

Conversion of Boolean and Integer FlatZinc Builtins to Quadratic or Linear Integer Problems*

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Abstract

Constraint satisfaction or optimisation models — even if they are formulated in high-level modelling languages — need to be reduced into an equivalent format before they can be solved by the use of Quantum Computing. In this paper we show how Boolean and integer FlatZinc builtins over finite-domain integer variables can be equivalently reformulated as linear equations, linear inequalities or binary products of those variables, i.e. as finite-domain quadratic integer programs. Those quadratic integer programs can be further transformed into equivalent Quadratic Unconstrained Binary Optimisation problem models, i.e. a general format for optimisation problems to be solved on Quantum Computers especially on Quantum Annealers.

1 Introduction

Our work in the EniQmA project¹ aims to transform MiniZinc programs² defining constraint satisfaction or optimisation problems (CSP/COP) into equivalent Quadratic Unconstrained Binary Optimisation (QUBO) problems. The intention is to offer thereby a tool to support the solving of a large set of satisfaction and optimisation problems by the use of Quantum Computing. In particular, to use Quantum Annealers beyond other solvers in the MiniZinc context.

Thanks to the developers of MiniZinc [?] and the according software tools we can focus on FlatZinc programs³ because any MiniZinc program can be transformed automatically by the MiniZinc compiler into an equivalent FlatZinc

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¹see <https://www.eniqma-quantum.de/>

²see <https://www.minizinc.org/>

³The language FlatZinc is a proper subset of the language MiniZinc.

program. A large set of FlatZinc programs are representing *quadratic or linear constraint optimisation problems* over finite domain integer variables. Therefore we focus in this document on the transformation of Boolean and integer FlatZinc builtins to finite-domain quadratic integer problems, which can be further transformed into QUBO problems [?].

2 Finite-Domain Quadratic Integer Problems

Here we focus on *Finite Domain Quadratic Integer Problems (QIP(FD))*, i.e. on finite integer optimisation problems of the form

$$\text{minimize} \quad \sum_{i=1}^n g_i \cdot x_i \quad (1)$$

subject to

$$\sum_{i=1}^n a_{j,i} \cdot x_i + \sum_{k=1}^m b_{j,k} \cdot y_k + c_j \leq 0 \quad \text{for } j = 1, \dots, p \quad (2)$$

$$\sum_{i=1}^n d_{j,i} \cdot x_i + \sum_{k=1}^m e_{j,k} \cdot y_k + f_j = 0 \quad \text{for } j = p+1, \dots, q \quad (3)$$

$$y_k = z_v \cdot z_w \quad \begin{array}{l} z_v, z_w \in \{x_1, \dots, x_n, y_1, \dots, y_{k-1}\} \\ \text{for } k = 1 \dots, m \end{array} \quad (4)$$

$$x_i \in D(x_i) \subset \mathbb{Z} \quad \text{for } i = 1, \dots, n \quad (5)$$

$$y_k \in D(y_k) \subset \mathbb{Z} \quad \text{for } k = 1, \dots, m, \quad (6)$$

where $g_i, a_{j,i}, b_{j,k}, c_j, d_{j,i}, e_{j,k}, f_j$ are rational or integer values, x_i, y_k are uniquely defined variables with non-empty finite domains $D(x_i)$ resp. $D(y_k)$. We assume that the finite domains of the variables are *bounds-consistent integer intervals*, i.e. $D(x_i) = [\min(x_i), \max(x_i)]$ resp. $D(y_k) = [\min(y_k), \max(y_k)]$, e.g. according to the pruning rules defined in [4] (Figure 1).

In the special case that the objective in Equation (1) is missing (or zero) the problem is a satisfaction problem and if the binary variable products, i.e. the Equations (4) are missing, the problem is linear.

3 Representing FlatZinc Builtins

Here we show how to transform Bool(ean) and integer FlatZinc builtins [?], i.e. these FlatZinc constraints into equivalent conjunctions of linear equations and inequalities as well as binary variable products, i.e. into QIP(FD) constraints. Therefore, we assume that each variable x has an integer domain $D(x) = [\min(D(x)), \max(D(x))] = [\min(x), \max(x)]$ where $\min(x), \max(x) \in \mathbb{N}$ with $\min(x) \leq \max(x)$, either defined by a statement `var min(x)..max(x): x;` or by `var bool: x;` in FlatZinc.

Boolean values are represented by `false` and `true` in FlatZinc, which can be explicitly converted to integer values 0 and 1 by the use of the `bool2int` builtin. Therefore, we assume that this builtin is (implicitly) applied to each Boolean variable resulting in a binary variable, i.e. any Boolean value $b \in \{\text{false}, \text{true}\}$ is considered as a binary (integer) value $b \in \{0, 1\} = [0, 1]$.

It should be noted that we do not consider special cases, when we transform FlatZinc builtins into QIP(FD) constraints, e.g. the fact that $(a = b) \leftrightarrow r$ can be reduced to $a = b$ if $r = 1$ applies, is ignored in the following considerations.

3.1 Representing Integer FlatZinc Builtins

3.1.1 The predicate `array_int_element`

The semantics of `array_int_element(var int: i, array [int] of int: a, var int: c)` is $c = a_i$, i.e. the value of the integer variable c is the i -th value in the integer array a where i is an integer index variable, $a = a_1, \dots, a_n$ are integer values. Then, we can restrict the domains of the variables:

$$\begin{aligned} i &\in [\max(1, \min(i)), \min(n, \max(i))] \\ c &\in \left[\min_{i \in [\min(i), \max(i)]} \min(a_i), \max_{i \in [\min(i), \max(i)]} \max(a_i) \right] \end{aligned}$$

and the predicate can be represented by linear equations:

$$1 = \sum_{j=\min(i)}^{\max(i)} b_j^i, \quad i = \sum_{j=\min(i)}^{\max(i)} j \cdot b_j^i, \quad c = \sum_{j=\min(i)}^{\max(i)} a_j \cdot b_j^i, \quad (7)$$

where $b_{\min(i)}^i \in [0, 1], \dots, b_{\max(i)}^i \in [0, 1]$ are new auxiliary binary variables.

It should be noted that the first two equations in (7) define a *one-hot-encoding* of the integer variable i . This should be respected when transforming the resulting QIP(FD) into a QUBO problem.

3.1.2 The predicate `array_int_maximum`

The semantics of `array_int_maximum(var int: m, array [int] of var int: x)` is $m = \max_i x_i$, i.e. the value of the integer variable m is the maximum value of the values of the integer variables array $x_1, \dots, x_n$. Then we can restrict the domains of the variables:

$$\begin{aligned} m &\in [\max(\min(m), \max_{i=1, \dots, n} \min(x_i)), \min(\max(m), \max_{i=1, \dots, n} \max(x_i))] \\ x_i &\in [\min(x_i), \min(\max(x_i), \max(m))] \quad \text{for } i = 1, \dots, n. \end{aligned}$$

This can be presented by linear equations and inequalities as well as binary variable products:

$$\begin{aligned} 1 &= \sum_{j=1}^n b_j \quad \text{and} \quad m = \sum_{j=1}^n z_j \\ z_j &= x_j \cdot b_j \quad \text{and} \quad m \geq x_j \quad \text{for } j = 1, \dots, n. \end{aligned}$$

where $b_1 \in [0, 1], \dots, b_n \in [0, 1]$ are new auxiliary binary variables and $z_1 \in [\min(0, \min(a_1)), \max(a_1)], \dots, z_n \in [\min(0, \min(a_n)), \max(a_n)]$ are new auxiliary integer variables.

3.1.3 The predicate `array_int_minimum`

The semantics of `array_int_minimum(var int: m, array [int] of var int: x)` is $m = \min_i x_i$, i.e. the value of the integer variable m is minimum value of the integer variables array $x_1, \dots, x_n$. Then we can restrict the domains of the variables:

$$\begin{aligned} m &\in [\max(\min(m), \min_{i=1, \dots, n} \min(x_i), \min(\max(m), \max_{i=1, \dots, n} \max(x_i))] \\ x_i &\in [\max(\min(x_i), \min(m)), \max(x_i)] \quad \text{for } i = 1, \dots, n. \end{aligned}$$

This can be presented by linear equations and inequalities as well as binary variable products:

$$\begin{aligned} 1 &= \sum_{j=1}^n b_j \quad \text{and} \quad m = \sum_{j=1}^n z_j \\ z_j &= x_j \cdot b_j \quad \text{and} \quad m \leq x_j \quad \text{for } j = 1, \dots, n, \end{aligned}$$

where $b_1 \in [0, 1], \dots, b_n \in [0, 1]$ are new auxiliary binary variables and $z_1 \in [\min(0, \min(a_1)), \max(a_1)], \dots, z_n \in [\min(0, \min(a_n)), \max(a_n)]$ are new auxiliary integer variables.

3.1.4 The predicate `array_var_int_element`

The semantics of `array_var_int_element(var int: i, array [int] of var: a, var int: c)` is $c = a_i$, i.e. the value of the integer variable c is equal to the value of i -th variable in the integer variable array a where i is an integer index variable, $a = a_1, \dots, a_n$ are integer variables. We can restrict the domain of i to $i \in [\max(1, \min(i)), \min(n, \max(i))]$ and the domain of c to $c \in [\min_{i \in [\min(i), \max(i)]} \min(a_i), \max_{i \in [\min(i), \max(i)]} \max(a_i)]$. Then the predicate can be represented by linear equations and binary variable products:

$$\begin{aligned} 1 &= \sum_{j=\min(i)}^{\max(i)} \beta_j^i, \quad i = \sum_{j=\min(i)}^{\max(i)} j \cdot \beta_j^i, \quad c = \sum_{j=\min(i)}^{\max(i)} z_j, \\ z_j &= a_j \cdot \beta_j^i \quad \text{for } j = \min(i), \dots, \max(i), \end{aligned}$$

where $\min(i)$ and $\max(i)$ are the updated boundaries of the restricted domain of i (see above), $\beta_{\min(i)}^i \in [0, 1], \dots, \beta_{\max(i)}^i \in [0, 1]$ are new auxiliary binary variables, and the variables $z_j \in [\min(0, \min(a_j)), \max(0, \max(a_j))]$ with $j = \min(i), \dots, \max(i)$ are new auxiliary integer variables.

It should be noted that the first two equations in (8) define a *one-hot encoding* of the integer variable i . This should be respected when transforming the resulting QIP(FD) into a QUBO problem.

3.1.5 The predicate `int_abs`

The semantics of `int_abs(var int: x, var int: y)` is $y = |x|$, i.e. the value of the integer variable y is equal to the absolute value of the integer variable x . We can restrict the domain of y to $y \in [\max(0, \min(y)), \max(0, \max(y))]$ reflecting $y \geq 0$. Then the predicate can be represented by a linear equation and a binary variable product:

$$y = x - 2 \cdot z \quad \text{and} \quad z = b \cdot x$$

where $z \in [\min(0, \min(x)), \max(0, \max(x))]$ is a new auxiliary integer variable and $b \in [0, 1]$ is a new auxiliary binary variable.

3.1.6 The predicate `int_div`

The semantics of `int_div(var int: n, var int: d, var int: q)` is as follows: The quotient q produced for operand values n and d is an integer value q whose magnitude is as large as possible while satisfying $|d \cdot q| \leq |n|$. Moreover, q is positive when $|n| \geq |d|$ and n and d have the same sign, but q is negative when $|n| \geq |d|$ and n and d have opposite signs. Obviously, the constraint is not satisfiable for $d = 0$ (division by zero) such that $d \neq 0$ respective $|d| > 0$ must hold (see below) and it holds that $q = 0$ when $|n| < |d|$, in particular if $n = 0$. In all other cases $|n| - |d| < |d \cdot q| \leq |n|$ must hold such that we distinguish the following four cases:

1. $n > 0$ and $d > 0$: It must hold that $n - d < d \cdot q$ and $d \cdot q \leq n$.
2. $n < 0$ and $d < 0$: It must hold that $n - d > d \cdot q$ and $d \cdot q \geq n$.
3. $n > 0$ and $d < 0$: It must hold that $n + d < d \cdot q$ and $d \cdot q \leq n$.
4. $n < 0$ and $d > 0$: It must hold that $n + d > d \cdot q$ and $d \cdot q \geq n$.

In the general case when neither $n \geq 0$, $n \leq 0$, $d > 0$ nor $d < 0$ holds in advance, we introduce two additional binary variables $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ to encode the preconditions of the four cases:

- Let $\alpha = 1$ if $n > 0$ and $\alpha = 0$ if $n < 0$:

$$\begin{aligned} n &> (1 - \alpha) \cdot (\min(n) - 1) \\ n &< \alpha \cdot (\max(n) + 1) \end{aligned}$$

- Let $\beta = 1$ if $d > 0$ and $\beta = 0$ if $d < 0$:

$$\begin{aligned} d &> (1 - \beta) \cdot (\min(d) - 1) \\ d &< \beta \cdot (\max(d) + 1) \end{aligned}$$

Using this encoding we are able to encode the four cases in one conjunction of linear or quadratic inequalities

$$\begin{aligned}
n &> (1 - \alpha) \cdot (\min(n) - 1) \quad \wedge \quad n < \alpha \cdot (\max(n) + 1) \\
d &> (1 - \beta) \cdot (\min(d) - 1) \quad \wedge \quad d < \beta \cdot (\max(d) + 1) \\
d \cdot q &\leq n + (1 - \alpha) \cdot M \quad \wedge \quad d \cdot q \geq n - \alpha \cdot M \\
n - d &< d \cdot q + (1 - \alpha \cdot \beta) \cdot M \quad \wedge \quad n - d > d \cdot q - (1 - (1 - \alpha) \cdot (1 - \beta)) \cdot M \\
n + d &< d \cdot q + (1 - \alpha \cdot (1 - \beta)) \cdot M \quad \wedge \quad n + d > d \cdot q - (1 - \beta \cdot (1 - \alpha)) \cdot M
\end{aligned}$$

where M is a sufficiently large integer value. These inequalities can be further simplified by introducing an auxiliary integer variable p representing the product $d \cdot q$ where

$$\begin{aligned}
D(p) = & [\min(\min(d) \cdot \min(q), \min(d) \cdot \max(q), \max(d) \cdot \min(q), \max(d) \cdot \max(q)), \\
& \max(\min(d) \cdot \min(q), \min(d) \cdot \max(q), \max(d) \cdot \min(q), \max(d) \cdot \max(q))]
\end{aligned}$$

and an auxiliary binary variable γ representing the product $\alpha \cdot \beta$:

$$\begin{aligned}
n &\geq \min(n) - \alpha \cdot (\min(n) - 1) \quad \wedge \quad n \leq \alpha \cdot (\max(n) + 1) - 1 \\
d &\geq \min(d) - \beta \cdot (\min(d) - 1) \quad \wedge \quad d \leq \beta \cdot (\max(d) + 1) - 1 \\
p &\leq n + M - \alpha \cdot M \quad \wedge \quad p \geq n - \alpha \cdot M \\
n - d &\leq p + M - \gamma \cdot M - 1 \quad \wedge \quad n - d \geq p - \alpha \cdot M - \beta \cdot M + \gamma \cdot M + 1 \\
n + d &\leq p + M - \alpha \cdot M + \gamma \cdot M - 1 \quad \wedge \quad n + d \geq p - M + \beta \cdot M + \gamma \cdot M + 1 \\
p &= d \cdot q \quad \wedge \quad \gamma = \alpha \cdot \beta
\end{aligned}$$

The integer value M is sufficiently large if

$$\begin{aligned}
\max(n - p) &\leq M \\
\min(p - n) &\geq -M \\
\max(n - d - p) &\leq M - 1 \\
\min(n - d - p) &\geq 1 - M \\
\max(n + d - p) &\leq M - 1 \\
\min(n + d - p) &\geq 1 - M
\end{aligned}$$

respective

$$\begin{aligned}
\max(n) - \min(p) &\leq M \\
-\min(p) + \max(n) &\leq M \\
\max(n) - \min(d) - \min(p) + 1 &\leq M \\
-\min(n) + \max(d) + \max(p) + 1 &\leq M \\
\max(n) + \max(d) - \min(p) + 1 &\leq M \\
-\min(n) - \min(d) + \max(p) + 1 &\leq M
\end{aligned}$$

i.e. M must be at least as large as the maximum value of the right-hand-sides of these inequalities.

3.1.7 The Predicate `int_eq`

The semantics of `int_eq(var int: a, var int: b)` is $a = b$, i.e. the value of the integer variable a is equal to the value of the integer variable b which is a linear equation.

3.1.8 The predicate `int_eq_reif`

The semantics of `int_eq_reif(var int: a, var int: b, var bool: r)` is $a = b \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the values integer variables a and b are equal and 0 if they are different. This can be represented by binary variable products, linear equations and linear inequalities:

$$\begin{aligned} x &= r \cdot a \\ y &= r \cdot b \\ x &= y \\ (1 - r) &\leq (1 - r) \cdot (|a| + |b|) \end{aligned}$$

where $x \in [\min(0, \min(a)), \max(0, \max(a))]$ and $y \in [\min(0, \min(b)), \max(0, \max(b))]$ are new auxiliary integer variables. The last inequality is necessary, to cover the case where $a = b = 0$, i.e. to force that $r = 1$. The modelling of the absolutes $|a|$ and $|b|$ is already described in Section 3.1.5, such that the resulting constraints are:

$$\begin{aligned} x &= r \cdot a \\ y &= r \cdot b \\ x &= y \\ r &\geq s + t - u - v + 1 \\ s &= r \cdot u \\ t &= r \cdot v \\ u &= a - 2 \cdot p \\ p &= c \cdot a \\ v &= b - 2 \cdot q \\ q &= d \cdot b \end{aligned}$$

where $c \in [0, 1]$ and $d \in [0, 1]$ are new auxiliary binary variables and where

$$\begin{aligned} \alpha(u) &= \begin{cases} \min(a) & \text{if } 0 \leq \min(a), \\ -\max(a) & \text{if } \max(a) \leq 0, \\ 0 & \text{otherwise.} \end{cases} \\ \beta(u) &= \begin{cases} \max(a) & \text{if } 0 \leq \min(a), \\ -\min(a) & \text{if } \max(a) \leq 0, \\ \max(-\min(a), \max(a)) & \text{otherwise.} \end{cases} \end{aligned}$$

$$\alpha(v) = \begin{cases} \min(b) & \text{if } 0 \leq \min(b), \\ -\max(b) & \text{if } \max(b) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta(v) = \begin{cases} \max(b) & \text{if } 0 \leq \min(b), \\ -\min(b) & \text{if } \max(b) \leq 0, \\ \max(-\min(b), \max(b)) & \text{otherwise.} \end{cases}$$

are integer values and the variables $u \in [\alpha(u), \beta(u)]$, $v \in [\alpha(v), \beta(v)]$ as well as $s \in [\min(0, \min(u)), \max(0, \max(u))]$, $t \in [\min(0, \min(v)), \max(0, \max(v))]$, $p \in [\min(0, \min(a)), \max(0, \max(a))]$ and $q \in [\min(0, \min(b)), \max(0, \max(b))]$ are new auxiliary integer variables.

3.1.9 The Predicate `int_le`

The semantics of `int_le(var int: a, var int: b)` is $a \leq b$, i.e. the value of the integer variable a is less than or equal to the value of the integer variable b which is a linear inequality.

3.1.10 The predicate `int_le_reif`

The semantics of `int_le_reif(var int: a, var int: b, var bool: r)` is $a \leq b \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the value of the integer variable a is less than or equal to the value of the integer variable b and 0 if the value of the integer variable a is greater than the value of the integer variable b . This can be represented by linear inequalities:

$$\begin{aligned} a &\leq b + (1 - r) \cdot (\max(a) - \min(b)) \\ b &\leq a - 1 + r \cdot (\max(b) - \min(a) + 1) . \end{aligned}$$

3.1.11 The predicate `int_lin_eq`

The semantics of `int_lin_eq(array [int] of int: as, array [int] of var int: bs, int: c)` is $\sum_i as_i \cdot bs_i = c$, i.e. the sum of the values of the integer variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is equal to the integer value c . This is a linear equation.

3.1.12 The predicate `int_lin_eq_reif`

The semantics of `int_lin_eq_reif(array [int] of int: as, array [int] of var int: bs, int: c, var bool: r)` is $\sum_i as_i \cdot bs_i = c \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the sum of the values of the variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is equal to the integer value c and the value of r is 0 in all other cases. Following the approach in Section 3.1.8

this can be represented by binary variable products, linear equations and linear inequalities:

$$\begin{aligned} x &= \sum_{i=1}^n as_i \cdot bs_i - c \\ |x| &\leq \max(|x|) \cdot (1 - r) \\ 1 - r &\leq |x| \end{aligned}$$

where $x \in [l(x), u(x)]$ with

$$\begin{aligned} l(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \max(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \min(bs_i) - c \\ u(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \min(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \max(bs_i) - c \end{aligned}$$

and where the modelling of the absolute $|x|$ described in Section 3.1.5 is used.

3.1.13 The predicate `int_lin_le`

The semantics of `int_lin_le(array [int] of int: as, array [int] of var int: bs, int: c)` is $\sum_i as_i \cdot bs_i \leq c$, i.e. the sum of the values of the variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is less than or equal to the integer value c which is a linear inequality.

3.1.14 The predicate `int_lin_le_reif`

The semantics of `int_lin_le_reif(array [int] of int: as, array [int] of var int: bs, int: c, var bool: r)` is $\sum_i as_i \cdot bs_i \leq c \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the sum of the values of the variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is less than or equal to the integer value c and the value of r is 0 in all other cases. Following the approach in Section 3.1.10 this can be represented by linear equations and linear inequalities:

$$\begin{aligned} x &= \sum_{i=1}^n as_i \cdot bs_i - c \\ x &\leq (1 - r) \cdot \max(x) \\ 1 &\leq x + r - r \cdot \min(x) \end{aligned}$$

where $x \in [l(x), u(x)]$ is a new auxiliary integer variable with

$$\begin{aligned} l(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \max(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \min(bs_i) - c \\ u(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \min(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \max(bs_i) - c . \end{aligned}$$

3.1.15 The predicate `int_lin_ne`

The semantics of `int_lin_ne(array [int] of int: as, array [int] of var int: bs, int: c)` is $\sum_i as_i \cdot bs_i \neq c$, i.e. the sum of the values of the variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is not equal to the integer value c . Following the approach in Section 3.1.5 this can be represented by linear equations, linear inequalities and binary variable products:

$$x = \sum_{i=1}^n as_i \cdot bs_i - c \quad \text{and} \quad 1 \leq |x|$$

where $x \in [l(x), u(x)]$ is a new auxiliary integer variable with

$$\begin{aligned} l(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \max(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \min(bs_i) - c \\ u(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \min(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \max(bs_i) - c \end{aligned}$$

and $|x| \in [\max(1, \min(|\min(x)|, |\max(x)|)), \max(|\min(x)|, |\max(x)|)]$. For the representation of $|x|$ see Section 3.1.5.

3.1.16 The predicate `int_lin_ne_reif`

The semantics of `int_lin_ne_reif(array [int] of int: as, array [int] of var int: bs, int: c, var bool: r)` is $\sum_i as_i \cdot bs_i \neq c \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the sum of the values of the variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is not equal to the integer value c and the value of r is 0 in all other cases. Following the approaches in Section 3.1.8 and Section 3.1.12 this can be represented by

$$\begin{aligned} x &= \sum_{i=1}^n as_i \cdot bs_i - c \\ x &= r \cdot x \\ r &\leq r \cdot |x| \end{aligned}$$

respective

$$\begin{aligned} x &= \sum_{i=1}^n as_i \cdot bs_i - c \\ x &= y \\ y &= r \cdot x \\ r &\leq z \\ z &= r \cdot |x| \end{aligned}$$

where $x \in [l(x), u(x)]$ is a new auxiliary integer variable with

$$\begin{aligned} l(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \max(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \min(bs_i) - c \\ u(x) &= \sum_{i=1 \wedge as_i < 0}^n as_i \cdot \min(bs_i) + \sum_{i=1 \wedge as_i > 0}^n as_i \cdot \max(bs_i) - c \end{aligned}$$

and $|x| \in [l(|x|), u(|x|)]$ with

$$\begin{aligned} l(|x|) &= \begin{cases} 0 & \text{if } l(x) \leq 0 \\ l(x) & \text{otherwise} \end{cases} \\ u(|x|) &= \max(|l(x)|, |u(x)|) \end{aligned}$$

$y \in [\min(0, \min(x)), \max(0, \max(x))]$, $z \in [\min(0, \min(|x|)), \max(0, \max(|x|))]$ are new auxiliary integer variables. For the representation of $|x|$ see Section 3.1.5.

3.1.17 The Predicate `int_lt`

The semantics of `int_lt(var int: a, var int: b)` is $a < b$, i.e. the value of the integer variable a is less than the value of the integer variable b . This can be represented by a linear inequality:

$$a \leq b - 1.$$

3.1.18 The predicate `int_lt_reif`

The semantics of `int_lt_reif(var int: a, var int: b, var bool: r)` is $a < b \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the value of the integer variable a is less than the value of the integer variable b and 0 if the value of the integer variable a is greater than or equal to the value of the integer variable b . This can be represented by linear inequalities:

$$\begin{aligned} a &\leq b - 1 + (1 - r) \cdot (\max(a) - \min(b) + 1) \\ b &\leq a + r \cdot (\max(b) - \min(a)). \end{aligned}$$

3.1.19 The predicate `int_max`

The semantics of `int_max(var int: a, var int: b, var int: c)` is the equation $\max(a, b) = c$, i.e. the value of the integer variable c is the maximum of the values of the integer variables a and b . This can be represented by the linear inequalities:

$$\begin{aligned} a &\leq c \\ b &\leq c \\ c &\leq a + r \cdot (\max(c) - \min(a)) \\ c &\leq b + (1 - r) \cdot (\max(c) - \min(b)) \end{aligned}$$

where $r \in [0, 1]$ is a new auxiliary binary variable.

3.1.20 The predicate `int_min`

The semantics of `int_min(var int: a, var int: b, var int: c)` is the equation $\min(a, b) = c$, i.e. the value of the integer variable c is the minimum of the values of the integer variables a and b . This can be represented by the linear inequalities:

$$\begin{aligned} c &\leq a \\ c &\leq b \\ a &\leq c + r \cdot (\max(a) - \min(c)) \\ b &\leq c + (1 - r) \cdot (\max(b) - \min(c)) \end{aligned}$$

where $r \in [0, 1]$ is a new auxiliary binary variable.

3.1.21 The predicate `int_mod`

The semantics of `int_mod(var int: n, var int: d, var int: r)` is $n \bmod d = r$, i.e. the value of the integer variable r is the modulo — i.e. the integer remainder of the integer division — of the values of the integer variables n and d . If n is non-negative then r is non-negative, too. The operator `mod` returns the remainder (modulus) of the nominator n divided by the denominator d . If n is non-positive then r is non-positive, too. This can be represented by

$$\text{int_div}(n, d, q) \wedge d \cdot q + r = n$$

(cf. Section 3.1.6) where q is a new auxiliary integer variable and r is the “slack” concerning the inequalities $d \cdot q + r \leq n$ respective $d \cdot q + r \geq n$ there.

3.1.22 The Predicate `int_ne`

The semantics of `int_ne(var int: a, var int: b)` is $a \neq b$, i.e. the value of the integer variable a is not equal to (different from) the value of the integer variable b . This can be represented by linear inequalities:

$$\begin{aligned} a &\leq b - 1 + r \cdot (\max(a) - \min(b) + 1) \\ b &\leq a - 1 + (1 - r) \cdot (\max(b) - \min(a) + 1) \end{aligned}$$

where $r \in [0, 1]$ is a new auxiliary binary variable.

3.1.23 The predicate `int_ne_reif`

The semantics of `int_ne_reif(var int: a, var int: b, var bool: r)` is $a \neq b \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the values of the integer variables a and b are not equal (different) and 0 if they are equal.

Following the approach in Section 3.1.16 this can be represented by

$$\begin{aligned} z &= a - b \\ z &= r \cdot z \\ r &\leq r \cdot |z| \end{aligned}$$

where $z \in [\min(a) - \max(b), \max(a) - \min(b)]$ is a new integer variable and $|z| \in [l(|z|), u(|z|)]$ with

$$\begin{aligned} l(|z|) &= \begin{cases} 0 & \text{if } \min(z) \leq 0 \\ \min(z) & \text{otherwise} \end{cases} \\ u(|z|) &= \max(|\min(z)|, |\max(z)|) . \end{aligned}$$

For the representation of $|z|$ see Section 3.1.5.

3.1.24 The predicate `int_plus`

The semantics of `int_plus(var int: a, var int: b, var int: c)` is $a+b=c$ which is a linear equation.

3.1.25 The predicate `int_pow`

The semantics of `int_pow(var int: x, var int: y, var int: z)` is $z = x^y$ if $y \geq 0$ and $z = 1 \div x^{-y}$ if $y < 0$. In general this cannot be represented by polynomial systems and thus it is beyond the scope of our work. However in the special case that y is determined, i.e. $y = n$ respective $y \in \{n\}$ with $n > 0$ then it holds $z = x^n$ which is polynomial. If n is even, i.e. $n = 2 \cdot k$ we can set $z = u^2$ and $u = x^k$. If n is odd, i.e. $n = 2 \cdot k + 1$ we can set $z = x \cdot v$, $v = u^2$ and $u = x^k$. In both cases we can apply this transformation steps recursively to $u = x^k$ until the considered result becomes quadratic or linear.

3.1.26 The predicate `int_times`

The semantics of `int_times(var int: a, var int: b, var int: c)` is $a \cdot b = c$ which is a binary variable product.

3.2 Representing Bool(ean) FlatZinc Builtins

3.2.1 The predicate `array_bool_and`

The semantics of the predicate `array_bool_and(array [int] of var bool: as, var bool: r)` is $r \leftrightarrow \bigwedge_i as_i$, i.e. the value of the binary variable r is 1 if all binary variables $as = as_1, \dots, as_n$ are 1 and the value of r is 0 if at least one value of $as_1, \dots, as_n$ is 0. This can be represented by linear inequalities:

$$\begin{aligned} r &\leq as_i \quad \text{for } i = 1, \dots, n \\ r &\geq \sum_{i=1}^n as_i - n + 1 . \end{aligned}$$

3.2.2 The predicate `array_bool_element`

The semantics of the predicate `array_bool_element(var int: i, array [int] of bool: as, var bool: c)` is $c = as_i$, i.e. the value of the binary variable c is the i -th value in the binary values array $as = as_1, \dots, as_n$ where i is an integer index variable i with $i \in [\max(1, \min(i)), \min(n, \max(i))]$. In the special case that $as_{\min(i)} = \dots = as_{\max(i)} = 1$ applies it must hold that $c = 1$. In the special case that $as_{\min(i)} = \dots = as_{\max(i)} = 0$ applies it must hold that $c = 0$. In all other cases this can be represented by linear equations, following Section 3.1.1:

$$1 = \sum_{j=\min(i)}^{\max(i)} b_j^i \quad (8)$$

$$i = \sum_{j=\min(i)}^{\max(i)} j \cdot b_j^i \quad (9)$$

$$c = \sum_{j=\min(i)}^{\max(i)} as_j \cdot b_j^i$$

where $b_{\min(i)}^i \in [0, 1], \dots, b_{\max(i)}^i \in [0, 1]$ are new auxiliary binary variables.

It should be noted that Equations 8 and 9 define the *one-hot-encoding* of the integer variable i . This means that any other occurrence of i should be substituted/replaced by the sum $\sum_{j=\min(i)}^{\max(i)} j \cdot b_j^i$ and any other binary encoding of i should be avoided.

3.2.3 The predicate `array_bool_xor`

The semantics of `array_bool_xor(array [int] of var bool: as)` is $\oplus_i as_i = 1$, i.e. exactly one value of the binary variables in the array $as = as_1, \dots, as_n$ must be 1. This can be represented by a linear equation:

$$1 = \sum_{i=1}^n as_i .$$

3.2.4 The predicate `array_var_bool_element`

The semantics of `array_var_bool_element(var int: i, array [int] of var bool: as, var bool: c)` is $c = as_i$, i.e. the value of the binary variable c is the i -th value in the binary variable array $as = as_1, \dots, as_n$ where i is an integer index variable i with $i \in [\max(1, \min(i)), \min(n, \max(i))]$. Following Section 3.1.1 this can be represented by linear equations and binary variable products:

$$1 = \sum_{j=\min(i)}^{\max(i)} b_j^i \quad (10)$$

$$\begin{aligned}
i &= \sum_{j=\min(i)}^{\max(i)} j \cdot b_j^i \\
c &= \sum_{j=\min(i)}^{\max(i)} z_j \\
z_j &= as_j \cdot b_j^i \quad \text{for } j = \min(i), \dots, \max(i)
\end{aligned} \tag{11}$$

where $b_{\min(i)}^i \in [0, 1], \dots, b_{\max(i)}^i \in [0, 1]$ are new auxiliary binary variables and for $j = \min(i), \dots, \max(i)$ the variables $z_j \in [\min(0, \min(a_j)), \max(0, \max(a_j))]$ are new auxiliary integer variables.

It should be noted that Equations 10 and 11 define the *one-hot-encoding* of the integer variable i . This means that any other occurrence of i should be substituted/replaced by the sum $\sum_{j=\min(i)}^{\max(i)} j \cdot b_j^i$ and any other binary encoding of i should be avoided.

3.2.5 The predicate `bool2int`

The semantics of `bool2int(var bool: a, var int: b)` is $b \in [0, 1]$ and $a \leftrightarrow b = 1$, i.e. the value of integer variable b is 0 if the value of the Boolean variable a is `false` and the value of b is 1 if the value of a is `true`. Due to the fact that we consider Boolean variables as binary integer variables, this constraint can be represented by a linear equation:

$$a = b$$

(see considerations at the beginning of Section 3).

3.2.6 The predicate `bool_and`

The semantics of `bool_and(var bool: a, var bool: b, var bool: r)` is $r \leftrightarrow a \wedge b$. The value of the binary variable r is 0 if at least one value of the binary variables a and b is zero and the value of r is 1 if both values of a and b are 1. Following Section 3.2.1 this can be represented by linear inequalities:

$$\begin{aligned}
r &\leq a \\
r &\leq b \\
a + b &\leq 1 + r
\end{aligned}$$

or by the binary variable product:

$$r = a \cdot b$$

which requires that r does not occur in the another variable product as multiplier.

3.2.7 The predicate `bool_clause`

The semantics of `bool_clause(array [int] of var bool: as, array [int] of var bool: bs)` is $1 = \bigvee_i as_i \vee \bigvee_j \neg bs_j$ where $as = as_1, \dots, as_n$ and $bs = bs_1, \dots, bs_m$ are arrays of binary variables. This can be represented by a linear inequality:

$$1 \leq \sum_{i=1}^n as_i + \sum_{j=1}^m (1 - bs_j) .$$

3.2.8 The predicate `bool_eq`

The semantics of `bool_eq(var bool: a, var bool: b)` is $a = b$, i.e. the value of the binary variable a is equal to the value of the integer variable b which is a linear equation.

3.2.9 The predicate `bool_eq_reif`

The semantics of `bool_eq_reif(var bool: a, var bool: b, var bool: r)` is $a = b \leftrightarrow r$, i.e. the value of the Boolean (binary) variable r is 1 if the values integer variables a and b are equal and 0 if they are different. This can be represented by

$$\begin{aligned} x &= r \cdot a \\ y &= r \cdot b \\ x &= y \\ (1 - r) &= (1 - r) \cdot (a + b) \end{aligned}$$

where $x \in [0, 1]$ and $y \in [0, 1]$ are new auxiliary binary variables.

3.2.10 The predicate `bool_le`

The semantics of `bool_le(var bool: a, var bool: b)` is $a \leq b$, i.e. the value of the binary variable a is less than or equal to the value of the binary variable b which is a linear inequality.

3.2.11 The predicate `bool_le_reif`

The semantics of `bool_le_reif(var bool: a, var bool: b, var bool: r)` is $a \leq b \leftrightarrow r$, i.e. the value of the binary variable r is 1 if the value of the binary variable a is less than or equal to the value of the binary variable b and the value of r is 0 if the value of a is greater than the value of b . This can be represented by linear inequalities:

$$\begin{aligned} a &\leq b + 1 - r \\ b &\leq a - 1 + 2 \cdot r . \end{aligned}$$

3.2.12 The predicate `bool_lin_eq`

The semantics of `bool_lin_eq(array [int] of int as, array [int] of var bool bs, var int: c)` is $\sum_i as_i \cdot bs_i = c$, i.e. the sum of the values of the binary variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is equal to the value of the integer variable c . This is a linear equation.

3.2.13 The predicate `bool_lin_le`

The semantics of `bool_lin_le(array [int] of int: as, array [int] of var bool: bs, int: c)` is $\sum_i as_i \cdot bs_i \leq c$, i.e. the sum of the values of the binary variables $bs_1, \dots, bs_n$ weighted by the integer values $as_1, \dots, as_n$ — assuming that both arrays have the same length n — is less than or equal to the integer value c . This is a linear inequality.

3.2.14 The predicate `bool_lt`

The semantics of `bool_lt(var bool: a, var bool: b)` is $a < b$, i.e. the value of the binary variable a is less than the value of the binary variable b which means that $a = 0$ and $b = 1$ must hold, i.e. these are two simple linear equations.

3.2.15 The predicate `bool_lt_reif`

The semantics of `bool_lt_reif(var bool: a, var bool: b, var bool: r)` is $a < b \leftrightarrow r$, i.e. the value of the binary variable r is 1 if the value of the binary variable a is less than the value of the binary variable b and the value of r is 0 if the value of a is greater than or equal to the value of b . This can be represented by linear inequalities:

$$\begin{aligned} a &\leq b + 1 - 2 \cdot r \\ b &\leq a + r \end{aligned}$$

or by a binary variable product:

$$r = (1 - a) \cdot b$$

which requires that r does not occur in the another variable product as multiplier.

3.2.16 The predicate `bool_not`

The semantics of `bool_not(var bool: a, var bool: b)` is $a \neq b$, i.e. the value of the binary variable a is not equal to (different from) the value of the binary variable b . This can be represented by a linear equation:

$$a = 1 - b$$

3.2.17 The predicate `bool_or`

The semantics of `bool_or(var bool: a, var bool: b, var bool: r)` is $a \vee b \leftrightarrow r$ i.e. the value of the binary variable r is 1 if the value of at least one of the binary variable a and b is 1 and the value of r is 0 if both values of a and b are 0. This can be represented by linear inequalities:

$$\begin{aligned} r &\leq a + b \\ a + b &\leq 2 \cdot r . \end{aligned}$$

3.2.18 The predicate `bool_xor` (ternary)

The semantics of `bool_xor(var bool: a, var bool: b, var bool: r)` is $a \oplus b \leftrightarrow r$ i.e. the value of the binary variable r is 1 if exactly one value of the binary variables a and b is 1 and the value of r is 0 if both values of a and b are either 0 or 1. This can be represented by

$$\begin{aligned} (1 - r) \cdot a &= (1 - r) \cdot b \\ r \cdot (a + b) &= r \end{aligned}$$

and thus by linear equations, linear inequalities and binary variable products:

$$\begin{aligned} x &= r \cdot a \\ y &= r \cdot b \\ a - x &= b - y \\ x + y &= r \end{aligned}$$

where $x \in [0, 1]$ and $y \in [0, 1]$ are new auxiliary binary variables.

3.2.19 The predicate `bool_xor` (binary)

The semantics of `bool_xor(var bool: a, var bool: b)` is $a \oplus b$ which can be represented by a linear equation:

$$a + b = 1 .$$

3.3 Representing Set FlatZinc Builtins

Set constraints, more precisely finite integer set constraints are not completely covered by our conversion from FlatZinc programs to Quadratic Integer Programs. However, there are two set constraints over integer and binary variables and fixed sets of integer values which we will consider here.

3.3.1 The predicate `set_in`

The semantics of `set_in(var int: x, set of int: S)` is $x \in S$. This can be represented by linear equations:

$$1 = \sum_{s \in S \cap D(x)} b_s^x \quad (12)$$

$$x = \sum_{s \in S \cap D(x)} s \cdot b_s^x \quad (13)$$

where $b_s^x \in [0, 1]$ are new auxiliary binary variables.

It should be noted that Equations 12 and 13 define the *one-hot-encoding* of the integer variable x . This should be respected when transforming the resulting QIP(FD) into a QUBO problem.

3.3.2 The predicate `set_in_reif`

The semantics of the predicate `set_in_reif(var int: x, set of int: S, var bool: r)` is $r \leftrightarrow (x \in S)$, i.e. the value of the binary variable r is 1 if and only if the value of x is in the finite integer values set S . This can be represented by linear equations:

$$1 = \sum_{j \in D(x)} b_j^x \quad (14)$$

$$x = \sum_{j \in D(x)} j \cdot b_j^x \quad (15)$$

$$r = \sum_{j \in D(x) \cap S} b_j^x$$

where $b_j^x \in [0, 1]$ are new auxiliary binary variables.

It should be noted that Equations 16 and 15 define the *one-hot-encoding* of the integer variable x . This should be respected when transforming the resulting QIP(FD) into a QUBO problem.

4 Conclusion and Future Work

We have shown how Boolean and integer FlatZinc builtins can be represented by linear equations, linear inequalities and/or binary variable products. This is the basis for the transformation of finite domain integer FlatZinc programs via their intermediate presentation as QIP(FD) into equivalent QUBO problems. This gives us the opportunity to transform a large class of MiniZinc programs into QUBO problems and solve them also by Quantum Computing, e.g. with Quantum Annealers. Our future work will focus on the implementation of a MiniZinc-to-QUBO workflow.

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