Characterizations of open and semi-open maps of compact Hausdorff spaces by induced maps

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Abstract

Let $f\colon X\to Y$ be a continuous surjection of compact Hausdorff spaces. By

$$f_*\colon \mathfrak{M}(X)\to\mathfrak{M}(Y),\ \mu\mapsto\mu\circ f^{-1}\quad\text{and}\quad 2^f\colon 2^X\to 2^Y,\ A\mapsto f[A]$$

we denote the induced continuous surjections on the probability measure spaces and hyperspaces, respectively. In this paper we mainly show the following facts:

(1) If f_* is semi-open, then f is semi-open.
(2) If f is semi-open densely open, then f_* is semi-open densely open.
(3) f is open iff 2^f is open.
(4) f is semi-open iff 2^f is semi-open.
(5) f is irreducible iff 2^f is irreducible.

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Let $f\colon X\to Y$ be a continuous map from a topological space X onto another Y. As usual, f is *open* iff the image of every open subset of X is open in Y; f is *semi-open*, or *almost open*, iff for every non-empty open subset U of X, the interior of $f[U]$, denoted $\mathrm{int}\, f[U]$, is non-empty in Y. The "open" and "semi-open" properties are important for the structure theory of compact minimal dynamics (see, e.g., [21, 20, 8, 9]).

By $\mathfrak{M}(X)$, it means the set of all regular Borel probability measures on X equipped with the weak-$*$ topology. Then, there exists a naturally induced continuous surjective map:

$$f_*\colon \mathfrak{M}(X)\to\mathfrak{M}(Y),\quad \mu\mapsto\mu\circ f^{-1}.$$

First of all there are equivalent descriptions of openness and semi-openness of f with the help of the induced map of $\mathfrak{M}(X)$ to $\mathfrak{M}(Y)$ as follows:

Theorem A (Ditor-Eifler [7])**.** *Let $f\colon X\to Y$ be a continuous surjection of compact Hausdorff spaces. Then f is open iff the induced map $f_*\colon\mathfrak{M}(X)\to\mathfrak{M}(Y)$ is an open surjection.*

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Theorem B′ (Glasner [8, 9])**.** *Let $f: X \to Y$ be a continuous surjection between compact metric spaces. If f is semi-open, then $f_*: \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is a semi-open surjection.*

In fact, it turns out that condition "f is semi-open" is also necessary for that "f_* is semi-open" as follows:

Theorem B″. *Let $f: X \to Y$ be a continuous surjection of compact Hausdorff spaces. If the induced map $f_*: \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is semi-open, then f is semi-open.*

Proof. Let $U \neq \emptyset$ be an open set in X. We shall prove that $\operatorname{int} f[U] \neq \emptyset$ in Y. For that, let

$$\mathcal{U} = \{\mu \in \mathfrak{M}(X) \,|\, U \cap \operatorname{supp}(\mu) \neq \emptyset\}.$$

Since $\mu \in \mathfrak{M}(X) \mapsto \operatorname{supp}(\mu) \in 2^X$ is lower semi-continuous (cf. [20, Lem. VII.1.4]), $\mathcal{U}$ is open in $\mathfrak{M}(X)$. As f_* is semi-open by hypothesis, it follows that there exists an open subset $\mathcal{V}$ of $\mathfrak{M}(Y)$ with $\emptyset \neq \mathcal{V} \subseteq f_*[\mathcal{U}]$.

Because $\overline{\mathrm{co}}(\delta_Y) = \mathfrak{M}(Y)$ where $\delta_Y = \{\delta_y \,|\, y \in Y\}$ is the set of Dirac measures in Y, we can choose a measure $\nu = \sum_{i=1}^n \alpha_i \delta_{y_i} \in \mathcal{V}$ with $\alpha_i > 0$ for $i = 1, \dots, n$ and $\sum_i \alpha_i = 1$. Further, we can choose an $\varepsilon \in \mathscr{U}_Y$, the uniformity structure of Y, such that for all $(y'_1, \dots, y'_n) \in \varepsilon[y_1] \times \cdots \times \varepsilon[y_n]$, there exists an irreducible convex combination $\nu' = \alpha'_1 \delta_{y'_1} + \cdots + \alpha'_n \delta_{y'_n} \in \mathcal{V}$. By $\mathcal{V} \subseteq f_*[\mathcal{U}]$, it follows that there is some $\mu' \in \mathcal{U}$ with $f_*(\mu') = \nu'$. As $\operatorname{supp}(\nu') = \{y'_1, \dots, y'_n\}$, $U \cap \operatorname{supp}(\mu') \neq \emptyset$ and $\operatorname{supp}(\mu') \subseteq f^{-1}[\operatorname{supp}(\nu')] \subseteq \bigcup_{i=1}^n f^{-1}(y'_i)$, it follows that

$$\{y'_1, \dots, y'_n\} \cap f[U] \neq \emptyset.$$

Clearly, this implies that $\varepsilon[y_i] \subseteq f[U]$ for some i with $1 \leq i \leq n$. Thus, $\operatorname{int} f[U] \neq \emptyset$. The proof is completed. □

Consequently we have concluded the following theorem by a combination of Theorem B′ and Theorem B″:

Theorem B. *Let $f: X \to Y$ be a continuous surjection of compact metric spaces. Then f is semi-open iff $f_*: \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is semi-open.*

On the other hand, recall that the (largest) *hyperspace*, denoted 2^X, of X is defined to be the collection of all non-empty closed subsets of X equipped with the Vietoris topology (see [20, §II.1] or [13, Thm. I.1.2]). Note here that a base for the Vietoris topology is formed by the sets of the form

$$\langle U_1, \dots, U_n \rangle := \left\{K \in 2^X \,|\, K \subseteq U_1 \cup \cdots \cup U_n, K \cap U_i \neq \emptyset, 1 \leq i \leq n\right\}$$

for all $n \geq 1$ and all non-empty open sets $U_1, \dots, U_n$ in X. Then X is a compact Hausdorff space iff so is 2^X, and X is metrizable iff 2^X is metrizable (cf. [19], [20, Thm. II.1.1], [13, Thm. I.3.3]).

Since Kelley 1942 [15] the hyperspace theory became an important way of obtaining information on the structure of a topological space X (in continua–compact connected metric spaces) by studying properties of the hyperspace 2^X and its hyperspace 2^{2^X}. In this note we shall give other characterizations of open and semi-open maps with the help of the hyperspaces 2^X and 2^Y (see Thm. 3 and Thm. 4). Moreover, we shall consider the interrelation of the irreducibility of f and its induced map 2^f (see Thm. 9B), and improve Theorem B′ (see Thm. 10C).

1. Let X, Y be compact Hausdorff spaces. Let $\phi\colon X \to Y$ be a continuous surjection. Then ϕ induces maps

$$2^\phi\colon 2^X \to 2^Y \text{ by } K \in 2^X \mapsto \phi[K] \in 2^Y \quad \text{and} \quad \phi_{\mathrm{ad}}\colon 2^Y \to 2^X \text{ by } K \in 2^Y \mapsto \phi^{-1}[K] \in 2^X.$$

Then, 2^ϕ is a continuous surjection. Moreover, ϕ_{ad} is continuous iff $\phi_{\mathrm{ad}}|_Y = \phi^{-1}$ is continuous where Y is identified with $\{\{y\}\,|\,y \in Y\} \subset 2^Y$ by Lemma below, iff ϕ is open; see [19] and [20, Thm. II.1.3].

Lemma (cf. [20, Rem. II.1.4])**.** *Let X be a compact Hausdorff space and let $n \geq 1$ be an integer. Then the map*

$$i_n\colon X^n \to 2^X \textit{ defined by } (x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$$

is continuous. Moreover, it is locally 1-1 in the points $(x_1, \dots, x_n)$ with $x_i \neq x_j$ for all $i \neq j$. Also note that $\bigcup\{i_n[X^n]\,|\,n \in \mathbb{N}\}$ is dense in 2^X.

It is natural to wonder about which properties are transmitted between ϕ and 2^ϕ. This problem has been addressed by several authors (cf., e.g., [19, 6, 5]). We shall be concerned with the "openness" and "semi-openness" here.

2 Lemma. *Let $\phi\colon X \to Y$ be a continuous surjection between Hausdorff spaces with X locally compact. Then:*

(a) *ϕ is semi-open iff the preimage of every dense subset of Y is dense in X (cf. [9, Lem. 2.1]).*
(b) *ϕ is semi-open iff the preimage of every residual subset of Y is residual in X.*

Proof.

(a): If ϕ is semi-open and let $A \subset Y$ be a dense set and set $U = X \setminus \overline{\phi^{-1}[A]}$, then $\phi[U] \cap A \neq \emptyset$ whenever $U \neq \emptyset$, a contradiction. Thus, $U = \emptyset$ so that $\phi^{-1}[A]$ is dense in X. Conversely, suppose the preimage of every dense subset of Y is dense in X. Let $U \neq \emptyset$ be open in X. If $\operatorname{int}\phi[U] = \emptyset$, then $\phi^{-1}[Y \setminus \phi[U]] \cap U \neq \emptyset$, a contradiction. Thus, $\operatorname{int}\phi[U] \neq \emptyset$.

(b)-*Necessity:* Let $A = \bigcap_{i=1}^\infty A_i$, where A_i, $i = 1, 2, \dots$ are open dense subsets of Y. Then, obviously, $\phi^{-1}[A] = \bigcap_i \phi^{-1}[A_i]$. It follows from (a) that $\phi^{-1}[A_i]$ are open dense in X. Thus, $\phi^{-1}[A]$ is a residual subset of X.

(b)-*Sufficiency:* Let $U \neq \emptyset$ be open in X. If $\operatorname{int}\phi[U] = \emptyset$, then there exists an open set V with $\emptyset \neq V \subseteq \bar{V} \subseteq U$ such that $\bar{V}$ is compact in X. Further,

$$\phi^{-1}[Y \setminus \phi[\bar{V}]] \cap \bar{V} \neq \emptyset \quad \text{and} \quad \phi[\bar{V}] \cap (Y \setminus \phi[\bar{V}]) \neq \emptyset,$$

a contradiction. Thus, $\operatorname{int}\phi[U] \neq \emptyset$. The proof is completed. □

3 Theorem (cf. [11, Thm. 4.3] for X, Y in continua)**.** *Let $f\colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. Then f is open iff $2^f\colon 2^X \to 2^Y$ is open.*

Proof.

Necessity: Let $\mathcal{A}$ be a member of the base of the hyperspace 2^X. Then by definition of the Vietoris topology, it follows that there exist nonempty open sets $U_1, \dots, U_n$ in X such that

$$\mathcal{A} = \{K \in 2^X \mid K \subseteq U_1 \cup \cdots \cup U_n, K \cap U_i \neq \emptyset \text{ for } 1 \leq i \leq n\}.$$

Let $U = U_1 \cup \cdots \cup U_n$. Since f is open, $f[U]$ and $f[U_i]$, $1 \leq i \leq n$, are all open subsets of Y. Let

$$\mathcal{B} = \{K \in 2^Y \mid K \subseteq f[U], K \cap f[U_i] \neq \emptyset \text{ for } 1 \leq i \leq n\}.$$

Clearly, $2^f[\mathcal{A}] \subseteq \mathcal{B}$. In order to prove that 2^f is open, it suffices to prove that $2^f[\mathcal{A}] = \mathcal{B}$. For that, we need only prove $\mathcal{B} \subseteq 2^f[\mathcal{A}]$.

Let $B \in \mathcal{B}$ be arbitrarily given. By definitions, there exist points $y_i \in B \cap f[U_i]$ for $i = 1, \dots, n$. So we can select points $x_i \in U_i$ with $f(x_i) = y_i$ for $1 \leq i \leq n$. Moreover, as f is open and X, Y compact, it follows that there exists a closed set $A' \in 2^X$ with $A' \subseteq U$ such that $f[A'] = B$. Let

$$A = A' \cup \{x_1, \dots, x_n\}.$$

Then $A \in 2^X$ such that $A \subseteq U$ and $x_i \in A \cap U_i \neq \emptyset$ for $i = 1, \dots, n$. Thus, $A \in \mathcal{A}$ and then $2^f[\mathcal{A}] = \mathcal{B}$.

Sufficiency: Let $V \subseteq X$ be an open nonempty set. We need prove that $f[V]$ is open in Y. As

$$\langle V \rangle = \{K \in 2^X \mid K \subseteq V\}$$

is an open subset of 2^X and $2^f \colon 2^X \to 2^Y$ is open, it follows that $2^f[\langle V \rangle] \subseteq 2^Y$ is open. Since $V = \bigcup\{F \mid F \in \langle V \rangle\}$, hence $f[V] = \bigcup\{K \mid K \in 2^f[\langle V \rangle]\}$. Given $y \in f[V]$, $\{y\} \in 2^f[\langle V \rangle]$ and there exists an open neighborhood $\langle V_{y,1}, \dots, V_{y,n} \rangle$ of $\{y\}$ in 2^Y such that $\{y\} \in \langle V_{y,1}, \dots, V_{y,n} \rangle \subseteq 2^f[\langle V \rangle]$. Then

$$f[V] \subseteq \bigcup_{y \in f[V]} (\langle V_{y,1}, \dots, V_{y,n} \rangle) \subseteq \bigcup\{K \mid K \in 2^f[\langle V \rangle]\} = f[V].$$

Thus, $f[V] = \bigcup_{y \in f[V]} (\langle V_{y,1}, \dots, V_{y,n} \rangle)$ is open in Y, for each $\langle V_{y,1}, \dots, V_{y,n} \rangle$ is open in Y. The proof is completed. □

Let $f \colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. Now define

$$2^{X,f} = \{A \in 2^X \mid \exists y \in Y \text{ s.t. } A \subseteq f^{-1}(y)\},$$

which is called the *quasifactor representation* of Y in X; and, as $i_1 \colon Y \to 2^Y$ is an embedding (see Lem. 1), we may identify Y with $i_1[Y] = \{\{y\} \mid y \in Y\} \subseteq 2^Y$ as mentioned before. Thus,

$$f' = 2^f|_{2^{X,f}} \colon 2^{X,f} \to Y.$$

is a well-defined continuous surjection, called the *quasifactor* of f. It is of interest to know when f' is actually a factor of f. Then Theorem 3 has an interesting variation as follows:

3′. Theorem. *Let $f \colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. Then f is open iff f' is open iff f_* is open.*

Proof. In view of Theorem A we need only prove the first "iff". Suppose f is open. Then 2^f is open by Theorem 3. Set $\mathcal{Y} = \{\{y\} \mid y \in Y\}$. Then $2^{X,f} = (2^f)^{-1}[\mathcal{Y}]$. Thus, f' is open. Now conversely, if $f' \colon 2^{X,f} \to Y$ is open, then f is obviously open. The proof is completed. □

4 Theorem (cf. [12, Lem. 2.3] for "only if" part)**.** *Let $f \colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. Then f is semi-open iff $2^f \colon 2^X \to 2^Y$ is semi-open.*

Proof.

Necessity (different with [12]): Let $\mathcal{B} \subseteq 2^Y$ be a dense subset. By Lemma 2a, we need only prove that $\mathcal{A} := (2^f)^{-1}[\mathcal{B}]$ is dense in 2^X. For that, let $\langle U_1, \dots, U_n\rangle$ be a basic open set in 2^X. It is enough to prove that $\mathcal{A} \cap \langle U_1, \dots, U_n\rangle \neq \emptyset$. Indeed, since f is semi-open, hence

$$V_i := \operatorname{int} f[U_i] \neq \emptyset,\ i = 1, \dots, n.$$

Let $V = V_1 \cup \cdots \cup V_n$, $U = U_1 \cup \cdots \cup U_n$. Then $\langle V_1, \dots, V_n\rangle$ is an open non-empty subset of 2^Y. So $\mathcal{B} \cap \langle V_1, \dots, V_n\rangle \neq \emptyset$. Let $B \in \mathcal{B} \cap \langle V_1, \dots, V_n\rangle$. As $V \subseteq f[U]$ and $B \in 2^Y$ is compact, it follows that we can select $A \in 2^X$ with $A \subseteq U$ such that $f[A] = B$. Moreover, by $B \cap V_i \neq \emptyset$, we can take a point $x_i \in U_i$ with $f(x_i) \in B$ for $i = 1, \dots, n$. Now set

$$K = A \cup \{x_1, \dots, x_n\}.$$

Then $K \subseteq U$, $K \cap U_i \neq \emptyset$ for all $i = 1, \dots, n$ and $f[K] = B$. Thus, $K \in \mathcal{A}$ and further $\mathcal{A} \cap \langle U_1, \dots, U_n\rangle \neq \emptyset$.

Sufficiency: Let $U \neq \emptyset$ be an open set in X. Since $\langle U\rangle = \{K \in 2^X \,|\, K \subseteq U\}$ is open in 2^X and 2^f is semi-open, it follows that $2^f[\langle U\rangle] = \{f[K] \,|\, K \in \langle U\rangle\}$ has a non-empty interior. Therefore, there exists a non-empty basic open subset of 2^Y, say $\langle V_1, \dots, V_n\rangle \subseteq 2^f[\langle U\rangle]$. Put

$$B = V_1 \cup \cdots \cup V_n.$$

Then $B \neq \emptyset$ is open in Y such that $B \subseteq f[U]$. For, as $B \in \langle V_1, \dots, V_n\rangle \subseteq 2^f[\langle U\rangle]$, there exists $A \in \langle U\rangle$ such that $f[A] = B$; thus, $A \subseteq U$ implies $B \subseteq f[U]$. The proof is completed. □

Note here that we do not know whether or not Theorem 4 has a variation similar to Theorem 3′. That is, we do not know how to characterize the semi-openness of $f' \colon 2^{X,f} \to Y$.

5. A *flow* is a triple (T, X, π), simply denoted $\mathscr{X}$ or $T \curvearrowright X$ if no confusion, where T is a topological group, called the phase group; X is a topological space, called the phase space; and where $\pi\colon T \times X \xrightarrow{(t,x)\mapsto x} X$, the action, is a jointly continuous map, such that $ex = x$ and $(st)x = s(tx)$ for all $x \in X$ and $s, t \in T$. Here e is the identity of T.

Every flow $\mathscr{X}$ can induce a so-called *hyperflow* $(T, 2^X, 2^\pi)$ (cf. [16] or [20, Thm. II.1.6]), denoted $2^{\mathscr{X}}$, where the phase group is T, the phase space is 2^X and the action is the induced mapping $2^\pi\colon T \times 2^X \xrightarrow{(t,K)\mapsto tK} 2^X$. Here $tK = \{tx \,|\, x \in K\}$ for $t \in T$ and $K \in 2^X$.

If $\overline{Tx} = X\ \forall x \in X$, then $\mathscr{X}$ is called a *minimal flow*. Let $\mathscr{X}$ be a compact flow; that is, $\mathscr{X}$ is a flow with compact Hausdorff phase space X. If $x \in X$ such that $\overline{Tx}$ is a minimal subset of $\mathscr{X}$, then x is refereed to as an *almost periodic* (a.p.) *point* for $\mathscr{X}$. The induced affine flow $T \curvearrowright \mathfrak{M}(X)$ and hyperflow $2^{\mathscr{X}}$ of a compact minimal flow $\mathscr{X}$ are not minimal unless $X = \{pt\}$ a singleton. So, the dynamics on $2^{\mathscr{X}}$ is more richer than that on $\mathscr{X}$. See, e.g., [3, 10, 18, 1, 17, 14].

Let $\mathscr{X}, \mathscr{Y}$ be two compact flows. Then $\phi\colon \mathscr{X} \to \mathscr{Y}$ is called an *extension* if $\phi\colon X \to Y$ is a continuous surjection such that $\phi(tx) = t\phi(x)$ for all $t \in T$ and $x \in X$. If $\phi\colon \mathscr{X} \to \mathscr{Y}$ is an extension of compact flows, then $2^\phi\colon 2^{\mathscr{X}} \to 2^{\mathscr{Y}}$ is an extension of compact hyperflows (see [20, Thm. II.1.8]).

6 Lemma (cf. [4, Lem. 3.12.15] or [20, Thm. I.1.4])**.** *If $\phi\colon \mathscr{X} \to \mathscr{Y}$ is an extension of compact flows with $\mathscr{X}$ having a dense set of a.p. points and $\mathscr{Y}$ minimal, then ϕ is semi-open.*

6A. In Definition 5, if T is only a topological semigroup, then $\mathscr{X}$ or $T \curvearrowright X$ is refereed to as a *semiflow*. In this case, $x \in X$ is called *a.p.* for $\mathscr{X}$ iff $x \in \overline{Tx}$ and $\overline{Tx}$ is a minimal subset of $\mathscr{X}$. In fact, it is not known whether or not Lemma 6 is still true if T is only a semigroup. See Theorem 6C for a confirmative conditional case.

6B. Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ be an extension of compact semiflows. We say that ϕ is *highly proximal* (h.p.) iff, for all $y \in Y$, there is a net $t_n \in T$ with $t_n\phi^{-1}(y) \to \{pt\}$ in 2^X (cf. [20, p.104]).

6C. Theorem. *Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ be an extension of compact semiflows such that $t\phi^{-1}(y) = \phi^{-1}(ty)$ for all $t \in T, y \in Y$. If ϕ is h.p. and $\mathscr{X}$ has a dense set of a.p. points, then ϕ is semi-open.*

Proof. Firstly we claim that every nonempty open subset of X contains a ϕ-fiber. Indeed, let $U \subset X$ be open with $U \neq \emptyset$. Then there is an a.p. point $x_0 \in U$. Since ϕ is h.p., there is a net $t_n \in T$ such that $t_n\phi^{-1}\phi(x_0) \to \{x'\}$, with $x' = \lim t_n x_0$, in 2^X. As x_0 is a.p., it follows that there is a net $s_j \in T$ with $s_j x' \to x_0$. Thus, there is a net $\tau_i \in T$ such that $\tau_i\phi^{-1}\phi(x_0) \to L \subseteq U$, and so that $\phi^{-1}\phi(\tau_i x_0) \subseteq U$ eventually.

Now to prove that ϕ is semi-open, let $U \subset X$ be open with $U \neq \emptyset$. By our claim above, there exists a point $y_0 \in Y$ such that $\phi^{-1}(y_0) \subseteq U$. Since $\phi_{\mathrm{ad}}|_Y\colon Y \to 2^X$, $y \mapsto \phi^{-1}(y)$ is upper semi-continuous and U is open, there is an open neighborhood V of y_0 in Y such that $\phi^{-1}[V] \subseteq U$. Thus, $\phi[U] \supseteq V$ so that $\operatorname{int}\phi[U] \neq \emptyset$. The proof is completed. □

Therefore, if $\phi\colon \mathscr{X} \to \mathscr{Y}$ is an h.p. extension of compact flows with $\mathscr{X}$ having a dense set of a.p. points, then ϕ is semi-open. A point of Theorem 6C is that $\mathscr{Y}$ need be minimal here.

As was mentioned before, $2^{\mathscr{X}}$ need not have a dense set of a.p. points and $2^{\mathscr{Y}}$ are generally not minimal, so Lemma 6 is not applicable straightforwardly for $2^\phi\colon 2^{\mathscr{X}} \to 2^{\mathscr{Y}}$. However, using Theorem 4 we can obtain the following:

7 Corollary. *Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ be an extension of compact minimal flows. Then:*

(1) $2^\phi\colon 2^{\mathscr{X}} \to 2^{\mathscr{Y}}$ *and* $2^{2^\phi}\colon 2^{2^{\mathscr{X}}} \to 2^{2^{\mathscr{Y}}}$ *both are semi-open extensions of compact hyperflows.*

(2) *If X and Y both are compact metric spaces, then*

$$\phi_*\colon \mathfrak{M}(X) \to \mathfrak{M}(Y) \quad \textit{and} \quad (\phi_*)_*\colon T \curvearrowright \mathfrak{M}(\mathfrak{M}(X)) \to T \curvearrowright \mathfrak{M}(\mathfrak{M}(Y))$$

both are semi-open extensions of compact flows.

Proof. By Lemma 6, Theorem 4 and Theorem B′. □

8 Corollary. *Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ be an extension of compact flows. Then ϕ is open iff $2^\phi\colon 2^{\mathscr{X}} \to 2^{\mathscr{Y}}$ is open iff $2^{2^\phi}\colon 2^{2^{\mathscr{X}}} \to 2^{2^{\mathscr{Y}}}$ is open.*

Proof. By Theorem 3. □

9 (Irreducibility of maps)**.** In what follows, let $\phi\colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. We say that ϕ is *irreducible* if the only member $A \in 2^X$ with $\phi[A] = Y$ is X itself. This notion is closely related to "highly proximal" in extensions of minimal flows [20].

9A. Lemma. *ϕ is irreducible iff every non-empty open subset U of X contains a fiber $\phi^{-1}(y)$ for some point $y \in Y$.*

Proof. It is straightforward. □

9B. Theorem. *ϕ is irreducible iff $2^\phi\colon 2^X \to 2^Y$ is irreducible.*

Proof.

Sufficiency: Assume 2^ϕ is irreducible. Let $A \in 2^X$ with $\phi[A] = Y$. Then $2^A \subseteq 2^X$ is a closed set such that $2^\phi[2^A] = 2^Y$. Thus, $2^A = 2^X$ and $A = X$. This shows that ϕ is irreducible.

Necessity: Suppose ϕ is irreducible. Let $\langle U_1, \dots, U_n\rangle$ be a basic open set in 2^X. In view of Lemma 9A, it is sufficient to prove that $\langle U_1, \dots, U_n\rangle$ includes a fiber of 2^ϕ. Indeed, as ϕ is irreducible, it follows by Lemma 9A that there is a point $y_i \in \phi[U_i]$ with $\phi^{-1}(y_i) \subseteq U_i$ for all $i = 1, \dots, n$. Let $B = \{y_1, \dots, y_n\} \in 2^Y$. Then $(2^\phi)^{-1}(B) \subseteq \langle U_1, \dots, U_n\rangle$. Thus, 2^ϕ is irreducible. The proof is completed. □

10. Let $f\colon X \to Y$ be a semi-open continuous surjection of compact Hausdorff spaces, where X is not metrizable. In view of Theorem B″ we naturally wonder whether or not $f_*\colon \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is semi-open. The following lemma seems to be helpful for the this question, which is of interest independently.

10A. Lemma. *Let $f\colon X \to Y$ be a continuous surjection between compact Hausdorff spaces, where X is not necessarily metrizable. Let $\psi\colon X \to \mathbb{R}$ be a continuous function and define functions*

$$\psi^*\colon Y \xrightarrow{y\mapsto\psi^*(y)=\sup_{x\in f^{-1}(y)}\psi(x)} \mathbb{R} \quad \textit{and} \quad \psi_*\colon Y \xrightarrow{y\mapsto\psi_*(y)=\inf_{x\in f^{-1}(y)}\psi(x)} \mathbb{R}.$$

Then there is a residual set $Y_c(\psi) \subseteq Y$ such that ψ^* and ψ_* are continuous at each point of $Y_c(\psi)$.

Proof. Let ρ be a continuous pseudo-metric on X and let $\tilde\rho$ be the naturally induced one on 2^X. Since $f^{-1}\colon Y \to 2^X$, defined by $y \mapsto f^{-1}y$, is upper semi-continuous, hence $f^{-1}\colon Y \to (2^X, \tilde\rho)$ is also upper semi-continuous. Thus, there exists a residual set $Y_\rho \subseteq Y$ such that f^{-1} is continuous at every point of Y_ρ. Now, for every $\varepsilon > 0$, there exists a continuous pseudo-metric ρ on X and a positive $r > 0$ such that if $x, x' \in X$ with $\rho(x, x') < r$, then $|\psi(x) - \psi(x')| < \varepsilon/3$. Then there exists a residual set $Y_\varepsilon = Y_\rho \subseteq Y$ such that for every $y \in Y_\varepsilon$, we have that $|\psi^*(y)-\psi^*(y')|+|\psi_*(y)-\psi_*(y')| < \varepsilon$ as $y' \in Y$ close sufficiently to y. Let $Y_c = \bigcap_{n=1}^\infty Y_{1/n}$. Clearly, Y_c is a residual subset of Y as desired. The proof is completed. □

10B (Densely open mappings)**.** Let $f\colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. We say that f is *densely open* if there exists a dense set $Y_o \subseteq Y$ such that $f^{-1}\colon Y \to 2^X$ is continuous at each point of Y_o. For example, if X is a separable metric space, then f is always densely open. We notice here that the “densely open” is different with “almost open” considered in [12, Def. 1.5] and [2].

10C. Theorem. *Let $f\colon X \to Y$ be a continuous surjection between compact Hausdorff spaces. Then:*

(1) *If f is semi-open densely open, then f_* is semi-open densely open.*
(2) *If f is densely open, then f_* is densely open.*

Proof. Based on Def. 10B, let Y_o be the dense set of points of Y at which the set-valued map $f^{-1}\colon Y \to 2^X$ is continuous. Let $\delta_Y = \{\delta_y \,|\, y \in Y\}$.

(1): As f is semi-open, it follows from Lemma 2 that $X_o := f^{-1}[Y_o]$ is dense in X. In view of (2), we need only prove that f_* is semi-open. For that let $\mathcal{U} \subset \mathfrak{M}(X)$ be a closed set with non-empty interior. We need only show that $\operatorname{int} f_*[\mathcal{U}] \neq \emptyset$. Suppose to the contrary that $\operatorname{int} f_*[\mathcal{U}] = \emptyset$. We now fix a measure

$$\mu_0 = \textstyle\sum_{i=1}^m c_i\delta_{x_i} \in \operatorname{int}\mathcal{U} \text{ with } x_i \in X_o,\ 0 < c_i \leq 1,\ \sum_i c_i = 1.$$

Set

$$\nu_0 = f_*(\mu_0) = \textstyle\sum_{i=1}^m c_i\delta_{y_i} \text{ where } y_i = f(x_i) \in Y_o.$$

Let

$$\nu_j = \textstyle\sum_{i=1}^{m_j} c_{j,i}\delta_{y_{j,i}} \in \operatorname{co}(\delta_Y) \setminus f_*[\mathcal{U}] \text{ be a net with } \nu_j \to \nu_0.$$

So $\nu_j \to \nu_0$. Each Q_j is a closed convex subset of $\mathfrak{M}(X)$ and, with no loss of generality, we may assume that $Q = \lim Q_j$ exists in $2^{\mathfrak{M}(X)}$. Then Q is a compact convex subset of $\mathfrak{M}(X)$ with $f_*[Q] = \{\nu_0\}$.

If $\mu_0 \in Q$, then $Q_j \cap \mathcal{U} \neq \emptyset$ eventually so that $\nu_j \in f_*[\mathcal{U}]$, contradicting our choice of ν_j. Thus, $\mu_0 \notin Q$ and by the Separation Theorem there exists a $\psi \in C(X)$ and $\epsilon > 0$ such that

$$\mu_0(\psi) \geq q(\psi) + \epsilon\ \forall q \in Q.$$

Define the associated function

$$\psi^*\colon Y \to \mathbb{R}, \text{ by } y \mapsto \psi(y) = \sup\{\psi(x) \,|\, x \in f^{-1}y\}, \text{ such that } \psi^* \circ f \geq \psi.$$

We can choose points $x_{j,i} \in X$ with $f(x_{j,i}) = y_{j,i}$ and $\psi^*(y_{j,i}) = \psi(x_{j,i})$. Now form the measures $\mu_j = \sum_{i=1}^{m_j} c_{j,i}\delta_{x_{j,i}}$ and assume, with no loss of generality, that $\mu = \lim_j \mu_j$ exists in $\mathfrak{M}(X)$. Since $\mu_j \in Q_j$ for each j, we have $\mu \in Q$. Thus, $\mu_0(\psi) \geq \mu(\psi) + \epsilon$. By our construction $\nu_j(\psi^*) = \mu_j(\psi)$ for every j. By assumption we have that $\operatorname{supp}(\nu_0) = \{y_1, \dots, y_m\}$ is a subset of Y_o and therefore, each y_i is a continuity point of ψ^*. Thus, $\lim \nu_j(\psi^*) = \nu_0(\psi^*)$. It then follows that

$$\mu(\psi) = \lim_j \mu_j(\psi) = \lim_j \nu_j(\psi^*) = \nu_0(\psi^*) = \sum_{i=1}^m c_i\psi^*(y_i) \geq \sum_{i=1}^m c_i\psi(x_i) = \mu_0(\psi).$$

This contradicts the choice of ψ. Therefore f_* is semi-open.

(2): We will now prove that f_* is densely open. Suppose to the contrary that f_* is not densely open. Then there exists an open set $\mathcal{V} \neq \emptyset$ in $\mathfrak{M}(Y)$ such that $f_*^{-1}\colon \mathfrak{M}(Y) \to 2^{\mathfrak{M}(X)}$ is not continuous at every point of $\mathcal{V}$. Since $\operatorname{co}(\delta_{Y_o})$ is dense in $\mathfrak{M}(Y)$, there exists a measure of the form

$$\nu_o = \textstyle\sum_{i=1}^m c_i\delta_{y_i}, \text{ where } y_i \in Y_o \text{ and } 0 \leq c_i \leq 1 \text{ with } \sum c_i = 1,$$

such that $\nu_0 \in \mathcal{V}$. This implies that there is a measure

$$\mu_o \in f_*^{-1}(\nu_o) \text{ such that there is } \mathcal{U} \in \mathfrak{N}_{\mu_o}(\mathfrak{M}(X)) \text{ with } f_*[\mathcal{U}] \notin \mathfrak{N}_{\nu_o}(\mathfrak{M}(Y)).$$

Then we can select a net $\nu_j = \sum_{i=1}^{m_j} c_{j,i}\delta_{y_{j,i}} \in \operatorname{co}(\delta_{Y_o}) \setminus f_*[\mathcal{U}]$ such that $\nu_j \to \nu_o$. Next by an argument as in (1) we can reach a contradiction. The proof is completed. ☐

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