# ATYPICAL VALUES AT INFINITY OF REAL POLYNOMIAL MAPS WITH 2-DIMENSIONAL FIBERS 

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#### Abstract

We characterize atypical values at infinity of a real polynomial function of three variables by a certain sum of indices of the gradient vector field of the function restricted to a sphere with a sufficiently large radius. This is an analogy of a result of Coste and de la Puente for real polynomial functions with two variables. We also give a characterization of atypical values at infinity of a real polynomial map whose regular fibers are 2-dimensional surfaces.


## 1. Introduction

Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be a real polynomial map, $\operatorname{Sing}(f)$ be the set of singular points of $f$ in $\mathbb{R}^{n}$, and $K_{0}(f)=f(\operatorname{Sing}(f))$. A bifurcation set of $f$ is the smallest set of values in $\mathbb{R}^{m}$ outside which $f$ is a locally trivial fibration. This is a semialgebraic set of codimension at least one [12, 14, 9]. A regular value $\lambda \in f\left(\mathbb{R}^{n}\right) \backslash K_{0}(f)$ is called a typical value at $\infty$ of $f$ if there is an open neighborhood over which $f$ is a trivial fibration. Otherwise, $\lambda$ is called an atypical value at $\infty$ of $f$. For example, the polynomial map $f(x, y)=x(x y+1)$ has no critical value but its bifurcation set is $\{0\}$. There are several studies about the bifurcation sets of real polynomial maps, see for instance [13, 2, 7, 6, 3].

Suppose $m=1$, that is, $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a real polynomial function. Let $B_{a, R}$ be the closed ball in $\mathbb{R}^{n}$ centered at a point $a \in \mathbb{R}^{n}$ and of radius $R>0$. Set

$$
\Gamma=\left\{x \in \mathbb{R}^{n} \left\lvert\, \operatorname{rank}\binom{x-a}{\operatorname{grad} f} \leq 1\right.\right\} .
$$

Note that $\operatorname{Sing}(f) \subset \Gamma$. We choose a center $a \in \mathbb{R}^{n}$ and a sufficiently large $R>0$ so that $\Gamma$ is transverse to $\partial B_{a, r}$ for any $r>R$ and $\Gamma \backslash \operatorname{Int} B_{a, R}$ is homeomorphic to $\Gamma \cap \partial B_{a, R} \times[0,1)$. Each connected component of $\Gamma \backslash \operatorname{Int} B_{a, R}$ is contained in either $\operatorname{Sing}(f)$ or $\Gamma \backslash \operatorname{Sing}(f)$. Throughout the paper, we always choose the center

[^0]$a$ generic so that each connected component of $\Gamma \backslash\left(\operatorname{Sing}(f) \cup \operatorname{Int} B_{a, R}\right)$ is a curve. These curves are called tangency branches at $\infty$ of $f$.

For each point $p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}$, let $\Gamma_{p}$ denote the tangency branch at $\infty$ of $f$ passing through $p$. Set $x_{p}(r)=\Gamma_{p} \cap \partial B_{a, r}$ for $r \geq R$ and define

$$
\lambda_{p}=\lim _{r \rightarrow \infty} f\left(x_{p}(r)\right) \in \mathbb{R} \cup\{ \pm \infty\}
$$

Let $T_{\infty}(f)$ denote the set of values $\lambda \in \mathbb{R}$ for which there exists a curve $x$ : $[R, \infty) \rightarrow \Gamma$ with $x(r) \in \Gamma \cap \partial B_{a, r}$ and $\lim _{r \rightarrow \infty} f(x(r))=\lambda$. Note that

$$
T_{\infty}(f) \subset\left\{\lambda_{p} \in \mathbb{R} \mid p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}\right\} \cup K_{0}(f)
$$

The aim of this paper is to characterize atypical values of $f$ by observing its behavior on the sphere $\partial B_{a, R}$ with a sufficiently large radius $R>0$. Specifically, we focus on the vector field $X_{a, R}$ on $\partial B_{a, R}$ defined by the gradient vector field of the restriction of $f$ to $\partial B_{a, R}$. For each isolated zero $p$ of $X_{a, R}$, the index $\operatorname{Ind}_{p}\left(X_{a, R}\right)$ is defined by the degree of the map from $\partial B_{p, \varepsilon}$ to the $(n-1)$-dimensional sphere given by $x \mapsto \frac{X_{a, R}(x)}{\left\|X_{a, R}(x)\right\|}$, where $\varepsilon>0$ is a sufficiently small real number. Note that each point of $(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}$ is an isolated zero of $X_{a, R}$. For each $\lambda \in T_{\infty}(f)$, let $\Gamma^{(\lambda)}$ be the union of tangency branches $\Gamma_{p}$ with $\lambda_{p}=\lambda$. For each connected component $\Omega$ of $\partial B_{a, R} \backslash f^{-1}(\lambda)$, set

$$
\operatorname{Ind}(\lambda, \Omega)=\sum_{p \in \Gamma^{(\lambda)} \cap \Omega} \operatorname{Ind}_{p}\left(X_{a, R}\right)
$$

We focus on the case $n=3$. In this case, since regular fibers of $f$ are of dimension 2, their topology can be determined by the indices of the vector field $X_{a, R}$. In consequence, we obtain the following theorem. For the definition of a vanishing component, see Section 2.1.

Theorem 1.1. Let $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$ be a polynomial function and $\lambda \in T_{\infty}(f) \backslash K_{0}(f)$. If $\operatorname{Ind}(\lambda, \Omega) \neq 0$ for some connected component $\Omega$ of $\partial B_{a, R} \backslash f^{-1}(\lambda)$ then $\lambda$ is an atypical value at $\infty$ of $f$. Conversely, if there does not exist a vanishing component at $\infty$ when $t$ tends to $\lambda$ and $\operatorname{Ind}(\lambda, \Omega)=0$ for any connected component $\Omega$ of $\partial B_{R} \backslash f^{-1}(\lambda)$ then $\lambda$ is a typical value at $\infty$ of $f$.

In the proof, it is shown that if $\operatorname{Ind}(\lambda, \Omega) \neq 0$ for some $\Omega$ then, for $t$ sufficiently close to $\lambda$, there exists a connected component of $f^{-1}(t) \backslash \operatorname{Int} B_{a, R}$ diffeomorphic to a disk. This interpretation can be used when we generalize the assertion in Theorem 1.1 to polynomial maps $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-2}$ for $n \geq 3$. The statement is the following.

Theorem 1.2. Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-2}$ be a polynomial map, where $n \geq 3$. Suppose that the radius $R>0$ of $B_{a, R}$ is sufficiently large. Then, $\lambda \in F\left(\mathbb{R}^{n}\right) \backslash K_{0}(F)$ is a typical value at $\infty$ of $F$ if and only if the following are satisfied:
(1) There is no vanishing component at $\infty$ when $t$ tends to $\lambda$;
(2) There exists a neighborhood $D$ of $\lambda$ in $\mathbb{R}^{n-2}$ such that, for any $t \in D$,
(2-1) $F^{-1}(t) \backslash \operatorname{Int} B_{R}$ has no compact, connected component, and
(2-2) $\chi\left(F^{-1}(t)\right)=\chi\left(F^{-1}(\lambda)\right)$ holds.
The above theorem is stated again in Section 5 (Theorem 5.1), where a precise condition for the radius $R$ is given. The condition (2-1) is added instead of the condition about the indices in Theorem 1.1. Note that atypical values of an algebraic family of real curves, which can be seen as a restriction of a polynomial map from $\mathbb{R}^{n}$ to $\mathbb{R}^{n-1}$, are characterized by the conditions (1) and (2-2) [13]. See also [7]. Atypical values of a holomorphic map between connected complex manifolds $M \rightarrow B$ with $\operatorname{dim}_{\mathbb{C}} M=\operatorname{dim}_{\mathbb{C}} B+1$ are also characterized by the conditions (1) and (2-2) [8].

This paper is organized as follows. In Section 2, we prove a few lemmas concerning a choice of the center $a$ and the radius $R$ of the ball $B_{a, R}$. In Section 2.5, two examples of polynomial functions $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$, which are based on examples in [13] (also [2]), are given. In Section 3, we prove a theorem that characterizes a vanishing component at infinity of a real polynomial function. Using this theorem, we can obtain some argument for detecting a vanishing component at infinity, see Remark 3.2. Section 4 is devoted to the proof of Theorem 1.1, and Section 5 is devoted to the proof of Theorem 1.2 ,

## 2. Preliminaries

2.1. Vanishing component. In this section we give the definition of a vanishing component at $\infty$ for a polynomial map from $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$ with $n>m \geq 1$.

Definition 2.1. Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be a polynomial map. It is said that there is a vanishing component at $\infty$ when $t$ tends to $\lambda$ if there exists a sequence of points $\left\{t_{k}\right\}$ in $\mathbb{R}^{m}$ such that

$$
\lim _{k \rightarrow \infty} t_{k}=\lambda \quad \text { and } \quad \lim _{k \rightarrow \infty} \max _{i} \inf \left\{\|x\| \in \mathbb{R} \mid x \in Y_{t_{k}, i}\right\}=\infty
$$

where $Y_{t, 1}, \ldots, Y_{t, n_{t}}$ are the connected components of $F^{-1}(t)$.
Remark 2.2. The existence of a vanishing component at $\infty$ does not change even if the distance function $\|x\|$ is replaced by $\|x-a\|$ for any point $a \in \mathbb{R}^{n}$.
2.2. The center of the ball $B_{a, R}$. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a polynomial function, Sing $(f)$ be the set of critical points of $f$ in $\mathbb{R}^{n}$, and $K_{0}(f)=f(\operatorname{Sing}(f))$.

Lemma 2.3. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a polynomial function, $K$ be a finite set in $f\left(\mathbb{R}^{n}\right) \backslash K_{0}(f)$, and $A_{K}$ be the set of points a in $\mathbb{R}^{n}$ satisfying that, for each $\lambda \in K$, there exists an open interval $I_{\lambda}$ in $\mathbb{R}$ containing $\lambda$ such that the function on $f^{-1}(t)$ defined by $x \mapsto\|x-a\|^{2}$ has only non-degenerate critical points for any $t \in I_{\lambda} \backslash\{\lambda\}$. Then the set $A_{K}$ is dense in $\mathbb{R}^{n}$.

Proof. Set

$$
S=\left\{(x, v, t) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \times\left(\mathbb{R} \backslash K_{0}(f)\right) \mid f(x)=t, \operatorname{rank}\binom{v}{\operatorname{grad} f} \leq 1\right\}
$$

It is easy to check that $S$ is a semialgebraic set of dimensional $n+1$ having no singular points.

Consider the "endpoint" map (see [11]):

$$
E: S \rightarrow \mathbb{R}^{n} \times\left(\mathbb{R} \backslash K_{0}(f)\right), \quad(x, v, t) \mapsto(x+v, t)
$$

By the Sard Theorem, the set $E(\operatorname{Sing}(E))$ of singular values of $E$ has measure 0 . We can also check that $E(\operatorname{Sing}(E))$ is a semialgebraic set in $\mathbb{R}^{n} \times\left(\mathbb{R} \backslash K_{0}(f)\right)$ of dimension at most $n$. By [11, Lemma 6.5], $(a, t) \in E(\operatorname{Sing}(E))$ if and only if the function on $f^{-1}(t)$ defined by $x \mapsto\|x-a\|^{2}$ has a degenerate critical point.

We will prove the following claim: For each point $a \in \mathbb{R}^{n}$, any neighborhood of $a$ in $\mathbb{R}^{n}$ contains at least one point $x \neq a$ such that the intersection $(\{x\} \times \mathbb{R}) \cap$ $E(\operatorname{Sing}(E))$ is an isolated set. This implies that $A_{K}$ is dense in $\mathbb{R}^{n}$.

For a contradiction, we assume that there exist a point $a \in \mathbb{R}^{n}$ and a small neighborhood $U$ of $a$ in $\mathbb{R}^{n}$ satisfying that, for each $x \in U \backslash\{a\}$, there is an open interval $I_{x} \subset \mathbb{R}$ such that $\{x\} \times I_{x} \subset E(\operatorname{Sing}(E))$.

Since $E(\operatorname{Sing}(E))$ is a semialgebraic set in $E(S)$ of codimension at least one, its Zariski closure $V$ in $\mathbb{R}^{n} \times \mathbb{R}$ is an algebraic subset of dimension at most $n$. Let $\pi: V \rightarrow \mathbb{R}^{n}$ be the projection from $V \subset \mathbb{R}^{n} \times \mathbb{R}$ to $\mathbb{R}^{n}$ defined by $(x, t) \mapsto x$. Since $\{x\} \times I_{x} \subset V$ for $x \in U \backslash\{a\}$, the inclusion $U \backslash\{a\} \subset \pi(V)$ holds.

On the other hand, it implies from [12, 14] that there exists an open ball $B \subset U \backslash$ $\{a\}$ such that $\pi$ is trivial on $B$, which means that $\pi^{-1}(B) \subset V$ is diffeomorphic to $B \times \pi^{-1}(x)$ for $x \in B$. From the inclusion $U \backslash\{a\} \subset \pi(V)$, we get $\{x\} \times I_{x} \subset \pi^{-1}(x)$. Therefore $\operatorname{dim} \pi^{-1}(B)=\operatorname{dim} U+1=n+1$. This contradicts $\operatorname{dim} V \leq n$.
Remark 2.4. In Lemma 2.3, a point in $f^{-1}(t)$ around which the function on $f^{-1}(t)$ defined by $x \mapsto\|x-a\|^{2}$ is locally constant is regarded as a degenerate critical point.
2.3. Topology of fibers and indices of vector fields on the sphere. Choose a center $a \in \mathbb{R}^{n}$ of $B_{a, R}$ generic and the radius $R>0$ sufficiently large. The interior of $B_{a, R}$ is denoted by $\operatorname{Int} B_{a, R}$ and its boundary is by $\partial B_{a, R}$. For each point $p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}$, let $\Gamma_{p}$ denote the tangency branch at $\infty$ of $f$ passing through $p$. Set $x_{p}(r)=\Gamma_{p} \cap \partial B_{a, r}$, then $f\left(x_{p}(r)\right)$ is monotone with respect to the parameter $r$. We use the following notations:

- $f \nearrow \lambda$ along $\Gamma_{p}$ means that $f\left(x_{p}(r)\right)$ is monotone increasing for $r \geq R$ and $\lim _{r \rightarrow \infty} f\left(x_{p}(r)\right)=\lambda$.
- $f \searrow \lambda$ along $\Gamma_{p}$ means that $f\left(x_{p}(r)\right)$ is monotone decreasing for $r \geq R$ and $\lim _{r \rightarrow \infty} f\left(x_{p}(r)\right)=\lambda$.

Remark 2.5. Let $\Gamma_{p}$ be the tangency branch at $\infty$ of $f$ passing through $p \in(\Gamma \backslash$ Sing $(f)) \cap \partial B_{a, R}$. We have the following remarks.
(1) The point $p$ is a critical point of the following two functions:

$$
\begin{aligned}
& \left.f\right|_{\partial B_{a, r_{a}(p)}}: \partial B_{a, r_{a}(p)} \rightarrow \mathbb{R}, \text { where } r_{a}(p)=\|p-a\|, \\
& \left.r_{a}\right|_{f^{-1}(f(p))}: f^{-1}(f(p)) \rightarrow \mathbb{R}, \text { where } r_{a}(x)=\|x-a\| .
\end{aligned}
$$

(2) Suppose that $f \nearrow \lambda_{p}$ along $\Gamma_{p}$. Then, $p$ is a local maximum (resp. minimum) point of $\left.f\right|_{\partial B_{a, r_{a}(p)}}$ if and only if it is a local minimum (resp. maximum) point of $\left.r_{a}\right|_{f^{-1}(f(p))}$.
(3) Suppose that $f \searrow \lambda_{p}$ along $\Gamma_{p}$. Then, $p$ is a local maximum (resp. minimum) point of $\left.f\right|_{\partial B_{a, r_{a}(p)}}$ if and only if it is a local maximum (resp. minimum) point of $\left.r_{a}\right|_{f^{-1}(f(p))}(\mathrm{cf}$. Example 2.7).
For simplicity, we denote by $\mathcal{P}$ one of the properties "local maximum", "local minimum", "neither local maximum nor local minimum".

Lemma 2.6. There exists a sufficiently large radius $R>0$ such that, for each $p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}$, the property $\mathcal{P}$ of $\left.f\right|_{\partial B_{a, r}}$ is constant on $\Gamma_{p}$.

Proof. For each property $\mathcal{P}$, define the subset $V_{\mathcal{P}}$ of $\mathbb{R}^{n}$ by

$$
V_{\mathcal{P}}=\left\{x \in \mathbb{R}^{n} \mid x \text { is a } \mathcal{P} \text { point of }\left.f\right|_{\partial B_{a, r_{a}(x)}} \text { for } r_{a}(x)=\|x-a\| \geq R\right\}
$$

We will show that, for each $\mathcal{P}$, the set $V_{\mathcal{P}}$ is a semi-algebraic set. If $\mathcal{P}$ is local maximum, the set $V_{\mathcal{P}}$ is represented in terms of the first-order formulas as follows (for the definitions of first-order formulas, see [1, 4]):

$$
V_{\mathcal{P}}=\left\{x \in \mathbb{R}^{n} \mid \exists \varepsilon \in \mathbb{R}\left(\left(y \in \mathbb{R}^{n},\|y\|=\|x\|,\|y-x\|<\varepsilon\right) \Rightarrow f(y) \leq f(x)\right)\right\} .
$$

Hence, it implies from the Tarski-Seidenberg Theorem (see [1, Proposition 2.2.4] or [4, Theorem 1.6]) that $V_{\mathcal{P}}$ is a semialgebraic set. The set $V_{\mathcal{P}}$ for $\mathcal{P}$ being
local minimum is also semialgebraic by a similar argument. If $\mathcal{P}$ is neither local maximum nor local minimum, then the set $V_{\mathcal{P}}$ is the complement of the above two semialgebraic sets. Therefore it is also semialgebraic.

Now, $\Gamma_{p} \cap V_{\mathcal{P}}$ is a semialgebraic subset of a curve for each $\mathcal{P}$. Hence we can choose $R>0$ sufficiently large so that each $\Gamma_{p}$ is contained in some of $V_{\mathcal{P}}$.
2.4. Choice of the radius $R$. Define the set $K_{\infty}(f)$ by

$$
K_{\infty}(f)=\left\{t \in \mathbb{R} \mid \text { there exists a sequence }\left\{x_{k}\right\} \text { in } \mathbb{R}^{n} \text { such that }\left\|x_{k}\right\| \rightarrow \infty\right.
$$

$$
\left.f\left(x_{k}\right) \rightarrow t, \text { and }\left\|x_{k}\right\|\left\|\operatorname{grad} f\left(x_{k}\right)\right\| \rightarrow 0 \text { as } k \rightarrow \infty\right\}
$$

Note that $K_{\infty}(f)$ is a finite set and satisfies $T_{\infty}(f) \subset K_{0}(f) \cup K_{\infty}(f)$. We choose a generic point $a \in \mathbb{R}^{n}$ as in Lemma 2.3 with respect to the set $K=K_{\infty}(f) \backslash K_{0}(f)$. Choose an open interval $I_{\lambda}$ for each $\lambda \in K$ so that $I_{\lambda} \cap I_{\lambda^{\prime}}=\emptyset$ for $\lambda \neq \lambda^{\prime} \in T_{\infty}(f)$. We choose the radius $R>0$ sufficiently large so that the following properties hold:
(i) $\Gamma \backslash \operatorname{Int} B_{a, R}$ is homeomorphic to $\left(\Gamma \cap \partial B_{a, R}\right) \times[0,1)$ and, for each $p \in$ $(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}$,

$$
\Gamma_{p} \cap \bigcup_{\lambda \in T_{\infty}(f)} f^{-1}(\lambda)=\emptyset
$$

(ii) $R>0$ satisfies the condition in Lemma 2.6. Since the center $a$ is chosen as in Lemma 2.3, the property "neither local maximum nor local minimum" for tangency branches is replaced by "saddle".
(iii) For each $\lambda \in K$, each connected component $Y$ of $f^{-1}(\lambda) \backslash \operatorname{Int} B_{a, r}$ intersects $\partial B_{a, r}$ transversely for any $r \geq R$. In particular, $Y$ is diffeomorphic to $\left(Y \cap \partial B_{a, r}\right) \times[0,1)$ for any $r \geq R$.
(iv) $\left\{f(x) \mid x \in \Gamma_{p}\right\} \subset I_{\lambda_{p}}$ holds for any $p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{a, R}$.

In the following sections, we always assume that the radius $R>0$ is sufficiently large so that these properties hold.
2.5. Examples. We give two examples of polynomial functions $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$ of the form $f(x, y, z)=g(x, y)$, where $g(x, y)$ is a polynomial function of two variables.
Example 2.7. Let $g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be the following polynomial function:

$$
g(x, y)=2 y^{5}+4 x y^{4}+\left(2 x^{2}-9\right) y^{3}-9 x y^{2}+12 y .
$$

This example is given in [13, Example 3.4]. The shapes of fibers around the infinity is studied in [2] explicitly, which is given as in Figure 1. There are eight tangency branches, four of which are on the right-hand side and the other four are on the left-hand side. The arrow on each tangency branch represents the direction in
which the value of $f$ increases. For example, for the right-top tangency branch $\Gamma_{p_{1}}$, we have $g \searrow 0$ along $\Gamma_{p_{1}}, p_{1}$ is a local minimum of $\left.g\right|_{\left.B_{a, r_{a}\left(p_{1}\right)}\right)}$ and it is a local minimum of $\left.r_{a}\right|_{g^{-1}\left(g\left(p_{1}\right)\right.}$, where $r_{a}(x)=\|x-a\|$. This function has no vanishing component at $\infty$.


Figure 1. Fibers around the infinity in Example 2.7. The oriented dotted curves are tangency branches at $\infty$.

Let $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$ be a polynomial function given by $f(x, y, z)=g(x, y)$. In [13, Example 3.4], the function $g$ is obtained from $h(x, y)=y\left(2 x^{2} y^{2}-9 x y+12\right)$ as $g(x, y)=h(x+y, y)$. From this form, we can see that $g^{-1}(0) \cap \partial B_{a, R}$ is given by $\{y=0\} \cap \partial B_{a, R}$, which is a connected, simple closed curve on the 2-sphere $\partial B_{a, R}$. The complement of this curve in $\partial B_{a, R}$ consists of two open disks. We denote the one where $y$ is positive by $\Omega_{1}$ and the other by $\Omega_{2}$. By choosing the center $a$ of $B_{a, R}$ on $z=0$, we may assume that all tangency branches in Figure 1 are on the plane $z=0$. Then, for example, the point $p_{1}$ is local minimum of $\left.f\right|_{B_{a, r_{a}\left(p_{1}\right)}}$ and also local minimum of $\left.r_{a}\right|_{f^{-1}\left(f\left(p_{1}\right)\right.}$, where $r_{a}(x)=\|x-a\|$. This is in the case (3) of Remark 2.5. The index is $\operatorname{Ind}_{p_{1}}\left(X_{a, R}\right)=1$. On the other hand, the singularity of $\left.r_{a}\right|_{f^{-1}\left(f\left(p_{2}\right)\right.}$ on the tangency branch $\Gamma_{p_{2}}$ passing through the point $p_{2}$ in the figure becomes a saddle, and therefore its index is $\operatorname{Ind}_{p_{2}}\left(X_{a, R}\right)=-1$. The union $\Gamma^{(0)}$ of tangency branches at $\infty$ of $f$ along which either $f \searrow 0$ or $f \nearrow 0$ has no other
tangency branch passing through the region $\Omega_{1}$. Hence we have

$$
\begin{aligned}
\operatorname{Ind}\left(\lambda, \Omega_{1}\right) & =\sum_{p \in \Gamma^{(0)} \cap \Omega_{1}} \operatorname{Ind}_{p}\left(X_{a, R}\right) \\
& =\operatorname{Ind}_{p_{1}}\left(X_{a, R}\right)+\operatorname{Ind}_{p_{2}}\left(X_{a, R}\right) \\
& =1+(-1)=0 .
\end{aligned}
$$

By the same observation, we have $\operatorname{Ind}\left(\lambda, \Omega_{2}\right)=0$. Then, by Theorem 1.1, we can conclude that 0 is a typical value at $\infty$ of $f$.

Example 2.8. Let $g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be the following polynomial function:

$$
g(x, y)=x^{2} y^{3}\left(y^{2}-25\right)^{2}+2 x y\left(y^{2}-25\right)(y+25)-y^{4}-y^{3}+50 y^{2}+51 y-575 .
$$

This example is given in [13, Example 3.1]. The shapes of fibers around the infinity is studied in [2] explicitly after replacing $x$ by $x+y$ to avoid vertical tangency at infinity. The fibers are given as in Figure 2. There are two component vanishing at $\infty$ when $t$ tends to 0 . The word "cleaving" means that the point on the tangency branch goes to $\infty$ when $t$ tends to 0 , so that the curve cleaves locally into two curves. There are two cleaving curves.


Figure 2. Fibers around the infinity in Example 2.8. The oriented dotted curves are tangency branches at $\infty$. All horizontal solid lines are curves representing $g^{-1}(0)$.

Let $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$ be a polynomial function given by $f(x, y, z)=g(x, y)$. Using Mathematica, we can see that the curve of $g^{-1}(0)$ inside the dotted circle is as shown in Figure 2, Note that it is explained in [13, Example 3.1] that if $|t|$ is sufficiently small then $g^{-1}(t)$ is a disjoint union of five non-compact connected components. Thus, the curves $f^{-1}(0) \cap \partial B_{a, R}$ on the sphere $\partial B_{a, R}$ becomes as shown in Figure 3. There are five circles. Let $X_{a, R}$ be the gradient vector field of $\left.f\right|_{\partial B_{a, R}}$. We have $\operatorname{Ind}_{p_{1}}\left(X_{a, R}\right)=\operatorname{Ind}_{p_{3}}\left(X_{a, R}\right)=-1$ and $\operatorname{Ind}_{p_{2}}\left(X_{a, R}\right)=\operatorname{Ind}_{p_{4}}\left(X_{a, R}\right)=1$. On the region $\Omega_{1}$ depicted in the figure, we have

$$
\begin{aligned}
\operatorname{Ind}\left(0, \Omega_{1}\right) & =\operatorname{Ind}_{p_{1}}\left(X_{a, R}\right)+\operatorname{Ind}_{p_{2}}\left(X_{a, R}\right)+\operatorname{Ind}_{p_{4}}\left(X_{a, R}\right) \\
& =(-1)+1+1=1 \neq 0 .
\end{aligned}
$$

Hence 0 is an atypical value at $\infty$ of $f$ by Theorem 1.1. We can get the same conclusion from the region $\Omega_{2}$ depicted in the figure since $\operatorname{Ind}\left(0, \Omega_{2}\right)=\operatorname{Ind}_{p_{3}}\left(X_{a, R}\right)=$ $-1 \neq 0$.


Figure 3. The curves $f^{-1}(0) \cap \partial B_{a, R}$ on the sphere $\partial B_{a, R}$.

## 3. A Characterization of vanishing component at infinity

Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a polynomial function. Hereafter we omit $a$ in the suffix of $B_{a, r}$ for $r>0$ and denote it by $B_{r}$ for simplicity. Each critical point $p \in \partial B_{R}$ of $\left.f\right|_{\partial B_{R}}$ not lying on $\operatorname{Sing}(f)$ is a point in $(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{R}$. Hence it has a tangency branch $\Gamma_{p}$.

Theorem 3.1. Suppose $n \geq 2$ and $\lambda \in T_{\infty}(f) \backslash K_{0}(f)$. There is a vanishing component at $\infty$ when $t$ tends to $\lambda$ with $t>\lambda$ (resp. $t<\lambda$ ) if and only if there exists a local minimum (resp. maximum) point $p \in \partial B_{R}$ of $\left.f\right|_{\partial B_{R}}$ with $f \searrow \lambda$ (resp. $f \nearrow \lambda$ ) along $\Gamma_{p}$ such that the intersection of the connected component of $f^{-1}(f(p))$ containing $p$ with $\partial B_{R}$ consists of isolated points.

Proof. We first prove the "only if" assertion. We only prove the assertion in the case where $t$ tends to $\lambda$ with $t>\lambda$. The proof for the other case is similar. Let $\left\{Y_{t}\right\}$ be a continuous family of connected components of $f^{-1}(t)$ that vanishes at $\infty$ when $t$ tends to $\lambda$. Tangency branches intersecting $\left\{Y_{t}\right\}$ are contained in a connected component $H$ of $\mathbb{R}^{n} \backslash\left(f^{-1}(\lambda) \cup \operatorname{Int} B_{R}\right)$ by the property (i) about the choice of the radius $R$ in Section 2.4. Remark that $H$ is possibly $\mathbb{R}^{n} \backslash \operatorname{Int} B_{R}$. Set $\Omega=H \cap \partial B_{R}$. Either $\Omega=\partial B_{R}$, or $\Omega \subset \partial B_{R}$ is bounded by a finite number of circles belonging to $f^{-1}(\lambda) \cap \partial B_{R}$. Since $t$ tends to $\lambda$ with $t>\lambda$, we have $\Omega \subset\left\{x \in \partial B_{R} \mid f(x)>\lambda\right\}$. Let $S_{\Omega}$ denote the set of local minimum points of $\left.f\right|_{\partial B_{R}}$ in $\Omega \cap \Gamma^{(\lambda)}$, where $\Gamma^{(\lambda)}$ is the union of tangency branches at $\infty$ of $f$ along which either $f \nearrow \lambda$ or $f \searrow \lambda$. By the definition of a vanishing component at $\infty$ in Section 2.1 and Remark 2.5 (1), the function $r_{a}(x)=\|x-a\|$ restricted to $f^{-1}(t)$ has a local minimum point $y$ on $\Omega \cap \Gamma^{(\lambda)}$. Since $t>\lambda$, it satisfies that $f \searrow \lambda$ along $\Gamma_{y}$. Hence, by Remark 2.5 (3), $y$ is a local minimum point of $\left.f\right|_{\partial B_{R}}$, that is, $y$ is a point in $S_{\Omega}$. In particular, $S_{\Omega}$ is non-empty.

Set $\delta=\min _{x \in S_{\Omega}} f(x)$ and let $p$ be a point in $S_{\Omega}$ such that $f(p)=\delta$ and $f \searrow \lambda$ along $\Gamma_{p}$. Let $(\lambda, \delta]$ be the range of the parameter $t$ of $Y_{t}$. We will show that $Y_{\delta} \cap \partial B_{R}$ consists of isolated points.

Assume for a contradiction that $Y_{\delta} \cap \partial B_{R}$ is not isolated.
Claim 1. $Y_{\delta} \cap \Omega$ is not isolated.
Proof. Assume that $Y_{\delta} \cap \Omega$ is isolated. Then, all points in $Y_{\delta} \cap \Omega$ are local minima of $\left.r_{a}\right|_{Y_{\delta}}: Y_{\delta} \rightarrow \mathbb{R}$, where $r_{a}(x)=\|x-a\|$. The inequality $\lambda<\delta$ implies that $Y_{\delta} \cap f^{-1}(\lambda)=\emptyset$. Hence $Y_{\delta} \subset H$, see Figure 4. This inclusion implies $Y_{\delta} \cap \partial B_{R}=$ $Y_{\delta} \cap \Omega$. However, the right-hand side is isolated while the left-hand is not. This is a contradiction.


Figure 4. $Y_{\delta} \cap f^{-1}(\lambda)=\emptyset$ implies $Y_{\delta} \subset H$.
We continue the proof of Theorem 3.1. Let $Y_{[\lambda, \delta]}$ be the connected component of $f^{-1}([\lambda, \delta])$ containing $p$. There are two cases:

Case 1: $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda)=\emptyset$ (cf. Figure 5). Set $\Omega_{[\lambda, \delta]}=Y_{[\lambda, \delta]} \cap \bar{\Omega}$, where $\bar{\Omega}$ is the closure of $\Omega$ in $\partial B_{R}$. Since $Y_{\delta} \cap \Omega$ is not isolated by Claim 1, $Y_{\delta} \cap \Omega$ has a connected component $C$ of dimension at least 1. A point in $\Omega$ at which $Y_{\delta}$ is tangent to $\Omega$ belongs to a tangency branch at $\infty$ of $f$ and hence it is isolated in $\Omega$. In particular, it cannot be in $C$. This means that $Y_{\delta}$ and $\Omega$ intersect along $C$ transversely. Therefore, since $f$ is continuous on $\bar{\Omega}, Y_{[\lambda, \delta]} \cap \Omega$ has a connected component $C$ of dimension $n-1 \geq 1$. This set $C$ is a compact subset of $\Omega$. Due to a generic choice of the center $a$ in Lemma 2.3 , the restriction of $f$ to $C$ cannot be a constant function. Hence $f$ is not a constant function on $\Omega_{[\lambda, \delta]}$. Since $\partial \bar{\Omega} \subset f^{-1}(\lambda)$ (possibly $\partial \bar{\Omega}=\emptyset$ ) and $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda)=\emptyset$, we have $\partial \Omega_{[\lambda, \delta]} \subset f^{-1}(\delta)$ (possibly $\left.\partial \Omega_{[\lambda, \delta]}=\emptyset\right)$. Hence, there exists a local minimum point $q$ of $\left.f\right|_{\partial B_{R}}$ in the interior of $\Omega_{[\lambda, \delta]}$ with $\lambda<f(q)<\delta$.


Figure 5. A schematic picture for the proof in Case 1.
Since $\lambda<f(q)<\delta=f(p)$, there exists a point $q^{\prime}$ on $\Gamma_{p}$ such that $f(q)=f\left(q^{\prime}\right)$. If $f \searrow \lambda_{q}$ along $\Gamma_{q}$ with $\lambda_{q} \neq \lambda$, then the two sets $\left\{f(x) \mid x \in \Gamma_{p}\right\}$ and $\left\{f(x) \mid x \in \Gamma_{q}\right\}$ should be disjoint by the property (iv). However $f(q)=f\left(q^{\prime}\right)$ is a common element of these two sets. If $f \searrow \lambda$ along $\Gamma_{q}$, then $q \in S_{\Omega}$. However, this and $f(q)<f(p)$ contradict $f(p)=\delta=\min _{x \in S_{\Omega}} f(x)$. Thus, in either case, a contradiction arises.

Case 2: $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda) \neq \emptyset$. Take one point $q \in Y_{[\lambda, \delta]} \cap f^{-1}(\lambda)$, then $q$ belongs to the connected component of $f^{-1}([\lambda, \varepsilon]) \backslash \operatorname{Int} B_{a, R}$ contained in $Y_{[\lambda, \delta]}$ for any $\lambda<\varepsilon<\delta$. This contradicts the fact that $\left\{Y_{t}\right\}$ vanishes at $\infty$ when $t$ tends to $\lambda$.

Next we prove the "if" assertion. Assume that there exists a local minimum point $p \in \Gamma_{\lambda} \cap \partial B_{R}$ of $\left.f\right|_{\partial B_{R}}$ with $f \searrow \lambda$ along $\Gamma_{p}$ such that the intersection of the connected component $Z_{f(p)}$ of $f^{-1}(f(p))$ containing $p$ with the sphere $\partial B_{R}$ consists of isolated points. Since $p$ is a local minimum point of $\left.f\right|_{\partial_{R}}, p$ is also a
local minimum point of $\left.r_{a}\right|_{Z_{f(a)}}$ by Remark 2.5 (3). This and the isolatedness of $Z_{f(p)} \cap \partial B_{R}$ imply that $Z_{f(p)} \subset \mathbb{R}^{n} \backslash \operatorname{Int} B_{R}$.

Put $\delta=f(p)$. Let $Z_{(\lambda, \delta]}$ be the connected component of $f^{-1}((\lambda, \delta])$ containing $p$. We will show that $Z_{(\lambda, \delta]} \cap \partial B_{R}=Z_{\delta} \cap \partial B_{R}$. It is easy to see that the two connected components $\Gamma_{p} \backslash\{p\}$ and $\partial B_{R} \backslash Z_{\delta}$ are subsets of different connected components of $\mathbb{R} \backslash Z_{\delta}$. Assume that there exists a point $x \in Z_{(\lambda, \delta]} \cap\left(\partial B_{R} \backslash Z_{\delta}\right)$. Choose a point $y \in \Gamma_{p} \backslash\{p\}$. Note that the values $f(x)$ and $f(y)$ are in $(\lambda, \delta)$. Since $Z_{(\lambda, \delta]}$ is connected, there exists a path in $Z_{(\lambda, \delta]}$ connecting $x$ and $y$. Furthermore, since $f$ is a trivial fibration on $(\lambda, \delta]$, we can isotope this path so that it is in $Z_{(\lambda, \delta]} \backslash Z_{\delta}$. However, this is impossible since $x$ and $y$ belong to different connected components of $\mathbb{R}^{n} \backslash Z_{\delta}$. Therefore, $Z_{(\lambda, \delta]} \cap \partial B_{R}=Z_{\delta} \cap \partial B_{R}$.

Now it follows that for any $t \in(\lambda, \delta)$, the connected component $Z_{t}$ of $f^{-1}(t)$ intersecting $\Gamma_{p}$ does not intersect $B_{R}$. Hence the distance function $\left.r_{a}\right|_{Z_{t}}$ on $Z_{t}$ attains a minimum value at some point belonging to a tangency branch in $\mathbb{R}^{n} \backslash$ Int $B_{R}$. Thus, we can find a sequence $\left\{t_{k}\right\}$ on $(\lambda, \delta)$ with $\lim _{k \rightarrow \infty} t_{k}=\lambda$ and a point $q \in \Gamma \cap \partial B_{R}$ such that

$$
\min \left\{r_{a}(x) \mid x \in Z_{t_{k}}\right\}=r_{a}\left(q_{k}\right)
$$

where $q_{k}=Z_{t_{k}} \cap \Gamma_{q}$. The distance $r_{a}\left(q_{k}\right)$ goes to $\infty$ as $k \rightarrow \infty$, otherwise $\Gamma_{q}$ intersects $f^{-1}(\lambda)$ and this contradicts the property (i). Hence $Z_{t_{k}}$ vanishes at $\infty$ as $k \rightarrow \infty$.

The proof for the case where $p$ is a local maximum point is similar.
Remark 3.2. Using Theorem 3.1, a vanishing component at $\infty$ of $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is detected as follows:
(Step 1) Choose a generic center $a$, calculate all tangency branches, and fix a sufficiently large radius $R>0$ that satisfies the conditions written in Section 2.4.
(Step 2) For each $p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{R}$, calculate $\lambda_{p}=\lim _{r \rightarrow \infty} f\left(x_{p}(r)\right)$, where $x_{p}(r)=\Gamma_{p} \cap \partial B_{r}$. Then, make the following lists of finite sets:

$$
\begin{aligned}
P_{\min }(\lambda) & =\left\{p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{R} \mid p \text { is local minimum of }\left.f\right|_{\partial B_{R}} \text { with } f \searrow \lambda\right\} \\
P_{\max }(\lambda) & =\left\{p \in(\Gamma \backslash \operatorname{Sing}(f)) \cap \partial B_{R} \mid p \text { is local maximum of }\left.f\right|_{\partial B_{R}} \text { with } f \nearrow \lambda\right\} \\
\Lambda_{\min } & =\left\{\lambda \in \mathbb{R} \mid P_{\min }(\lambda) \neq \emptyset\right\} \\
\Lambda_{\max } & =\left\{\lambda \in \mathbb{R} \mid P_{\max }(\lambda) \neq \emptyset\right\} .
\end{aligned}
$$

(Step 3) For each element $\lambda \in \Lambda_{\min }$ (resp. $\lambda \in \Lambda_{\max }$ ), check if there exists $p \in$ $P_{\min }(\lambda)\left(\right.$ resp. $\left.\quad p \in P_{\max }(\lambda)\right)$ such that the intersection $f^{-1}(f(p)) \cap \partial B_{R}$ consists of isolated points.
(3-1) If it exists, then there exists a vanishing component at $\infty$ when $t$ tends to $\lambda$ by Theorem 3.1.
(3-2) If it does not exist, then $\operatorname{dim} f^{-1}(f(p)) \cap \partial B_{R} \geq 1$ for any $p \in P_{\min }(\lambda) \cup$ $P_{\max }(\lambda)$. For each $p \in P_{\min }(\lambda) \cup P_{\max }(\lambda)$, calculate all critical values $c_{1}, \ldots, c_{k}$ of $r_{a}(x)=\|x-a\|$ on $f^{-1}(f(p))$ and then choose a real number $R^{\prime}$ greater than $\max \left\{R, c_{1}, \ldots, c_{k}\right\}$, see Figure 6. Make a list $L^{\prime}$ of the connected components of $\partial B_{R^{\prime}} \backslash f^{-1}(f(p))$ and find a component $\Omega_{p}^{\prime} \in L^{\prime}$ intersecting $\Gamma_{p}$. If $f^{-1}(\lambda) \cap \Omega_{p}^{\prime}=\emptyset$ then there exists a vanishing component at $\infty$ when $t$ tends to $\lambda$ as shown in the next lemma (Lemma 3.3).
All vanishing components at $\infty$ are detected by the above steps, which is proved in Lemma 3.4 below.


Figure 6. A schematic picture for Step (3-2).

Lemma 3.3. If $f^{-1}(\lambda) \cap \Omega_{p}^{\prime}=\emptyset$ then there exists a vanishing component at $\infty$ when $t$ tends to $\lambda$.

Proof. Consider the case where $p \in P_{\min }(\lambda)$. Let $H^{\prime}$ be the connected component of $\mathbb{R}^{n} \backslash\left(f^{-1}(f(p)) \cup \operatorname{Int} B_{R^{\prime}}\right)$ intersecting $\Gamma_{p}$ and $\bar{H}^{\prime}$ be its closure. Let $\bar{\Omega}_{p}^{\prime}$ be the closure of $\Omega_{p}^{\prime}$ in $\partial B_{R^{\prime}}$. Since $R^{\prime}>\max \left\{R, c_{1}, \ldots, c_{k}\right\}, \bar{H}^{\prime}$ is diffeomorphic to $\bar{\Omega}_{p}^{\prime} \times[0,1)$. Let $Y_{(\lambda, f(p)]}$ be the connected component of $f^{-1}((\lambda, f(p)]) \backslash \operatorname{Int} B_{R^{\prime}}$ intersecting $\Gamma_{p}$ and $\bar{Y}_{(\lambda, f(p)]}$ be its closure. The inclusion $\bar{Y}_{(\lambda, f(p)]} \subset \bar{H}^{\prime}$, the property (iii), and the assumption $f^{-1}(\lambda) \cap \Omega_{p}^{\prime}=\emptyset$ imply that $\bar{Y}_{(\lambda, f(p)]} \cap f^{-1}(\lambda)=\emptyset$. Since $f \searrow \lambda$ along $\Gamma_{p}$ and $f^{-1}(\lambda) \cap \Omega_{p}^{\prime}=\emptyset$, we have $\lambda<f(x)$ for $x \in \bar{\Omega}_{p}^{\prime}, f(x)=f(p)$ for $x \in \partial \bar{\Omega}_{p}^{\prime}$, and there exists a point $x^{\prime} \in \Omega_{p}^{\prime}$ such that $f\left(x^{\prime}\right)<f(p)$. Set $\delta=\min \left\{f(x) \mid x \in \bar{\Omega}_{p}^{\prime}\right\}$. Note that $\lambda<\delta<f(p)$. For $t \in(\lambda, \delta)$, the connected component $Y_{t}$ of $f^{-1}(t)$
intersecting $\Gamma_{p}$ does not intersect $\Omega_{p}^{\prime}$, and therefore it is contained in $Y_{(\lambda, f(p)]}$. Since $\bar{Y}_{(\lambda, f(p)]} \cap f^{-1}(\lambda)=\emptyset,\left\{Y_{t}\right\}$ is a vanishing component at $\infty$ when $t$ tends to $\lambda$.

The assertion for the case $p \in P_{\max }(\lambda)$ is proved similarly.
Lemma 3.4. If there exists a vanishing component at $\infty$ when $t$ tends to $\lambda$, then there exists a point $p \in(\Gamma \backslash(\operatorname{Sing}(f))) \cap \partial B_{R}$ with $f^{-1}(\lambda) \cap \Omega_{p}^{\prime}=\emptyset$.

Proof. Suppose that there exists a vanishing component at $\infty$ when $t$ tends to $\lambda$. We prove only the case $t>\lambda$. By Theorem 3.1, there exists a local minimum point $p \in \Gamma_{p} \cap \partial B_{R}$ of $\left.f\right|_{\partial B_{R}}$ such that the intersection of the connected component $Z_{f(p)}$ of $f^{-1}(f(p))$ containing $p$ with the sphere $\partial B_{R}$ consists of isolated points. Put $\delta=f(p)$ and let $Z_{(\lambda, \delta]}$ be the connected component of $f^{-1}((\lambda, \delta])$ containing $p$. Then, as shown in the proof of the "if" assertion of Theorem 3.1, we have $Z_{(\lambda, \delta]} \cap \partial B_{R}=Z_{\delta} \cap \partial B_{R}$. Let $R^{\prime}$ be the radius chosen as in (3-2) and $\Omega_{p}^{\prime}$ be the connected component of $\partial B_{R^{\prime}} \backslash Z_{\delta}$ intersecting $\Gamma_{p}$. Assume that there exists an intersection point $x \in f^{-1}(\lambda) \cap \Omega_{p}^{\prime}$. By the property (iii), there exists an arc on $f^{-1}(\lambda)$ connecting $x$ and a point on $f^{-1}(\lambda) \cap \partial B_{R}$, but such an arc should intersect $Z_{\delta}$. This contradicts the fact that the image of this arc is $\lambda$.

## 4. Proof of Theorem 1.1

Now, we restrict our setting to the case of polynomial functions with three variables. Let $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$ be a polynomial function. For each $\lambda \in T_{\infty}(f) \backslash K_{0}(f)$, there exists a sufficiently small $\varepsilon>0$ such that, for $I_{\lambda}^{-}=(\lambda-\varepsilon, \lambda)$ and $I_{\lambda}^{+}=$ $(\lambda, \lambda+\varepsilon)$, the restriction of $f$ to $f^{-1}\left(I_{\lambda}^{*}\right)$ and the restriction of $f$ to $f^{-1}\left(I_{\lambda}^{*}\right) \cap B_{R}$ are trivial fibrations unless $f^{-1}\left(I_{\lambda}^{*}\right)=\emptyset$, where $* \in\{-,+\}$. Here $\varepsilon$ is chosen so that $f^{-1}(t)$ intersects $\partial B_{R}$ transversely for $t \in I_{\lambda}^{*}$. Then the restriction of $f$ to $f^{-1}\left(I_{\lambda}^{*}\right) \cap\left(\mathbb{R}^{n} \backslash \operatorname{Int} B_{R}\right)$ is also a trivial fibration.

The surface $f^{-1}(\lambda) \backslash \operatorname{Int} B_{R}$ divides $\mathbb{R}^{3} \backslash \operatorname{Int} B_{R}$ into a finite number of connected components $H_{\lambda, 1}^{*}, \ldots, H_{\lambda, n_{\lambda}}^{*}$ by the property (iii), where $*=-$ if $f(x)<\lambda$ on $H_{\lambda, i}^{*}$ and $*=+$ if $f(x)>\lambda$ on $H_{\lambda, i}^{*}$. Each $H_{\lambda, i}^{*}$ is homeomorphic to $\Omega_{\lambda, i}^{*} \times[0,1)$, where $\Omega_{\lambda, i}^{*}=H_{\lambda, i}^{*} \cap \partial B_{R}$.

Let $\operatorname{Ind}\left(\lambda, \Omega_{\lambda, i}^{*}\right)$ be the sum of indices of the gradient vector field of $f$ restricted to $\partial B_{R}$ for all zeros belonging to $\Gamma^{(\lambda)}$ on $\Omega_{\lambda, i}^{*}$ as defined in the introduction.

Lemma 4.1. $\chi\left(f^{-1}(t) \cap H_{\lambda, i}^{*}\right)=\operatorname{Ind}\left(\lambda, \Omega_{\lambda, i}^{*}\right)$ for any $t \in I_{\lambda}^{*}$.
Proof. Choose $R^{\prime}>R$ sufficiently large so that $f^{-1}(t)$ intersects $\partial B_{R^{\prime}}$ transversely and $f^{-1}(t) \backslash \operatorname{Int} B_{R^{\prime}}$ is diffeomorphic to $\left(f^{-1}(t) \cap \partial B_{R}\right) \times[0,1)$ for $t \in I_{\lambda}^{*}$. Then $f^{-1}(t) \cap H_{\lambda, i}^{*}$ has the same homotopy type as $f^{-1}(t) \cap H_{\lambda, i}^{*} \cap B_{R}^{R^{\prime}}$, where $B_{R}^{R^{\prime}}=$
$\left\{x \in \mathbb{R}^{3} \mid R \leq\|x-a\| \leq R^{\prime}\right\}$. Hence we have

$$
\chi\left(f^{-1}(t) \cap H_{\lambda, i}^{*}\right)=\chi\left(f^{-1}(t) \cap H_{\lambda, i}^{*} \cap B_{R}^{R^{\prime}}\right) .
$$

Consider the distance function $r_{a}(x)=\|x-a\|$ on $f^{-1}(t) \cap H_{\lambda, i}^{*} \cap B_{R}^{R^{\prime}}$. Due to a generic choice of the center of $B_{R}$ in Section 2.2, this function has only non-degenerate critical points and has no critical point on the boundary. Hence, there is a one-to-one correspondence between critical points of $r_{a}$ on $f^{-1}(t) \cap H_{\lambda, i}^{*}$ and the tangency branches $\Gamma_{p}$ passing through $p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda, i}^{*}$ as mentioned in Remark 2.5 (1). If $p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda, i}^{*}$ is local minimum or maximum of $\left.f\right|_{\partial B_{R}}$ then $\operatorname{Ind}_{p}\left(X_{a, R}\right)=1$ and the Morse index $i\left(p_{t}\right)$ of the distance function $r_{a}$ on $f^{-1}(t) \cap H_{\lambda, i}^{*}$ at the intersection point $p_{t}$ of $\Gamma_{p}$ with $f^{-1}(t)$ is 0 or 2 . If $p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda, i}$ is a saddle point of $\left.f\right|_{\partial B_{R}}$ then $\operatorname{Ind}_{p}\left(X_{a, R}\right)=-1$ and the Morse index $i\left(p_{t}\right)$ at the intersection point $p_{t}$ of $\Gamma_{p}$ with $f^{-1}(t)$ is 1 . Hence we have $\operatorname{Ind}_{p}\left(X_{a, R}\right)=(-1)^{i\left(p_{t}\right)}$. Since the Euler characteristic of $f^{-1}(t) \cap \Omega_{\lambda, i}^{*}$ is 0 , by the Morse Theory, we have

$$
\begin{aligned}
\chi\left(f^{-1}(t) \cap H_{\lambda, i}^{*} \cap B_{R}^{R^{\prime}}\right) & =\sum_{p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda, i}}(-1)^{i\left(p_{t}\right)} \\
& =\sum_{p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda, i}} \operatorname{Ind}_{p}\left(X_{a, R}\right)=\operatorname{Ind}\left(\lambda, \Omega_{\lambda, i}^{*}\right) .
\end{aligned}
$$

This completes the proof.
Proof of Theorem 1.1. We prove the first assertion by contraposition. Assume that $\lambda$ is a typical value of $f$. There exists a sufficiently small $\varepsilon>0$ such that $f$ is a trivial fibration on $I_{\varepsilon}=(\lambda-\epsilon, \lambda+\epsilon)$. Let $\Omega_{\lambda, i}^{*}$ be a connected component of $\partial B_{R} \backslash f^{-1}(\lambda)$ and $\partial \bar{\Omega}_{\lambda, i}^{*}$ be the boundary of the closure of $\Omega_{\lambda, i}^{*}$ in $\partial B_{R}$, which is a union of circles. By the property (iii), the connected component $Y$ of $f^{-1}(\lambda) \backslash \operatorname{Int} B_{R}$ intersecting $\partial \bar{\Omega}_{\lambda, i}^{*}$ is diffeomorphic to $\partial \bar{\Omega}_{\lambda, i}^{*} \times[0,1)$, and hence $\chi(Y)=0$. This and the triviality of $f$ on $I_{\varepsilon}$ imply that $\chi\left(f^{-1}(t) \cap H_{\lambda, i}^{*}\right)=0$ for $t \in I_{\lambda}^{*}$, where $H_{\lambda, i}^{*}$ is the connected component of $\mathbb{R}^{3} \backslash\left(f^{-1}(\lambda) \cup \operatorname{Int} B_{R}\right)$ intersecting $\Omega_{\lambda, i}^{*}$. Combining this with Lemma 4.1 we obtain $\operatorname{Ind}\left(\lambda, \Omega_{\lambda, i}^{*}\right)=0$. This completes the proof of the first assertion.

Next we prove the second assertion. Because there does not exist a component of $f^{-1}(t)$ vanishing at $\infty$ when $t$ tends to $\lambda$, there exists a sufficiently small $\varepsilon>0$ such that each connected component of $f^{-1}(t)$ intersects $\partial B_{R}$ for all $t \in I_{\lambda}^{-} \cup I_{\lambda}^{+}$. Let $H_{\lambda, i}^{*}$ be a connected component of $\mathbb{R}^{3} \backslash\left(f^{-1}(\lambda) \cup \operatorname{Int} B_{R}\right)$ and $\left\{Y_{t}^{1}, \ldots, Y_{t}^{s}\right\}$ be the connected components of $f^{-1}(t) \cap H_{\lambda, i}^{*}$. Set $\Omega_{\lambda, i}^{*}=H_{\lambda, i}^{*} \cap \partial B_{R}$. Since
$\operatorname{Ind}\left(\lambda, \Omega_{\lambda, i}^{*}\right)=0$, we have $\chi\left(f^{-1}(t) \cap H_{\lambda, i}^{*}\right)=0$ by Lemma 4.1. Hence

$$
\sum_{j=1}^{s} \chi\left(Y_{t}^{j}\right)=0
$$

Choose $R^{\prime}>R$ sufficiently large so that $f^{-1}(t)$ intersects $\partial B_{R^{\prime}}$ transversely and $f^{-1}(t) \backslash \operatorname{Int} B_{R^{\prime}}$ is diffeomorphic to $\left(f^{-1}(t) \cap \partial B_{R^{\prime}}\right) \times[0,1)$ for $t \in I_{\lambda}^{*}$, and set $B_{R}^{R^{\prime}}=\left\{x \in \mathbb{R}^{3} \mid R \leq\|x-a\| \leq R^{\prime}\right\}$. Then, as mentioned at the beginning of the proof of Lemma 4.1, $\chi\left(Y_{t}^{j} \cap B_{R}^{R^{\prime}}\right)=\chi\left(Y_{t}^{j}\right)$ holds. Hence

$$
\begin{equation*}
\sum_{j=1}^{s} \chi\left(Y_{t}^{j} \cap B_{R}^{R^{\prime}}\right)=0 \tag{4.1}
\end{equation*}
$$

Here each $Y_{t}^{j} \cap B_{R}^{R^{\prime}}$ is a compact, connected, orientable surface embedded in $\mathbb{R}^{3}$.
We claim that $\chi\left(Y_{t}^{j} \cap B_{R}^{R^{\prime}}\right)=0$ for any $j=1, \ldots, s$. If $s=1$ then it follows from equation 4.1). Suppose that $s \geq 2$. Assume that $\chi\left(Y_{t}^{j_{0}} \cap B_{R}^{R^{\prime}}\right) \neq 0$ for some $j_{0} \in\{1, \ldots, s\}$. Then there exists a connected component $Y_{t}^{j_{1}}$ with $\chi\left(Y_{t}^{j_{1}} \cap\right.$ $\left.B_{R}^{R^{\prime}}\right)>0$ by (4.1). Since $Y_{t}^{j_{1}} \cap B_{R}^{R^{\prime}}$ is a compact, connected, orientable surface, it is diffeomorphic to a disk. The boundary of this disk lies on $\partial B_{R}$ since $\varepsilon>$ 0 is chosen so that $Y_{t}^{j_{1}} \cap \partial B_{R} \neq \emptyset$. Moreover, this boundary is parallel to a boundary component of the closure of $\Omega_{\lambda_{i}}^{*}$ due to the property (iii). Since $H_{\lambda, i}^{*}$ is homeomorphic to $\Omega_{\lambda, i}^{*} \times[0,1)$ and the disk $Y_{t}^{j_{1}} \cap B_{R}^{R^{\prime}}$ is relatively embedded in $H_{\lambda, i}^{*}, \Omega_{\lambda_{i}}^{*}$ should be a disk. Since the boundary of $\Omega_{\lambda_{i}}^{*}$ is connected, $f^{-1}(t) \cap H_{\lambda, i}^{*}$ is also connected. This contradicts $s \geq 2$.

Now we have $\chi\left(Y_{t}^{j} \cap B_{R}^{R^{\prime}}\right)=0$ for any $t \in I_{\lambda}^{*}$ and $j=1, \ldots, s$. This means that all of these connected components are diffeomorphic to $S^{1} \times[0,1]$. Therefore, the relative homotopy groups $\pi_{i}\left(f^{-1}\left(I_{\lambda}^{*} \cup\{\lambda\}\right), f^{-1}(\lambda), x\right)$ are trivial for all $i \in \mathbb{N}$ and any base point $x \in f^{-1}(\lambda)$. Note that this conclusion holds for both of the cases $*=-$ and $*=+$. Hence, by [5, Proposition 3.3 and Theorem 1.2], for $I_{\lambda}=(\lambda-\varepsilon, \lambda+\varepsilon)$, the map

$$
\left.f\right|_{f^{-1}\left(I_{\lambda}\right)}: f^{-1}\left(I_{\lambda}\right) \rightarrow I_{\lambda}
$$

is a Serre fibration. Then, this implies that $\left.f\right|_{f^{-1}\left(I_{\lambda}\right)}$ is a trivial fibration by [10, Corollary 32]. Hence $\lambda$ is a typical value at $\infty$ of $f$.

## 5. Typical values of polynomial maps with 2-dimensional fibers

In this section, we study polynomial maps from $\mathbb{R}^{n}$ to $\mathbb{R}^{n-2}$. The case $n=3$ is studied in the previous section.

Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-2}$ be a polynomial map, where $n \geq 3$, and $\lambda$ be a point in $F\left(\mathbb{R}^{n}\right) \backslash \bar{K}_{0}(F)$, where $\bar{K}_{0}(F)$ is the closure of $K_{0}(F)$ in $\mathbb{R}^{n-2}$. Let $B_{R}$ be the $n$-dimensional ball in $\mathbb{R}^{n}$ centered at $a \in \mathbb{R}^{n}$ and with radius $R>0$. As shown in [5, Lemma 3.2], we can choose a sufficiently large radius $R>0$ satisfying the following property:
(v) Each connected component $Y$ of $F^{-1}(\lambda) \backslash \operatorname{Int} B_{R}$ intersects $\partial B_{r}$ transversely for any $r \geq R$. In particular, $Y \backslash \operatorname{Int} B_{r}$ is diffeomorphic to $\left(Y \cap \partial B_{r}\right) \times[0,1)$ for any $r \geq R$.
In particular, there is a deformation-retract from $F^{-1}(\lambda)$ to $F^{-1}(\lambda) \cap B_{R}$.
Theorem 5.1. Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-2}$ be a polynomial map, where $n \geq 3$. For $\lambda \in F\left(\mathbb{R}^{n}\right) \backslash \bar{K}_{0}(F)$, choose a radius $R$ so that the property $(\mathrm{v})$ holds. Then, $\lambda$ is a typical value at $\infty$ of $F$ if and only if the following are satisfied:
(1) There is no vanishing component at $\infty$ when $t$ tends to $\lambda$;
(2) There exists a neighborhood $D$ of $\lambda$ in $\mathbb{R}^{n-2}$ such that, for all $t \in D$,
(2-1) $F^{-1}(t) \backslash \operatorname{Int} B_{R}$ has no compact, connected component, and (2-2) $\chi\left(F^{-1}(t)\right)=\chi\left(F^{-1}(\lambda)\right)$ holds.

Proof. It is enough to show that if the conditions (1) and (2) are satisfied then $F$ is a trivial fibration over some neighborhood of $\lambda$. Assume that the two conditions are satisfied. Let $D$ be a small neighborhood of $\lambda$ as in the condition (2). We can choose $D$ small enough so that the fibers $F^{-1}(t)$ are regular and intersect $\partial B_{R}$ transversely for all $t \in D$. The map $\left.F\right|_{F^{-1}(D) \cap B_{R}}: F^{-1}(D) \cap B_{R} \rightarrow D$ is a trivial fibration.

By the conditions (1) and (2-1), $F^{-1}(t) \backslash \operatorname{Int} B_{R}$ does not have a connected component which is contractible for any $t \in D$. Hence we have $\chi\left(F^{-1}(t) \backslash \operatorname{Int} B_{R}\right) \leq$ 0 . Then, by the condition (2-2) and the property (v), we have

$$
\begin{aligned}
\chi\left(F^{-1}(\lambda)\right) & =\chi\left(F^{-1}(t)\right)=\chi\left(F^{-1}(t) \cap B_{R}\right)+\chi\left(F^{-1}(t) \backslash \operatorname{Int} B_{R}\right) \\
& \leq \chi\left(F^{-1}(t) \cap B_{R}\right)=\chi\left(F^{-1}(\lambda) \cap B_{R}\right)=\chi\left(F^{-1}(\lambda)\right),
\end{aligned}
$$

which implies that $\chi\left(F^{-1}(t) \backslash \operatorname{Int} B_{R}\right)=0$. Here we used the fact that $F^{-1}(t) \cap \partial B_{R}$ is a disjoint union of circles and its Euler characteristic is 0 . By the condition (2-1), $F^{-1}(t) \backslash \operatorname{Int} B_{R}$ is diffeomorphic to a disjoint union of a finite number of copies of $S^{1} \times[0,1)$.

The rest of the proof is same as the last argument in the proof of Theorem 1.1. Since there exists a deformation-retract from $F^{-1}(t)$ to $F^{-1}(t) \cap B_{R}$ for each $t \in$ $D$ and the map $\left.F\right|_{F^{-1}(D) \cap B_{R}}$ is a trivial fibration, the relative homotopy groups $\pi_{i}\left(F^{-1}(D), F^{-1}(\lambda), x\right)$ are trivial for all $i \in \mathbb{N}$ and any base point $x \in f^{-1}(\lambda)$.

Then, by [5, Proposition 3.3 and Theorem 1.2], the map $\left.F\right|_{F^{-1}(D)}: F^{-1}(D) \rightarrow D$ is a Serre fibration and hence it is a trivial fibration by [10, Corollary 32]. Hence $\lambda$ is a typical value at $\infty$ of $F$.

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[^0]:    The first author is supported by JSPS KAKENHI Grant numbers JP19K03499, JP23K03098, JP23H00081 and Keio University Academic Development Funds for Individual Research. This work is supported by JSPS-VAST Joint Research Program, Grant number JPJSBP120219602.

