

STABILITY FOR A CLASS OF THREE-TORI WITH SMALL NEGATIVE SCALAR CURVATURE

EDWARD BRYDEN AND LIZHI CHEN

ABSTRACT. We define a flexible class of Riemannian metrics on the three-torus. Then, using Stern's inequality relating scalar curvature to harmonic one-forms, we show that any sequence of metrics in this family has a subsequence which converges to some flat metric on the torus in the sense of Dong-Song.

1. INTRODUCTION

General relationships between geometric quantities are as beautiful as they are useful, and they are very useful. There are many famous examples of such relationships, all of which have far reaching implications. Especially relevant for the current work are the volume growth and isoperimetric inequalities implied by lower Ricci curvature bounds, the systolic inequality of Gromov giving a lower bound on the volume of a Riemannian manifold in terms of its systole, and the relationship between the negative part of the scalar curvature and the integral norm of the Hessian of harmonic maps into $\mathbb{S}^1$ given by Stern in [Ste19].

It is natural to wonder what the extreme geometries are with respect to these relationships, and in what sense, if at all, they are unique. This is the question of rigidity. For example, we have the classical fact that any metric on the torus with non-negative scalar curvature must be flat, see for example [GL80] and [SY79]. In fact, for three-tori this follows from Stern's inequality in [Ste19], mentioned above.

Once the hard work of establishing such a rigidity result has been done, it is natural to wonder what can be said about those metrics which are nearly extremal, which is the question of *stability*. In the context of this paper, the question is whether metrics on the three-torus whose scalar curvature has small negative part must be close to a flat metric in some sense. This is a subtle question; for a more in depth discussion of the ideas and difficulties involved one can read Sormani's survey article [Sor23].

So far there seem to be at least three geometric phenomena that complicate the study of tori with almost non-negative scalar curvature. The first two, *other worlds* and *splines*, have been expected to occur since the work of Gromov-Lawson [GL80] and Schoen-Yau [SY79]. Rigorous examples showing the existence and ubiquity of such objects have recently been constructed by Sweeney [Jr23]. The third, *drawstrings*, was first observed in dimensions greater than 3 by Lee-Naber-Neumayer, see [LNN23]. Later, Lee-Topping showed that drawstrings can be used to produce counter intuitive convergence results, see [LT22].

Roughly speaking, the spaces and notions of convergence proposed to study stability problems involving scalar curvature lower bounds correspond to which of these three phenomena should be considered "small perturbations", and which are to be eliminated, or controlled, by hypothesis. Take for example the amazing result of Dong-Song on the stability of the Positive Mass Theorem [DS23]. One way to interpret this result is to say that for metrics with nonnegative scalar curvature and small mass, such wild geometric phenomena as bubbles, splines, and drawstrings must be hidden behind a surface with small area. In this way, they are small perturbations of the geometry.

At the opposite end of the spectrum, we may make an hypothesis which severely controls bubbles, splines, and drawstrings. For example, one may restrict attention to metrics satisfying a uniform

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lower bound on their Ricci curvature. Results of this flavor are the stability of the Positive Mass Theorem proven by Kazaras-Khuri-Lee in [KKL21], and the stability for tori proven by Honda-Ketterer-Mondello-Perales-Rigoni in [HKM⁺23]. Results in a similar vein were obtained in [ABK22] through controlling the geometry by assuming isoperimetric and integral Ricci curvature bounds.

Another approach has been to assume some geometric condition which eliminates the existence of one, or perhaps two, of the wild geometries. This approach has seen some success using the Intrinsic Flat distance on integral currents, which contain Riemannian manifolds as a subset. See for example the work of Allen, Kazaras, and the first named author in [ABK23] and the work of Hirsch and Zhang in [HZ23]. The specific hypotheses assumed in these results control bubbles, and eliminate drawstrings, but allow splines to exist. Concerning the stability of tori, there is the work of Allen-Vazquez Hernandez-Parise-Payne-Wang [AHVP⁺19].

In a different direction, Lee-Naber-Neumayer [LNN23] give conditions under which splines and bubbles are eliminated, but drawstrings are allowed to persist. They are then able to establish a scalar curvature stability result using the d_p distance, also defined in [LNN23]. In [MY24] Mazurowski and Yao use the d_p distance to study the stability of the Yamabe invariant on $\mathbb{S}^3$.

In the present work we will study a family of metrics which allow splines, and a tamer version of drawstrings to persist, but which eliminates bubbles. For this family of metrics we will prove that metrics with small negative scalar curvature are close to a flat metric in the Dong-Song sense [Don22], see Definition 2.2.

Theorem 1.1. *Fix $V, R, \Lambda, \eta, M > 0$ and let $\mathcal{F}(V, R, \Lambda, \eta, M)$ be the family of Riemannian metrics on $\mathbb{T}^3$ such that*

- (1) $|\mathbb{T}^3|_g \leq V$;
- (2) $\|R_g^-\|_{L^2(g)} \leq R$;
- (3) $\text{IN}_1(g) \geq \Lambda$;
- (4) $\min\{\text{stabsys}_1(g), \text{stabsys}_2(g)\} \geq \sigma$;
- (5) $\kappa(g, \eta) \leq M$, see Definition 4.4.

Let g_i be a sequence of metrics in $\mathcal{F}$ such that

$$(1) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0.$$

Then, there is a subsequence, also denoted g_i , and a flat metric g_{F_∞} on $\mathbb{T}^3$ such that g_i converges to g_{F_∞} in the sense of Dong-Song. That is, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $i \geq N$ there is an open sub manifold $\tilde{\Omega}_i$ with smooth boundary such that

$$(2) \quad |\tilde{\Omega}_i^c|_{g_i} + |\partial \tilde{\Omega}_i|_{g_i} \leq \varepsilon,$$

and

$$(3) \quad d_{GH} \left(\left(\tilde{\Omega}_i, d_{\tilde{\Omega}_i}^{g_i} \right), \left(\mathbb{T}^3, d^{g_{F_\infty}} \right) \right) \leq \varepsilon.$$

As a consequence of the above, we obtain the following two theorems.

Theorem 1.2. *Let $\sigma, K, V > 0$, and define $\mathcal{R}(\sigma, K, D)$ to be the collection of Riemannian metrics g on $\mathbb{T}^3$ such that*

- (1) $\min\{\text{stsys}_1(g), \text{stsys}_2(g)\} \geq \sigma$;
- (2) $\text{Ric}_g \geq -K$;
- (3) $\text{diam}_g(\mathbb{T}^3) \leq D$.

Then, for any sequence of metrics $\{g_i\}_{i=1}^\infty \subset \mathcal{R}(\sigma, K, D)$ such that

$$(4) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0,$$

$\lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0$, there exists a subsequence $\{g_{i_j}\}_{j=1}^\infty$ and a flat metric g_{F_∞} on $\mathbb{T}^3$ such that $g_{i_j} \rightarrow g_{F_\infty}$ in the sense of Dong-Song.

Theorem 1.3. *Let g_0 be a fixed Riemannian metric on $\mathbb{T}^3$, and let $\Lambda, V > 0$. We denote by $\mathcal{V}(g_0, \Lambda, V)$ the collection of Riemannian metrics g on $\mathbb{T}^3$ satisfying the following properties:*

- (1) $g \geq g_0$;
- (2) $\text{IN}_1(g) \geq \Lambda$;
- (3) $|\mathbb{T}^3|_g \leq V$.

Then, for any sequence of metrics $\{g_i\}_{i=1}^\infty \subset \mathcal{V}(\sigma, \Lambda, V)$ such that

$$(5) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0,$$

there exists a subsequence $\{g_{i_j}\}_{j=1}^\infty$ and a flat metric g_{F_∞} on $\mathbb{T}^3$ such that $g_{i_j} \rightarrow g_{F_\infty}$ in the sense of Dong-Song.

The theorem above has the following corollary for Volume Above Distance Below (VADB) convergence, a notion of convergence which implies volume preserving intrinsic flat convergence, see [APS21].

Corollary 1.4. *Let g_0 be a fixed Riemannian metric on $\mathbb{T}^3$, let $\Lambda, R, V > 0$, and let $\mathcal{V} = \mathcal{V}(g_0, \Lambda, R, V)$ denote the collection of Riemannian metrics g on $\mathbb{T}^3$ which satisfy the following properties:*

- (1) $g \geq g_0$;
- (2) $\text{IN}_1(g) \geq \Lambda$;
- (3) $\|R^-\|_{L^2(g)} \leq R$;
- (4) $|\mathbb{T}^3|_g \leq V$.

Suppose that g_i is a sequence of metrics in $\mathcal{V}$ such that

$$(6) \quad |\mathbb{T}^3|_{g_i} \rightarrow |\mathbb{T}^3|_{g_0},$$

and

$$(7) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0.$$

Then, the metric g_0 must be flat.

2. BACKGROUND

2.1. Convergence in the sense of Dong-Song. In [DS23] Dong-Song established the stability of the Positive Mass Theorem with respect to a novel notion of convergence. Here we will offer a slight modification of this notion. Let us begin by recalling the intrinsic length metric associated with a subset of a Riemannian manifold.

Definition 2.1. Let (M, g) be a Riemannian manifold, and let $\Omega \subset M$ be a subset of M . Furthermore, let $L_g(\gamma)$ denote the length of γ as measured by g . For any two points x, y in Ω let us define

$$(8) \quad \hat{d}_\Omega^g(x, y) = \inf\{L_g(\gamma) : \gamma \text{ connects } x \text{ and } y\}$$

We can now use the above definition to define convergence in the sense of Dong-Song.

Definition 2.2. Let (M, g) and (N, h) be two closed Riemannian manifolds. We say that (M, g) is ε -close to (N, h) in the Dong-Song sense if there exists a connected open domain $\Omega_M \subset M$ with smooth boundary such that

$$(9) \quad |\Omega^c| + |\partial\Omega| \leq \varepsilon$$

and

$$(10) \quad d_{GH}\left((N, h), \left(\Omega_M, \hat{d}_{\Omega_M}^g\right)\right) \leq \varepsilon.$$

Remark 2.3. Note the asymmetry in the above definition. If the results of [LT22] hold for three tori, then the results of this paper indicate that the above is not in general symmetric. However, notions of nearness like the above do seem to be useful for stability problems, see [Don22] and [Don24].

The following lemma is crucial for establishing the type of convergence given in Definition 2.2.

Lemma 2.4. *Let $(\mathbb{T}^3, h)$ be a flat Riemannian metric on the three Torus. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\Omega \subset \mathbb{T}^3$ has smooth boundary and*

$$(11) \quad |\Omega| \geq |\mathbb{T}^3| - \delta;$$

$$(12) \quad |\partial\Omega| \leq \delta,$$

then there exists a connected subset $\Omega' \subset \Omega$ with smooth boundary such that

$$(13) \quad |\Omega'| \geq |\mathbb{T}^3| - \varepsilon;$$

$$(14) \quad |\partial\Omega'| \leq \varepsilon;$$

$$(15) \quad d_{\text{GH}}((\Omega', d_{\Omega'}^h), (\Omega', d^h)) \leq \varepsilon.$$

2.2. Harmonic maps and Stern's inequality. Let (M, g) be an arbitrary closed and oriented Riemannian manifold. We begin by recalling the Hodge star map $\star : \Omega^p(M) \rightarrow \Omega^{n-p}(M)$.

Definition 2.5. Let (M, g) be a closed and oriented Riemannian manifold, and let $a \in \Omega^p(M)$. Then, we may uniquely define an element $\star a \in \Omega^{n-p}(M)$ as follows. Set $\star a$ to be the unique differential form such that

$$(16) \quad \int_M g(a, b) dV_g = \int_M \star a \wedge b$$

for all $b \in \Omega^p(M)$.

Recall that for each cohomology class in $H^k(M; \mathbb{R})$ there is a unique harmonic representative. In particular, for any cohomology class the harmonic representative has minimal L^2 norm among the representatives of the class, see [Pet06] for a quick introduction to harmonic forms and Hodge decomposition. In the case that $\alpha \in H^1(M; \mathbb{Z})$, then there is a harmonic map $u : (M, g) \rightarrow \mathbb{S}$ such that du is the harmonic representative of α .

The existence and uniqueness of harmonic representatives of cohomology classes provides $H^p(M; \mathbb{R})$ with an inner product structure, as is defined below.

Definition 2.6. Let (M, g) be a closed oriented Riemannian manifold, and let α and β be two cohomology classes in $H^p(M; \mathbb{R})$. Furthermore, let a and b be the corresponding unique harmonic representatives. Then, we define $g(\alpha, \beta)$ to be

$$(17) \quad g(\alpha, \beta) = \int_M g(a, b) dV_g.$$

Notation 2.7. Following Hebda [Heb23], for $\alpha \in H^p(M; \mathbb{R})$ we let $|\alpha|_2^*$ denote $\sqrt{g(\alpha, \alpha)}$.

Specializing to three dimensions, Stern [Ste19] connected scalar curvature to harmonic forms with the following powerful inequality.

Theorem 2.8 (Stern's inequality). *Let (M^3, g) be a closed and oriented 3-manifold, let $u : (M, g) \rightarrow \mathbb{S}$ be a nontrivial harmonic map, and let R_g denote the scalar curvature of g . For the level sets $\Sigma_\theta = u^{-1}\{\theta\}$ we let $\chi(\Sigma_\theta)$ denote the Euler characteristic of Σ_θ . Then, we have that*

$$(18) \quad 2\pi \int_{\mathbb{S}} \chi(\Sigma_\theta) d\theta \geq \frac{1}{2} \int_{\mathbb{S}} \int_{\Sigma_\theta} (|du|^{-2} |\nabla du|^2 + R_g) dA_g d\theta.$$

By inspection, we see that if we could control $\chi(\Sigma_\theta)$, then we would have a very strong relationship between R_g^- and the Hessian of non-trivial maps into $\mathbb{S}$. The following lemma is crucial in this regard.

Lemma 2.9 ([Ste19]). *Let (M, g) be a closed oriented Riemannian manifold, and let $u : (M, g) \rightarrow \mathbb{S}$ be a nontrivial harmonic map. Then, for almost every $\theta \in \mathbb{S}$ we have that $\Sigma_\theta = u$ is smooth, and each component is a non-trivial element of $H_{n-1}(M)$.*

Proof. That Σ_θ is smooth for almost every $\theta \in \mathbb{S}$ is a standard consequence of Sard's Lemma. In particular, we may also assume that $|\nabla u|_{\Sigma_\theta} > 0$ for almost every $\theta \in \mathbb{S}$. Therefore, we will focus on the second statement. Let $\theta_0 \in \mathbb{S}$ be such that $\Sigma = \Sigma_{\theta_0}$ is smooth and is the disjoint union of its connected components: $\Sigma = \bigsqcup \Sigma_i$. Let Σ_{i_0} be an arbitrary component of Σ , and suppose that it is a trivial element in $H_{n-1}(M)$. Since u is harmonic, we have that $\star du$ is also harmonic, and in particular is a closed element of $H^{n-1}(M; \mathbb{R})$. Therefore, since Σ_{i_0} is trivial, we have that

$$(19) \quad \int_{\Sigma_{i_0}} \star du = 0.$$

However, from the definition of $\star du$ we see that

$$(20) \quad \int_{\Sigma_{i_0}} \star du = \int_{\Sigma_{i_0}} |\nabla u| dA_g.$$

Since $|\nabla u|_\Sigma > 0$, this leads us to a contradiction, and so it follows that Σ_{i_0} could not have been a trivial element of $H_{n-1}(M)$. $\square$

With this lemma and Stern's inequality in hand, we immediately get the following important corollary.

Corollary 2.10. *Let (M^3, g) be a closed oriented three dimensional Riemannian manifold, let $u : (M, g) \rightarrow \mathbb{S}$ be a nontrivial harmonic map, and let R_g^- be the negative part of the scalar curvature. Finally, suppose that $H_2(M)$ has no non-separating 2-spheres. Then, we have that*

$$(21) \quad \|R_g^-\|_{L^2(g)} \|du\|_{L^2(g)} \geq \int_M \frac{|\nabla du|^2}{|du|} dV_g.$$

Proof. It follows from our hypotheses, Lemma 2.9, and the classification of surfaces that for almost every $\theta \in \mathbb{S}$ we have

$$(22) \quad \chi(\Sigma_\theta) \leq 0.$$

Therefore, we may rearrange Stern's inequality to obtain

$$(23) \quad - \int_{\mathbb{S}} \int_{\Sigma_\theta} R_g dA_g d\theta \geq \int_{\mathbb{S}} \int_{\Sigma_\theta} \frac{|\nabla du|^2}{|du|^2} dA_g d\theta.$$

The result now follows from an application of the coarea formula on both sides, the definition of R_g^- and an application of Hölder's inequality on the left hand side. $\square$

At this stage it is convenient to introduce the following notation.

Notation 2.11. For $\mathbb{S}^1$ let $d\theta^2$ denote the metric for which $\mathbb{S}^1$ has length 1. Then, we let $h = (d\theta^1)^2 + \dots + (d\theta^n)^2$ be the product metric on $\mathbb{T}^n$.

Recall that to each element of $H^1(\mathbb{T}^3; \mathbb{Z})$ we may associate a map to $\mathbb{S}$, and so we see that every element of $H^1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}} \subset H^1(\mathbb{T}^3; \mathbb{R})$ corresponds to an harmonic map from (M, g) to $\mathbb{S}$, which is unique up to translation. Therefore, to any three elements $\alpha^i \in H^1(M; \mathbb{R})$ we may find three harmonic maps u^i such that $[du^i] = \alpha^i$, and from these three maps we get an harmonic map $\mathbb{U} : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ defined by

$$(24) \quad \mathbb{U}(x) = (u^1(x), u^2(x), u^3(x))$$

for $x \in \mathbb{T}^3$. From this expression we see that

$$(25) \quad d\mathbb{U} = (du^1, du^2, du^3).$$

This immediately leads us to the following proposition, which we will often make use of without further comment.

Proposition 2.12. *Let $\mathbb{U} : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$, let $x \in M$, and let $\nu \in T_x \mathbb{T}^3$. Then, we have that*

$$(26) \quad d\mathbb{U}(\nu) = du^i(\nu) \frac{\partial}{\partial \theta^i} = g(\nabla u^i, \nu) \frac{\partial}{\partial \theta^i}$$

and

$$(27) \quad \nabla d\mathbb{U} = (\nabla du^1, \nabla du^2, \nabla du^3).$$

Due to the above proposition, we see that the Hessian of $\mathbb{U}$ will be controlled in terms of Stern's inequality in the form of Corollary 2.10, which is the heart of this paper.

2.3. Lattices and Successive Minima. As may be suspected from the above section, it is important to analyze how $H^1(\mathbb{T}^n; \mathbb{Z})$ sits inside of $H^1(\mathbb{T}^n; \mathbb{R})$ as a lattice. Let us now fix some notation and terminology which will be helpful in this pursuit.

Notation 2.13. Let (M, g) be a closed and oriented Riemannian manifold. We denote by $H^p(M; \mathbb{Z})_{\mathbb{R}}$ the lattice in $H^p(M; \mathbb{R})$ generated by $H^p(M; \mathbb{Z})$. If $H^p(M; \mathbb{Z})$ is free, then $H^p(M; \mathbb{Z}) \simeq H^p(M; \mathbb{Z})_{\mathbb{R}}$. Let us give $H^p(M; \mathbb{R})$ the inner product structure mentioned in Definition 2.6. Then, we define $\det(H^p(M; \mathbb{Z})_{\mathbb{R}})$ to be the determinant of the lattice $H^p(M; \mathbb{Z})_{\mathbb{R}} \subset H^p(M; \mathbb{R})$ with respect to the inner product on $H^p(M; \mathbb{R})$.

To begin analyzing $H^p(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$ as a lattice of $H^p(\mathbb{T}^n; \mathbb{R})$, one may use Poincare Duality and the free-ness of $H^p(\mathbb{T}^n; \mathbb{Z})$ to show that tori have the following property.

Lemma 2.14 (Berger [Ber72]). *For any p and any $\alpha \in H^p(\mathbb{T}^n; \mathbb{R})$ we have that if $(\alpha \cup \beta)[\mathbb{T}^n]$ is in $\mathbb{Z}$ for all β in $H^{n-p}(\mathbb{T}^n; \mathbb{Z})$, then α is in $H^p(\mathbb{T}^n; \mathbb{Z})$.*

Definition 2.15. Let M be any smooth closed manifold, and fix $p \in \mathbb{N}$. We say that M satisfies the dual lattice condition in degree p if the conclusion of the above lemma holds for p -forms.

We now come to an important result relating the determinants of different cohomology groups to each other.

Lemma 2.16 (Berger [Ber72]). *Let (M, g) be a closed oriented n -dimensional Riemannian manifold which satisfies the dual lattice condition in degree p . Then, we have that*

$$(28) \quad 1 = \det(H^p(M; \mathbb{Z})_{\mathbb{R}}) \det(H^{n-p}(M; \mathbb{Z})_{\mathbb{R}}).$$

In order to make good use of Lemma 2.16, we need a few results from the Geometry of Numbers and Systolic Geometry. Let us begin by reviewing the Geometry of Numbers. In what follows, most definitions and results have generalizations which are not needed for this paper, but can be found in the relevant sections of [Cas12].

Definition 2.17. Let L be a lattice in $\mathbb{R}^n$, and let $F_0(x)$ denote the Euclidean norm. Then, we define

$$(29) \quad F_0(L) = \inf\{F_0(a) : a \in L\}.$$

We can go a step further to define the following quantity associated with F_0 .

Definition 2.18. For F_0 as above, we define δ_0 as follows:

$$(30) \quad \delta_0 = \sup \left\{ \frac{F_0(L)}{\det(L)} : L \text{ is a lattice in } \mathbb{R}^n \right\}$$

We have the following result bounding δ_0 .

Lemma 2.19 ([Cas12]). *Let $|B(0, 1)|$ be the volume of the unit ball in $\mathbb{R}^n$. Then, we have the following upper bound on δ_0 :*

$$(31) \quad \delta_0 \leq \frac{2^n}{|B(0, 1)|}.$$

Proof. From [Cas12, Theorem 1 Chapter 3], we know that if $|B(0, r)|$ is greater than $\det(L)$, then there exists two points in $B(0, r)$, say x_1 and x_2 , such that $x_1 - x_2 \in B(0, r) \cap L$, and so we see that $F_0(L) \leq 2r$. Therefore, if we choose $r_L = \sqrt{n} \frac{\det(L)}{|B(0, 1)|}$, then we get that $F_0(L) \leq 2r_L$ for all lattices L . This gives the desired bound. $\square$

For any given lattice $L \subset \mathbb{R}^n$ the quantity $F_0(L)$ has the following useful generalization.

Definition 2.20. Let $L \subset \mathbb{R}^n$ be a lattice, and let F_0 denote the Euclidean norm. We denote by λ_k the following quantity:

$$(32) \quad \lambda_k = \inf \{ \lambda : \exists \text{ linearly independent } \{\nu_i\}_{i=1}^k \subset L \text{ such that } F_0(\nu_i) \leq \lambda \forall i \}.$$

We refer to λ_k as the k^{th} successive minima of L with respect to F_0 . Observe that $\lambda_1 = F_0(L)$.

The importance of successive minima for this paper is contained in the following lemma.

Lemma 2.21. Let L be a lattice in $\mathbb{R}^n$, and let $\lambda_1, \dots, \lambda_n$ be the successive minima of L with respect to F_0 . Then, there is a basis of L , say $\{b_i\}_{i=1}^n$ such that

$$(33) \quad |b_1| = \lambda_1$$

$$(34) \quad |b_j| \leq \frac{1}{2} j \lambda_j \quad (2 \leq j \leq n).$$

Proof. One may apply [Cas12, Chapter 8 Lemma 1] to find n linearly independent elements of the lattice $a_1, \dots, a_n$ such that $|a_j| = \lambda_j$ for all $j = 1, \dots, n$. Then, we may apply [Cas12, Chapter 5 Lemma 1] to find a basis $b_1, \dots, b_n$ such that $|b_1| = \lambda_1$ and for each $j \geq 2$ we have

$$(35) \quad |b_j| \leq \max \left\{ |a_j|, \frac{1}{2} \sum_{i=1}^j |a_i| \right\}.$$

$\square$

Remark 2.22. In fact, more can be said since we are working with the Euclidean norm. See [Rem38] and [vdW56].

The above shows that we may always find a basis for a lattice whose norms are bounded by the successive minima of the lattice. The following result is the key to estimating these successive minima.

Theorem 2.23 (Chapter 8 Theorem 1 [Cas12]). Let L be a lattice in $\mathbb{R}^n$, and let $\lambda_1, \dots, \lambda_n$ be its successive minima with respect to F_0 , the Euclidean norm. Then, we have that

$$(36) \quad \det(L) \leq \lambda_1 \cdots \lambda_n \leq \delta_0 \cdot \det(L).$$

Combining this theorem with Lemma 2.19 gives us the following useful corollary.

Corollary 2.24. Let $L \subset \mathbb{R}^n$ be a lattice, and let $\lambda_1, \dots, \lambda_n$ be its successive minima with respect to the Euclidean norm. Then, we have that

$$(37) \quad \lambda_1 \cdots \lambda_n \leq \frac{2^n}{|B(0, 1)|} \det(L).$$

2.4. Stable Systoles. We now turn to a quick review of some concepts in Systolic Geometry.

Definition 2.25 (Stable Norm of a Real Homology Class). The volume of a real k -dimensional Lipschitz cycle $c = \sum_i r_i \sigma_i$ is given by

$$\text{Vol}_k(c) = \sum_i |r_i| \text{Vol}_k(\Delta^k, \sigma_i^* g).$$

The stable norm $\|\alpha\|$ of a real homology class $\alpha \in H_k(M; \mathbb{R})$ is defined as the infimum of the volumes of all real Lipschitz cycles representing α .

Definition 2.26 (Stable Systoles). The **stable k -systole**, denoted $\text{stabsys}_k(M, g)$, is defined to be the minimum of the stable norm on the nonzero classes of the integral lattice $H_k(M; \mathbb{Z})_{\mathbb{R}}$ in $H_k(M; \mathbb{R})$.

Notation 2.27. Let $\alpha \in H^{n-p}(M; \mathbb{R})$, we shall denote by $\text{PD}(\alpha)$ its Poincaré dual in $H_p(M; \mathbb{R})$.

The following proposition is [Heb86, Corollary 3], see [Heb23, Proposition 3.2] for its statement using the stable norm.

Proposition 2.28. *Let $\text{PD}(\alpha) \in H_p(M; \mathbb{R})$ be the Poincaré dual of the cohomology class $\alpha \in H^{n-p}(M; \mathbb{R})$, and let $\|\text{PD}(\alpha)\|$ be the stable norm of $\text{PD}(\alpha)$. Then*

$$\|\text{PD}(\alpha)\| \leq |M|_g^{1/2} C(n, p) |\alpha|_2^*,$$

where $C(n, p)$ is a constant depending only on n and p , and $|\alpha|_2^*$ is the L^2 norm of α , see Definition 2.6 and Notation 2.7.

The above proposition has a simple, but important corollary:

Corollary 2.29. *Let (M, g) be a closed Riemannian manifold. Then we have that*

$$(38) \quad \text{stabsys}_p(M, g) \leq |M|_g^{\frac{1}{2}} \min \left\{ |\alpha|_2^* : \alpha \neq 0; \alpha \in H^{n-p}(M; \mathbb{Z})_{\mathbb{R}} \right\} = |M|_g^{\frac{1}{2}} F_0(H^{n-p}(M; \mathbb{Z})_{\mathbb{R}}).$$

Proof. According to Proposition 2.28, for $\alpha \in H^p(M; \mathbb{Z})_{\mathbb{R}}$, $1 \leq p \leq n-1$,

$$\|\text{PD}(\alpha)\| \leq |M|_g^{1/2} C(n, p) |\alpha|_2^*,$$

Since the Poincaré dual map is an isomorphism, we have that $\text{PD}(\alpha) \neq 0$ in $H_p(M; \mathbb{Z})_{\mathbb{R}}$. Therefore, by the definition of the stable p -systole, we have that

$$(39) \quad \text{stabsys}_p(M, g) \leq \|\text{PD}(\alpha)\|.$$

This gives the result. □

2.5. Isoperimetric constants. On a smooth Riemannian manifold, functions are bounded in terms of their gradients. The character and quality of this bound can be determined by the character and quality of isoperimetric bounds. Here we recall the definition of the Cheeger and Sobolev constants, and the fact that they are closely related.

Definition 2.30. Let (M, g) be a given n -dimensional Riemannian manifold, and let $\alpha \in [1, \frac{n}{n-1}]$. We denote by $\text{IN}_{\alpha}(M, g)$ the following quantity:

$$(40) \quad \text{IN}_{\alpha}(M, g) = \inf \left\{ \frac{|\partial\Omega|}{\min\{|\Omega|, |\Omega^c|\}} : \Omega \subset M \right\}.$$

When $\alpha = 1$, we call $\text{IN}_1(M, g)$ the Cheeger constant of (M, g) .

Next, we have the Sobolev constant of a Riemannian manifold.

Definition 2.31. Let (M, g) be a given n -dimensional Riemannian manifold, and let $\alpha \in [1, \frac{n}{n-1}]$. Let us denote by $\text{SN}_{\alpha}(M, g)$ the following quantity:

$$(41) \quad \text{SN}_{\alpha}(M, g) = \inf \left\{ \frac{\int_M |\nabla f| dV_g}{\inf_{k \in \mathbb{R}} \|f - k\|_{L^{\alpha}(g)}} : f \in W^{1,1}(M, g) \right\}.$$

It is standard, see [Li12, Theorem 9.6], that the Cheeger constant and SN_1 are equivalent:

Proposition 2.32. *Let (M, g) be a given n -dimensional Riemannian manifold, then we have that*

$$(42) \quad \text{IN}_1(M, g) = \text{SN}_1(M, g).$$

3. STABLE SYSTOLE BOUNDS AND L^2 ESTIMATES

In this section we will show that lower bounds on $\text{stabsys}_1(M, g)$ and $\text{stabsys}_{n-1}(M, g)$ guarantee the existence of a good basis for $H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$. The following lemma and its proof are modeled on [Heb86, Proposition 6].

Lemma 3.1. *Let g be a Riemannian metric on $\mathbb{T}^n$ such that*

$$(43) \quad \text{stabsys}_1(g) \geq \sigma > 0$$

and let $\lambda_1, \dots, \lambda_n$ be the successive minima of the lattice $H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$ in $H^1(\mathbb{T}^n; \mathbb{R})$ with respect to the inner product on $H^1(\mathbb{T}^n; \mathbb{R})$ induced by g , see Definition 2.6. Then, we have that

$$(44) \quad \lambda_1 \cdots \lambda_n \leq 2^{2n} |B(0, 1)|^2 \sigma^{-n} |\mathbb{T}^n|_g^{\frac{n}{2}}.$$

In particular,

$$(45) \quad \lambda_1 \leq 4 |B(0, 1)|^{\frac{2}{n}} \sigma^{-1} |\mathbb{T}^n|_g^{\frac{1}{2}}.$$

Proof. From Corollary 2.24 we see that

$$(46) \quad \frac{|B(0, 1)|}{2^n} \lambda_1 \cdots \lambda_n \leq \det(H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}).$$

Letting $\mu_1, \dots, \mu_n$ be the successive minima of $H^{n-1}(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$, we get the following as well:

$$(47) \quad \frac{|B(0, 1)|}{2^n} \mu_1 \cdots \mu_n \leq \det(H^{n-1}(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}).$$

Since $\mathbb{T}^n$ satisfies the dual lattice condition in all degrees, it follows from Lemma 2.16 that

$$(48) \quad 2^{-2n} |B(0, 1)|^2 \lambda_1 \cdots \lambda_n \cdot \mu_1 \cdots \mu_n \leq 1.$$

Since $\mu_1 \leq \mu_j$ for all $j \geq 1$, it follows that

$$(49) \quad \lambda_1 \cdots \lambda_n \leq 2^{2n} |B(0, 1)|^2 \mu_1^{-n}.$$

In fact, since $\lambda_1 \leq \lambda_j$ for all $j \geq 1$ as well, we have

$$(50) \quad \lambda_1^n \leq 2^{2n} |B(0, 1)|^2 \mu_1^{-n}.$$

Therefore, a lower bound on μ_1 will give us our desired upper bound on λ_1 . From Corollary 2.29, we have that

$$(51) \quad \text{stabsys}_1(\mathbb{T}^n, g) \leq |\mathbb{T}^n|_g^{\frac{1}{2}} \min\{|\alpha|_2^* : \alpha \neq 0; \alpha \in H^{n-1}(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}\} = |\mathbb{T}^n|_g^{\frac{1}{2}} \mu_1.$$

This gives the desired lower bound on μ_1 , and so the result follows. $\square$

We now know that $\lambda_1 = \inf\{|b|_2^* : b \neq 0; b \in H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}\}$ is bounded above in terms of $|\mathbb{T}^n|_g$ and σ^{-1} . However, in order to construct a useful harmonic map, it seems reasonable to suppose that we need to use n one-forms, which together form a basis for $H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$. It turns out that a lower bound on $\min\{\text{stabsys}_1(g), \text{stabsys}_{n-1}(g)\}$ is sufficient to ensure the existence of such a basis.

Lemma 3.2. *Let g be a Riemannian metric on $\mathbb{T}^n$ such that*

$$(52) \quad \min\{\text{stabsys}_1(g), \text{stabsys}_{n-1}(g)\} \geq \sigma > 0.$$

Then, there exists a basis $\alpha_1, \dots, \alpha_n$ of $H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$ with harmonic representatives a_j such that

$$(53) \quad \|a_j\|_{L^2(g)} \leq j \times \sqrt[{}^{(n-j+1)}]{2^{2n} |B(0, 1)| \sigma^{-(n+j-1)} |\mathbb{T}^n|_g^{\frac{1}{2}(n+j-1)}}.$$

Proof. We wish to apply Lemma 2.21, however first we must bound each successive minima. Equation (45) gives the desired upper bound on λ_1 . Therefore we must focus our attention on the higher successive minima. Using that $\lambda_j \leq \lambda_i$ for $i \leq j$, we may apply Lemma 3.1 to obtain

$$(54) \quad \lambda_j^{n-j+1} \leq \lambda_j \cdots \lambda_n \leq \lambda_1^{-(j-1)} 2^{2n} |B(0, 1)|^2 \sigma^{-n} |\mathbb{T}^n|_g^{\frac{n}{2}}.$$

From Corollary 2.29, we have that

$$(55) \quad \text{stabsys}_{n-1}(\mathbb{T}^n, g) \leq |\mathbb{T}^n|_g^{\frac{1}{2}} \min\left\{|\alpha|_2^* : \alpha \neq 0; H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}\right\} = |\mathbb{T}^n|_g^{\frac{1}{2}} \lambda_1.$$

Thus, using Equation (55), we see that

$$(56) \quad \lambda_j^{n-j+1} \leq 2^{2n} |B(0, 1)|^2 \sigma^{-(n+j-1)} |\mathbb{T}^n|_g^{\frac{1}{2}(n+j-1)}.$$

Taking $(n - j + 1)^{th}$ roots and then applying Lemma 2.21 gives the desired result. $\square$

Using the above results and notation, we can now establish the existence of an harmonic map $\mathbb{U} : (\mathbb{T}^n, g) \rightarrow (\mathbb{T}^n, h)$ with several desirable properties.

Corollary 3.3. *Let g be a Riemannian metric on $\mathbb{T}^n$ such that*

$$(57) \quad \min\{\text{stabsys}_1(g), \text{stabsys}_{n-1}(g)\} \geq \sigma > 0.$$

Then, there exists a surjective harmonic function $\mathbb{U} : (\mathbb{T}^n, g) \rightarrow (\mathbb{T}^n, h)$ such that $\deg(\mathbb{U}) = 1$ and

$$(58) \quad \|d\mathbb{U}\|_{L^2(g)} \leq \sum_{j=1}^n j \times \sqrt{4^n |B(0, 1)|^2 \sigma^{-(n+j-1)} |\mathbb{T}^n|_g^{\frac{1}{2}(n+j-1)}}.$$

Proof. Let α^i be a basis for the lattice $H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$ with harmonic representatives a^i as in Lemma 2.21, and let $u^i : (\mathbb{T}^n, g) \rightarrow \mathbb{S}$ be the harmonic map such that $du^i = \alpha^i$. Then, if we let $\mathbb{U} = (u^1, \dots, u^n)$, the estimate on $\|d\mathbb{U}\|_{L^2(g)}$ follows from Proposition 2.12.

Next, since the α^i form a basis of $H^1(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$, it follows that their wedge product $\bigwedge_{i=1}^n \alpha^i$ forms a basis of $H^n(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$. Let θ^i be the i^{th} coordinate function for $\mathbb{T}^n$, then we also have that $\bigwedge_{i=1}^n d\theta^i$ is a basis for $H^n(\mathbb{T}^n; \mathbb{Z})_{\mathbb{R}}$. As such, we may calculate the degree of $\mathbb{U}$ as follows:

$$(59) \quad \deg(\mathbb{U}) = \int_{\mathbb{T}^n} \mathbb{U}^* \bigwedge_{i=1}^n d\theta^i = \int_{\mathbb{T}^n} \bigwedge_{i=1}^n a^i = \pm 1.$$

We may take $\mathbb{U} = (-u^1, u^2, \dots, u^n)$ as necessary to ensure that $\deg(\mathbb{U}) = 1$. The surjectivity of $\mathbb{U}$ follows from the general fact that degree one maps between Riemannian manifolds are surjective [Eps66]. $\square$

For the sake of completeness, let us give a simple proof of the fact that degree one maps between tori are surjective.

Proposition 3.4. *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a degree one map. Then, it must be that f is surjective.*

Proof. Since the result is purely topological, we may assume that $\mathbb{T}^n$ has the product metric $\sum_{i=1}^n (d\theta^i)^2$. Let a be an arbitrary point in $\mathbb{T}^n$. Using the flat metric, we can see that we may give $\mathbb{T}^n$ the structure of a CW-complex with one n -cell which contains a in its interior.

With the above reduction in place, let $\alpha \in H_n(\mathbb{T}^n; \mathbb{Z})$ be a fundamental class with respect to which $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is a degree one map:

$$(60) \quad f_* \alpha = \alpha.$$

Suppose that f is not surjective, and so there is a point $a \in \mathbb{T}^n$ such that a is not in the image of $\mathbb{T}^n$ under f . Since $f(\mathbb{T}^n)$ is compact, it follows that there is an $\varepsilon > 0$ such that

$$(61) \quad d(a, f(\mathbb{T}^n)) \geq \varepsilon.$$

As stated above, we may give $\mathbb{T}^n$ the structure of a CW complex with one n cell, which contains a in its interior. Then, by shrinking ε as necessary, we may assume that $B(a, \varepsilon) \subset \text{int} D^n$, where D^n is the n -cell. We also have that $f(\mathbb{T}^n) \subset B(a, \varepsilon)^c$. As such, we see that there is a homotopy of the map, say f_t , such that $f_0 = f$ and $f_0(\mathbb{T}^n)$ is contained in the $n - 1$ skeleton of $\mathbb{T}^n$.

Let X^{n-1} denote the $n-1$ skeleton of $\mathbb{T}^n$ and let $\iota : X^{n-1} \rightarrow \mathbb{T}^n$ denote the inclusion map; we have that $H_n(X^{n-1}; \mathbb{Z}) = 0$. In particular, we see that $(f_1)_*\alpha = 0$, in X^{n-1} . Using the inclusion map, we have that $f_1 : \mathbb{T}^n \rightarrow \mathbb{T}^n$ has the following decomposition, which is almost tautological:

$$(62) \quad \mathbb{T}^n \xrightarrow{f_1} X^{n-1} \xrightarrow{\iota} \mathbb{T}^n$$

This shows that $(f_1)_*\alpha = 0 \in H_n(\mathbb{T}^n; \mathbb{Z})$. However, since f_1 is homotopic to $f_0 = f$, this shows that $\alpha = f_*\alpha = 0$. This is a contradiction, since α was assumed to be a fundamental class. $\square$

4. FROM L^2 TO L^3 BOUNDS

From the previous section we know that given a Riemannian metric g on $\mathbb{T}^n$, we may find an harmonic map $\mathbb{U} : (\mathbb{T}^n, g) \rightarrow (\mathbb{T}^n, h)$ whose L^2 energy is controlled in terms of the volume of $|\mathbb{T}^n|_g$ and $\min\{\text{stabsys}_1(g), \text{stabsys}_{n-1}(g)\}$. In this section we will ultimately focus on the case that $n = 3$, and study the relationship between the universal cover $\mathbb{R}^3$ with its pullback metric π^*g and $(\mathbb{T}^3, g)$. The goal is to use this relationship along with Stern's inequality, in particular Corollary 2.10, to improve the L^2 bounds of the previous section to L^3 bounds.

Of fundamental importance to this discussion is the notion of fundamental domain, which we recall now.

Definition 4.1. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the covering map. Suppose that $\mathbb{V} \subset \mathbb{R}^n$ is a closed subset such that $\pi(\mathbb{V}) = \mathbb{T}^n$ and $\pi|_{\text{int}\mathbb{V}}$ is injective. If in addition $\mathbb{V}$ is path connected, and $\partial\mathbb{V}$ has measure zero, then we call $\mathbb{V}$ a fundamental domain of $\mathbb{T}^n$. Given a fundamental domain $\mathbb{V}$, for $\nu \in \mathbb{Z}^n$ we let $\mathbb{V}^\nu$ denote the image of $\mathbb{V}$ under the deck transformation associated to $\nu \in \mathbb{Z}^n = \pi_1(\mathbb{T}^n)$.

The following proposition lists some of the basic properties of fundamental domains.

Proposition 4.2. *Let g be a Riemannian metric on $\mathbb{T}^n$, let $\mathbb{U} : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a map, and let u^k denote the components of $\mathbb{U}$. Then, there exists lifts $\hat{u}^k$, and so a lift $\hat{\mathbb{U}}$, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\hat{u}^k} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{T}^n & \xrightarrow{u^k} & \mathbb{S}^1 \end{array}$$

Furthermore, for any integrable function $f : \mathbb{T}^n \rightarrow \mathbb{R}$ and fundamental domain $\mathbb{V}$ we have that

$$(63) \quad \int_{\mathbb{V}} f \circ \pi dV_{\pi^*g} = \int_{\mathbb{T}^n} f dV_g.$$

Proof. Note that $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. For the moment, let us denote elements of $\mathbb{T}^n$ by $[x]$, where $x \in \mathbb{R}^n$: we have that $[x] = \mathbb{Z}^n x$. We denote elements in $\mathbb{S}^1$ similarly. We can define a lift $\hat{u}^k$ as follows. First, denote by $u_*^k : \mathbb{Z}^n \rightarrow \mathbb{Z}$ the homomorphism between fundamental groups induced by u^k . Next, fix 0 in $\mathbb{R}^n$, and consider the points $[0]$ in $\mathbb{T}^n$ and $u^k([0])$. Pick any point $y_0 \in \mathbb{R}$ such that $u^k([0]) = \mathbb{Z}y_0$. Abusing notation, we may use u_*^k to define $\hat{u}^k(\alpha)$ for any $\alpha \in \mathbb{Z}^n$ as $\hat{u}^k(\alpha) = u_*^k(\alpha) + y_0$. Every element $x \in \mathbb{R}^n$ can be written as $x = \tilde{x} + \alpha$, where $\tilde{x}$ is in the same fundamental domain as 0, and $\alpha \in \mathbb{Z}^n$. Then, we may set $\hat{u}^k(x) = u_*^k(\alpha) + y_0$. $\hat{u}^k(\alpha(x)) = u_*^k(\alpha)u^k(x)$ for $\alpha \in \mathbb{Z}^n$, where $u_*^k : \mathbb{Z}^n \rightarrow \mathbb{Z}$ is the homomorphism between fundamental groups induced by u^k . The second part follows from the fact that $\partial\mathbb{V}$ has measure zero, that $\pi|_{\text{int}\mathbb{V}}$ is injective, and from the fact that the image of measure zero sets under π have measure zero, since π is a smooth map. $\square$

Working with fundamental domains allows us to treat maps $\mathbb{U} : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ as maps $\hat{\mathbb{U}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. The next lemma is an example of this, and will play a vital role in strengthening our L^2 bounds to L^3 bounds.

Lemma 4.3. *Let g be a Riemannian metric on $\mathbb{T}^3$ such that $\text{stabsys}_2(g) \geq \sigma > 0$, let $u : (\mathbb{T}^3, g) \rightarrow \mathbb{S}^1$ be a nontrivial harmonic map, let $\hat{u}$ denote its lift, and let $\mathbb{V}$ be any fundamental domain in $\mathbb{R}^3$. Then, we have that*

$$(64) \quad \max_{\mathbb{V}} \hat{u} - \min_{\mathbb{V}} \hat{u} \leq \sigma^{-1} |\mathbb{T}^3|_g^{\frac{1}{2}} \|du\|_{L^2(g)}.$$

Proof. Here $\chi_{\mathbb{V}}$ will denote the indicator function for the set $\mathbb{V}$. From the coarea formula, we have that

$$(65) \quad \int_{\mathbb{V}} |d\hat{u}| dV_{\pi^*g} = \int_{\min_{\mathbb{V}} \hat{u}}^{\max_{\mathbb{V}} \hat{u}} \int_{\hat{u}^{-1}\{t\}} \chi_{\mathbb{V}} dA_{\pi^*g} dt.$$

For every t , let $\theta(t) = t \bmod 1$, then it follows from the commutativity of the Diagram 4.2 and the fact that $\pi(\mathbb{V}) = \mathbb{T}^3$ that for all t we have

$$(66) \quad \int_{\hat{u}^{-1}\{t\}} \chi_{\mathbb{V}} dA_{\pi^*g} \geq \int_{u^{-1}\{\theta(t)\}} dA_g.$$

In particular, observe that for almost every t we have that t is a regular value for $\hat{u}$ and $\theta(t)$ is a regular value for u . From Lemma 2.9 it follows that for almost every t the surface $u^{-1}\{\theta(t)\}$ is a nontrivial element of $H_2(\mathbb{T}^3)$. Therefore, by hypothesis we have that

$$(67) \quad |u^{-1}\{\theta(t)\}|_g \geq \sigma.$$

Furthermore, since $\pi : \mathbb{V} \rightarrow \mathbb{T}^3$ is surjective, it follows from the above inequality that for almost every t we have

$$(68) \quad |\hat{u}^{-1}\{t\} \cap \mathbb{V}|_{\pi^*g} \geq |u^{-1}\{\theta(t)\}|_g \geq \sigma.$$

It now follows from the coarea formula that

$$(69) \quad \int_{\mathbb{V}} |d\hat{u}| dV_{\pi^*g} \geq \sigma (\max_{\mathbb{V}} \hat{u} - \min_{\mathbb{V}} \hat{u}).$$

We observe that $|d\hat{u}| = |d(u \circ \pi)| = |du| \circ \pi$, and so from Proposition 4.2 we have that

$$(70) \quad \int_{\mathbb{T}^3} |du| dV_g \geq \sigma (\max_{\mathbb{V}} \hat{u} - \min_{\mathbb{V}} \hat{u}).$$

After applying Hölder's inequality to the left hand side, we get the desired result. $\square$

We will actually need to understand $\sup_{\mathbb{V}_\eta} \hat{u} - \inf_{\mathbb{V}_\eta} \hat{u}$, where $\eta > 0$ and $\mathbb{V}_\eta$ denotes the η neighborhood of $\mathbb{V}$ with respect to the distance function induced by the metric π^*g . In order to obtain such information, we are led to consider the following quantity.

Definition 4.4. Let g be a Riemannian metric on $\mathbb{T}^3$, let $\pi : \mathbb{R}^3 \rightarrow \mathbb{T}^3$ be the covering map, and let $\eta > 0$. We define the constant $\kappa(g, \eta)$ to be the smallest integer such that there exists a fundamental domain $\mathbb{V}$ such that

$$(71) \quad \sup_{x \in \mathbb{T}^3} |\pi^{-1}\{x\} \cap \mathbb{V}_\eta| \leq \kappa(g, \eta).$$

Let us refer to $\kappa(g, \eta)$ as the η -covering constant of g , and refer to $\mathbb{V}$ as a test domain for $\kappa(g, \eta)$. See Figure 1 below.

Before we can estimate $(\sup_{\mathbb{V}_\eta} \hat{u} - \inf_{\mathbb{V}_\eta} \hat{u})$ more precisely, we need to understand how the different copies of $\mathbb{V}$ cover $\mathbb{V}_\eta$. This is the content of the following lemma.

Lemma 4.5. *Let g be a Riemannian metric on $\mathbb{T}^3$ and let $\mathbb{V}$ be a test domain for $\kappa(g, \eta)$, see Definition 4.4 above. Then there are $\kappa(g, \eta)$ copies of $\mathbb{V}$, say $\mathbb{V}^{\nu_i}$, generated by $\pi_1(\mathbb{T}^3)$ such that the following statements are true.*

$$(72) \quad \mathbb{V}_\eta \subset \bigcup_{i=1}^m \mathbb{V}^{\nu_i}.$$

$$(73) \quad \mathbb{V}_\eta \cap \mathbb{V}^{\nu_i} \neq \emptyset \forall i.$$

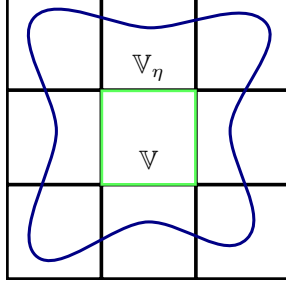


FIGURE 1. A fundamental domain $\mathbb{V}$ with neighborhood $\mathbb{V}_\eta$. In the case loosely depicted here, we have $\kappa(g, \eta) \leq 9$.

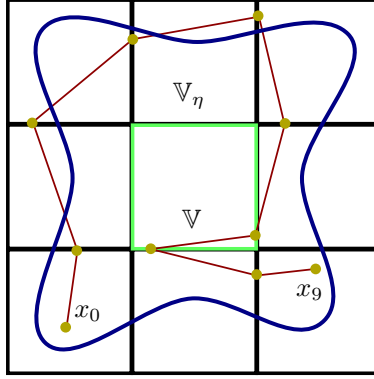


FIGURE 2. An example of a curve whose existence is established in Lemma 4.5.

Finally, for any x_0 and x_1 in $\mathbb{V}_\eta$ there exists a curve c in $\bigcup \mathbb{V}^i$ and times $\{t_j\}_1^l$ with $l \leq \kappa(g, \eta)$ satisfying the following properties:

- (1) $c(0) = x_0$;
- (2) $c(t_j) \in \mathbb{V}^{\nu_{i_j}} \cap \mathbb{V}^{\nu_{i_{j-1}}}$
- (3) once c leaves a domain $\mathbb{V}^{\nu_i}$, it does not re-enter it.

See Figure 2

Proof. By definition, the fundamental domain $\mathbb{V}$ is path connected, and so we have that $\mathbb{V}_\eta$ is as well. Let $c_0 : [0, 1] \rightarrow \mathbb{R}^3$ be a curve in $\mathbb{V}_\eta$ connecting x_0 to x_1 . We will describe a process for modifying this curve to fit the criteria laid out above. Let i_0 be the smallest index such that $\mathbb{V}^{i_0} = \mathbb{V}^{\nu_{i_0}}$ contains $x_0 = c_0(0)$. We define t_0 as follows:

$$(74) \quad t_0 = \sup\{t : c_0(t) \in \mathbb{V}^{i_0}\}$$

Since c_0 is continuous and $\mathbb{V}^{i_0}$ is closed, it follows that

$$(75) \quad c_0(t_0) \in \mathbb{V}^{i_0}.$$

Since $\mathbb{V}^{i_0}$ is path connected, we may replace $c_0|_{[0, t_0]}$ with a curve which lies entirely in $\mathbb{V}^{i_0}$. Let c_1 be the curve which results from this substitution, and let $\tilde{t}_1$ be as follows

$$(76) \quad \tilde{t}_1 = \inf\{t : c_1(t) \in \mathbb{V}^i; i \neq i_0\}$$

Since there are only finitely many $\mathbb{V}^i$, we find a smallest i_1 such that $c_1(\tilde{t}_1) \in \mathbb{V}^{i_1}$. By continuity, and the fact that all $\mathbb{V}^\nu$ are closed, we also have that $c_1(\tilde{t}_1) \in \mathbb{V}^{i_0}$, and actually $\tilde{t}_1$ is equal to t_0 . Continuing as before, we let t_1 be defined as follows:

$$(77) \quad t_1 = \sup\{t : c_1(t) \in \mathbb{V}^{i_1}\}$$

Since $\mathbb{V}^{i_1}$ is path connected, we may replace $c_1|_{[t_0, t_1]}$ by a path contained entirely in $\mathbb{V}^{i_1}$. Let c_2 be the resulting curve, and continue in this manner. Since there are only finitely many $\mathbb{V}^i$, this process will terminate. The resulting curve has the desired properties. $\square$

We are now in a position to obtain a bound on $\sup \mathbb{V}_\eta \hat{u} - \inf \mathbb{V}_\eta \hat{u}$.

Lemma 4.6. *Let g be a Riemannian metric on $\mathbb{T}^3$ such that $\text{stabsys}_2(g) \geq \sigma$, let $\mathbb{V}$ be a test domain for $\kappa(g, \eta)$, and let $u : (\mathbb{T}^3, g) \rightarrow \mathbb{S}^1$ be a nontrivial harmonic map. Then, we have that*

$$(78) \quad \sup_{\mathbb{V}_\eta} \hat{u} - \inf_{\mathbb{V}_\eta} \hat{u} \leq \kappa(g, \eta) \sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)}.$$

Proof. By definition, the subset $\mathbb{V}_\eta$ is contained in $\kappa(g, \eta)$ copies $\mathbb{V}^{\nu_i}$ of $\mathbb{V}$. Let x_1 and x_0 be any two points in $\mathbb{V}_\eta$, and connect them by a curve $c \subset \bigcup_{i=1}^{\kappa(g, \eta)} \mathbb{V}^{\nu_i}$ such as in Lemma 4.5. Let $\{t_{i_j}\}_{j=1}^l$ be the times such that $c(t_{i_j}) \in \mathbb{V}^{\nu_{i_j}} \cap \mathbb{V}^{\nu_{i_{j-1}}}$, and let $x_j = c(t_{i_j})$. Then, we may calculate as follows:

$$(79) \quad \hat{u}(x_1) - \hat{u}(x_0) \leq \hat{u}(x_1) - \hat{u}(x_l) + \sum_{j=1}^l \hat{u}(x_j) - \hat{u}(x_{j-1}).$$

From Lemma 4.3 each element in the sum on the right hand side is bounded above by $\sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)}$. Therefore, we see that

$$(80) \quad \hat{u}(x_1) - \hat{u}(x_0) \leq \kappa(g, \eta) \sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)}$$

$\square$

The above sup bound is the key to the integration by parts argument in the proof of the following lemma, as it allows us to avoid a Hölder like inequality, and so obtain control over higher L^p norms of non-trivial harmonic maps.

Lemma 4.7. *Let g be a Riemannian metric on $\mathbb{T}^3$, and suppose that*

$$(81) \quad \min\{\text{stabsys}_1(g), \text{stabsys}_2(g)\} \geq \sigma.$$

Given any nontrivial harmonic map, say $u : (\mathbb{T}^3, g) \rightarrow \mathbb{S}^1$ we have that

$$(82) \quad \|du\|_{L^3(g)} \leq 1 + \left(\left(1 + \kappa(g, \eta) \sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)} \right) \eta^{-1} \kappa(g, \eta) \|du\|_{L^2(g)}^2 \right)^{\frac{2}{3}}$$

$$(83) \quad + \left(\left(1 + \kappa(g, \eta) \sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)} \right) \kappa(g, \eta) \|du\|_{L^3(g)}^{\frac{3}{2}} \|R_g^-\|_{L^2(g)}^{\frac{1}{2}} \right)^{\frac{2}{3}}.$$

Proof. Let $\mathbb{V}$ be a test domain for $\kappa(g, \eta)$, let $\mathbb{V}_\eta$ be the η neighborhood of $\mathbb{V}$, let $\tilde{u}$ be a lift of u . From Lemma 4.6 we know that

$$(84) \quad \sup_{\mathbb{V}_\eta} \tilde{u} - \inf_{\mathbb{V}_\eta} \tilde{u} \leq \kappa(g, \eta) \sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)}.$$

Let $N = \lceil \inf_{\mathbb{V}_\eta} \tilde{u} \rceil$, and let $\hat{u} = \tilde{u} - N$. Then, $\hat{u}$ covers u , $d\hat{u} = d\tilde{u}$, and

$$(85) \quad \|u\|_{L^\infty(\mathbb{V}_\eta)} \leq \kappa(g, \eta) \sigma^{-1} |\mathbb{T}^3|^{\frac{1}{2}} \|du\|_{L^2(g)} + 1.$$

Finally, let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a cutoff function such that

- (1) $\chi_{\mathbb{V}} \leq f \leq 1$;
- (2) $\text{Lip}(f) \leq \eta^{-1}$;
- (3) $\text{supp}(f) \subset \mathbb{V}_\eta$.

With these elements we can calculate as follows:

$$(86) \quad \int_{\mathbb{T}^3} |du|^3 dV_g \leq \int_{\mathbb{V}_\eta} g(f|d\hat{u}|d\hat{u}, d\hat{u}) dV_{\pi^*g}$$

$$(87) \quad = - \int_{\mathbb{V}_\eta} \hat{u} \left(g(df, |d\hat{u}|d\hat{u}) + fg(d|d\hat{u}|, d\hat{u}) \right) dV_{\pi^*g},$$

where in the second line we integrated by parts and used the fact that $\hat{u}$ is an harmonic function. Taking the absolute value of both sides, and then using the Cauchy-Schwarz and Kato inequalities, we obtain

$$(88) \quad \|du\|_{L^3(g)}^3 \leq \|\hat{u}\|_{L^\infty(\mathbb{V}_\eta)} \left(\eta^{-1} \|d\hat{u}\|_{L^2(\mathbb{V}_\eta)}^2 + \int_{\mathbb{V}_\eta} |d\hat{u}| |\nabla d\hat{u}| dV_{\pi^*g} \right).$$

The right most term of the above can be estimated as follows

$$(89) \quad \int_{\mathbb{V}_\eta} |d\hat{u}| |\nabla d\hat{u}| dV_{\pi^*g} = \int_{\mathbb{V}_\eta} |d\hat{u}|^{\frac{3}{2}} \frac{|\nabla d\hat{u}|}{|d\hat{u}|^{\frac{1}{2}}} dV_{\pi^*g}$$

$$(90) \quad \leq \|d\hat{u}\|_{L^3(\mathbb{V}_\eta)}^{\frac{3}{2}} \left(\int_{\mathbb{V}_\eta} \frac{|\nabla d\hat{u}|^2}{|d\hat{u}|} dV_{\pi^*g} \right)^{\frac{1}{2}}.$$

Since $\mathbb{V}_\eta \subset \bigcup_{i=1}^{\kappa(g,\eta)} \mathbb{V}^i$, it follows from Proposition 4.2 that

$$(91) \quad \|d\hat{u}\|_{L^2(\pi^*g, \mathbb{V}_\eta)}^2 \leq \kappa(g, \eta) \|du\|_{L^2(g)}^2,$$

$$(92) \quad \|d\hat{u}\|_{L^3(\pi^*g, \mathbb{V}_\eta)}^3 \leq \kappa(g, \eta) \|du\|_{L^3(g)}^3,$$

and

$$(93) \quad \int_{\mathbb{V}_\eta} \frac{|\nabla d\hat{u}|^2}{|d\hat{u}|} dV_{\pi^*g} \leq \kappa(g, \eta) \int_{\mathbb{T}^3} \frac{|\nabla du|^2}{|du|} dV_g.$$

Therefore, we may apply Stern's inequality to the right hand side of Equation (93), see Corollary 2.10, and Hölder's inequality to obtain

$$(94) \quad \int_{\mathbb{V}_\eta} |d\hat{u}| |\nabla d\hat{u}| dV_{\pi^*g} \leq \kappa(g, \eta) \|du\|_{L^3(g)}^{\frac{3}{2}} \|du\|_{L^2(g)}^{\frac{1}{2}} \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

Putting everything together gives us that

$$(95) \quad \|du\|_{L^3(g)}^3 \leq \|\hat{u}\|_{L^\infty(\mathbb{V}_\eta)} \eta^{-1} \kappa(g, \eta) \|du\|_{L^2(g)}^2$$

$$(96) \quad + \|\hat{u}\|_{L^\infty(\mathbb{V}_\eta)} \kappa(g, \eta) \|du\|_{L^3(g)}^{\frac{3}{2}} \|du\|_{L^2(g)}^{\frac{1}{2}} \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

If $\|du\|_{L^3(g)} \geq 1$, then we may divide both sides by $\|du\|_{L^3(g)}^{\frac{3}{2}}$, otherwise we already have a good bound. Therefore, we see that

$$(97) \quad \|du\|_{L^3(g)}^{\frac{3}{2}} \leq 1 + \|\hat{u}\|_{L^\infty(\mathbb{V}_\eta)} \eta^{-1} \kappa(g, \eta) \|du\|_{L^2(g)}^2$$

$$(98) \quad + \|\hat{u}\|_{L^\infty(\mathbb{V}_\eta)} \kappa(g, \eta) \|du\|_{L^2(g)}^{\frac{1}{2}} \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

Including the bound on $\|\hat{u}\|_{L^\infty(\mathbb{V}_\eta)}$ from Equation (85), and then taking the $\frac{2}{3}$ root gives the result. $\square$

5. APPROXIMATION BY CONSTANT MATRICES

The importance of obtaining L^3 estimates on non-trivial harmonic functions is twofold. The second reason is in some sense the heart of Lemma 6.6 below. In this section, we will explore the first reason for their importance. Consider two nontrivial harmonic functions from $(\mathbb{T}^3, g)$ to $\mathbb{S}^1$, say u^j and u^k . Then, we can control $\|\nabla g(du^j, du^k)\|_{L^1(g)}$ in terms of $\max\{\|du^j\|_{L^3(g)}, \|du^k\|_{L^3(g)}\}$ and $\max\left\{\left\|\frac{|\nabla du^j|^2}{|du^j|}\right\|_{L^1(g)}, \left\|\frac{|\nabla du^k|^2}{|du^k|}\right\|_{L^1(g)}\right\}$. This will be carried out for smooth maps in general, and will be applied to harmonic maps in particular in Section 6. We begin with the following lemma.

Lemma 5.1. *Let g be a Riemannian metric on $\mathbb{T}^3$ such that $\text{IN}_1(g) \geq \Lambda$, let $\mathbb{U} : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ be a smooth map, and let u^k denote the components of $\mathbb{U}$. Then, we have that*

$$(99) \quad \|\nabla g(du^j, du^k)\|_{L^1(g)} \leq 2 \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}}.$$

Furthermore, letting g^{jk} denote $g(du^j, du^k)$, there exists a constant symmetric and non-negative matrix $a = a^{jk}$ such that

$$(100) \quad \|g^{jk} - a^{jk}\|_{L^1(g)} \leq 2\Lambda^{-1} \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}}.$$

Finally, we have that

$$(101) \quad \begin{aligned} \sup_{jk} |a^{jk}| &\leq 2|\mathbb{T}^3|_g^{-1} \Lambda^{-1} \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}} \\ &\quad + |\mathbb{T}^3|_g^{-1} \sup_{jk} \|du^j\|_{L^2(g)} \|du^k\|_{L^2(g)}. \end{aligned}$$

Proof. Once we establish (99), we have that (100) follows from the definition of $\text{IN}_1(g)$. Furthermore (101) also follows by writing $|a^{jk}| = \frac{1}{|\mathbb{T}^3|_g} \int_{\mathbb{T}^3} |a^{jk}| dV_g$, then adding and subtracting g^{jk} , applying the triangle inequality, and then using (100).

Therefore, we need only establish (99). Using the Cauchy-Schwarz inequality and rearranging terms, we estimate as follows

$$(102) \quad \begin{aligned} \int_{\mathbb{T}^3} |\nabla g(du^j, du^k)| dV_g &\leq \int_{\mathbb{T}^3} |du^k| |du^j|^{\frac{1}{2}} \frac{|\nabla du^j|}{|du^j|^{\frac{1}{2}}} dV_g \\ &\quad + \int_{\mathbb{T}^3} |du^j| |du^k|^{\frac{1}{2}} \frac{|\nabla du^k|}{|du^k|^{\frac{1}{2}}} dV_g. \end{aligned}$$

Both integrals may be estimated using Hölder's inequality for three terms with exponents 3, 6, and 2, respectively. This gives (99), and so the other results as well. $\square$

The above result can be used to show that on a subset of $\mathbb{T}^3$ the sup norm of $|g^{jk} - a^{jk}|$ is controlled.

Corollary 5.2. *Let g be a metric on $\mathbb{T}^3$ such that $\text{IN}_1(g) \geq \Lambda$, let $\mathbb{U} : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a smooth map, let u^j be the components of $\mathbb{U}$, let g^{jk} denote $g(du^j, du^k)$, and let a^{jk} be as in Lemma 5.1. Finally, let us denote by $E^1(g, \tau)$ the following set*

$$(103) \quad E^1(g, \tau) = \left\{ x : \sum_{jk} |g^{jk} - a^{jk}| < \tau \right\}.$$

Set τ to be as follows

$$(104) \quad \tau = \left(\frac{36}{|\mathbb{T}^3|_g \Lambda} \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

We have that if $\tau < 1$, then

$$(105) \quad |E^1(g, \tau)|_g \geq \frac{1}{2} |\mathbb{T}^3|_g.$$

Proof. The proof is shorter than the statement. After summing (100) over the indices j, k we can apply Chebyshev's inequality to the result to obtain

$$(106) \quad |E^1(g, \tau)^c|_g \leq \frac{1}{\tau} \frac{18}{|\mathbb{T}^3|_g \Lambda} \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}}$$

$$(107) \quad = \frac{\tau}{2} |\mathbb{T}^3|_g.$$

Thus, if $\tau < \frac{1}{2}$, we get the result. $\square$

Once we have a good subset to begin with, we can use it to show that there exists an open connected submanifold with smooth boundary on which $|g^{jk} - a^{jk}|$ is controlled.

Lemma 5.3. *Let g be a metric on $\mathbb{T}^3$ be such that $\text{IN}_1(g) \geq \Lambda$, let $\mathbb{U} : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a smooth map, let u^j be the components of $\mathbb{U}$, let g^{jk} denote $g(du^j, du^k)$, and let a^{jk} be as in Lemma 5.1. Next, let us denote by $E^2(g, \tau)$ the following set*

$$(108) \quad E^2(g, \tau) = \left\{ x : \sum_{jk} |g^{jk} - a^{jk}|^2 < \tau^2 \right\}$$

Finally let τ be given by

$$(109) \quad \tau = \left(\frac{36}{|\mathbb{T}^3|_g \Lambda} \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

If $\tau \leq \frac{1}{8}$, then there exists an open connected submanifold $\Omega(g, \tau)$ with smooth boundary and the following properties:

- (1) $\Omega(g, \tau) \subset \{x : \sup_{jk} |g^{jk} - a^{jk}| \leq 2\tau\};$
- (2) $|\Omega(g, \tau)| \geq \frac{1}{2} |\mathbb{T}^3|_g;$
- (3) $|\partial\Omega| \leq 2|\mathbb{T}^3|_g \Lambda \tau;$
- (4) $|\Omega(g, \tau)^c| \leq 2|\mathbb{T}^3|_g \tau.$

Proof. From the coarea formula, we have that

$$(110) \quad \int_{\tau^2}^{4\tau^2} |\partial E^2(g, \sqrt{s})| ds \leq 2 \int_{E^2(g, 2\tau)} \sum_{jk} |g^{jk} - a^{jk}| |\nabla g^{jk}| dV_g.$$

Since we are working inside of $E^2(g, 2\tau)$, it follows that for all j, k we have that $|g^{jk} - a^{jk}| \leq 2\tau$. Therefore, the righthand side of the above is bounded by

$$(111) \quad 36\tau \sup_{jk} \int_{\mathbb{T}^3} |\nabla g^{jk}| dV_g.$$

Therefore, we may apply (99) to see that

$$(112) \quad \int_{\tau^2}^{4\tau^2} |\partial E^2(g, \sqrt{s})| ds \leq 2\tau |\mathbb{T}^3|_g \Lambda \tau^2 = 2|\mathbb{T}^3|_g \Lambda \tau^3.$$

Using Chebyshev's inequality on the above equation shows us that

$$(113) \quad |\{s : |\partial E^2(g, \sqrt{s})| \leq 2|\mathbb{T}^3|_g \Lambda \tau\} \cap [\tau^2, 4\tau^2]| \geq 2\tau^2.$$

Since $\sum_{jk} |g^{jk} - a^{jk}|^2$ is smooth, we may apply Sard's Lemma to conclude that almost every value is regular. In particular, if the image of this function contains all of $[\tau^2, 4\tau^2]$, then we may find a regular value $t_0 \in [\tau^2, 4\tau^2]$ such that

$$(114) \quad |\partial E^2(g, \sqrt{t_0})| \leq 2|\mathbb{T}^3|_g \Lambda \tau.$$

So, suppose that the image of $\sum_{jk} |g^{jk} - a^{jk}|$ does not contain $[\tau^2, 4\tau^2]$. Since the function is continuous, and so has connected image, there is an $\varepsilon > 0$ such that the image lies either in $[0, 4\tau^2 - \varepsilon]$ or in $[4\tau^2, \infty)$.

We now observe that $\sum_{jk} |g^{jk} - a^{jk}|^2 \leq \left(\sum_{jk} |g^{jk} - a^{jk}| \right)^2$, and so

$$(115) \quad E^1(g, \tau) \subset E^2(g, \tau)$$

Since $\tau \leq \frac{1}{8}$ by assumption, we know from Corollary 5.2 that $|E^1(g, \tau)| \geq \frac{1}{2} |\mathbb{T}^3|_g$, and so we must have that $|E^2(g, \tau)| \geq \frac{1}{2} |\mathbb{T}^3|_g$. In this case, we must have that the image of $\sum_{jk} |g^{jk} - a^{jk}|^2$ must lie in $[0, 4\tau^2 - \varepsilon]$. Then, we see that $E^2(g, 2\tau) = \mathbb{T}^3$, and the result is clear.

So, let us continue by supposing that the image of $\sum_{jk} |g^{jk} - a^{jk}|^2$ does indeed contain $[\tau^2, 4\tau^2]$, and use Sard's Lemma to find a t_0 in $[\tau^2, 4\tau^2]$ such that $\partial E^2(g, \sqrt{t_0})$ is smooth and satisfies

$$(116) \quad |\partial E^2(g, \sqrt{t_0})| \leq 2|\mathbb{T}^3|_g \Lambda \tau.$$

$E^2(g, \sqrt{t_0})$ can be decomposed into a union of disjoint connected components, say G_i . Since

$$(117) \quad \nabla \sum_{jk} |g^{jk} - a^{jk}|^2$$

is nonzero and outward pointing everywhere on $\partial E^2(g, \sqrt{t_0})$, it follows that

$$(118) \quad \partial E^2(g, \sqrt{t_0}) = \bigsqcup_i \partial G_i.$$

From this, we immediately see that if there exists an i_0 such that $|G_{i_0}| \geq \frac{1}{2}|\mathbb{T}^3|_g$, then we may take $\Omega(g, \tau)$ to be G_{i_0} .

So, suppose that no such index exists. Then, we have that

$$(119) \quad \frac{1}{2}|\mathbb{T}^3|_g \leq |E(g, \sqrt{t_0})| = \sum_i |G_i| \leq \Lambda^{-1} \sum_i |\partial G_i| = \Lambda^{-1} |\partial E^2(g, \sqrt{t_0})|.$$

However, by the work done above, we know that the last term on the right has the bound

$$(120) \quad \Lambda^{-1} |\partial E^2(g, \sqrt{t_0})| \leq 2|\mathbb{T}^3| \tau.$$

Since we assumed that $\tau \leq \frac{1}{8}$, this last inequality gives us a contradiction. Therefore, there does indeed exist some i_0 such that $|G_{i_0}| \geq \frac{1}{2}|\mathbb{T}^3|_g$, and we let $\Omega(g, \tau) = G_{i_0}$. $\square$

6. CONVERGENCE

In this section we will combine the estimates for smooth functions coming from control on $\text{IN}_1(g)$ with the estimates on harmonic maps coming from the L^2 bounds obtained in terms of stable systoles, and L^3 bounds which result from control on the covering constant of g , see Definition 4.4. This combined control will lead to convergence in the sense of Dong-Song for certain sequences of Riemannian metrics whose negative part of their scalar curvature tends to zero.

We begin by defining the family of metrics on $\mathbb{T}^3$ for which the convergence result will hold.

Definition 6.1. Let $V, R, \Lambda, \sigma, \eta, M > 0$, and define $\mathcal{F} = \mathcal{F}(V, R, \Lambda, \sigma, \eta, M)$ to be the family of Riemannian metrics on $\mathbb{T}^3$ such that for all $g \in \mathcal{F}$ we have that

- (1) $|\mathbb{T}^3|_g \leq V$;
- (2) $\|R_g^-\|_{L^2(g)} \leq R$;
- (3) $\text{IN}_1(g) \geq \Lambda$;
- (4) $\min\{\text{stabsys}_1(g), \text{stabsys}_2(g)\} \geq \sigma$;
- (5) $\kappa(g, \eta) \leq M$.

The results of the previous section imply strong controls on metrics in $\mathcal{F}$. In order to make this clear, we will summarize the results obtained so far as they apply to $\mathcal{F}$.

Notation 6.2. In order to avoid ever expanding equations and terms, we shall denote by B any constant which depends only on $V, R, \Lambda, \sigma, \eta$ and M . It may be that from line to line B will change, increasing to be as large as necessary. This will only happen a finite number of times.

Let us begin with a volume lower bound and the existence of well controlled harmonic maps for the metrics in $\mathcal{F}$.

Proposition 6.3. *There exists a constant $B > 0$ depending only on $V, R, \Lambda, \sigma, \eta$ and M such that for any $g \in \mathcal{F}$ we have*

$$(121) \quad |\mathbb{T}^3|_g \geq B^{-1} > 0.$$

Furthermore, if $\mathbb{U}_g : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ is the harmonic map produced in Corollary 3.3 with components u^j , then for all j we have that

$$(122) \quad \|du^j\|_{L^2(g)} \leq B;$$

$$(123) \quad \int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \leq B \|R^-\|_{L^2(g)};$$

$$(124) \quad \|du^j\|_{L^3(g)} \leq B.$$

Proof. Since $\text{stabsys}_1(g) \leq \text{sys}_1(g)$, the first result follows from Gromov's systolic inequality. The next inequality follows from applying Corollary 3.3 to metrics in $\mathcal{F}$. Then, we have from Stern's inequality in the form of Corollary 2.10 that

$$(125) \quad \int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \leq \int_{\mathbb{T}^3} |du^j| R_g^- dV_g \leq \|du^j\|_{L^2(g)} \|R_g^-\|_{L^2(g)}.$$

The final bound on the L^3 norms of the coordinate functions follows from substituting the above into Lemma 4.7. $\square$

Next, we see that for any $g \in \mathcal{F}$ there is a constant matrix which approximates g in an integral sense, and the quality of the approximation depends on $\|R_g^-\|_{L^2(g)}$.

Proposition 6.4. *Fix $V, R, \Lambda, \sigma, \eta$, and M greater than zero, and for every $g \in \mathcal{F}$ let $\mathbb{U} = \mathbb{U}_g$ be the degree 1 harmonic map $\mathbb{U}_g : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ given in Corollary 3.3, let u^j denote its components, and let $g^{jk} = g(du^j, du^k)$. Then there exists a constant B depending only on $V, R, \Lambda, \sigma, \eta$ and M such that for every $g \in \mathcal{F}$ there exists a symmetric and non-negative matrix a^{jk} with the following properties:*

$$(126) \quad \int_{\mathbb{T}^3} |g^{jk} - a^{jk}| dV_g \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}$$

and

$$(127) \quad \sup_{jk} |a^{jk}| \leq B.$$

Proof. This result will follow from Lemma 5.1 if we can control $\sup_j \|du^j\|_{L^3(g)}$ and $\sup_j \int_{\mathbb{T}^3} \frac{|du^j|^2}{|du^j|} dV_g$ uniformly in terms of $\|R_g^-\|_{L^2(g)}$ and some constant B . Luckily, this is the content of Proposition 6.3. $\square$

In fact, as we show in the following proposition, for any metric $g \in \mathcal{F}$ whose negative part of the scalar curvature is small enough we can find a large connected open sub-manifold with smooth and small boundary on which g^{jk} is uniformly well approximated by a constant matrix. Before diving in, let us recall an elementary estimate on determinants, which will be a useful tool in the proof of Lemma 6.6 below.

Proposition 6.5. *Let a and b be any two $n \times n$ matrices. Then, there exists a constant $C = C(n)$, depending only on n , such that*

$$(128) \quad |\det(a) - \det(b)| \leq C(\|a\| + \|b\|)^{n-1} \|a - b\|.$$

Proof. Let $c(t) = (1-t)a + tb$, and use the cofactor expansion to calculate

$$(129) \quad \frac{d}{dt} \det(c(t)) = \sum_{j=1}^n \left((a-b)_{ij} \text{Cof}(c(t))_{ij} + c(t)_{ij} \frac{d}{dt} \text{Cof}(c(t))_{ij} \right).$$

Therefore, we may use an inductive argument to see that

$$(130) \quad \left| \frac{d}{dt} \det(c(t)) \right| \leq C(n)(\|a\| + \|b\|)^{n-1} \|a - b\|.$$

$\square$

We are now in the position to prove the following Lemma.

Lemma 6.6. Fix $V, R, \Lambda, \sigma, \eta$, and M greater than zero, and for every $g \in \mathcal{F}$ let $\mathbb{U} = \mathbb{U}_g$ be the degree 1 harmonic map $\mathbb{U} : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ given in Corollary 3.3, let u^j denote its components, and let $g^{jk} = g(du^j, du^k)$. For every $g \in \mathcal{F}$ let a^{jk} be the constant symmetric non-negative matrix given in Proposition 6.4. Then, there is a B such that for any $g \in \mathcal{F}$ with

$$(131) \quad \|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$$

there is a connected open submanifold $\Omega(g)$ with smooth boundary, and which satisfies the following properties:

$$(132) \quad \Omega(g) \subset \left\{ x : \sup_{jk} |g^{jk} - a^{jk}| \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}} \right\};$$

$$(133) \quad |\Omega(g)^c| \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}};$$

$$(134) \quad |\partial\Omega| \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}};$$

$$(135) \quad \|d\mathbb{U}\|_{L^3(g; \Omega(g)^c)}^3 \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}};$$

$$(136) \quad 1 + B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}} \geq \int_{\Omega(g)} \det(d\mathbb{U}) dV_g \geq 1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}};$$

$$(137) \quad \int_{\Omega(g)} |\det(d\mathbb{U})| dV_g = \int_{\Omega(g)} \det(d\mathbb{U}) dV_g;$$

$$(138) \quad |\mathbb{U}(\Omega(g)^c)|_h \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}};$$

$$(139) \quad \det(g^{jk})|_{\Omega(g)} \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}}\right)^2 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}};$$

$$(140) \quad \det(a^{jk}) \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}}\right)^2 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

Proof. We begin by proving the first three results. In order to do this, we will apply Lemma 5.3 to metrics in $\mathcal{F}$. Let us recall that this lemma was stated in terms of a parameter τ given as follows:

$$(141) \quad \tau = \left(\frac{36}{|\mathbb{T}^3|_g \Lambda} \sup_{jk} \|du^j\|_{L^3(g)}^{\frac{1}{2}} \|du^k\|_{L^3(g)} \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} dV_g \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

We know from Proposition 6.3 that for τ as above, we have that $\tau \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}$. Thus, the requirement that $\tau \leq \frac{1}{8}$ for Lemma 5.3 follows from assuming that $\|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$, where B is chosen to be sufficiently large. The first three results now follow from Lemma 5.3 and the expression of τ in terms of $B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}$.

We will begin by establishing an estimate for $\|du^j\|_{L^3(g; \Omega(g)^c)}$. This will involve another integration by parts argument, very similar to the one in the proof of Lemma 4.7. Let $\mathbb{V}$ be a test domain for $\kappa(g, \eta)$ as in Definition 4.4. Using the L^2 bound in Proposition 6.3 together with Lemma 4.6, for any j we find a lift of u^j , say $\tilde{u}^j$, such that

$$(142) \quad \sup_{\mathbb{V}_\eta} \tilde{u}^j - \inf_{\mathbb{V}_\eta} \tilde{u}^j \leq B.$$

Let $\hat{u}^j = \tilde{u}^j - \lceil \inf_{\mathbb{V}_\eta} \tilde{u}^j \rceil$. Then, $\hat{u}^j$ is a lift of u^j such that

$$(143) \quad \|\hat{u}^j\|_{L^\infty(\mathbb{V}_\eta)} \leq B.$$

Finally, let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a cutoff function such that

- (1) $\chi_{\mathbb{V}} \leq f \leq 1$;
- (2) $\text{Lip}(f) \leq \eta^{-1}$;
- (3) $\text{supp}(f) \subset \mathbb{V}_\eta$.

Then, letting $\vec{n}$ denote the outwards unit normal to $\partial\pi^{-1}\Omega(g)^c$, we may use integration by parts to calculate as follows:

$$(144) \quad \int_{\Omega(g)^c} |du^j|^3 dV_g \leq \int_{\pi^{-1}(\Omega(g)^c)} \pi^* g(f|d\hat{u}^j|d\hat{u}^j, d\hat{u}^j) dV_{\pi^*g}$$

$$(145) \quad = \int_{\partial\pi^{-1}\Omega(g)^c} \hat{u}^j f|d\hat{u}^j|d\hat{u}^j(\vec{n}) dA_{\pi^*g}$$

$$(146) \quad - \int_{\pi^{-1}\Omega(g)^c} \hat{u}^j \pi^* g(df, |d\hat{u}^j|d\hat{u}^j) dV_{\pi^*(g)}$$

$$(147) \quad - \int_{\pi^{-1}\Omega(g)^c} \hat{u}^j f \pi^* g(d|d\hat{u}^j|, d\hat{u}^j) dV_{\pi^*g}.$$

We now observe that from the properties of the covering map and $\Omega(g)$, we have that

$$(148) \quad \partial\pi^{-1}\Omega(g)^c = \pi^{-1}\partial\Omega(g),$$

where we are looking at the full preimage. Therefore, taking the absolute value of these terms gives us that

$$(149) \quad \|du^j\|_{L^3(g;\Omega(g)^c)}^3 \leq \|\hat{u}^j\|_{L^\infty(\mathbb{V}_\eta)} \int_{\mathbb{V}_\eta \cap \pi^{-1}\partial\Omega(g)^c} |d\hat{u}^j|^2 dA_{\pi^*g}$$

$$(150) \quad + \eta^{-1} \|\hat{u}^j\|_{L^\infty(\mathbb{V}_\eta)} \int_{\mathbb{V}_\eta \cap \pi^{-1}\Omega(g)^c} |d\hat{u}^j|^2 dV_{\pi^*g}$$

$$(151) \quad + \|\hat{u}^j\|_{L^\infty(\mathbb{V}_\eta)} \int_{\mathbb{V}_\eta \cap \pi^{-1}\Omega(g)^c} |d\hat{u}^j| |\nabla d\hat{u}^j| dV_{\pi^*(g)}.$$

It now follows from the definition of $\kappa(g, \eta)$ and the fact that $\hat{u}^j$ covers u^j that we have the following inequalities:

$$(152) \quad \int_{\mathbb{V}_\eta \cap \pi^{-1}\partial\Omega(g)} |d\hat{u}^j|^2 dA_{\pi^*g} \leq \kappa(g, \eta) \int_{\partial\Omega(g)} |du^j|^2 dA_g;$$

$$(153) \quad \int_{\mathbb{V}_\eta \cap \pi^{-1}\Omega(g)^c} |d\hat{u}^j|^2 dV_{\pi^*g} \leq \kappa(g, \eta) \int_{\Omega(g)^c} |du^j|^2 dV_g;$$

$$(154) \quad \int_{\mathbb{V}_\eta \cap \pi^{-1}\Omega(g)^c} |d\hat{u}^j| |\nabla d\hat{u}^j| dV_{\pi^*g} \leq \kappa(g, \eta) \int_{\mathbb{T}^3} |du^j| |\nabla du^j| dV_g.$$

We now recall that $\kappa(g, \eta) \leq M$ for all $g \in \mathcal{F}$ by assumption.

Since $|g^{jj} - a^{jj}|$ is small, see the first property of $\Omega(g)$ in Lemma 6.6, and a^{jj} is bounded, see the second conclusion of Proposition 6.4, we have that $|du^j| \leq B$ on $\partial\Omega(g)^c$. As the area of $\partial\Omega(g)$ is bounded in terms of $\|R_g^-\|_{L^2(g)}^{\frac{1}{4}}$, from the second property of $\Omega(g)$ in Lemma 6.6, we have that

$$(155) \quad \int_{\partial\Omega(g)} |du^j|^2 dA_g \leq |\partial\Omega(g)| B \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

Next, we may apply Hölder's inequality to obtain

$$(156) \quad \int_{\Omega(g)^c} |du^j|^2 dV_g \leq |\Omega(g)^c|^{\frac{1}{3}} \|du^j\|_{L^3(g)}^2.$$

So, from the third property of $\Omega(g)$ appearing in Lemma 6.6 and Proposition 6.3, we see that we have the bound

$$(157) \quad \int_{\Omega(g)^c} |du^j|^2 dV_g \leq B \|R_g^-\|^{\frac{1}{12}}.$$

Finally, rewriting $|du^j||\nabla du^j|$ as $|du^j|^{\frac{3}{2}}|\frac{|\nabla du^j|}{|du^j|^{\frac{1}{2}}}|$ and using Proposition 6.3 we have

$$(158) \quad \int_{\mathbb{T}^3} |du^j||\nabla du^j| dV_g \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

Thus, putting everything together results in the following inequality:

$$(159) \quad \|du^j\|_{L^3(g;\Omega(g)^c)}^3 \leq \kappa(g, \eta) B \left(\|R_g^-\|_{L^2(g)}^{\frac{1}{2}} + 2 \|R_g^-\|_{L^2(g)}^{\frac{1}{2}} \right)$$

$$(160) \quad \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}},$$

where we got the last inequality by absorbing the bound $\kappa(g, \eta) \leq M$ and the fact that $\|R^-\|_{L^2(g)}^{\frac{5}{12}}$ is less than $R^{\frac{5}{12}}$ into the constant B . This gives us (135).

We will now use (135) to estimate $\int_{\Omega(g)} |\det(d\mathbb{U})| dV_g$. To do this, recall that because $\mathbb{U}$ is a degree 1 map, we have that

$$(161) \quad \int_{\mathbb{T}^3} \det(d\mathbb{U}) dV_g = 1.$$

So, we see that

$$(162) \quad \int_{\Omega(g)} \det(d\mathbb{U}) dV_g = 1 - \int_{\Omega(g)^c} \det(d\mathbb{U}) dV_g.$$

Furthermore, we have the estimate $|\det(d\mathbb{U})| \leq |d\mathbb{U}|^3$, so

$$(163) \quad \int_{\Omega(g)^c} |\det(d\mathbb{U})| dV_g \leq \|d\mathbb{U}\|_{L^3(g;\Omega(g)^c)}^3 \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

Combined with the above, we get

$$(164) \quad 1 + B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}} \geq \int_{\Omega(g)} \det(d\mathbb{U}) dV_g \geq 1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}},$$

which is (136).

Next, we can use (163) and the area formula to see that

$$(165) \quad \int_{\mathbb{U}(\Omega(g)^c)} |\mathbb{U}^{-1}\{y\}| dV_h = \int_{\Omega(g)^c} |\det(d\mathbb{U})| dV_g \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

Thus, it follows that

$$(166) \quad |\mathbb{U}(\Omega(g)^c)|_h \leq \int_{\mathbb{U}(\Omega(g)^c)} |\mathbb{U}^{-1}\{y\}| dV_h \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

This gives us (138).

Since $\det(g^{jk}) = \det(d\mathbb{U})^2$, we have that

$$(167) \quad \int_{\Omega(g)} \det(g^{jk}) dV_g \geq \frac{1}{|\Omega(g)|} \left(\int_{\Omega(g)} |\det(d\mathbb{U})| dV_g \right)^2.$$

As $\int_{\Omega(g)} |\det(d\mathbb{U})| dV_g \geq \int_{\Omega(g)} \det(d\mathbb{U}) dV_g$, it follows that

$$(168) \quad \int_{\Omega(g)} \det(g^{jk}) dV_g \geq \frac{1}{|\Omega(g)|} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}} \right)^2.$$

From the mean value inequality, we know that there exists a point $x_0 \in \Omega(g)$ such that

$$(169) \quad \det(g^{jk}(x_0)) \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{2}} \right)^2.$$

In order to turn this into a lower bound for all $x \in \Omega(g)$, we recall the first property of $\Omega(g)$, namely that $|g^{jk} - a^{jk}| \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}$ for all $x \in \Omega(g)$. We may combine this with Proposition 6.5 to see that

$$(170) \quad |\det(g^{jk}(x_0)) - \det(a^{jk})| \leq C(n) (\|g^{jk}\| + \|a^{jk}\|)^{n-1} \|g^{jk} - a^{jk}\|$$

$$(171) \quad \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

One may use the triangle inequality to see that

$$(172) \quad \det(a^{jk}) \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}}\right)^2 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}},$$

and

$$(173) \quad \det(g^{jk}(x)) \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}}\right)^2 - 2B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

Thus, we have (140) and (139), respectively.

Now, recall that $\Omega(g)$ is connected, and so if $\det(d\mathbb{U})$ changes signs on $\Omega(g)$, then there must be a point $x_0 \in \Omega(g)$ such that

$$(174) \quad 0 = \det(d\mathbb{U}(x_0))^2 = \det(g^{jk}(x_0)).$$

Since $|\Omega(g)|^2 \leq |\mathbb{T}^3|_g \leq V$, there is a B depending only on $V, R, \Lambda, \sigma, \eta$ and M such that if

$$(175) \quad \|R_g^-\|_{L^2(g)} \leq \frac{1}{B},$$

then

$$(176) \quad \det(g^{jk}(x)) \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}}\right)^2 - 2B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}} \geq \frac{1}{2}$$

for all $x \in \Omega(g)$. In particular, the point $x_0 \in \Omega(g)$ cannot exist in this case. Therefore, we see that $|\det(d\mathbb{U}(x))| > 0$ for all $x \in \Omega(g)$. As a consequence, we have that

$$(177) \quad \int_{\Omega(g)} |\det(d\mathbb{U})| dV_g = \pm \int_{\Omega(g)} \det(d\mathbb{U}) dV_g.$$

It now follows from (136) that, once again, there is a B depending only on $V, R, \Lambda, \sigma, \eta$ and M such that if

$$(178) \quad \|R_g^-\|_{L^2(g)} \leq \frac{1}{B},$$

then

$$(179) \quad \int_{\Omega(g)} |\det(d\mathbb{U})| dV_g = \int_{\Omega(g)} \det(d\mathbb{U}) dV_g.$$

Thus, we have established (137). This finishes the proof of the result. $\square$

Now that we have established that the set $\Omega(g)$ has quite a few good properties with respect to the metrics $g \in \mathcal{F}$ and harmonic maps $\mathbb{U} : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$, we can begin to show that sequences of metrics whose negative part of their scalar curvatures tend to zero have sub-sequences converging to flat metrics.

Lemma 6.7. *Fix $V, R, \Lambda, \eta, M > 0$, and for every $g \in \mathcal{F}(V, R, \Lambda, \sigma, \eta, M)$ let $\mathbb{U} = \mathbb{U}_g$ be the harmonic map $\mathbb{U} : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ given in Corollary 3.3, let w^j be the components of $\mathbb{U}$, and let g^{jk} denote $g(dw^j, dw^k)$. Finally, for each g in $\mathcal{F}$, let a be the associated symmetric nonnegative matrix as in Proposition 6.4, and let $\Omega(g)$ be the set described in Lemma 6.6. Then, there exists a B depending only on $V, R, \Lambda, \eta, M > 0$ such that for any $g \in \mathcal{F}$ with*

$$(180) \quad \|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$$

we have that there is an open neighborhood $W(g)$ containing $\overline{\Omega}(g)$ such that $\mathbb{U}$ restricted to $\overline{W}(g)$ is injective. In particular, we have that

$$(181) \quad \mathbb{U}(\partial\Omega(g)) = \partial\mathbb{U}(\Omega(g)).$$

Proof. From the fact that $\det(g^{jk}) = \det(d\mathbb{U})$ and (139), we know that $\mathbb{U}$ is a local diffeomorphism around points of $\Omega(g)$ for all g with $\|R_g^-\|_{L^2(g)}$ small enough, by continuity this is actually true of a small connected neighborhood $W(g)$ of $\overline{\Omega}(g)$, and in fact true of the compact closure $\overline{W}(g)$ of $W(g)$. We will begin by showing that $y \mapsto |\mathbb{U}^{-1}\{y\} \cap \overline{W}(g)|$ is continuous on $\mathbb{U}(\overline{W}(g))$, and so locally constant. Fix $y_0 \in \mathbb{U}(\overline{W}(g))$. Since $\overline{W}(g)$ is compact and $\mathbb{U}$ is local diffeomorphism about every point in $\overline{W}(g)$, it must be that $\mathbb{U}^{-1}\{y_0\} \cap \overline{W}(g)$ is finite. As such, there is an open set V_{y_0} about y_0 and open sets G_i for $i = 1, \dots, n_{y_0}$ such that $\mathbb{U}$ is a diffeomorphism from G_i to V_{y_0} , and the G_i are pairwise disjoint and cover $\mathbb{U}^{-1}\{y_0\} \cap \overline{W}(g)$. This shows that for any $y \in V_{y_0}$ we have that $n(y) \geq n(y_0)$.

We will now argue by contradiction that n is actually continuous at y_0 . Suppose this were not the case, then we would be able to find a sequence of y_i converging to y_0 such that $n(y_i) > n(y_0)$ for all i . In particular, this implies that for each i there exists an x_i in $\mathbb{U}^{-1}\{y_i\} \cap \overline{W}(g)$ which is not in $\bigcup_{i=1}^{n(y_0)} G_i$. Since $\overline{W}(g)$ is compact, a subsequence of the x_i , say x_{i_j} , converges to some element x_0 . The continuity of $\mathbb{U}$ implies that $\mathbb{U}(x_0) = y_0$. However, this implies that $x_0 \in \bigcup_{i=1}^{n(y_0)} G_i$. Since $x_{i_j} \rightarrow x_0$, for all j big enough we must have that $x_{i_j} \in \bigcup_{i=1}^{n(y_0)} G_i$. This is a contradiction to how we chose the sequence x_i in the first place, and so we see that $y \mapsto n(y)$ is in fact continuous on $\overline{W}(g)$. Therefore, since $W(g)$, and so $\overline{W}(g)$, is connected, we see that $y \mapsto n(y)$ is in fact constant on $\overline{W}(g)$, and so constant on $W(g)$.

Let $n = n(y_0)$ be the value of n on $\overline{W}(g)$. From the area formula we have that

$$(182) \quad n|\mathbb{U}(\Omega(g))|_h = \int_{\mathbb{U}(\Omega(g))} n(y) dV_h(y) = \int_{\Omega(g)} |\det(d\mathbb{U})| dV_g.$$

Recalling (137) and (136), we get that

$$(183) \quad n|\mathbb{U}(\Omega(g))|_h \leq 1 + B\|R_g^-\|_{L^2(g)}^{\frac{1}{2}}.$$

Furthermore, from (138), we have that

$$(184) \quad |\mathbb{U}(\Omega(g))|_h \geq 1 - B\|R_g^-\|_{L^2(g)}^{\frac{1}{2}},$$

since the fact that $\deg(\mathbb{U}) = 1$ implies that it is surjective, and so $\mathbb{U}(\Omega(g))^c \subset \mathbb{U}(\Omega(g)^c)$. Thus, rearranging terms show us that

$$(185) \quad n \leq \frac{1 + B\|R_g^-\|_{L^2(g)}^{\frac{1}{2}}}{1 - B\|R_g^-\|_{L^2(g)}^{\frac{1}{2}}}.$$

Therefore, there is a constant B depending only on $V, R, \Lambda, \eta, M > 0$ such that if $\|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$, then $n = 1$, and $\mathbb{U}|_{\overline{W}(g)}$ is injective.

Since $\mathbb{U}$ is a local diffeomorphism around every point of $\overline{W}(g)$, it follows from the fact that $\mathbb{U}$ is injective when restricted to $\overline{W}(g)$ that $\mathbb{U}(W(g))$ is an open subset of $\mathbb{T}^3$ and $\mathbb{U}|_{W(g)}$ is a diffeomorphism. Since $\overline{\Omega}(g) \subset W(g)$, we see that $\mathbb{U}(\overline{\Omega}(g)) = \text{cl}(\mathbb{U}(\Omega(g)))$. As such, we have that $\partial\mathbb{U}(\Omega(g)) \subset \mathbb{U}(W(g))$. The result now follows from the fact that $\mathbb{U}|_{W(g)}$ is a diffeomorphism. $\square$

Consider a sequence of metrics $g_i \in \mathcal{F}$ such that $\|R_{g_i}^-\|_{L^2(g_i)}$ converges to zero, and let a_i be the corresponding sequence of matrices which approximate g_i as in Proposition 6.4. As may have been guessed, the constant matrices a_i will be used to show that a subsequence of the metrics g_i converge to a flat metric in the sense of Dong-Song. The following Corollary gives this idea a clearer form.

Corollary 6.8. *Fix $V, R, \Lambda, \sigma, \eta, M > 0$. For every $g \in \mathcal{F}$ let $\mathbb{U} = \mathbb{U}_g : (\mathbb{T}^3, g) \rightarrow (\mathbb{T}^3, h)$ be the harmonic map given in Corollary 3.3, let u^j denote its components, let $g^{jk} = g(du^j, du^k)$, let a be the symmetric and non-negative matrix approximating g^{jk} as in Proposition 6.4, and let $\Omega(g)$ be as in*

Lemma 6.6. *Then, there is a $B > 0$ depending only on $V, R, \Lambda, \sigma, \eta, M$ such that for all $g \in \mathcal{F}$ with $\|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$ we may find a flat metric g_F on $\mathbb{T}^3$ such that*

$$(186) \quad \|g - \mathbb{U}^* g_F\|_{g; \Omega(g)} \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

Proof. From (140) of Lemma 6.6, we know that there is a constant B depending only on $V, R, \Lambda, \sigma, \eta, M$ such that if $\|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$, then the approximating symmetric matrix has the upper bound $\|a\| \leq B$ and lower bound

$$(187) \quad \det(a) \geq \frac{1}{|\Omega(g)|^2} \left(1 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{12}}\right)^2 - B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

Therefore, from (133) and the lower volume bound in Proposition 6.3, there is a B depending only on $V, R, \Lambda, \sigma, \eta$ and M such that if $\|R_g^-\|_{L^2(g)} \leq \frac{1}{B}$, then we have that

$$(188) \quad \det(a) \geq \frac{1}{B}.$$

As such, we see that for some B not depending on $g \in F$, the approximating matrix a for g is invertible with a uniform bound on its inverse:

$$(189) \quad \max\{\|a\|, \|a^{-1}\|\} \leq B.$$

Letting θ^j denote the standard coordinates on $\mathbb{T}^3$, we may use the fact that a is invertible to define the following flat metric on $\mathbb{T}^3$:

$$(190) \quad g_F = (a^{-1})_{st} d\theta^s d\theta^t.$$

We see that

$$(191) \quad \mathbb{U}^* g_F = (a^{-1})_{st} du^s du^t.$$

At this point, we want to estimate $\|g - \mathbb{U}^* g_F\|_{g; \Omega(g)}$. To do this, observe that from the bound on $\max\{\|a\|, \|a^{-1}\|\}$ and from (132) in Lemma 6.6, we have that

$$(192) \quad \max\left\{\|g^{jk}\|_{L^\infty(\Omega(g))}, \|(g^{jk})^{-1}\|_{L^\infty(\Omega(g))}\right\} \leq B.$$

Thus, for any $x \in \Omega(g)$ and any two-form β over the point x , we have that

$$(193) \quad |\beta|_g^2 \leq B \sum_{jk} \beta(\nabla u^j, \nabla u^k).$$

In particular, we see that for any $x \in \Omega(g)$, we have that

$$(194) \quad |g - \mathbb{U}^* g_F|_g^2 \leq B \sum_{jk} (g(x) - \mathbb{U}^* g_F)(\nabla u^j, \nabla u^k)$$

$$(195) \quad \leq B \sum_{jk} (g^{jk} - (a^{-1})_{st} g^{sj} g^{tk})$$

Adding and subtracting a^{sj} gives

$$(196) \quad \sum_{jk} \left(g^{jk} - \left(\delta_t^j + (a^{-1})_{st} (g^{sj} - a^{sj}) \right) g^{tk} \right) = - \sum_{jk} g^{tk} (a)_{st}^{-1} (g^{sj} - a^{sj}).$$

Taking the absolute value of the right hand side, and using the bounds on $\|a^{-1}\|$ and $\|g^{jk}\|_{L^\infty(\Omega(g))}$, along with the estimate in (132), we see that

$$(197) \quad \|g - \mathbb{U}^* g_F\|_{g; \Omega(g)} \leq B \|R_g^-\|_{L^2(g)}^{\frac{1}{4}}.$$

□

The above shows that there is a large set with a small boundary on which metrics in $\mathcal{F}$ with small negative part of their scalar curvature are close to some flat metric in the C^0 sense. This is precisely the setting of Dong-Song's convergence and approximation result [DS23], which we recall here with minor modifications to suite the present situation a little better.

Lemma 6.9. *Let (M, g) be any smooth closed three dimensional Riemannian manifold. For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if Ω is any connected open submanifold of M with smooth boundary such that*

$$(198) \quad |\Omega^c| + |\partial\Omega| \leq \delta,$$

then we may find another open connected submanifold $\tilde{\Omega}$ with smooth boundary which satisfies the following properties:

$$(199) \quad \tilde{\Omega} \subset \Omega;$$

$$(200) \quad |\tilde{\Omega}^c| + |\partial\tilde{\Omega}| \leq \varepsilon;$$

for every $z \in M$ we have that

$$(201) \quad d(z, \tilde{\Omega}) \leq \varepsilon;$$

and for every $x, y \in \tilde{\Omega}$ there exists a curve $\gamma \subset \tilde{\Omega}$ connecting them such that

$$(202) \quad L(\gamma) \leq d^g(x, y) + \varepsilon.$$

The last two conditions imply the following:

$$(203) \quad d_{GH} \left((\tilde{\Omega}, \hat{d}_{\tilde{\Omega}}^g), (M, d^g) \right) \leq 2\varepsilon.$$

With all of the results up to now in hand, we can finally establish the main stability result of this paper.

Theorem 6.10. *Fix $V, R, \Lambda, \eta, M > 0$ and let $\mathcal{F}(V, R, \Lambda, \eta, M)$ be the family of Riemannian metrics on $\mathbb{T}^3$ given in Definition 6.1. Let g_i be a sequence of metrics in $\mathcal{F}$ such that*

$$(204) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0.$$

Then, there is a subsequence, also denoted g_i , and a flat metric g_{F_∞} on $\mathbb{T}^3$ such that g_i converges to g_{F_∞} in the sense of Dong-Song. That is, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $i \geq N$ there is an open submanifold $\tilde{\Omega}_i$ with smooth boundary such that

$$(205) \quad |\tilde{\Omega}_i^c|_{g_i} + |\partial\tilde{\Omega}_i|_{g_i} \leq \varepsilon,$$

and

$$(206) \quad d_{GH} \left((\tilde{\Omega}_i, \hat{d}_{\tilde{\Omega}_i}^{g_i}), (\mathbb{T}^3, d^{g_{F_\infty}}) \right) \leq \varepsilon.$$

Proof. Consider any sequence of metrics $g_i \in \mathcal{F}$ such that

$$(207) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0.$$

Let $\mathbb{U}_i$ denote the harmonic maps $\mathbb{U}_i : (\mathbb{T}^3, g_i) \rightarrow (\mathbb{T}^3, h)$ and let a_i denote the non-negative symmetric matrices which approximate g_i as in Proposition 6.4. Since the terms $\|R_{g_i}^-\|_{L^2(g)}$ are tending towards zero, we may always assume without loss of generality that $\|R_{g_i}^-\|_{L^2(g)}$ is small enough so that Lemma 6.6, Lemma 6.7, and Corollary 6.8 apply to g_i , $\mathbb{U}_i$, a_i , g_{F_i} , and $\Omega_i = \Omega(g_i)$.

Then, we see that on $\Omega_i = \Omega(g_i)$ we have that

$$(208) \quad \|g_i - \mathbb{U}_i^* g_{F_i}\|_{g_i, \Omega_i} \leq \|R_{g_i}^-\|_{L^2(g)}^{\frac{1}{4}}.$$

Furthermore, we have for all i that $\max\{\|a_i\|, \|a_i^{-1}\|\} \leq B$. As such, we see that there is a subsequence $a_{i(m)}$ and a symmetric positive definite matrix a_∞ such that

$$(209) \quad \lim_{m \rightarrow \infty} a_{i(m)} = a_\infty.$$

Letting g_{F_∞} be defined by

$$(210) \quad g_{F_\infty} = (a_\infty^{-1})_{st} d\theta^s d\theta^t,$$

we see that g_{F_i} converges to g_{F_∞} in the C^0 sense. Now, we will estimate

$$(211) \quad \|\mathbb{U}_{i(m)}^*(g_{F_{i(m)}} - g_{F_\infty})\|_{g_{i(m)}, \Omega_{i(m)}}.$$

From the bound on $\max\{\|a_i\|, \|a_i^{-1}\|\}$ and (132), we see that

$$(212) \quad \max\left\{\|g_{i(m)}^{jk}\|_{L^\infty(\Omega_{i(m)})}, \|(g_{i(m)}^{jk})^{-1}\|_{L^\infty(\Omega_{i(m)})}\right\} \leq B$$

So, we have that for all $x \in \Omega_i$ that

$$(213) \quad \|\mathbb{U}_{i(m)}^*(g_{F_{i(m)}} - g_{F_\infty})\|_{g_{i(m)}}^2 \leq B \sum_{jk} \mathbb{U}_{i(m)}^*(g_{F_i} - g_{F_\infty}) \left(\nabla u_{i(m)}^j, \nabla u_{i(m)}^k \right)$$

$$(214) \quad \leq B \sum_{jk} \left((a_{i(m)}^{-1})_{st} - (a_\infty^{-1})_{st} \right) g_{i(m)}^{sj} g_{i(m)}^{tk}.$$

So, combined with the bounds on $\max\{\|a_i\|, \|a_i^{-1}\|\}$ and (212), the above shows that

$$(215) \quad \lim_{m \rightarrow \infty} \|\mathbb{U}_{i(m)}^*(g_{F_{i(m)}} - g_{F_\infty})\|_{g_{i(m)}, \Omega_{i(m)}} = 0.$$

Therefore, using the triangle inequality, we see that

$$(216) \quad \lim_{m \rightarrow \infty} \|g_{i(m)} - \mathbb{U}_{i(m)}^* g_{F_\infty}\|_{g_{i(m)}, \Omega_{i(m)}} = 0.$$

Now, we recall that because $\lim_{i \rightarrow \infty} \|R_g^-\|_{L^2(g)} = 0$, we have that Lemma 6.7 applies to g_i for all i large enough. In particular, we may assume without loss of generality that $\mathbb{U}_{i(m)}|_{\Omega_{i(m)}}$ is injective for all m . Thus, it follows from (216) that

$$(217) \quad \lim_{m \rightarrow \infty} \max \left\{ \text{Lip}_{d_{\Omega_{i(m)}}^{g_{i(m)}}} (\mathbb{U}_{i(m)}), \text{Lip}_{d_{\mathbb{U}_{i(m)}(\Omega_{i(m)})}^{g_{F_\infty}}} (\mathbb{U}_{i(m)}^{-1}) \right\} = 1$$

From the fact that g_{F_∞} and h are two fixed Riemannian metrics on $\mathbb{T}^3$, they are uniformly comparable in the sense that there is a constant C such that for any $p \in \mathbb{T}^3$ and any $\nu \in T_p \mathbb{T}^3$ we have that

$$(218) \quad \frac{1}{C} \leq \frac{g_{F_\infty}(\nu, \nu)}{h(\nu, \nu)} \leq C$$

In particular, for any subset $W \subset \mathbb{T}^3$ we have that

$$(219) \quad \frac{1}{C^3} \leq \frac{|W|_{g_{F_\infty}}}{|W|_h} \leq C^3.$$

Since each $\mathbb{U}_{i(m)}$ has degree 1, and is therefore necessarily surjective by Proposition 3.4, we have that $\mathbb{U}_{i(m)}(\Omega_{i(m)})^c \subset \mathbb{U}_{i(m)}(\Omega_{i(m)}^c)$. Therefore, equation (133) of Lemma 6.6 shows that

$$(220) \quad |\mathbb{U}_{i(m)}(\Omega_{i(m)})|_{g_{F_\infty}} \leq B \|R_g^-\|_{L^2(g_{i(m)})}^{\frac{1}{4}}.$$

Similarly, for any 2 dimensional smooth submanifold $\Sigma \subset \mathbb{T}^3$ we have that

$$(221) \quad \frac{1}{C^2} \leq \frac{|\Sigma|_{g_{F_\infty}}}{|\Sigma|_h} \leq C^2,$$

and so Lemma 6.7, along with the bounds on $\left| du_{i(m)}^j \right|_{g_{i(m)}}$ for points in $\Omega_{i(m)}$, implies that

$$(222) \quad |\partial \mathbb{U}(\Omega_{i(m)})|_{g_{F_\infty}} = |\mathbb{U}(\partial \Omega_{i(m)})|_{g_{F_\infty}} \leq B |\partial \Omega_{i(m)}|_{g_{i(m)}} \leq B \|R_{g_{i(m)}}^-\|_{L^2(g_{i(m)})}^{\frac{1}{4}}.$$

Therefore, we may apply Lemma 6.9 to conclude that for all m large enough we may find open submanifolds $\Omega'_{i(m)} \subset \mathbb{U}_{i(m)}(\Omega_{i(m)})$ with smooth boundary so that

$$(223) \quad \lim_{m \rightarrow \infty} |\Omega'_{i(m)}|_{g_{F\infty}} = 0$$

$$(224) \quad \lim_{m \rightarrow \infty} |\partial\Omega'_{i(m)}|_{g_{F\infty}} = 0$$

$$(225) \quad \lim_{m \rightarrow \infty} d_{GH} \left(\left(\Omega'_{i(m)}, \hat{d}_{\Omega'_{i(m)}}^{g_{F\infty}} \right), (\mathbb{T}^3, d^{g_{F\infty}}) \right) = 0.$$

By the above work, we may also conclude that for all m large enough, we may set $\tilde{\Omega}_{i(m)} = \mathbb{U}_{i(m)}^{-1} \Omega'_{i(m)}$ and obtain

$$(226) \quad \lim_{m \rightarrow \infty} |\tilde{\Omega}_{i(m)}|_{g_{i(m)}} = 0$$

$$(227) \quad \lim_{m \rightarrow \infty} |\partial\tilde{\Omega}_{i(m)}|_{g_{i(m)}} = 0$$

$$(228) \quad \lim_{m \rightarrow \infty} d_{GH} \left(\left(\tilde{\Omega}_{i(m)}, \hat{d}_{\tilde{\Omega}_{i(m)}}^{g_{i(m)}} \right), \left(\Omega'_{i(m)}, \hat{d}_{\Omega'_{i(m)}}^{g_{F\infty}} \right) \right) = 0.$$

where the map $\mathbb{U}_{i(m)}$ gives the estimate on the above estimate on the Gromov-Hausdorff distance. Therefore, the result now follows from the triangle inequality and Equation (217). $\square$

The convergence results stated in the introduction will now follow if we can show that the two mentioned families of metrics both lie in $\mathcal{F}(V, R, \lambda, \sigma, \eta, M)$ for some values of $V, R, \lambda, \sigma, \eta$, and M .

Theorem 6.11. *Let $\sigma, K, V, D > 0$, and define $\mathcal{R} = \mathcal{R}(\sigma, K, D)$ to be the collection of Riemannian metrics g on $\mathbb{T}^3$ such that*

- (1) $\min\{\text{stsys}_1(g), \text{stsys}_2(g)\} \geq \sigma$;
- (2) $\text{Ric}_g \geq -K$;
- (3) $\text{diam}_g(\mathbb{T}^3) \leq D$.

Then, for any sequence of metrics $\{g_i\}_{i=1}^\infty \subset \mathcal{R}(\sigma, K, V)$ such that

$$(229) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0,$$

there exists a subsequence $\{g_{i_j}\}_{j=1}^\infty$ and a flat metric $g_{F\infty}$ on $\mathbb{T}^3$ such that $g_{i_j} \rightarrow g_{F\infty}$ in the sense of Dong-Song.

Proof. Once we show that $\mathcal{R}(\sigma, K, V) \subset \mathcal{F}(V_0, R_0, \Lambda_0, \sigma_0, \eta_0, M_0)$ for some values $V_0, R_0, \Lambda_0, \sigma_0, \eta_0, M_0$, then we are done. Using volume comparison and the diameter bound, we may find an appropriate V_0 . Furthermore, we may take $\sigma_0 = \sigma$, by definition. It is then standard theory for smooth manifolds with Ricci curvature lower bounds that $\text{IN}_1(g) \geq \Lambda_0$, where Λ_0 depends only on K and $\text{diam}_g(\mathbb{T}^3)$, see for example [Ber03, Theorem 114], [Gal88, Theorem 3], [PS98, Page 294]. Furthermore, see [DWZ18] for more related results.

Next, we have that $R_g \geq -nK$, so we may take $R_0 = nKV_0^{\frac{1}{2}}$. It remains to show that we can find $\eta_0 > 0$ and M_0 such that for all $g \in \mathcal{R}$ we have that

$$(230) \quad \kappa(g, \eta_0) \leq M.$$

Fix $a \in \mathbb{T}^3$ and $\hat{a}_0 \in \pi^{-1}\{a\}$, and let $\text{Dir}(\hat{a}_0)$ be the fundamental domain about $\hat{a}_0$ given in Lemma 7.11. We claim that $\text{diam}(\text{Dir}(\hat{a}_0)) \leq 2\text{diam}_g(\mathbb{T}^3)$. To see this, let $\hat{y}$ be an arbitrary element of $\text{Dir}(\hat{a}_0)$. By its construction we see that for any $\nu \in \mathbb{Z}^3$ we must have $d(\hat{a}_0, \hat{y}) \leq d(\hat{a}_\nu, \hat{y})$. Let γ be any length minimizing geodesic connecting a to $y = \pi(\hat{y})$, and let $\tilde{\gamma}$ be the geodesic lifting γ starting at $\hat{y}$. Then, we see that there is an a_{ν_0} such that

$$(231) \quad d(\hat{a}_{\nu_0}, \hat{y}) = d(a, y) \leq \text{diam}_g(\mathbb{T}^3).$$

But, then we must have that

$$(232) \quad d(\hat{a}_0, \hat{y}) \leq d(\hat{a}_{\nu_0}, \hat{y}) \leq \text{diam}_g(\mathbb{T}^3).$$

This gives the claim. As such, for all $g \in \mathcal{R}$, we have that

$$(233) \quad \text{diam}(\text{Dir}(\hat{a}_0)) \leq 2D.$$

Once again, let $\hat{y}$ be an arbitrary element of $\text{Dir}(\hat{a}_0)$, let $\eta_0 = \frac{\sigma}{100}$, and let

$$(234) \quad J_{\hat{y}} = \{\nu \in \mathbb{Z}^3 : B\left(\hat{y}, \frac{\sigma}{100}\right) \cap \text{Dir}(\hat{a}_\nu) \neq \emptyset\}.$$

For all $\nu \in J_{\hat{y}}$, let

$$(235) \quad \hat{z}_\nu \in B\left(\hat{y}, \frac{\sigma}{100}\right) \cap \text{Dir}(\hat{a}_\nu).$$

Then, we have that

$$(236) \quad d(\hat{y}, \hat{a}_\nu) \leq d(\hat{z}_\nu, \hat{a}_\nu) + \frac{\sigma}{100} \leq d(\hat{z}_\nu, \hat{a}_0) + \frac{\sigma}{100}$$

$$(237) \quad \leq d(\hat{y}, \hat{a}_0) + \frac{\sigma}{50}.$$

In particular, for all $\nu \in J_{\hat{y}}$ we have that

$$(238) \quad d(\hat{a}_\nu, \hat{a}_0) \leq 2D + \frac{\sigma}{25}.$$

Furthermore, as $\text{stabsys}_1(g) \leq \text{sys}_1(g)$, for all $\mu, \nu \in J$ we must have that

$$(239) \quad d(\hat{a}_\mu, \hat{a}_\nu) \geq \sigma,$$

and so $B(\hat{a}_\mu, \frac{\sigma}{4}) \cap B(\hat{a}_\nu, \frac{\sigma}{4}) = \emptyset$. Finally, from the diameter upper bound, and volume lower bound given by the systolic inequality and the lower bound on σ , we see that there are constants p_0 and P_0 not depending on $g \in \mathcal{R}$ such that any ball of radius $\frac{\sigma}{4}$ has volume at least p_0 . Thus, we see that

$$(240) \quad |J_{\hat{y}}| p_0 \leq |B(\hat{a}_0, 2D + \sigma)|_{\pi^*g} \leq P_0.$$

Rearranging terms shows that

$$(241) \quad |J_{\hat{y}}| \leq \frac{P_0}{p_0}.$$

Now, by the doubling property and the volume upper bound, we may find a cover of $\text{Dir}(\hat{a}_0)$ by at most G balls of radius $\frac{\sigma}{100}$, where G does not depend on $g \in \mathcal{R}$. Any such cover will also cover $\text{Dir}(\hat{a}_0)_{\frac{\sigma}{200}}$. In particular, we see that

$$(242) \quad \kappa(g, \frac{\sigma}{200}) \leq \frac{GP_0}{p_0}.$$

We can now apply Theorem 6.10 to $\mathcal{R}$ to obtain the result. $\square$

The result also follows for families of Riemannian metrics which have a uniform lower bound in terms of some background metric on $\mathbb{T}^3$, as in the following theorem.

Theorem 6.12. *Let g_0 be a fixed Riemannian metric on $\mathbb{T}^3$, and let $\Lambda, R, V > 0$. We denote by $\mathcal{V}(g_0, \Lambda, V)$ the collection of Riemannian metrics g on $\mathbb{T}^3$ satisfying the following properties:*

- (1) $g \geq g_0$;
- (2) $\text{IN}_1(g) \geq \Lambda$;
- (3) $\|R_g^-\|_{L^2(g)} \leq R$;
- (4) $|\mathbb{T}^3|_g \leq V$.

Then, for any sequence of metrics $\{g_i\}_{i=1}^\infty \subset \mathcal{V}(g_0, \Lambda, V)$ such that

$$(243) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0,$$

there exists a subsequence $\{g_{i_j}\}_{j=1}^\infty$ and a flat metric g_{F_∞} on $\mathbb{T}^3$ such that $g_{i_j} \rightarrow g_{F_\infty}$ in the sense of Dong-Song.

Proof. We need to show that there are $V_0, R_0, \Lambda_0, \sigma_0, \eta_0$, and M_0 such that $\mathcal{V} \subset \mathcal{F}$. We may immediately take $V_0 = V, R_0 = R$, and $\Lambda_0 = \Lambda$. Since $g \geq g_0$, we have that

$$(244) \quad \min\{\text{stabsys}_1(g), \text{stabsys}_2(g)\} \geq \min\{\text{stabsys}_1(g_0), \text{stabsys}_2(g_0)\}.$$

Therefore, we may set $\sigma_0 = \min\{\text{stabsys}_1(g_0), \text{stabsys}_2(g_0)\}$.

Let $\eta_0 > 0$ be arbitrary, then there is a M_0 such that $\kappa(g_0, \eta_0) \leq M_0$. Let $\mathbb{V}$ be a test domain for $\kappa(g_0, \eta_0)$. Since

$$(245) \quad \{x : d^g(x, \mathbb{V}) < \eta_0\} \subset \{x : d^{g_0}(x, \mathbb{V}) < \eta_0\},$$

it follows that $\kappa(g, \eta_0) \leq \kappa(g_0, \eta_0)$. Therefore, we may take M_0 for our final piece of the puzzle, and apply Theorem 6.10 to get the result. $\square$

If in addition to a metric lower bound the sequence has volumes converging to the volume of the metric lower bound, then we actually have that g_0 is flat.

Corollary 6.13. *Let g_0 be a fixed Riemannian metric on $\mathbb{T}^3$, let $\Lambda, R, V > 0$, and let $\mathcal{V} = \mathcal{V}(g_0, \Lambda, R, V)$ denote the collection of Riemannian metrics g on $\mathbb{T}^3$ which satisfy the following properties:*

- (1) $g \geq g_0$;
- (2) $\text{IN}_1(g) \geq \Lambda$;
- (3) $\|R_g^-\|_{L^2(g)} \leq R$;
- (4) $|\mathbb{T}^3|_g \leq V$.

Suppose that g_i is a sequence of metrics in $\mathcal{V}$ such that

$$(246) \quad |\mathbb{T}^3|_{g_i} \rightarrow |\mathbb{T}^3|_{g_0},$$

and

$$(247) \quad \lim_{i \rightarrow \infty} \|R_{g_i}^-\|_{L^2(g_i)} = 0.$$

Then, the metric g_0 must be flat.

Proof. By Theorem 6.12, we know that a subsequence of $\{g_i\}_{i=1}^\infty$, also denoted $\{g_i\}_{i=1}^\infty$, converges to a flat metric g_F in the Dong-Song sense. Let $\mathbb{U}_i$ and $\tilde{\Omega}_i$ be as in Theorem 6.12:

$$(248) \quad \lim_{m \rightarrow \infty} \left(|\tilde{\Omega}_i^c|_{g_i} + |\partial \tilde{\Omega}_i|_{g_i} \right) = 0,$$

$\mathbb{U}_i|_{\tilde{\Omega}_i}$ is injective, and it induces

$$(249) \quad \lim_{m \rightarrow \infty} d_{\text{GH}} \left(\left(\tilde{\Omega}_{i(m)}, d_{\tilde{\Omega}_i}^{g_i} \right), (\mathbb{T}^3, d^{g_F}) \right) = 0.$$

We will now observe several consequences of the inequality $g_i \geq g_0$:

$$(250) \quad \hat{d}_{\tilde{\Omega}_i}^{g_i} \geq \hat{d}_{\tilde{\Omega}_i}^{g_0};$$

$$(251) \quad |\tilde{\Omega}_i^c|_{g_i} \geq |\tilde{\Omega}_i^c|_{g_0}.$$

From (250) above we have the following sequence of inequalities:

$$(252) \quad d^{g_0}|_{\tilde{\Omega}_i \times \tilde{\Omega}_i} \leq \hat{d}_{\tilde{\Omega}_i}^{g_0} \leq \hat{d}_{\tilde{\Omega}_i}^{g_i}.$$

Furthermore, from (251) we have that

$$(253) \quad \lim_{i \rightarrow \infty} |\tilde{\Omega}_i^c|_{g_0} = 0.$$

Since g_0 is a smooth Riemannian manifold, balls have a lower bound on their volume growth. Combined with the above limit, this shows that the inclusion map gives

$$(254) \quad \lim_{i \rightarrow \infty} d_{\text{GH}} \left(\left(\tilde{\Omega}_i, d_{\tilde{\Omega}_i}^{g_i} \right), (\mathbb{T}^3, d^{g_0}) \right) = 0.$$

Next for each i , the identity map

$$(255) \quad \text{Id} : (\tilde{\Omega}_i, d_{\tilde{\Omega}_i}^{g_i}) \rightarrow (\tilde{\Omega}_i, d^{g_0})$$

is a Lipschitz one map. Since

$$(256) \quad \lim_{m \rightarrow \infty} d_{\text{GH}} \left(\left(\tilde{\Omega}_{i(m)}, \hat{d}_{\tilde{\Omega}_i}^{g_i} \right), (\mathbb{T}^3, d^{g_F}) \right) = 0.$$

and

$$(257) \quad \lim_{i \rightarrow \infty} |\tilde{\Omega}_i^c|_{g_0} = 0,$$

it follows from the Arzela-Ascoli theorem that there is a Lipschitz one map

$$(258) \quad F : (\mathbb{T}^3, h) \rightarrow (\mathbb{T}^3, g_0).$$

However, from the definition of g_F , we have that $|\mathbb{T}^3|_{g_F} = \lim_{i \rightarrow \infty} |\mathbb{T}^3|_{g_i}$. Furthermore, it was an hypothesis that $\lim_{i \rightarrow \infty} |\mathbb{T}^3|_{g_i} = |\mathbb{T}^3|_{g_0}$. Therefore, we must have that the map F is an isometry, and so g_0 is flat. $\square$

7. APPENDIX

7.1. Dong-Song Curve Approximation. The goal of this section is to prove the following approximation lemma. The proof presented here is a minor modification of the proof found in [DS23].

Lemma 7.1. *Let (M^3, g) be any smooth closed three dimensional Riemannian manifold. For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if Ω is a connected open sub-manifold of M with smooth boundary such that $|\Omega|^c + |\partial\Omega| \leq \varepsilon$, then we may find $\tilde{\Omega} \subset \Omega$, another open sub-manifold with smooth boundary, such that $|\tilde{\Omega}^c| + |\partial\tilde{\Omega}| \leq \varepsilon$, and for every $x, y \in \tilde{\Omega}$ there exists a curve $\gamma \subset \tilde{\Omega}$ connecting them such that*

$$(259) \quad L(\gamma) \leq d(x, y) + \varepsilon.$$

Lemma 7.2. *Let $\eta, \Lambda > 0$ be fixed, let (M, g) be a compact smooth n -dimensional Riemannian manifold, and suppose that $\text{IN}_1(M, g) \geq \Lambda$. Then, there exists a $\delta > 0$ depending only on η and Λ such that if E is a smooth n -dimensional submanifold of M such that $|E| \geq \eta$ and $|\partial E| \leq \delta$, then E has a connected component, say Ω , such that*

$$(260) \quad |\Omega^c| \leq \frac{1}{\Lambda} |\partial\Omega| \leq \frac{1}{\Lambda} |\partial E|.$$

Proof. Let E_i be the connected components of E . Since E is a smooth n dimensional sub manifold, it follows that

$$(261) \quad \partial E = \bigsqcup_i \partial E_i.$$

Suppose that there is an i_0 such that $|E_{i_0}| \geq \frac{1}{2} |M|$. Then, taking $\Omega = E_{i_0}$, we have from the definition of $\text{IN}_1(M, g)$ that

$$(262) \quad |\Omega^c| \leq \frac{1}{\Lambda} |\partial\Omega|.$$

Suppose on the contrary that there is no such i_0 : for all i we have that $|E_i| < \frac{1}{2} |M|$. Then, it follows from the definition of $\text{IN}_1(M, g)$ that

$$(263) \quad \eta \leq \sum_i |E_i| \leq \frac{1}{\Lambda} \sum_i |\partial E_i| = \frac{|\partial E|}{\Lambda}.$$

Choosing $|\partial E| < \eta\Lambda$ gives a contradiction, and hence we obtain the result. $\square$

Lemma 7.3. *Let (M, g) be a smooth, compact, closed, two-dimensional Riemannian manifold. Let A be a two-dimensional sub manifold of M with smooth boundary. Furthermore, suppose that every component of ∂A bounds a two-dimensional submanifold of M , and that A is connected. Then, for any x, y in A there is a curve γ lying entirely in A connecting x to y such that*

$$(264) \quad L(\gamma) \leq d(x, y) + |\partial A|.$$

Proof. By our hypotheses, we have that $M \setminus A$ is the disjoint and finite union of connected open two dimensional submanifolds B_i , with smooth boundaries ∂B_i . We claim that the boundaries ∂B_i must be connected, and so correspond to the connected components of ∂A . For any i , suppose that Σ_1 and Σ_2 are two connected components of ∂B_i . Such connected components can only bound connected regions. Therefore, without loss of generality, we may find disjoint connected regions Ω_1 and Ω_2 with boundaries Σ_1 and Σ_2 such that $\Omega_1 \cap B_i = \Omega_2 \cap B_i = \emptyset$ and such that $\Omega_1 \cap A \neq \emptyset$ as well as $\Omega_2 \cap A \neq \emptyset$. Let $x \in \Omega_1 \cap A$ and $y \in \Omega_2 \cap A$. Then, any curve connecting x to y must pass through B_i , which contradicts the assumption that A is connected.

Let γ_0 be a unit speed length minimizing geodesic connecting x to y . We will modify γ_0 so that it lies entirely in A . First, we let s_1 be the first time that γ_0 lies in $\overline{B}_1$ and we let t_1 be the last time that γ_0 lies in $\overline{B}_1$. Since ∂B_i is connected for all i , we may connect $\gamma_0(s_1) \in \partial B_1$ to $\gamma_0(t_1) \in \partial B_1$ by a curve c_1 lying in ∂B_1 which has length less than $|\partial B_1|$, though is not necessarily unit speed. We define γ_1 to be equal to γ_0 on $[0, L(\gamma_0)] \setminus [s_1, t_1]$ and equal to c_1 on $[s_1, t_1]$. We produce γ_{i+1} inductively as follows. Let s_{i+1} be the first time that γ_i lies in $\overline{B}_{i+1}$, and let t_{i+1} be the last time γ_i lies in $\overline{B}_{i+1}$. Since ∂B_i is connected for all i , we may connect $\gamma_i(s_{i+1}) \in \partial B_{i+1}$ to $\gamma_i(t_{i+1}) \in \partial B_{i+1}$ by a curve c_{i+1} lying in ∂B_{i+1} which has length less than $|\partial B_{i+1}|$. Then we let γ_{i+1} be γ_i on the interval $[0, L(\gamma_0)] \setminus [s_{i+1}, t_{i+1}]$, and equal to c_{i+1} on $[s_{i+1}, t_{i+1}]$. Since there are only finitely many regions B_i to consider, this process terminates at some γ_P . We then have that

$$(265) \quad L(\gamma_P) \leq L(\gamma_0) + \sum_{i=1}^P |\partial B_i| \leq d(x, y) + |\partial A|.$$

□

Lemma 7.4. *Let (M^3, g) be a smooth, compact, closed, three-dimensional Riemannian manifold. For any $L > 1$ let r_L be such that balls with radius less than or equal to r_L are geodesically convex, and the exponential map is L bi-Lipschitz. Let r be such that $24r \leq r_L$. Then, there is a constant $\delta(r)$, depending only on r , such that if $|\Omega^c| + |\partial\Omega| \leq \delta(r)$, then the following is true. In every ball $B(a, 20r)$ we may find a connected subset $D_a \subset \Omega \cap B(a, 20r)$ with the following properties.*

- (1) *For all $p, q \in D_a$ there is a curve $\gamma(p, q) = \gamma$ connecting them, which lies entirely in D_a . Furthermore, we have that*

$$L(\gamma) \leq 96r + 8rL\pi + 2r^{-1} |B(a, 20r) \cap \partial\Omega|.$$

- (2) *We have that*

$$(266) \quad |B(a, 20r) \setminus D_a| \leq 2(24^3 L^5) \left(\frac{2L^5}{\text{IN}_1(\mathbb{S}^2)} + L^4 \right) |B(a, 20r) \cap \partial\Omega|.$$

Proof. Let $B(a, 20r)$ be an arbitrary ball of radius $20r$, and let $\nu \in T_a M$ be an arbitrary unit vector. Set $x = \exp_a(-4r\nu)$ and $y = \exp_a(4r\nu)$. For the moment, we will focus our attention to a neighborhood of x . Let $f : \partial\Omega \rightarrow \mathbb{R}$ be given by $f(z) = d(z, x)$, and observe that f is smooth on $(B(x, 24r) \setminus B(x, r)) \cap \partial\Omega$. From the coarea formula, we have that

$$(267) \quad \int_r^{2r} |f^{-1}\{t\}| dt = \int_{f^{-1}[r, 2r]} |\nabla f| dA_g \leq |B(x, 2r) \cap \partial\Omega|.$$

Therefore, using Sard's Lemma and a mean-value inequality, we may find $\sigma_x \in [r, 2r]$ such that $\partial B(x, \sigma_x) \cap \partial\Omega = f^{-1}\{\sigma_x\}$ is a smooth submanifold of $\partial\Omega$ and

$$(268) \quad |\partial B(x, \sigma_x) \cap \partial\Omega| \leq \frac{1}{r} |B(x, 2r) \cap \partial\Omega| \leq \frac{1}{r} |B(a, 20r) \cap \partial\Omega|.$$

We will define two sets on $\partial B(x, \sigma_x)$, estimate their volumes, and then study their interaction. To begin, let $G_x = \Omega \cap \partial B(x, \sigma_x)$, which has smooth boundary given by

$$(269) \quad \partial G_x = \partial(\Omega \cap \partial B(x, \sigma_x)) = \partial\Omega \cap \partial B(x, \sigma_x).$$

From (268), we see that $|\partial G_x| \leq \frac{1}{r} |B(a, 20r) \cap \partial \Omega|$. Since $\exp_x$ is L bi-Lipschitz, it follows that

$$(270) \quad \text{IN}_1(\partial B(x, \sigma_x)) \geq \frac{\text{IN}_1(\partial B(0, \sigma_x))}{L^5} \geq \frac{1}{\sigma_x L^5} \text{IN}_1(\mathbb{S}^2) \geq \frac{1}{2r L^5} \text{IN}_1(\mathbb{S}^2).$$

We eventually want to apply Lemma 7.2 to G_x . In order to do this, it suffices to show that there is an $\eta > 0$ such that for all $|\Omega^c| + |\partial \Omega|$ small enough, we have $|G_x| \geq \eta$.

We may find such an η by estimating the number of radial geodesics which intersect $\partial \Omega$. Let us define $\text{proj}_{\sigma_x} : B(x, 24r) \setminus B(x, \sigma_x) \rightarrow \partial B(x, \sigma_x)$ to be the map projecting radial geodesics onto $\partial B(x, \sigma_x)$. Using $\exp_x$ to compare with the Euclidean case, we see that this map has a Lipschitz constant of at most L^2 . Now, consider $H_x = \text{proj}_{\sigma_x} \left((B(x, 24r) \setminus B(x, r) \cap B(a, 20r) \cap \partial \Omega) \right)$. We see that $|H_x| \leq L^4 |B(a, 20r) \cap \partial \Omega|$. For $\omega \in \partial B(x, \sigma_x)$ we shall abuse notation slightly by identifying ω with the radial geodesic emanating from x , say γ_ω , such that $\gamma_\omega(\sigma_x) = \omega$. We will then write $\omega(t)$ to denote $\gamma_\omega(t)$.

With this convention in mind, we observe that for any $t \in [\sigma_x, 16r]$ and $\omega \in G_x^c \cap H_x^c$ we have that $\omega(t)$ lies in Ω^c . Therefore, using that $\exp_x$ is L bi-Lipschitz, we have the following string of estimates:

$$(271) \quad |\Omega^c| \geq |\{\omega(t) : \omega \in G_x^c \cap H_x^c; t \in [\sigma_x, 16r]\}| \geq \frac{(16r - \sigma_x)}{L^5} |G_x^c \cap H_x^c|$$

$$(272) \quad \geq \frac{14r}{L^5} (|G_x^c| - |H_x|)$$

$$(273) \quad \geq \frac{14r}{L^5} (|G_x^c| - L^4 |B(a, 20r) \cap \partial \Omega|).$$

Rearranging terms and estimating shows us that

$$(274) \quad |\Omega^c| + \frac{14r}{L} |\partial \Omega| \geq \frac{14r}{L^5} |G_x^c|.$$

Since $|\partial B(x, \sigma_x)| \geq L^{-2} |\partial B(0, \sigma_x)| \geq \frac{r^2}{L^2} |\mathbb{S}^2|$, we see that

$$(275) \quad |G_x| \geq \frac{r^2}{L^2} |\mathbb{S}^2| - \left(\frac{L^5}{14r} |\Omega^c| + L^4 |\partial \Omega| \right)$$

Letting $\eta = \frac{r^2}{2L^2} |\mathbb{S}^2|$, we see that for all $|\Omega^c| + |\partial \Omega|$ small enough, depending only on r and L , we have that $|G_x| \geq \eta$. Therefore, we may apply Lemma 7.2 to G_x to find a connected subset $\tilde{G}_x$ with smooth boundary such that

$$(276) \quad |\partial \tilde{G}_x| \leq |\partial G_x| \leq \frac{1}{r} |B(a, 20r) \cap \partial \Omega|$$

and

$$(277) \quad |G_x^c| \leq \frac{|\partial \tilde{G}_x|}{\text{IN}_1(\partial B(x, \sigma_x))} \leq \frac{2L^5}{\text{IN}_1(\mathbb{S}^2)} |B(a, 20r) \cap \partial \Omega|.$$

Finally, we observe that since $\mathbb{S}^2$ is simply connected, and $\tilde{G}_x$ is connected, we may apply Lemma 7.3 to $A = \text{cl} \tilde{G}_x$ to obtain the following: for any $\omega, \zeta \in \text{cl} \tilde{G}_x$, there exists a curve $\gamma \subset \text{cl} \tilde{G}_x$ connecting them such that

$$(278) \quad L(\gamma) \leq d(\omega, \zeta) + |\partial \tilde{G}_x| \leq L\sigma_x\pi + \frac{1}{r} |B(a, 20r) \cap \partial \Omega|$$

$$(279) \quad \leq 2rL\pi + \frac{1}{r} |B(a, 20r) \cap \partial \Omega|.$$

Let us denote by D_x the set given below:

$$(280) \quad D_x = \left(\{\omega(t) : \omega \in \tilde{G}_x \cap H_x^c; t \in [\sigma_x, 24r]\} \cap B(a, 20r) \right) \cup \text{cl} \tilde{G}_x.$$

From looking at the definitions, we may see that any two points in D_x may be connected together by a curve in D_x which has length no greater than $48r + 4rL\pi + r^{-1} |B(a, 20r) \cap \partial \Omega|$.

Let us use the exact same construction in a neighborhood of y to construct the set D_y . Ultimately, we wish to set $D_a = D_x \cup D_y$, and conclude that for any two points in D_a , there is a curve lying in D_a which connects them, and which has length no greater than

$$(281) \quad 96r + 8rL\pi + 2r^{-1} |B(a, 20r) \cap \partial\Omega|.$$

In order to do this, it suffices to show that $D_x \cap D_y \neq \emptyset$. To do this, we will show that $|D_x \cap D_y| > 0$. Let us observe that

$$(282) \quad B(a, r) \subset \{\omega(t) : \omega \in \partial B(x, \sigma_x); t \in [\sigma_x, 5r]\}$$

and

$$(283) \quad B(a, r) \subset \{\nu(t) : \nu \in \partial B(y, \sigma_y); t \in [\sigma_y, 5r]\}$$

Therefore, we see that

$$(284) \quad |D_x \cap D_y| \geq |B(a, r)|$$

$$(285) \quad -125L^5 \left(\left| \left(\tilde{G}_x \cap H_x^c \right)^c \right| + \left| \left(\tilde{G}_y \cap H_y^c \right)^c \right| \right)$$

So, we see that

$$(286) \quad |D_x \cap D_y| \geq \frac{|B(0, r)|}{L^3} - 250L^5 \left(\frac{2L^5}{\text{IN}_1(\mathbb{S}^2)} + L^4 \right) |B(a, 20r) \cap \partial\Omega|.$$

Therefore, for all $|\Omega^c| + |\partial\Omega|$ small enough, we see that $D_x \cap D_y \neq \emptyset$, and so $D_a = D_x \cup D_y$ has the desired path connected property given in (281).

Now, we need to estimate $|B(a, 20r) \cap \Omega \setminus D_a|$. To do this, observe that $B(a, 20r)$ is contained in

$$(287) \quad \{\omega(t) : \omega \in \partial B(x, \sigma_x); t \in [\sigma_x, 24r]\} \cup \{\nu(t) : \nu \in \partial B(y, \sigma_y); t \in [\sigma_y, 24r]\}.$$

Therefore, we see that

$$(288) \quad |B(a, 20r) \setminus D_a| \leq \left| \{\omega(t) : \omega \in (\tilde{G}_x \cap H_x^c)^c; t \in [\sigma_x, 24r]\} \right|$$

$$(289) \quad + \left| \{\nu(t) : \nu \in (\tilde{G}_y \cap H_y^c)^c; t \in [\sigma_y, 24r]\} \right|$$

Using the estimates above, we see that

$$(290) \quad |B(a, 20r) \setminus D_a| \leq 2(24^3 L^5) \left(\frac{2L^5}{\text{IN}_1(\mathbb{S}^2)} + L^4 \right) |B(a, 20r) \cap \partial\Omega|.$$

□

Lemma 7.5 ([DS23]). *Let (M^3, g) be a smooth closed three dimensional Riemannian manifold. For every $\varepsilon > 0$ there exists a δ such that if $\Omega \subset M^3$ has smooth boundary, and we have that*

$$(291) \quad |\Omega| \geq |M| - \delta$$

$$(292) \quad |\partial\Omega| \leq \delta,$$

then there exists a connected subset $\Omega' \subset \Omega$ with smooth boundary such that

$$(293) \quad |\Omega'| \geq |M^3| - \varepsilon$$

$$(294) \quad |\partial\Omega'| \leq \varepsilon$$

and

$$(295) \quad d_{\text{GH}}\left((\Omega', \hat{d}_{\Omega'}^h), (\Omega', d^h)\right) \leq \varepsilon.$$

Proof. Fix $L = (1 + \frac{1}{2}\varepsilon)$, and fix $r > 0$ to be chosen later. We will however assume that $24r \leq r_L$, where r_L is the constant appearing in Lemma 7.4. We may additionally assume that $|\Omega^c| + |\partial\Omega|$ is small enough so that Lemma 7.4 applies to balls of radius $20r$. Since M is compact, we may cover M in finitely many balls of the form $B(a_i, r)$;

$$(296) \quad 1 \leq \sum_{i=1}^P \chi_{B(a_i, r)} \leq P.$$

For each i let $D_i \subset \Omega \cap B(a_i, 20r)$ be the set given in Lemma 7.4. Let $D = \bigcup_{i=1}^P D_i$, and consider $x, y \in D$ such that $d(x, y) \leq r$. Since $x, y \in D$, there exists $i(x)$ and $i(y)$ such that $x \in D_{i(x)}$ and $y \in D_{i(y)}$. Furthermore, since $\{B(a_i, r)\}$ covers M , there exists $m(x)$ and $m(y)$ such that $x \in B(a_{m(x)}, r)$ and $y \in B(a_{m(y)}, r)$. Finally there are $z_x \in B(a_{i(x)}, 20r)$ and $z_y \in B(a_{i(y)}, r)$ such that $d(x, z_x) = d(y, z_y) = r$, $B(z_x, r) \subset B(a_{m(x)}, 20r) \cap B(a_{i(x)}, 20r)$, and $B(z_y, r) \subset B(a_{m(y)}, 20r) \cap B(a_{i(y)}, 20r)$. Since $d(x, y) \leq r$, we also have that $B(x, r) \subset B(a_{m(x)}, 20r) \cap B(a_{m(y)}, 20r)$. As before, we can show that for $|\Omega^c| + |\partial\Omega|$ small enough, we have that

$$(297) \quad \min\{|D_{i(x)} \cap D_{m(x)}|, |D_{i(y)} \cap D_{m(y)}|, |D_{m(x)} \cap D_{m(y)}|\} > 0,$$

and in particular the intersections are not empty. As such, we see from Lemma 7.4 that there is a path $\gamma \subset D$ connecting x to y such that

$$(298) \quad L(\gamma) \leq 3r(96 + 8L\pi + 2r^{-2}|\partial\Omega|).$$

Lemma 7.4 also gives us the following volume estimate:

$$(299) \quad |M \setminus D| \leq |\Omega^c| + 2(24^3 L^5) \sum_{i=1}^P \left(\frac{2L^5}{\text{IN}_1(\mathbb{S}^2)} + L^4 \right) |B(a_i, 20r) \cap \partial\Omega|$$

$$(300) \quad \leq |\Omega^c| + 2P(24^3 L^5) \left(\frac{2L^5}{\text{IN}_1(\mathbb{S}^2)} + L^4 \right) |\partial\Omega|.$$

At this point, D satisfies the requirement of local connectivity, but it does not necessarily have a smooth boundary, nor need this boundary be small. Therefore, we must modify D . Following Dong-Song [DS23], let $f : \Omega \rightarrow \mathbb{R}$ be given by

$$(301) \quad f(x) = \hat{d}_\Omega(z, D).$$

Although the Lipschitz constant of f depends on Ω and can be rather large, since Ω is a domain with smooth boundary, the local Lipschitz constant of f is always bounded above by 2, independently of Ω . As such, for any $\eta > 0$, we may find a smooth map $\phi : \Omega \rightarrow \mathbb{R}$ such that $|\nabla\phi| \leq 2$ and $\|f - \phi\|_{L^\infty} < \frac{r}{64}$. Using the coarea formula, we have that

$$(302) \quad \int_{\frac{r}{32}}^{\frac{r}{16}} |\phi^{-1}\{t\} \cap \Omega| dt = \int_{\Omega \cap \phi^{-1}[\frac{r}{32}, \frac{r}{16}]} |\nabla\phi| dV_g \leq 2|\Omega \setminus D|.$$

Therefore, using Sard's lemma and a mean-value inequality, we may find a $t_0 \in [\frac{r}{32}, \frac{r}{16}]$ such that $\phi^{-1}\{t_0\} \subset \Omega$ is a submanifold of M , possibly with corners. Furthermore, we have that

$$(303) \quad |\phi^{-1}\{t_0\}| \leq \frac{32}{r} |\Omega \setminus D|.$$

Observe that $\phi^{-1}\{t_0\}$ bounds the region $\phi^{-1}[0, t_0]$. Let us smooth out $\phi^{-1}\{t_0\}$ in such a way that the result $\partial\Omega'$ bounds a region $\Omega' \subset \phi^{-1}[0, t_0] \subset \Omega$ and

$$(304) \quad |\partial\Omega'| \leq 2|\phi^{-1}\{t_0\}| \leq \frac{64}{r} |\Omega \setminus D|.$$

Next, observe that $D \subset \Omega'$, since $\|f - \phi\|_{L^\infty} < \frac{r}{64}$. Therefore, we have that

$$(305) \quad |\Omega^c| \leq |D^c|.$$

Ω' is very nearly the set we desire, however there is one more crucial property that it may not satisfy: we want to find a set such that if x, y are elements of the set, and $d(x, y) \leq \frac{r}{2}$, then there is a path from x to y in Ω' which isn't too long. Of course, this is true if x and y are in D , but for more general x and y , there is more work to be done. Since Ω and Ω' are domains with smooth boundary, the topology generated by $d(\cdot, \cdot)$, the topology generated by $\hat{d}_\Omega(\cdot, \cdot)$, and the topology generated by $\hat{d}_{\Omega'}(\cdot, \cdot)$ all agree. In particular, we have that Ω' is compact, and furthermore $\text{cl}(\Omega' \setminus D)$ is compact. Therefore, we may find a finite collection of points z_i in $\text{cl}(\Omega' \setminus D)$ which form an $\frac{r}{8}$ net with respect to $\hat{d}_{\Omega'}$. In particular, by the definition of Ω' , for each i we may find a curve c_i lying in Ω connecting z_i to some $\hat{z}_i \in D$ which has length less than or equal to $\frac{r}{8}$. Consider a small cylindrical neighborhood C_i of c_i . We may choose the area of its boundary to be arbitrarily small, and for every point to be within $\frac{r}{8}$ of the center curve, by choosing an arbitrarily small radius. Furthermore, we may perturb its boundary so that it intersects $\partial\Omega$ and $\partial\Omega'$ transversely. We may smooth out

$$(306) \quad \Omega' \bigcup_i (C_i \cap \Omega)$$

to produce a region $\tilde{\Omega}$ with the following properties. First, we may assume that $|\partial\tilde{\Omega}| \leq 2|\partial\Omega'|$. Next, we have that $|\tilde{\Omega}^c| \leq |(\Omega')^c|$. Finally, suppose that x and y are in $\tilde{\Omega}$ such that $d(x, y) \leq r$. Let us begin by assuming that $x, y \in \text{cl}(\tilde{\Omega} \setminus D)$. Then there are z_x and z_y in the $\frac{r}{8}$ net such that $\hat{d}_{\Omega'}(x, z_x)$ and $\hat{d}_{\Omega'}(y, z_y)$ are less than $\frac{r}{8}$. Furthermore, by construction of Ω' , there are $\hat{z}_x$ and $\hat{z}_y$ in D such that $\hat{d}_{\tilde{\Omega}}(z_x, \hat{z}_x)$ and $\hat{d}_{\tilde{\Omega}}(z_y, \hat{z}_y)$ are less than $\frac{r}{8}$. Thus, we see that

$$(307) \quad d(\hat{z}_x, \hat{z}_y) \leq d_{\tilde{\Omega}}(\hat{z}_x, z_x) + d_{\tilde{\Omega}}(z_x, x) + d(x, y) + d_{\tilde{\Omega}}(y, z_y) + d_{\tilde{\Omega}}(z_y, \hat{z}_y)$$

$$(308) \quad \leq r.$$

Let us recall that L denotes the bi-Lipschitz constant of $\exp$ for balls with radius less than or equal to r_L , which are also geodesically convex by assumption. Since $d(\hat{z}_x, \hat{z}_y) \leq r$, it follows from the construction of D , see Lemma 7.4, that we may find a curve γ in D connecting $\hat{z}_x$ and $\hat{z}_y$ which has length bounded as follows:

$$(309) \quad L(\gamma) \leq r(96 + 8L\pi + 2r^{-2}|B(a, 20r) \cap \partial\Omega|).$$

The other cases are similar, but use fewer applications of the triangle inequality.

To save space in the calculations below, we set $K = K(|\partial\Omega|, L, r)$, where

$$(310) \quad K(|\partial\Omega|, L, r) = 96 + 8L\pi + 2r^{-2}|B(a, 20r) \cap \partial\Omega|,$$

Let x, y be any two points in $\tilde{\Omega}$, and let γ be a length minimizing geodesic connecting them. We may split γ into at most $\left\lceil \frac{16 \text{diam}(M, g)}{r_L} \right\rceil$ segments of length l_i , where $\frac{1}{16}r_L \leq l_i \leq \frac{1}{4}r_L$. Let us consider any two successive sections of γ , say $\gamma(x_0, y_0)$ and $\gamma_{i_0+1}(x_1, y_1)$, with endpoints x_j, y_j for $j = 0, 1$, respectively. Let c_0 be the midpoint of $\gamma(x_0, y_0)$, and consider $\tilde{\gamma}_0 = \exp_{c_0}^{-1}(\gamma(x_0, y_0)) \subset T_{c_0}M$. Let $\tilde{C}_r$ be the corresponding cylinder about $\tilde{\gamma}_0$ of radius $\frac{r}{8L}$; it is foliated by curves, as seen below:

$$(311) \quad \tilde{C}_r = \left\{ \tilde{\gamma}_\omega : \tilde{\gamma}_\omega(t) = t + \omega; \omega \perp \frac{d}{dt}\tilde{\gamma}_1(t); |\omega| \leq \frac{r}{8L} \right\}.$$

Consider $\exp_{c_0}^{-1}(\partial\tilde{\Omega})$, and let $H = \{\omega : \tilde{\gamma}_\omega \cap \exp_{c_0}^{-1}(\partial\tilde{\Omega}) \neq \emptyset\}$. By projecting the curve $\tilde{\gamma}_\omega$ onto ω , we observe that

$$(312) \quad |H| \leq \left| \exp_{c_0}^{-1}(\partial\tilde{\Omega}) \right|$$

$$(313) \quad \leq L^2 |\partial\tilde{\Omega}|$$

$$(314) \quad \leq 2L^2 |\partial\Omega'|$$

$$(315) \quad \leq \frac{128}{r} |\Omega \setminus D|.$$

Now observe that for $\omega \in H^c$ we have either that $\exp_{c_0}(\tilde{\gamma}_\omega) \subset \tilde{\Omega}$ or $\exp_{c_0}(\tilde{\gamma}_\omega) \subset \tilde{\Omega}^c$. Suppose that for all $\omega \in H^c$ that $\exp_{c_0}(\tilde{\gamma}_\omega) \subset \tilde{\Omega}^c$. Then, it follows that

$$(316) \quad |\tilde{\Omega}^c| \geq \frac{r_L}{16L^5} \left(\pi \left(\frac{r}{8L} \right)^2 - |H| \right).$$

Rearranging terms shows us that

$$(317) \quad |\tilde{\Omega}^c| + \frac{r_L |H|}{15L^5} \geq \pi \left(\frac{r}{8L} \right)^2.$$

This gives a contradiction for $|\Omega^c| + |\partial\Omega|$ small enough depending only on r and L . Therefore, there is at least one $\omega \in H^c$ such that $\exp_{c_0}(\tilde{\gamma}_\omega) \subset \tilde{\Omega}$. Let ω_0 denote such an element, and let $\gamma_{\omega_0}(\tilde{x}_0, \tilde{y}_0)$ denote $\exp_{c_0}(\tilde{\gamma}_\omega)$, which has endpoints $\tilde{x}_0$ and $\tilde{y}_0$. We may similarly find $\gamma_1(\tilde{x}_1, \tilde{y}_1)$ with endpoints $\tilde{x}_1$ and $\tilde{y}_1$. Then, we have that

$$(318) \quad L(\gamma_{\omega_j}(\tilde{x}_j, \tilde{y}_j)) \leq Ld(x_j, y_j),$$

for $j = 0, 1$. Furthermore, we see that

$$(319) \quad d(\tilde{y}_0, \tilde{x}_1) \leq \frac{r}{4}.$$

As such, there is a curve connecting them, which lies entirely in $\tilde{\Omega}$ and which has length less than or equal to Kr . Let us modify γ along the section γ_0 and γ_1 by replacing γ_j with $\tilde{\gamma}_{\omega_j}$, and connecting their endpoints by the curve above. Doing this for all of the segments yields a curve $\tilde{\gamma}$ which lies entirely in $\tilde{\Omega}$ and has length bounded above as follows:

$$(320) \quad L(\tilde{\gamma}) \leq \left(\left\lceil \frac{16 \operatorname{diam}(M, g)}{r_L} \right\rceil + 2 \right) Kr + Ld(x, y),$$

where the $+2$ comes from the fact that we may have to perturb the endpoints x, y of γ in a similar way as above. Therefore, for r chosen small enough, and $|\Omega^c| + |\partial\Omega|$ small enough depending only on r and L , the above has the following estimate:

$$(321) \quad L(\tilde{\gamma}) \leq (1 + \varepsilon)d(x, y).$$

□

7.2. Fundamental Domains. In this section we give a construction of fundamental domains, which can be found in [Cha06, Section IV.3].

Definition 7.6 ([Cha06]). Let (M, g) be a closed Riemannian manifold, and let p be any point in M , and denote by $\mathcal{S}_p$ the collection of unit vectors in $T_p M$, and let $\mathcal{S}M$ denote the unit tangent bundle. Finally, let $\pi : TM \rightarrow M$ be the projection map. Let us define a map $c : \mathcal{S}M \rightarrow (0, \infty)$ as follows

$$(322) \quad c(\xi) = \sup \left\{ t : d(p, \exp_{\pi(\xi)}(t\xi)) = t \right\}$$

We have the following important result

Theorem 7.7 ([Cha06]). Let (M, g) be a complete Riemannian manifold without boundary. Then, the map $c : \mathcal{S}M \rightarrow (0, \infty]$ is continuous.

Proof. Fix ξ in $\mathcal{S}M$ and let ξ_k be a sequence approaching ξ . For notational convenience, let $p = \pi(\xi)$, $p_k = \pi(\xi_k)$, and $d_k = c(\xi_k)$.

We will first show that $\limsup_{k \rightarrow \infty} c(\xi_k) \leq c(\xi)$. Pick a subsequence $d_{k(i)}$ such that $\lim_{i \rightarrow \infty} d_{k(i)} = \limsup_{k \rightarrow \infty} d_k$. If $\limsup_{k \rightarrow \infty} d_k = \infty$, then, for any $T > 0$ one has that $d_{k(i)} > T$ for all i sufficiently large. Since $\xi_k \rightarrow \xi$, it follows that $\exp_{p_{k(i)}}(T\xi_{k(i)})$ converges to $\exp_p(T\xi)$. Since distance is a continuous function, we see that

$$(323) \quad d(p, \exp_p(T\xi)) = \lim_{i \rightarrow \infty} d(p_{k(i)}, T\xi_{k(i)}) = T.$$

Therefore, by the definition of c we see that $c(\xi) \geq T$. As T can be chosen to be arbitrarily large, we see that $c(\xi) = \infty$.

Next, suppose that $\limsup_{k \rightarrow \infty} d_k = \delta < \infty$. Then, for any ε we may find an N so that for all $i \geq N$ we have that $\delta - \varepsilon < d_{k(i)}$. It follows that

$$(324) \quad d\left(p, \exp_p((\delta - \varepsilon)\xi)\right) = \lim_{i \rightarrow \infty} d\left(p_{k(i)}, \exp_{p_{k(i)}}((\delta - \varepsilon)\xi_{k(i)})\right) = \delta - \varepsilon.$$

So, we see that $c(\xi) \geq \delta - \varepsilon$, where ε was arbitrary.

Continuity will now follow if we can show that $\liminf_{k \rightarrow \infty} d_k \geq c(\xi)$. By way of contradiction, suppose that $\liminf_{k \rightarrow \infty} d_k + 2\varepsilon < c(\xi)$. Let U_p denote the open star-shaped region about $p = \pi(\xi)$ on which $\exp_p$ is a diffeomorphism, let $T = \liminf_{k \rightarrow \infty} d_k$, and let $\xi_{k(i)}$ be a subsequence such that $\lim_{i \rightarrow \infty} d_{k(i)} = T$. Since $\xi_{k(i)} \rightarrow \xi$, we have that $\lim_{k \rightarrow \infty} \exp_{p_{k(i)}}((T + \varepsilon)\xi_{k(i)}) = \exp_p((T + \varepsilon)\xi)$. Since $\exp_p(T\xi) \in U_p$, it follows that for all i large enough, we have that $\exp_{p_{k(i)}}((T + \varepsilon)\xi_{k(i)}) \in U_p$. However, we also have for all i sufficiently large that $c(\xi_{k(i)}) = d_{k(i)} < T + \varepsilon$. This is a contradiction, and so we see that $\liminf_{k \rightarrow \infty} d_k \geq c(\xi)$. This establishes the continuity of c on $\mathcal{SM}$. $\square$

Remark 7.8. Here we avoided the second half of proof presented in [Cha06], since by the results in [Mar93] conjugate loci are in general ill behaved.

Definition 7.9. Given a complete Riemannian manifold (M, g) and a point p in M , we will let D_p denote the following set:

$$(325) \quad D_p = \{c(\xi)\xi : \xi \in \mathcal{S}_p M\},$$

the graph of the continuous function $c : \mathcal{S}_p \rightarrow (0, \infty]$. Furthermore, let

$$(326) \quad G_p = \{t\xi : t < c(\xi); \xi \in \mathcal{S}_p\}.$$

Proposition 7.10. *Let (M, g) be a closed Riemannian manifold, and let p in M . Then, D_p has measure zero.*

Proof. Observe that $\mathcal{S}_p$ and M are compact. As such, it follows that $c(\mathcal{S}_p)$ is contained in a compact interval bounded away from zero. Since D_p is the graph of c over $\mathcal{S}_p$, it follows from using polar coordinates and the Fubini-Tonelli theorem that

$$(327) \quad \int_{T_p M} \chi_{D_p} = \int_{\mathcal{S}_p} \int_a^b n \omega_n r^{n-1} \chi_{D_p}(r, \theta) dr d\theta = 0.$$

$\square$

Lemma 7.11. *Let (M, g) be a closed Riemannian manifold, and let $(\widetilde{M}, \pi^*g)$ be its universal cover. For any given $p \in M$ and $\tilde{p} \in \widetilde{M}$, let $\text{Dir}(\tilde{p})$ denote the following set*

$$(328) \quad \text{Dir}(\tilde{p}) = \exp_{\tilde{p}} \circ (d\pi_{\tilde{p}})^{-1} G_p.$$

Then, we have that $\text{Dir}(\tilde{p})$ is a fundamental domain.

Proof. We claim that $\text{Dir}(\tilde{p})$ is a fundamental domain. To begin, suppose that $x, y \in \text{intDir}(\tilde{p})$ are such that $\pi(x) = \pi(y)$. This would mean that there are at least two length minimizing geodesics from p to $\pi(x) = \pi(y)$, but this contradicts the definition of G_p . Observe that this implies that $\overline{\text{Dir}}(\tilde{p}) = \exp_{\tilde{p}} \circ (d\pi_{\tilde{p}})^{-1} \overline{G}_p$, and as a consequence we also have that $\partial \text{Dir}(\tilde{p}) \subset \exp_{\tilde{p}} \circ (d\pi_{\tilde{p}})^{-1} D_p$. Therefore, since D_p has measure zero and $\exp_{\tilde{p}} \circ (d\pi_{\tilde{p}})^{-1}$ is a smooth map, it follows that $\partial \text{Dir}(\tilde{p})$ has measure zero. Finally, since $\overline{G}_p$ is path connected, it follows that $\overline{\text{Dir}}(\tilde{p})$ is as well. $\square$

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Current address: Universiteit Antwerpen

Email address: etbryden@gmail.com

Current address: Lanzhou University

Email address: chenzhmath@gmail.com