_j^k \bar{u}_j^l d\mu_n = \int_{\mathbb{C}^n} u_j^k \bar{u}_j^l \prod_{j=1}^n \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}}$$

$$= \int_{\mathbb{C}^n} u_j^k \left(-\frac{\partial}{\partial u_j}\right)^l \prod_{j=1}^n \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = \int_{\mathbb{C}^n} \left(\frac{\partial^l}{(\partial u_j)^l} u_j^k\right) \prod_{j=1}^n \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = k! \delta_{k,l},$$

if $k > l$ we calculate

$$\int_{\mathbb{C}^n} \bar{u}_j^l \left(-\frac{\partial}{\partial \bar{u}_j}\right)^k \prod_{j=1}^n \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = \int_{\mathbb{C}^n} \left(\frac{\partial^k}{(\partial \bar{u}_j)^k} \bar{u}_j^l\right) \prod_{j=1}^n \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = 0.$$

The completeness follows from the fact that holomorphic functions $f(\bar{u})$ of $\bar{u}$ on $\mathbb{C}^n$ have power series expansions

$$f(\bar{u}) = \sum_{k_1,\ldots,k_n=0}^{\infty} c_{k_1\ldots k_n} \bar{u}_1^{k_1} \cdots \bar{u}_n^{k_n}.$$

Proposition 2.3. *Let $\bar{\alpha} \cdot u = \sum_{j=1}^n \bar{\alpha}_j u_j$. Then we have*

$$\int_{\mathbb{C}^n} \mathrm{e}^{\bar{\alpha}\cdot u} f(\bar{u}) d\mu_n = f(\bar{\alpha}).$$

Proof. It follows from the equalities

$$\int_{\mathbb{C}^n} \mathrm{e}^{\bar{\alpha}\cdot u} \bar{u}_1^{k_1} \cdots \bar{u}_n^{k_n} d\mu_n = \prod_{j=1}^n \int_{\mathbb{C}} \mathrm{e}^{\bar{\alpha}_j u_j} \bar{u}_j^{k_j} \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}}$$

$$= \prod_{j=1}^n \int_{\mathbb{C}} \sum_{l_j=0}^{\infty} \frac{(\bar{\alpha}_j u_j)^{l_j}}{l_j!} \bar{u}_j^{k_j} \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = \prod_{j=1}^n \bar{\alpha}_j^{k_j} = \bar{\alpha}_1^{k_1} \cdots \bar{\alpha}_n^{k_n}$$

that

$$\int_{\mathbb{C}^n} \mathrm{e}^{\bar{\alpha}\cdot u} f(\bar{u}) d\mu_n = \int_{\mathbb{C}^n} \mathrm{e}^{\bar{\alpha}\cdot u} \sum_{k_1,\ldots,k_n=0}^{\infty} c_{k_1\ldots k_n} \bar{u}_1^{k_1} \cdots \bar{u}_n^{k_n} d\mu_n$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} c_{k_1\ldots k_n} \bar{\alpha}^{k_1} \cdots \bar{\alpha}^{k_n} = f(\bar{\alpha}).$$

□

Corollary 2.4. $\mathrm{e}^{\bar{u}\cdot v}$ *is the integral kernel of the identity operator.*

Proof.

$$\mathcal{H} \ni f(\bar{v}) \mapsto \int_{\mathbb{C}^n} \mathrm{e}^{\bar{u}\cdot v} f(\bar{v}) \prod_{j=1}^n \mathrm{e}^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi \mathrm{i}} = f(\bar{u}) \in \mathcal{H}.$$

$\square$

Remark 2.5. The holomorphic representation of the coherent state

$$\prod_{j=1}^{n} \mathrm{e}^{-|\alpha_j|^2/2} \mathrm{e}^{\alpha_j c_j^*} |0\rangle, \ \alpha_j \in \mathbb{C}$$

is

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha \cdot \bar{u}}, \ |\alpha|^2 = \bar{\alpha} \cdot \alpha$$

and it follows from Proposition 2.3 that

$$\langle \alpha | f \rangle = \mathrm{e}^{-|\alpha|^2/2} f(\bar{\alpha})$$

for $|f\rangle$ represented by the holomorphic function $f(\bar{u})$ of $\bar{u}$ in the holomorphpic representation. If $A(\bar{u}, v)$ is the kernel function of an operator A in the holomorphic representation, then $\langle \alpha | A | \beta \rangle = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{-|\beta|^2/2} A(\bar{\alpha}, \beta)$.

Therefore, the holomorphic representation is sometimes called the coherent-state representation.

The following equalities for the coherent state follow from Proposition 2.3 and Corollary 2.4.

$$\langle \alpha | \beta \rangle = \int_{\mathbb{C}^n} \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\bar{\alpha} \cdot u} \mathrm{e}^{-|\beta|^2/2} \mathrm{e}^{\beta \cdot \bar{u}} \prod_{j=1}^{n} \mathrm{e}^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{-|\beta|^2/2} \mathrm{e}^{\bar{\alpha} \cdot \beta}, \tag{2.1}$$

$$\int_{\mathbb{C}^n} |\alpha\rangle \langle \alpha| \prod_{j=1}^{n} \frac{d\bar{u}_j du_j}{2\pi \mathrm{i}} = \int_{\mathbb{C}^n} \mathrm{e}^{\alpha \cdot \bar{u}} \mathrm{e}^{\bar{\alpha} \cdot v} \prod_{j=1}^{n} \mathrm{e}^{-\bar{\alpha}_j \alpha_j} \frac{d\bar{\alpha}_j d\alpha_j}{2\pi \mathrm{i}} = \mathrm{e}^{\bar{u} \cdot v} = I. \tag{2.2}$$

Proposition 2.6. *The operator*

$$\xi_j(\theta) = \mathrm{e}^{-\mathrm{i}\theta} c_j + \mathrm{e}^{\mathrm{i}\theta} c_j^* = \mathrm{e}^{-\mathrm{i}\theta} \frac{\partial}{\partial \bar{u}_j} + \mathrm{e}^{\mathrm{i}\theta} \bar{u}_j. \tag{2.3}$$

has

$$|\theta; r\rangle_j = \mathrm{e}^{-r^2/4} \mathrm{e}^{r \mathrm{e}^{\mathrm{i}\theta} \bar{u}_j} \mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta} \bar{u}_j^2/2} \tag{2.4}$$

as an eigen-vector with eigen-value r*, and these eigen-vectors satisfy*

$$\int_{-\infty}^{\infty} |\theta; r\rangle_{j\,j} \langle \theta; r| dr = \sqrt{2\pi} I \tag{2.5}$$

and

$${}_j\langle \theta; r | \theta; s \rangle_j = \sqrt{2\pi} \delta(r - s). \tag{2.6}$$

Proof. It is easy to see that

$$\left(\frac{\partial}{\partial \bar{u}_j} + \bar{u}_j\right) \mathrm{e}^{r\bar{u}_j}\mathrm{e}^{-\bar{u}_j^2/2} = r\mathrm{e}^{r\bar{u}_j}\mathrm{e}^{-\bar{u}_j^2/2},$$

and

$$\left(\mathrm{e}^{-\mathrm{i}\theta}\frac{\partial}{\partial \bar{u}_j} + \mathrm{e}^{\mathrm{i}\theta}\bar{u}_j\right) \mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}\bar{u}_j}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_j^2/2} = r\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}\bar{u}_j}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_j^2/2},$$

that is, the holomorphic representation of $|0;r\rangle_j$ is $C\mathrm{e}^{r\bar{u}_j}\mathrm{e}^{-\bar{u}_j^2/2}$ and $|\theta;r\rangle_j$ is $C\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}\bar{u}_j}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_j^2/2}$ for some constant C.

For $|\pi/2;r\rangle_j = \mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\bar{u}_j}\mathrm{e}^{\bar{u}_j^2/2}$, we have

$$\int_{-\infty}^{\infty} |\pi/2;r\rangle_{j\,j}\langle\pi/2;r|dr = \int_{-\infty}^{\infty} \mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\bar{u}_j}\mathrm{e}^{\bar{u}_j^2/2}\overline{\mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\bar{v}_j}\mathrm{e}^{\bar{v}_j^2/2}}dr$$
$$= \int_{-\infty}^{\infty} \mathrm{e}^{-r^2/2}\mathrm{e}^{\mathrm{i}r(\bar{u}_j - v_j)}\mathrm{e}^{\bar{u}_j^2/2}\mathrm{e}^{v_j^2/2}dr = \sqrt{2\pi}\mathrm{e}^{-(\bar{u}_j - v_j)^2/2}\mathrm{e}^{\bar{u}_j^2/2}\mathrm{e}^{v_j^2/2} = \sqrt{2\pi}\mathrm{e}^{\bar{u}_j v_j},$$

and

$$_j\langle\pi/2;r|\pi/2;s\rangle_j = \int_{\mathbb{C}} \overline{\mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\bar{u}_j}\mathrm{e}^{\bar{u}_j^2/2}}\mathrm{e}^{-s^2/4}\mathrm{e}^{\mathrm{i}s\bar{u}_j}\mathrm{e}^{\bar{u}_j^2/2}\mathrm{e}^{-\bar{u}_j u_j}\frac{d\bar{u}_j du_j}{2\pi\mathrm{i}}$$
$$= \mathrm{e}^{-r^2/4}\mathrm{e}^{-s^2/4}\int_{\mathbb{C}} \mathrm{e}^{-\mathrm{i}ru_j}\mathrm{e}^{u_j^2/2}\mathrm{e}^{\mathrm{i}s\bar{u}_j}\mathrm{e}^{\bar{u}_j^2/2}\mathrm{e}^{-\bar{u}_j u_j}\frac{d\bar{u}_j du_j}{2\pi\mathrm{i}}$$
$$= \mathrm{e}^{-(r^2+s^2)/4}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}(r-s)x_j}\mathrm{e}^{(r+s)y_j}\mathrm{e}^{-2y_j^2}\frac{dx_j dy_j}{\pi}$$
$$= \mathrm{e}^{-(r-s)^2/8}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(r-s)x_j}dx_j = \mathrm{e}^{-(r-s)^2/8}\sqrt{2\pi}\delta(r-s) = \sqrt{2\pi}\delta(r-s).$$

By the change of variables $\bar{u}_j \to \mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{u}_j$, $\bar{v}_j \to \mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{v}_j$, we get

$$\int_{-\infty}^{\infty} |\theta;r\rangle_{j\,j}\langle\theta;r|dr =$$
$$\int_{-\infty}^{\infty} \mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{u}_j}\mathrm{e}^{\mathrm{e}^{2\mathrm{i}(\theta-\pi/2)}\bar{u}_j^2/2}\overline{\mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{v}_j}\mathrm{e}^{\mathrm{e}^{2\mathrm{i}(\theta-\pi/2)}\bar{v}_j^2/2}}dr$$
$$= \sqrt{2\pi}\mathrm{e}^{\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{u}_j\mathrm{e}^{-\mathrm{i}(\theta-\pi/2)}v_j} = \sqrt{2\pi}\mathrm{e}^{\bar{u}_j v_j} = \sqrt{2\pi}I,$$

and

$$_j\langle\theta;r|\theta;s\rangle_j =$$
$$\int_{\mathbb{C}} \overline{\mathrm{e}^{-r^2/4}\mathrm{e}^{\mathrm{i}r\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{u}_j}\mathrm{e}^{\mathrm{e}^{2\mathrm{i}(\theta-\pi/2)}\bar{u}_j^2/2}}\mathrm{e}^{-s^2/4}\mathrm{e}^{\mathrm{i}s\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\bar{u}_j}\mathrm{e}^{\mathrm{e}^{2\mathrm{i}(\theta-\pi/2)}\bar{u}_j^2/2}\mathrm{e}^{-\bar{u}_j u_j}\frac{d\bar{u}_j du_j}{2\pi\mathrm{i}}$$
$$= \sqrt{2\pi}\delta(r-s),$$

where used the fact that $\mathrm{e}^{-\bar{u}_j u_j}d\bar{u}_j du_j$ is invariant under the above change of variables. This completes the proof. $\square$

The above proposition shows the spectral resolution of $\xi_j(\theta)$:

$$\xi_j(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r\mathrm{e}^{-r^2/4} \mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}\bar{u}_j} \mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_j^2/2} \overline{\mathrm{e}^{-r^2/4}\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}\bar{v}_j}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_j^2/2}} dr.$$

Remark 2.7. Since $x_0 = (c_0 + c_0^*)/\sqrt{2} = \xi_0(0)/\sqrt{2}$ and $p_1 = (c_1 - c_1^*)/\sqrt{2}\mathrm{i} = \xi_1(\pi/2)/\sqrt{2}$, we have $|0; \sqrt{2}x_-\rangle_0 = |x_-\rangle_0$ and $|\pi/2; \sqrt{2}p_+\rangle_1 = |p_+\rangle_1$ (see (1.2)).

Let

$$\mathbb{R} \ni \theta \mapsto \exp\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in U(2)$$

be a one-parameter subgroup in $U(2)$, the group of 2×2 unitary matrices, and its action on $\mathcal{H}_0 \hat{\otimes} \mathcal{H}_1$ is

$$U(\theta) : \mathcal{H}_0 \hat{\otimes} \mathcal{H}_1 \ni f(\bar{u}_0, \bar{u}_1) \mapsto f(\bar{u}_0 \cos\theta + \bar{u}_1 \sin\theta, -\bar{u}_0 \sin\theta + \bar{u}_1 \cos\theta) \in \mathcal{H}_0 \hat{\otimes} \mathcal{H}_1. \tag{2.7}$$

Its infinitesimal generator is

$$\frac{d}{d\theta} U(\theta) f_{|\theta=0} = \left(-\bar{u}_0 \frac{\partial}{\partial \bar{u}_1} + \bar{u}_1 \frac{\partial}{\partial \bar{u}_0} \right) f = (-c_0^* c_1 + c_1^* c_0) f,$$

and therefore $U(\theta) = \exp\theta(-c_0^* c_1 + c_1^* c_0)$ and $U(\pi/4) = \exp(\pi/4)(-c_0^* c_1 + c_1^* c_0) = \mathrm{e}^{\mathrm{i}H_{hbs}^{01}}$ (see (1.2)). In the same reasoning we get (see (1.1))

$$\mathcal{H}_0 \ni f(\bar{u}_0) \mapsto \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha\bar{u}_0} f(\bar{u}_0 - \bar{\alpha}) = \mathrm{e}^{\alpha c_0^* - \bar{\alpha} c_0} f(\bar{u}_0) = D_0(\alpha) f(\bar{u}_0),$$

and

$$\sum_{n=0}^{\infty} (D_0(\alpha)|n\rangle_0) \otimes |n\rangle_1 = \sum_{n=0}^{\infty} \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha\bar{u}_0} \frac{(\bar{u}_0 - \bar{\alpha})^n \bar{u}_1^n}{n!} = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha\bar{u}_0} \mathrm{e}^{\bar{u}_0\bar{u}_1} \mathrm{e}^{-\bar{\alpha}\bar{u}_1}.$$

Applying $U(\pi/4) = \mathrm{e}^{\mathrm{i}H_{hbs}^{01}}$ gives

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}H_{hbs}^{01}} \sum_{n=0}^{\infty} (D_0(\alpha)|n\rangle_0) \otimes |n\rangle_1 = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha(\bar{u}_0 + \bar{u}_1)/\sqrt{2}} \mathrm{e}^{(\bar{u}_0 + \bar{u}_1)(-\bar{u}_0 + \bar{u}_1)/2} \mathrm{e}^{-\bar{\alpha}(-\bar{u}_0 + \bar{u}_1)/\sqrt{2}} \\ &= \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{(\alpha + \bar{\alpha})\bar{u}_0/\sqrt{2}} \mathrm{e}^{(-\bar{u}_0^2 + \bar{u}_1^2)/2} \mathrm{e}^{(\alpha - \bar{\alpha})\bar{u}_1/\sqrt{2}} = \pi^{1/2} |x_-\rangle_0 \otimes |p_+\rangle_1, \end{aligned}$$

where $\alpha = x_- + \mathrm{i}p_+$. This proves (1.3).

3. Balanced homodyne measurement

Let

$$\Xi = c_2^* c_1 + c_1^* c_2 = \bar{u}_2 \frac{\partial}{\partial \bar{u}_1} + \bar{u}_1 \frac{\partial}{\partial \bar{u}_2} \tag{3.1}$$

be a symmetric operator on $\mathcal{H} = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ generated by $\bar{u} = (\bar{u}_1, \bar{u}_2)$ where the space $\mathcal{H}_j$ is generated by $\bar{u}_j$ $(j = 1, 2)$. Then the function

$$(\bar{u}_1 + \bar{u}_2)^m (-\bar{u}_1 + \bar{u}_2)^n$$

is an eigen function of Ξ with eigen value $m - n$. In addition, the set of functions

$$|m, n\rangle = \frac{1}{\sqrt{2^m 2^n m! n!}} (\bar{u}_1 + \bar{u}_2)^m (-\bar{u}_1 + \bar{u}_2)^n, \quad 0 \leq m, n \in \mathbb{Z}$$

constitutes an orthonormal basis of the Fock space $\mathcal{H}$ of c_1 and c_2, which follows from Remark 2.2 since the above functions are expressed as

$$\frac{1}{\sqrt{m! n!}} \bar{v}_1^m \bar{v}_2^n, \quad 0 \leq m, n \in \mathbb{Z}$$

by the change of variables

$$\bar{v}_1 = (\bar{u}_1 + \bar{u}_2)/\sqrt{2}, \quad \bar{v}_2 = (-\bar{u}_1 + \bar{u}_2)/\sqrt{2} \tag{3.2}$$

and the measure μ_n $(n = 2)$ is invariant under the change of variables (3.2).

The eigen space of the operator Ξ associated with the eigen-value $l \in \mathbb{Z}$ is the space spanned by the vectors

$$|l + j, j\rangle, \ 0 \leq j \in \mathbb{Z}$$

for $l \geq 0$ and

$$|j, j - l\rangle, \ 0 \leq j \in \mathbb{Z}$$

for $l \leq 0$.

The orthogonal projection Π^l onto the eigen-space with eigen-value l is defined by

$$\Pi^l = \sum_{j=0}^{\infty} |l + j, j\rangle\langle l + j, j|$$

for $l \geq 0$, and

$$\Pi^l = \sum_{j=0}^{\infty} |j, j - l\rangle\langle j, j - l|$$

for $l \leq 0$.

The operator $\Xi = c_2^* c_1 + c_1^* c_2$ has the spectral resolution

$$\Xi = \sum_{l=-\infty}^{\infty} l \Pi^l,$$

and therefore Ξ is a self-adjoint operator. The operator $X_\alpha = \Xi / |\alpha|$ has the spectral resolution

$$X_\alpha = \Xi / |\alpha| = \sum_{l=-\infty}^{\infty} (l/|\alpha|)\Pi^l.$$

Now we have expressed the projections Π^l and the generalized vectors $|\theta; r\rangle_1$ in the holomorphic representation. But in order to prove (see (1.8))

$$|\alpha|\,\Pi^l(|\phi\rangle_1 \otimes |\alpha\rangle_2) \to \frac{1}{\sqrt{2\pi}}|\theta; x\rangle_{11}\langle\theta; x|\phi\rangle_1 \otimes |\alpha\rangle_2$$

for a coherent state $|\alpha\rangle_2 \in \mathcal{H}_2$ ($\alpha \in \mathbb{C}$) and $|\phi\rangle_1 \in \mathcal{H}_1$, we need some preparation. We begin with the useful expression of the projection operator Π^l.

Proposition 3.1. *Introduce*

$$\begin{aligned} |\varphi, l\rangle &= \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi}|l+j, j\rangle = f(\varphi, l, \bar{u}_1, \bar{u}_2) \\ &= \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{\sqrt{2^{l+j}2^j(l+j)!j!}}(\bar{u}_1+\bar{u}_2)^{l+j}(-\bar{u}_1+\bar{u}_2)^j \\ &= \frac{1}{\sqrt{2^l}}(\bar{u}_1+\bar{u}_2)^l\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{\sqrt{j!}}{\sqrt{(l+j)!}}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{u}_2^2)^j \end{aligned} \tag{3.3}$$

for $l \geq 0$ and

$$\begin{aligned} |\varphi, l\rangle &= \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi}|j, j-l\rangle = f(\varphi, l, \bar{u}_1, \bar{u}_2) \\ &= \frac{1}{\sqrt{2^{|l|}}}(-\bar{u}_1+\bar{u}_2)^{|l|}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{\sqrt{j!}}{\sqrt{(|l|+j)!}}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{u}_2^2)^j \end{aligned} \tag{3.4}$$

for $l \leq 0$. Then the series (3.3) and (3.4) converge to entire functions of $\bar{u}_1$ and $\bar{u}_1$, and we have

$$\frac{1}{2\pi}\int_{a-\pi}^{a+\pi}|\varphi, l\rangle\langle\varphi, l|d\varphi = \Pi^l$$

for any $a \in \mathbb{R}$.

Proof. We present the proof explicitly for the case of $l \geq 0$, for $l \leq 0$, the proof is the same. The inequality

$$\sum_{j=0}^{n}\left|\mathrm{e}^{\mathrm{i}j\varphi}|l+j, j\rangle\right| \leq \frac{1}{\sqrt{2^l}}\left|\bar{u}_1+\bar{u}_2\right|^l\sum_{j=0}^{n}\frac{1}{2^j j!}\left|-\bar{u}_1^2+\bar{u}_2^2\right|^j \leq \frac{1}{\sqrt{2^l}}\left|\bar{u}_1+\bar{u}_2\right|^l\mathrm{e}^{|-\bar{u}_1^2+\bar{u}_2^2|/2}$$

implies the uniform convergence of (3.3) with respect to φ in $\mathbb{R}$ and $\bar{u}_1, \bar{u}_2$ in any compact subset of $\mathbb{C}$. Integration term by term implies

$$\begin{aligned}\int_{a-\pi}^{a+\pi} |\varphi, l\rangle\langle\varphi, l| d\varphi &= \int_{a-\pi}^{a+\pi} \left[\sum_{i=0}^{\infty} \mathrm{e}^{\mathrm{i}i\varphi}|l+i, i\rangle\right] \left[\sum_{j=0}^{\infty} \mathrm{e}^{-\mathrm{i}j\varphi}\langle l+j, j|\right] d\varphi \\ &= 2\pi \sum_{j=0}^{\infty} |l+j, j\rangle\langle l+j, j| = 2\pi\Pi^l,\end{aligned}$$

where we used the relations

$$\int_{a-\pi}^{a+\pi} \mathrm{e}^{\mathrm{i}i\varphi}\mathrm{e}^{-\mathrm{i}j\varphi} d\varphi = 2\pi\delta_{i,j}$$

for $i, j = 0, 1, 2, \cdots$. $\square$

Proposition 3.2.

$$|\varphi, 0\rangle = f(\varphi, 0, \bar{u}_1, \bar{u}_2) = \mathrm{e}^{\mathrm{i}\varphi(-\bar{u}_1^2+\bar{u}_2^2)/2}.$$

Proof.

$$\begin{aligned}|\varphi, 0\rangle &= \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi} \frac{1}{\sqrt{2^j 2^j j! j!}} (\bar{u}_1 + \bar{u}_2)^j (-\bar{u}_1 + \bar{u}_2)^j \\ &= \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi} \frac{1}{2^j j!} (-\bar{u}_1^2 + \bar{u}_2^2)^j = \mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\bar{u}_2^2)/2}.\end{aligned}$$

$\square$

It follows from Proposition 2.3 that

$${}_2\langle\alpha|\varphi, 0\rangle = \mathrm{e}^{-|\alpha|^2/2} \int \mathrm{e}^{\bar{\alpha}u_2} f(\varphi, 0, \bar{u}_1, \bar{u}_2) \mathrm{e}^{-\bar{u}_2 u_2} \frac{d\bar{u}_2 du_2}{2\pi\mathrm{i}} = \mathrm{e}^{-|\alpha|^2/2} f(\varphi, 0, \bar{u}_1, \bar{\alpha}).$$

With $\alpha = \mathrm{e}^{\mathrm{i}\theta}|\alpha|$ we get

$${}_2\langle\alpha|\varphi, 0\rangle = \mathrm{e}^{-|\alpha|^2/2} f(\varphi, 0, \bar{u}_1, \mathrm{e}^{-\mathrm{i}\theta}|\alpha|) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)/2}. \tag{3.5}$$

If $\mathrm{e}^{\mathrm{i}\varphi} = \mathrm{e}^{2\mathrm{i}\theta}$ then

$$\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)/2} = \mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}.$$

It is interesting that $\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}$ is a generalized-eigen function of $\xi_1(\theta) = \mathrm{e}^{-\mathrm{i}\theta}c_1 + \mathrm{e}^{\mathrm{i}\theta}c_1^*$ with eigen-value 0. For other φ,

$${}_2\langle\alpha|\varphi, 0\rangle = \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2/2} \to 0$$

as $|\alpha| \to \infty$. This suggests the convergence of

$$|\alpha|\,{}_2\langle\alpha|\Pi^0|\alpha\rangle_2 = \frac{|\alpha|}{2\pi}\int_{2\theta-\pi}^{2\theta+\pi} {}_2\langle\alpha|\varphi,0\rangle\langle\varphi,0|\alpha\rangle_2\,d\varphi$$

to $\dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\overline{\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}}$ as $|\alpha| \to \infty$ (see (1.10)).

4. Proof of the convergence of $|\alpha|\,{}_2\langle\alpha|\Pi^{[x|\alpha|]}|\alpha\rangle_2$

Our main aim is to prove the formula (1.11) in Section 1: For $\alpha = \mathrm{e}^{\mathrm{i}\theta}\,|\alpha|$

$$\begin{aligned} |\alpha|\,{}_2\langle\alpha|\Pi^{[x|\alpha|]}|\alpha\rangle_2 &\longrightarrow \frac{1}{\sqrt{2\pi}}|\theta;x\rangle_{11}\langle\theta;x| \\ &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\overline{\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{v}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}\mathrm{e}^{-x^2/4}} \end{aligned} \tag{4.1}$$

as $|\alpha| \to \infty$. Note that the above formula immediately gives the convergence of the probability distribution (see (1.7))

$$|\alpha|\,({}_1\langle\beta|\otimes{}_2\langle\alpha|)\Pi^{[x|\alpha|]}(|\beta\rangle_1\otimes|\alpha\rangle_2) \to \frac{1}{\sqrt{2\pi}}{}_1\langle\beta|\theta;x\rangle_{11}\langle\theta;x|\beta\rangle_1 = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\{x-(\bar{\alpha}\beta+\alpha\bar{\beta})/|\alpha|\}^2/2}.$$

Define projections $\Pi^{(a,b]|\alpha|}$, $P_R^{(a,b]}$ and $P^{(a,b]}$ by

$$\Pi^{(a,b]|\alpha|} = \sum_{a|\alpha|<l\le b|\alpha|}\Pi^l,\quad P_R^{(a,b]} = \frac{1}{\sqrt{2\pi}}\sum_{a|\alpha|<l\le b|\alpha|}|\theta;l/\,|\alpha|\rangle_{11}\langle\theta;l/\,|\alpha|\,|\frac{1}{|\alpha|}$$

and

$$P^{(a,b]} = \frac{1}{\sqrt{2\pi}}\int_a^b|\theta;r\rangle_{11}\langle\theta;r|dr$$

respectively.

Before we prove (4.1), we show that formula (4.1) implies the following proposition:

Proposition 4.1. *Let $|\beta\rangle_1$ be a coherent state, and $|\psi\rangle = |\beta\rangle_1\otimes|\alpha\rangle_2$. Then*

$$\left\|\Pi^{(a,b]|\alpha|}|\psi\rangle - (P^{(a,b]}\otimes I)|\psi\rangle\right\|^2 \to 0$$

as $|\alpha| \to \infty$.

Proof. In order to prove the proposition, we will show that each term of the right hand side of the following formula converges to ${}_1\langle\beta|P^{(a,b]}|\beta\rangle_1$ as $|\alpha| \to \infty$.

$$\begin{aligned}
&\left\|\Pi^{(a,b]|\alpha|}|\psi\rangle - P^{(a,b]}(|\beta\rangle_1 \otimes |\alpha\rangle_2)\right\|^2 \\
&= \langle\psi|\Pi^{(a,b]|\alpha|}|\psi\rangle + {}_1\langle\beta|P^{(a,b]}|\beta\rangle_{1\,2}\langle\alpha|\alpha\rangle_2 \\
&- \langle\psi|\Pi^{(a,b]|\alpha|}(P^{(a,b]}|\beta\rangle_1 \otimes |\alpha\rangle_2) - ({}_1\langle\beta|P^{(a,b]} \otimes {}_2\langle\alpha|)\Pi^{(a,b]|\alpha|}|\psi\rangle.
\end{aligned}$$

Since the formula (4.1) implies

$$ {}_1\langle\beta|\,|\alpha|\,{}_2\langle\alpha|\Pi^{[x|\alpha|]}|\alpha\rangle_2|\beta\rangle_1 \to \frac{1}{\sqrt{2\pi}}{}_1\langle\beta|\theta;x\rangle_{1\,1}\langle\theta;x|\beta\rangle_1 $$

as $|\alpha| \to \infty$, we have

$$\begin{aligned}
&\langle\psi|\Pi^{(a,b]|\alpha|}|\psi\rangle = \sum_{a|\alpha|<l\le b|\alpha|} {}_1\langle\beta|_2\langle\alpha|\Pi^l|\alpha\rangle_2|\beta\rangle_1 \\
&\to \frac{1}{\sqrt{2\pi}} \sum_{a|\alpha|<l\le b|\alpha|} {}_1\langle\beta|\theta;l/\,|\alpha|\rangle_{1\,1}\langle\theta;l/\,|\alpha|\,|\beta\rangle_1 \frac{1}{|\alpha|} = {}_1\langle\beta|P_R^{(a,b]}|\beta\rangle_1 \to {}_1\langle\beta|P^{(a,b]}|\beta\rangle_1
\end{aligned}$$

as $|\alpha| \to \infty$. Next observe

$$ \langle\psi|\Pi^{(a,b]|\alpha|}(P^{(a,b]}|\beta\rangle_1 \otimes |\alpha\rangle_2) = {}_1\langle\beta|({}_2\langle\alpha|\Pi^{(a,b]|\alpha|}|\alpha\rangle_2)P^{(a,b]}|\beta\rangle_1 $$

$$ \to {}_1\langle\beta|P_R^{(a,b]}P^{(a,b]}|\beta\rangle_1 = {}_1\langle\beta|P_R^{(a,b]}|\beta\rangle_1 \to {}_1\langle\beta|P^{(a,b]}|\beta\rangle_1 $$

where we used the relation (see (2.6))

$$\begin{aligned}
&|\theta;l/\,|\alpha|\rangle_{1\,1}\langle\theta;l/\,|\alpha|\,|P^{(a,b]} = \frac{1}{\sqrt{2\pi}}\int_a^b |\theta;l/\,|\alpha|\rangle_{1\,1}\langle\theta;l/\,|\alpha|\,|\theta;r\rangle_{1\,1}\langle\theta;r|dr \\
&= \int_a^b |\theta;l/\,|\alpha|\rangle_{1\,1}\delta(l/\,|\alpha| - r)_1\langle\theta;r|dr = |\theta;l/\,|\alpha|\rangle_{1\,1}\langle\theta;l/\,|\alpha|\,|.
\end{aligned}$$

Similarly we find

$$\begin{aligned}
&({}_1\langle\beta|P^{(a,b]} \otimes {}_2\langle\alpha|)\Pi^{(a,b]|\alpha|}|\psi\rangle = {}_1\langle\beta|P^{(a,b]}({}_2\langle\alpha|\Pi^{(a,b]|\alpha|}|\alpha\rangle_2)|\beta\rangle_1 \\
&\to {}_1\langle\beta|P^{(a,b]}P_R^{(a,b]}|\beta\rangle_1 = {}_1\langle\beta|P_R^{(a,b]}|\beta\rangle_1 \to {}_1\langle\beta|P^{(a,b]}|\beta\rangle_1.
\end{aligned}$$

This proves the proposition. □

Corollary 4.2. *Let* $|\phi\rangle_1 \in \mathcal{H}_1$*, and* $|\psi\rangle = |\phi\rangle_1 \otimes |\alpha\rangle_2$*. Then*

$$ \left\|\Pi^{(a,b]|\alpha|}|\psi\rangle - (P^{(a,b]} \otimes I)|\psi\rangle\right\|^2 \to 0 $$

as $|\alpha| \to \infty$.

Proof. Since $\Pi^{(a,b]|\alpha|}$ and $P^{(a,b]}$ are projections (and consequently, bounded operators) and linear combinations of coherent states constitute a dense subset of $\mathcal{H}_1$ (see (2.2)), the corollary follows from the previous proposition. $\square$

Remark 4.3. We can modify the postulates of projective measurement of Definition 1.1 as follows (see [11]):

Postulate 1) When one measures the observable $R = \int_{-\infty}^{\infty} r dE_r$ on the state $|\psi\rangle$, the probability for obtaining the outcome r in the interval $(a, b]$ is $\langle\psi|E(a,b]|\psi\rangle/\langle\psi|\psi\rangle$, where $E(a,b] = \int_a^b dE_r$.

Postulate 2) After the measurement the initial state $|\psi\rangle$ changes to $E(a,b]|\psi\rangle$.

Note that for the observable $\Xi = \sum_{l=-\infty}^{\infty} l\Pi^l$, $E_r = \Pi^{(-\infty,r]}$ and $E(a,b] = \Pi^{(a,b]}$, for the observable $\xi(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r|\theta;r\rangle_{11}\langle\theta;r|dr$, $E_r = P^{(-\infty,r]}$ and $E(a,b] = P^{(a,b]}$.

Remark 4.4. Let $a = x - \epsilon$ and $b = x + \epsilon$ for a small $\epsilon > 0$, and $|\phi\rangle_1 \in \mathcal{H}_1$. Then the relation

$$\Pi^{(a,b]|\alpha|}|\phi\rangle_1 \otimes |\alpha\rangle_2 \approx (P^{(a,b]}|\phi\rangle_1) \otimes |\alpha\rangle_2$$

shows the following: If the balanced homodyne detector detects the difference $N_2 - N_1$ of the photon numbers N_1 and N_2 on the state $e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}|\phi\rangle_1 \otimes |\alpha\rangle_2$, composed of the signal $|\phi\rangle_1$, the strong laser beam $|\alpha\rangle_2$ and the operation of beam-splitter $e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}$, and the result is $l = x\,|\alpha|$ within an error $\epsilon\,|\alpha|$, then the state $e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}|\phi\rangle_1 \otimes |\alpha\rangle_2$ changes to $e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}P^{(a,b]}|\phi\rangle_1 \otimes |\alpha\rangle_2$.

If ${}_1\langle\theta;x|\phi\rangle_1$ is meaningful (e.g., $|\phi\rangle_1$ is a linear combination of coherent states), $(P^{(a,b]}|\phi\rangle_1) \otimes |\alpha\rangle_2 \approx 2\epsilon|\theta;x\rangle_{11}\langle\theta;x|\phi\rangle_1 \otimes |\alpha\rangle_2$ and the state $e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}|\phi\rangle_1 \otimes |\alpha\rangle_2$ changes to $2\epsilon e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}|\theta;x\rangle_{11}\langle\theta;x|\phi\rangle_1 \otimes |\alpha\rangle_2$, which is equivalent to $e^{\mathrm{i}H^{12}_{\mathrm{hbs}}}|\theta;x\rangle_{11}\langle\theta;x|\phi\rangle_1 \otimes |\alpha\rangle_2$ by the normalization (of generalized state).

For $\epsilon = 1/2\,|\alpha|$ the relations $\Pi^{(a,b]|\alpha|}|\phi\rangle_1 \otimes |\alpha\rangle_2 = \Pi^{[x|\alpha|+1/2]}|\phi\rangle_1 \otimes |\alpha\rangle_2$ and $(P^{(a,b]}|\phi\rangle_1) \otimes |\alpha\rangle_2 \approx (1/\,|\alpha|)|\theta;x\rangle_{11}\langle\theta;x|\phi\rangle_1 \otimes |\alpha\rangle_2$ show (1.8).

First, we attend to the case of $x = 0$, i.e., $\lim_{|\alpha|\to\infty} |\alpha|\,{}_2\langle\alpha|\Pi^0|\alpha\rangle_2$. We begin with the following proposition.

Proposition 4.5. *For a continuous function $g(\varphi)$ we have*

$$\frac{|\alpha|}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(\varphi)\mathrm{e}^{(\cos\varphi - 1)|\alpha|^2} d\varphi \to g(0) \tag{4.2}$$

as $|\alpha| \to \infty$.

Proof. Taylor's theorem tells us that the relation

$$\cos\varphi - 1 = -\frac{1}{2}\varphi^2 + \frac{\cos\theta\varphi}{4!}\varphi^4$$

holds for some $0 < \theta < 1$, and consequently, for $-\pi/2 \le -\delta \le \varphi \le \delta \le \pi/2$,

$$-\frac{1}{2}\varphi^2 \le -\frac{1}{2}\varphi^2 + \frac{\cos\varphi}{4!}\varphi^4 \le \cos\varphi - 1 \le -\frac{1}{2}\varphi^2 + \frac{1}{4!}\varphi^4 \le -\frac{1}{2}\varphi^2 + \frac{1}{4!}\delta^4$$

and

$$e^{-\varphi^2|\alpha|^2/2} \le e^{(\cos\varphi - 1)|\alpha|^2} \le e^{\delta^4|\alpha|^2/4!} e^{-\varphi^2|\alpha|^2/2}.$$

Therefore we have

$$\int_{-\delta}^{\delta} e^{-\varphi^2|\alpha|^2/2} d\varphi \le \int_{-\delta}^{\delta} e^{(\cos\varphi - 1)|\alpha|^2} d\varphi \le e^{\delta^4|\alpha|^2/4!} \int_{-\delta}^{\delta} e^{-\varphi^2|\alpha|^2/2} d\varphi. \tag{4.3}$$

Let $\delta = |\alpha|^{-3/4}$. Then, for $\alpha \to \infty$,

$$e^{\delta^4|\alpha|^2/4!} = e^{|\alpha|^{-3}|\alpha|^2/4!} = e^{|\alpha|^{-1}/4!} \to 1 \tag{4.4}$$

and

$$\begin{aligned} \frac{|\alpha|}{\sqrt{2\pi}} \int_{-\delta}^{\delta} e^{-\varphi^2|\alpha|^2/2} d\varphi &= \frac{1}{\sqrt{2\pi}} \int_{-\delta|\alpha|}^{\delta|\alpha|} e^{-x^2/2} dx \\ = \quad \frac{1}{\sqrt{2\pi}} \int_{-|\alpha|^{1/4}}^{|\alpha|^{1/4}} e^{-x^2/2} dx &\to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1 \end{aligned} \tag{4.5}$$

hold as $|\alpha| \to \infty$. Combining the relations (4.3) - (4.5) we get

$$\frac{|\alpha|}{\sqrt{2\pi}} \int_{-\delta}^{\delta} e^{(\cos\varphi - 1)|\alpha|^2} d\varphi \to 1$$

as $|\alpha| \to \infty$. This and the following relations show the desired relation (4.2).

$$\begin{aligned} 0 &\le \int_{\delta}^{\pi} e^{(\cos\varphi - 1)|\alpha|^2} d\varphi + \int_{-\pi}^{-\delta} e^{(\cos\varphi - 1)|\alpha|^2} d\varphi \\ &\le 2e^{\delta^4|\alpha|^2/4!} \int_{\delta}^{\pi} e^{-\varphi^2|\alpha|^2/2} d\varphi \\ &\le 2e^{\delta^4|\alpha|^2/4!} e^{-\delta^2|\alpha|^2/2} (\pi - \delta) \to 0 \end{aligned}$$

for $0 < \delta \le \pi$ and $|\alpha| \to \infty$. □

Corollary 4.6. *For a continuous function $f(\varphi)$ we have*

$$\frac{|\alpha|}{\sqrt{2\pi}} \int_{a-\pi}^{a+\pi} f(\varphi) e^{(\cos(\varphi - a) - 1)|\alpha|^2} d\varphi \to f(a)$$

as $|\alpha| \to \infty$.

Proof. By changing the variable $\varphi - a \to \varphi$, we have the corollary. □

Now we can prove the formula (4.1) for $x = 0$.

Theorem 4.7. *Let Π^0 be the projection operator onto the eigen-space of the observable Ξ of (3.1) with eigen-value 0 and $\alpha = \mathrm{e}^{\mathrm{i}\theta}|\alpha|$. Then*

$$\lim_{|\alpha|\to\infty} |\alpha|\,{}_2\langle\alpha|\Pi^0|\alpha\rangle_2 = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\overline{\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}},$$

where $\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}$ is the eigen-function of the observable $\xi_1(\theta)$ of (2.3) with eigen value 0.

Proof. According to Proposition 3.1, we calculate

$$\lim_{|\alpha|\to\infty}\frac{|\alpha|}{2\pi}\int_{a-\pi}^{a+\pi} {}_2\langle\alpha|\varphi,0\rangle\langle\varphi,0|\alpha\rangle_2\, d\varphi.$$

Let $\alpha = \mathrm{e}^{\mathrm{i}\theta}|\alpha|$. Then from the formula (3.5) we have

$$\begin{aligned}{}_2\langle\alpha|\varphi,0\rangle\langle\varphi,0|\alpha\rangle_2 &= \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)/2}\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{-\mathrm{i}\varphi}(-v_1^2+\mathrm{e}^{2\mathrm{i}\theta}|\alpha|^2)/2}\\ &= \mathrm{e}^{-|\alpha|^2}\mathrm{e}^{-(\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2+\mathrm{e}^{-\mathrm{i}\varphi}v_1^2/2)+\cos(\varphi-2\theta)|\alpha|^2)}\end{aligned}$$

and we get $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\overline{\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}}$ by using Corollary 4.6 for $a = 2\theta$ and $f(\varphi) = \mathrm{e}^{-(\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2+\mathrm{e}^{-\mathrm{i}\varphi}v_1^2/2)}$. This completes the proof. □

Next, we calculate $\lim_{|\alpha|\to\infty}|\alpha|\,{}_2\langle\alpha|\Pi^l|\alpha\rangle_2$ for $l = [x\,|\alpha|]$, $x \neq 0$. In the holomorphic representation, ${}_2\langle\alpha|\varphi,l\rangle$ is expressed as

$${}_2\langle\alpha|\varphi,l\rangle = \mathrm{e}^{-|\alpha|^2/2}\left[\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{\sqrt{j!}}{\sqrt{2^l(l+j)!}}(\bar{u}_1+\bar{\alpha})^l\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j\right] \tag{4.6}$$

for $l \geq 0$ and

$${}_2\langle\alpha|\varphi,l\rangle = \mathrm{e}^{-|\alpha|^2/2}\left[\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{\sqrt{j!}}{\sqrt{2^{|l|}(|l|+j)!}}(-\bar{u}_1+\bar{\alpha})^{|l|}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j\right]$$

for $l \leq 0$. We calculate ${}_2\langle\alpha|\varphi,l\rangle$ only for the case of $l \geq 0$, because the calculation for $l \leq 0$ goes in the same way. Note that (see (3.5))

$$\begin{aligned}{}_2\langle\alpha|\varphi,0\rangle &= \mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)/2}\\ &= \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j(\varphi-2\theta)}\frac{(|\alpha|^2/2)^j}{j!}.\end{aligned}$$

Comparing this with (4.6), the terms $\frac{\sqrt{j!}}{\sqrt{2^l(l+j)!}}(\bar{u}_1+\bar{\alpha})^l$ change ${}_2\langle\alpha|\varphi,0\rangle$ into ${}_2\langle\alpha|\varphi,l\rangle$, and we will study how these terms change $\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}$ into $\mathrm{e}^{-x^2/4}\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}$.

Let l be an even positive integer. Note that the following change of the sum induces the interesting effect:

$$\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j \longrightarrow \mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j \tag{4.7}$$

$$\begin{aligned}&= \mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}(j+l/2)\varphi}\frac{1}{2^{(j+l/2)}(j+l/2)!}(-\bar{u}_1^2+\bar{\alpha}^2)^{(j+l/2)}\\ &= \mathrm{e}^{\mathrm{i}l\varphi/2}\frac{1}{\sqrt{2^l}}(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{j!}{(j+l/2)!}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j.\end{aligned} \tag{4.8}$$

The multiplication of each j-th term in (4.8) with

$$\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}}\frac{(j+l/2)!}{\sqrt{(j+l)!j!}}$$

induces the change

$$\to \mathrm{e}^{\mathrm{i}l\varphi/2}\frac{1}{\sqrt{2^l}}(\bar{u}_1+\bar{\alpha})^l\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{\sqrt{j!}}{\sqrt{(j+l)!}}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j = \mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi,l\rangle.$$

Equivalently,

$$\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j = \mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi,l\rangle. \tag{4.9}$$

Now we study the formula (4.9).

Lemma 4.8. *For any $\lambda, m > 0$ we have*

$$\sum_{0\le j\le m-\lambda\sqrt{m}}\mathrm{e}^{-m}\frac{m^j}{j!}+\sum_{m+\lambda\sqrt{m}\le j}\mathrm{e}^{-m}\frac{m^j}{j!}\le\frac{1}{\lambda^2}.$$

Proof. The above inequality is nothing but the Chebyshev's inequality $\Pr(|X-\mu|\ge\lambda\sigma)\le 1/\lambda^2$ for the Poisson random variable X with mean $\mu=m$ and standard deviation $\sigma=\sqrt{m}$. □

Proposition 4.9. *Let $M>0$ and $0<\theta<1$. Then*

$$\mathrm{e}^{-M}\sum_{0\le j\le\theta M}\frac{M^j}{j!}\to 0$$

as $M\to\infty$.

Proof. For any $\lambda > 0$, there exists $M > 0$ such that $\theta m \leq m - \lambda\sqrt{m}$ for $m \geq M$, and it follows from Lemma 4.8 that

$$\mathrm{e}^{-m} \sum_{0 \leq j \leq \theta m} \frac{m^j}{j!} \leq \sum_{0 \leq j \leq m - \lambda\sqrt{m}} \mathrm{e}^{-m} \frac{m^j}{j!} \leq \frac{1}{\lambda^2}.$$

This completes the proof. □

Proposition 4.10. *Let $l/2 = [x\,|\alpha|\,/2]$ for $x > 0$. Then*

$$\mathrm{e}^{-|\alpha|^2/2} \sum_{j=0}^{l/2} \mathrm{e}^{\mathrm{i}j\varphi} \frac{1}{2^j j!} (-\bar{u}_1^2 + \bar{\alpha}^2)^j \to 0$$

as $|\alpha| \to \infty$.

Proof. Let $\mu = \left|\bar{u}_1^2\right|/2$, $m = |\alpha|^2/2$ and $M = \mu + m$, and note that

$$l/2 = [x\sqrt{2m}/2] < (\mu + m)/2 = M/2$$

for sufficiently large m. Then

$$\mathrm{e}^{-\mu}\mathrm{e}^{-m} \left| \sum_{j=0}^{l/2} \mathrm{e}^{\mathrm{i}j\varphi} \frac{1}{2^j j!} (-\bar{u}_1^2 + \bar{\alpha}^2)^j \right| \leq \mathrm{e}^{-M} \sum_{0 \leq j \leq M/2} \frac{M^j}{j!} \to 0, \ (m \to \infty)$$

follows from the previous proposition. □

Corollary 4.11. *Let $l/2 = [x\,|\alpha|\,/2]$ for $x > 0$. Then*

$$\mathrm{e}^{-|\alpha|^2/2} \sum_{j=l/2}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi} \frac{1}{2^j j!} (-\bar{u}_1^2 + \bar{\alpha}^2)^j$$

can be replaced by

$$\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2} \mathrm{e}^{-|\alpha|^2/2} \sum_{j=l/2}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi} \frac{1}{2^j j!} \bar{\alpha}^{2j}$$

for sufficiently large $|\alpha|$.

Proof. It follows from the previous proposition that

$$
\begin{aligned}
&\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j \approx \mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j \\
&= \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}(-\bar{u}_1^2+\bar{\alpha}^2)/2} = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\varphi}\bar{\alpha}^2/2} = \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}\bar{\alpha}^{2j} \\
&\approx \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}\bar{\alpha}^{2j}.
\end{aligned}
$$

□

For the proof of Proposition 4.13 we need the following lemma.

Lemma 4.12. *For any $R > 0$, $\log\left(1+\frac{z}{m}\right)^{ma}$ converges to za as $m \to \infty$, uniformly on the sets $\{a \in \mathbb{C}; |a| \le R\}$ and $\{z \in \mathbb{C}; |z| \le R\}$.*

Proof. We begin with the following Taylor's expansion formula of $\log(1+z)$ for $|z| < \rho < 1$:

$$\log(1+z) = z + z^2\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=\rho}\frac{1}{\zeta^2(\zeta-z)}\log(1+\zeta)\,d\zeta.$$

Let $0 < \rho < 1$ and $|\zeta| = \rho$. Then

$$\log(1-\rho) \le \log|1+\zeta| \le \log(1+\rho), \;\; |\arg(1+\zeta)| \le \sin^{-1}\rho$$

and

$$|\log(1+\zeta)| \le |\log|1+\zeta|| + |\arg(1+\zeta)| \le |\log(1-\rho)| + \sin^{-1}\rho.$$

Let $|z| < \rho$. Then $|\zeta - z| \ge \rho - |z|$ and

$$
\begin{aligned}
\left|z^2\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=\rho}\frac{1}{\zeta^2(\zeta-z)}\log(1+\zeta)\,d\zeta\right| &\le \frac{1}{2\pi}\int_{|\zeta|=\rho}\frac{|z|^2}{\rho^2(\rho-|z|)}(|\log(1-\rho)| + \sin^{-1}\rho)d\,|\zeta| \\
&= \frac{|z|^2}{\rho(\rho-|z|)}(|\log(1-\rho)| + \sin^{-1}\rho).
\end{aligned}
$$

For any $R > 0$, there exists M_1 such that for $m \ge M_1$, $|z/m| < \rho < 1$ holds for $|z| \le R$. Therefore, for any $\epsilon > 0$, $R > 0$ and $|a|, |z| \le R$, there exists $M \ge M_1$ such that

$$
\begin{aligned}
&\left|\log\left(1+\frac{z}{m}\right)^{ma} - ma\frac{z}{m}\right| = \left|ma\frac{z^2}{m^2}\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=\rho}\frac{1}{\zeta^2(\zeta-z/m)}\log(1+\zeta)\,d\zeta\right| \\
&\le \frac{|a|\,|z|^2}{m}\frac{1}{\rho(\rho-|z/m|)}(|\log(1-\rho)| + \sin^{-1}\rho) < \epsilon
\end{aligned}
$$

for $m \ge M$. This completes the proof. □

Proposition 4.13. *Let l be a positive even integer and $x > 0$. For $\alpha = |\alpha| \mathrm{e}^{\mathrm{i}\theta}$ such that $l = x|\alpha|$*

$$\frac{(\bar{u}_1 + \bar{\alpha})^l}{(-\bar{u}_1^2 + \bar{\alpha}^2)^{l/2}} \to \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}$$

as $|\alpha| \to \infty$.

Proof. It follows from the above lemma that

$$\frac{(\bar{u}_1 + \bar{\alpha})^l}{(-\bar{u}_1^2 + \bar{\alpha}^2)^{l/2}} = \frac{(\bar{u}_1 + \bar{\alpha})^l}{(\bar{u}_1 + \bar{\alpha})^{l/2}(-\bar{u}_1 + \bar{\alpha})^{l/2}} = \frac{(\bar{u}_1 + \bar{\alpha})^{l/2}}{(-\bar{u}_1 + \bar{\alpha})^{l/2}}$$

$$= \frac{(\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1 + |\alpha|)^{l/2}}{(-\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1 + |\alpha|)^{l/2}} = \frac{(\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1/|\alpha| + 1)^{x|\alpha|/2}}{(-\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1/|\alpha| + 1)^{x|\alpha|/2}} \to \frac{\mathrm{e}^{\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1 x/2}}{\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1 x/2}} = \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}$$

as $|\alpha| \to \infty$. □

Proposition 4.14. *Let l be a positive integer and $\epsilon(l) = 0$ for even l and $\epsilon(l) = 1/2$ for odd l. For $j \geq l/2$,*

$$\frac{j!}{\sqrt{(j + (l/2 \pm \epsilon(l)))!(j - ((l/2) \pm \epsilon(l)))!}} \leq 1.$$

Let $m = |\alpha|^2/2$, $l/2 = [x\sqrt{2m}/2] + \epsilon$ and $j = [m + \lambda\sqrt{m}] + \delta = [(1 + \lambda/\sqrt{m})m] + \delta = [\mu m] + \delta$, where $\epsilon, \delta \in \{0, 1\}$. Then

$$\frac{j!}{\sqrt{(j + l/2)!(j - l/2)!}} \to \mathrm{e}^{-x^2/4\mu}$$

as $m \to \infty$ uniformly on the set $\left|x^2/\mu\right| \leq R$ for any $R > 0$.

Proof. The first inequality follows from the inequalities. For even integers l,

$$\frac{j!j!}{(j + l/2)!(j - l/2)!} = \frac{(j - l/2 + 1)\cdots j}{(j + 1)\cdots(j + l/2)} = \frac{j - l/2 + 1}{j + 1} \cdots \frac{j + 1}{j + 1 + l/2} < 1.$$

For odd integers l,

$$\frac{j!j!}{(j + l/2 \pm 1/2)!(j - l/2 \mp 1/2)!} = \frac{(j - l/2 \mp 1/2 + 1)\cdots j}{(j + 1)\cdots(j + l/2 \pm 1/2)}$$

$$= \frac{j - l/2 \mp 1/2 + 1}{j + 1} \cdots \frac{j + 1}{j + 1 + l/2 \pm 1/2} \leq 1.$$

Stirling's approximation $n! \sim \sqrt{2\pi n}\{n/\mathrm{e}\}^n$ implies

$$\frac{j!j!}{(j+l/2)!(j-l/2)!} \approx \frac{2\pi j\{j/\mathrm{e}\}^{2j}}{\sqrt{2\pi(j+l/2)}\{(j+l/2)/\mathrm{e}\}^{(j+l/2)}\sqrt{2\pi(j-l/2)}\{(j-l/2)/\mathrm{e}\}^{j-l/2}}$$
$$= \frac{2\pi j j^{2j}}{\sqrt{2\pi(j+l/2)}(j+l/2)^{(j+l/2)}\sqrt{2\pi(j-l/2)}(j-l/2)^{j-l/2}}$$
$$= \frac{j}{\sqrt{j+l/2}\sqrt{j-l/2}}\frac{j^{2j}}{(j+l/2)^{(j+l/2)}(j-l/2)^{j-l/2}}$$
$$= \frac{j}{\sqrt{j+l/2}\sqrt{j-l/2}}\frac{1}{(1+l/2j)^j(1-l/2j)^j}\frac{(1-l/2j)^{l/2}}{(1+l/2j)^{l/2}}.$$

The uniform convergence of the following three sequences, as $m \to \infty$,

$$\frac{j}{\sqrt{j+l/2}\sqrt{j-l/2}} = \frac{1}{\sqrt{1+l/2j}\sqrt{1-l/2j}}$$
$$= \frac{1}{\sqrt{1+([x\sqrt{2m}/2]+\epsilon)/([\mu m]+\delta)}\sqrt{1-([x\sqrt{2m}/2]+\epsilon)/([\mu m]+\delta)}}$$
$$= \frac{1}{\sqrt{1-([x\sqrt{2m}/2]+\epsilon)^2/([\mu m]+\delta)^2}} \longrightarrow 1,$$

$$\frac{(1-l/2j)^{l/2}}{(1+l/2j)^{l/2}} = \frac{(1-([x\sqrt{2m}/2]+\epsilon)/([\mu m]+\delta))^{[x\sqrt{2m}/2]+\epsilon}}{(1+([x\sqrt{2m}/2]+\epsilon)/([\mu m]+\delta))^{[x\sqrt{2m}/2]+\epsilon}}$$
$$= \frac{(1-\{([x\sqrt{2m}/2]+\epsilon)^2/([\mu m]+\delta)\}/([x\sqrt{2m}/2]+\epsilon))^{[x\sqrt{2m}/2]+\epsilon}}{(1+\{([x\sqrt{2m}/2]+\epsilon)^2/([\mu m]+\delta)\}/([x\sqrt{2m}/2]+\epsilon))^{[x\sqrt{2m}/2]+\epsilon}}$$
$$\to \frac{\mathrm{e}^{-x^2/2\mu}}{\mathrm{e}^{x^2/2\mu}} = \mathrm{e}^{-x^2/\mu},$$

where we used

$$([x\sqrt{2m}/2]+\epsilon)^2/([\mu m]+\delta) \to x^2/\mu$$

and

$$\frac{1}{(1+l/2j)^j(1-l/2j)^j} = \frac{1}{(1-([x\sqrt{2m}/2]+\epsilon)^2/([\mu m]+\delta)^2)^{[\mu m]+\delta}}$$

$$= \frac{1}{(1-\{([x\sqrt{2m}/2]+\epsilon)^2/([\mu m]+\delta)\}/([\mu m]+\delta))^{[\mu m]+\delta}} \to \frac{1}{\mathrm{e}^{-x^2/2\mu}} = \mathrm{e}^{x^2/2\mu}$$

as $m \to \infty$ on the set $\left|x^2/\mu\right| \le R$ for any $R > 0$ implies the uniform convergence of

$$\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}} \to \mathrm{e}^{-x^2/4\mu}.$$

$\square$

Remark 4.15. If l is an odd integer, $l/2$ is not an integer, so we can not use $l/2$ in the summation $\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j$ of (4.7). Instead of $l/2$ we must choose $[l/2]$ or $[l/2]+1$. Proposition 4.14 shows that the term $\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}}$ in the formula (4.9) is not sensitive to the choice. But the term $\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}}$ in (4.9) is very sensitive to the choice, i.e.,

$$\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{[l/2]}}\to\infty,\quad \frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{[l/2]+1}}\to 0$$

as $\alpha\to\infty$. We choose their geometric mean

$$\sqrt{\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{[l/2]}}\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{[l/2]+1}}}=\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}},$$

then it has a nonzero limit (see Proposition 4.13).

Now we have the main proposition.

Proposition 4.16. *For $\alpha=\mathrm{e}^{\mathrm{i}\theta}|\alpha|$ and $|\alpha|=l/x$ with a fixed $0\neq x\in\mathbb{R}$ we have*

$$\mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi,l\rangle\approx\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\mathrm{e}^{(\cos(\varphi-2\theta)-1)|\alpha|^2/2}\mathrm{e}^{\mathrm{i}(\sin(\varphi-2\theta))|\alpha|^2/2}$$

for sufficient large integer $l=x|\alpha|$.

Proof. It follows from (4.9), Remark 4.15, Proposition 4.13, Proposition 4.14 and Corollary 4.11 that

$$\begin{aligned}\mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi,l\rangle&=\frac{(\bar{u}_1+\bar{\alpha})^l}{(-\bar{u}_1^2+\bar{\alpha}^2)^{l/2}}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{j!}{\sqrt{(j+l/2)!(j-l/2)!}}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j\\&\approx\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\mathrm{e}^{-x^2/4\mu}\frac{1}{2^j j!}(-\bar{u}_1^2+\bar{\alpha}^2)^j\\&\approx\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\mathrm{e}^{-x^2/4\mu}\frac{1}{2^j j!}(\mathrm{e}^{-2\mathrm{i}\theta}|\alpha|^2)^j.\end{aligned}$$

Let $m=|\alpha|^2/2$. Then it follows from Lemma 4.8 that the summation $\sum_{j=l/2}^{\infty}$ can be replaced by $\sum_{j=m-\lambda\sqrt{m}}^{m+\lambda\sqrt{m}}$. If $j\in[m-\lambda\sqrt{m},m+\lambda\sqrt{m}]$, then $\mu=j/m\in[1-\lambda/\sqrt{m},1+$

$\lambda/\sqrt{m}]$. So, $\mathrm{e}^{-x^2/4\mu}$ can be replaced by $\mathrm{e}^{-x^2/4}$ for large m. Thus we have

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}l\varphi/2}{}_2\langle\alpha|\varphi,l\rangle &\approx \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=l/2}^{\infty}\mathrm{e}^{\mathrm{i}j\varphi}\frac{1}{2^j j!}(\mathrm{e}^{-2\mathrm{i}\theta}\,|\alpha|^2)^j \\
&\approx \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\mathrm{e}^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\mathrm{e}^{\mathrm{i}j(\varphi-2\theta)}\frac{|\alpha|^2/2}{j!} \\
&= \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\mathrm{e}^{(\cos(\varphi-2\theta)-1)|\alpha|^2/2}\mathrm{e}^{\mathrm{i}(\sin(\varphi-2\theta))|\alpha|^2/2}
\end{aligned}$$

for large $|\alpha|$. This completes the proof. □

Now our main result follows easily.

Theorem 4.17. *Let Π^l be the projection operator onto the eigen-space of the observable Ξ of (3.1) with eigen-value l and $\alpha = \mathrm{e}^{\mathrm{i}\theta}\,|\alpha|$ and $|\alpha| = l/x$. Then*

$$\lim_{l\to\infty}|\alpha|\,{}_2\langle\alpha|\Pi^l|\alpha\rangle_2 = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\overline{\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{v}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{v}_1^2/2}\mathrm{e}^{-x^2/4}},$$

where $\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{2\mathrm{i}\theta}\bar{u}_1^2/2}$ is the eigen-function of the observable $\xi_1(\theta)$ of (2.3) with eigen-value x.

Proof. Proposition 4.16 shows

$${}_2\langle\alpha|\varphi,l\rangle\langle\varphi,l|\alpha\rangle_2 \approx \mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{u}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{u}_1^2/2}\mathrm{e}^{-x^2/4}\overline{\mathrm{e}^{x\mathrm{e}^{\mathrm{i}\theta}\bar{v}_1}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi}\bar{v}_1^2/2}\mathrm{e}^{-x^2/4}}\mathrm{e}^{(\cos(\varphi-2\theta)-1)|\alpha|^2}$$

for sufficiently large $l = x\,|\alpha|$. Proposition 3.1 and Corollary 4.6 show the theorem. □

5. Conclusion

For the balanced homodyne detection of p_1, we prepare creation and annihilation operators c_4^* resp. c_4 of a local oscillator $|\gamma\rangle_4$ ($\alpha = \mathrm{e}^{\mathrm{i}\theta}\,|\alpha|$) and a half-beamsplitter

$$\mathrm{e}^{\mathrm{i}H_{hbs}^{14}},\ H_{hbs}^{14} = \mathrm{i}(\pi/4)(c_1^*c_4 - c_1c_4^*).$$

Let

$$c_3^*c_0 + c_0^*c_3 = \sum_{l=-\infty}^{\infty} l\Pi_{03}^l \quad\text{and}\quad c_4^*c_1 + c_1^*c_4 = \sum_{k=-\infty}^{\infty} k\Pi_{14}^k.$$

Case A): If we make the measurement of $c_3^*c_0 + c_0^*c_3$ and $c_4^*c_1 + c_1^*c_4$ for the state $(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I)|\Phi\rangle\otimes|\alpha\rangle_3\otimes|\gamma\rangle_4$ and get an outcome $(l,k) = (x_-\,|\alpha|\,, p_+\,|\gamma|)$ within an error $\epsilon(|\alpha|\,,|\gamma|)$, i.e., $l \in ((x_- - \epsilon)\,|\alpha|\,,(x_- + \epsilon)\,|\alpha|] = (a,b]\,|\alpha|$ and $k \in (p_+ - \epsilon, p_+ + \epsilon]\,|\gamma| =$

$(c,d]\,|\gamma|$, then the projective measurement changes the state to

$$
\begin{aligned}
&\Pi_{03}^{(a,b]|\alpha|}\Pi_{14}^{(c,d]|\gamma|}(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I)|\Phi\rangle\otimes|\alpha\rangle_3\otimes|\gamma\rangle_4\\
&\approx(2\epsilon)^2|0;x_-\rangle_{00}\langle 0;x_-|\otimes|\pi/2;p_+\rangle_{11}\langle\pi/2;p_+|(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I)|\Phi\rangle\otimes|\alpha\rangle_3\otimes|\gamma\rangle_4\\
&=(2\epsilon)^2|x_-\rangle_0\otimes|p_+\rangle_1\{{}_0\langle x_-|\otimes{}_1\langle p_+|(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I)|\Phi\rangle\}\otimes|\alpha\rangle_3\otimes|\gamma\rangle_4,
\end{aligned}
$$

which follows from Proposition 4.1 (see Remark 4.4). Note that Proposition 4.1 follows from Theorem 4.17.

Case B): If we measures N_3-N_0 and N_4-N_1 for the state $\mathrm{e}^{\mathrm{i}H_{hbs}^{03}}\mathrm{e}^{\mathrm{i}H_{hbs}^{14}}(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I)|\Phi\rangle\otimes|\alpha\rangle_3\otimes|\gamma\rangle_4$, then the collasped state is

$$
(2\epsilon)^2\mathrm{e}^{\mathrm{i}H_{hbs}^{03}}\mathrm{e}^{\mathrm{i}H_{hbs}^{14}}|x_-\rangle_0\otimes|p_+\rangle_1\{{}_1\langle x_-|\otimes{}_1\langle p_+|(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I)|\Phi\rangle\}\otimes|\alpha\rangle_3\otimes|\gamma\rangle_4.
$$

The essential part of the teleportation is represented in the following relations:

$$
\begin{aligned}
{}_0\langle x_-|\otimes{}_1\langle p_+|\otimes I)(\mathrm{e}^{\mathrm{i}H_{hbs}^{01}}\otimes I|\Phi\rangle)&=\pi^{-1/2}\sum_{k=0}^{\infty}{}_0\langle k|D_0(x_-+\mathrm{i}p_+)^*\otimes{}_1\langle k|\otimes I|\Phi\rangle\\
&\to D_2(\alpha)^*|\psi\rangle_2
\end{aligned}
$$

as $q\to 1$ for

$$
|\Phi\rangle=\pi^{1/2}|\psi\rangle_0\otimes\sum_{n=0}^{\infty}q^n|n\rangle_1\otimes|n\rangle_2\in\mathcal{H}_0\hat{\otimes}\mathcal{H}_1\hat{\otimes}\mathcal{H}_2,\ |q|<1
$$

see (1.3) and (1.4). The essential part $\{\cdots\}$ is the same in both cases A) and B), and so, the teleportation $|\psi\rangle_0\to D_2(\alpha)^*|\psi\rangle_2$ occurs in both cases.

Thus, Theorem 4.17 implies that in the framework of projective measurements, the balanced homodyne measurement causes quantum teleportation.

References

[1] Samuel L. Braunstein, Homodyne statistics. Phys. Rev. A., 42:474–481, 1990.

[2] A. Furusawa, J.L. Sørensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble and E.S. Polzik. Unconditional Quantum Teleportation. Science, 282:706–709, 1998.

[3] Nagamachi, S.; Brüning, E., Quantum Teleportation and Holomorphic Representation of CCR, Open System & Information Dynamics 2008, 15, 155–172.

[4] T. Gerrits, S. Glancy and S.W. Nam, A balanced homodyne detector and local oscilloator shaping for measuring optical Schrödinger cat states, Proc. of SPIE 2011, Vol. 8033, 80330X-1–80330X-7.

[5] Takahiro Serikawa and Akira Furusawa, 500 MHz resonant photo detector for high-quantum-efficiency, low-noise homodyne measurement, Review of Scientific Instruments 2018, 89, 063120-1–063120-8.

[6] Brüning, E.; Nagamachi, S., On foundational aspects of optical quantum communication, Journal of Modern Optics 2019, 66, 1476–1490.

[7] Takaya Matsuuura, Kento Maeda, Toshihiko Sasaki and Masato Koashi, Finite-size security of continuous-variable quantum key distribution with digtital signal processing, Nature Communications 2021, 12:252.

[8] Wen-Bo Liu, Chen-Long Li, Yuan-Mei Xie, Chen-Xun Weng, Jie Gu, Xiao-Yu Cao, Yu-Shuo Lu, Bing-Hong Li, Hua-Lei Yin and Zeng-Bing Chen, Homodyne Detection Quadrature Phase Shift Keying Continuous-Variable Quantum key Distribution with High Excess Noise Tolerance, PRX QUANTUM 2 2021, 040334.

[9] Kosuke Fukui and Shuntaro Takeda, Building a large-scale quantum computer with continuous-variable optical technologies, Journal of Physics B: Atomic, Molecular and Optical Physics 55 2022, 012001.

[10] L. D. Faddeev and A. A. Slavnov, Gauge Field: Introduction to Quantum Theory, Benjamin/Cummings Publishing Co. Inc., London-Amsterdam-Sydney-Tokyo, 1980

[11] H.P. Breuer and F. Petruccione, The theory of open quantum systems, Oxford University Press: Oxford New York, 2002.

[12] H.M. Wiseman and G.J. Milburn, Quantum Measurement and Control, Cambridge University Press: The Edinburgh Building, Cambridge CB2 8RU, UK, 2009.