A NOTE ON COMBINATORIAL INVARIANCE OF KAZHDAN–LUSZTIG POLYNOMIALS

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Abstract. We introduce the concepts of an amazing hypercube decomposition and a double shortcut for it, and use these new ideas to formulate a conjecture implying the Combinatorial Invariance Conjecture of the Kazhdan–Lusztig polynomials for the symmetric group. This conjecture has the advantage of being combinatorial in nature.

1. Introduction

The Combinatorial Invariance Conjecture for Kazhdan–Lusztig polynomials was formulated independently by Lusztig and Dyer in the '80's. Its version for the symmetric group W is the following.

Conjecture 1.1. *Let $u, v \in W$. The Kazhdan–Lusztig polynomial $P_{u,v}(q)$ (or, equivalently, the Kazhdan–Lusztig $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$) depends only on the isomorphism class of the Bruhat interval $[u, v]$ as a poset.*

It has been the focus of active research for the past forty years and has recently received important new inputs by [3] and [6], where the concept of a hypercube decomposition was introduced and used to give a conjectural formula for the Kazhdan–Lusztig polynomials of the symmetric group that implies Conjecture 1.1.

Successively, a conjectural formula for $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$ of the symmetric group was given in terms of shortcuts ([5]) of a hypercube decomposition. Also this formula would imply Conjecture 1.1. A further work studying hypercube decompositions is [1], where the authors give another conjecture implying Conjecture 1.1 and prove combinatorial invariance for elementary intervals, which marks an improvement of the result in [4] for lower Bruhat intervals.

In this short note, we introduce the concepts of an amazing hypercube decomposition and a double shortcut for it, and use these new ideas to formulate yet another conjecture implying the conjecture in [5] and thus ultimately Conjecture 1.1. This new conjecture has the advantage of being combinatorial in nature, with no mention of Kazhdan–Lusztig polynomials: a purely graph theoretic property of the Bruhat interval would imply the Combinatorial Invariance Conjecture for the symmetric group.

The conjecture presented in this paper has been verified (in a strong form) up to S_6 by computer calculations.

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2. Preliminaries

In this section we recall the basic definitions and set the notation to be used. We refer the reader to [2] for undefined terminology concerning Coxeter groups.

We recall the definition of a hypercube decomposition from [3]. Let (W, S) be a Coxeter group of type A. We denote by $B(W)$ the Bruhat graph of W.

Given a set E, let $\mathcal{P}(E)$ denote the directed Boolean algebra on E, i.e. the directed graph having the power set of E as vertex set and where $I \to J$ if I is obtained from J by removing one element.

Let $p \in W$ and E be a set of arrows of $B(W)$ all having p as target. Then E *spans a hypercube* if there exists a unique embedding of directed graphs $\theta : \mathcal{P}(E) \to W$ sending the directed edge $E \setminus \{\alpha\} \to E$ to α, for all $\alpha \in E$. Furthermore, E *spans a hypercube cluster* if every subset E' of E consisting of edges with pairwise incomparable sources (with respect to Bruhat order) spans a hypercube.

Let $u, v \in W$ and $z \in [u, v]$. We say that z is an *upper hypercube decomposition* of $[u, v]$ provided that:

(1) $[z, v]$ is *diamond complete* (with respect to $[u, v]$), meaning that, if there exist $x \in [u, v]$ and $a_1, a_2, y \in [z, v]$, $a_1 \neq a_2$, such that $x \to a_1 \to y$, $x \to a_2 \to y$, then $x \in [z, v]$;

(2) for all $p \in [z, v]$, the set $E^p = \{x \to p : x \notin [z, v]\}$ spans a hypercube cluster.

Similarly, taking the dual versions of the above definitions, we obtain the concept of a *lower hypercube decomposition*.

Following [5], we give the next definition. We denote by $\mathrm{d}(x, y)$ the distance function of $B(W)$, i.e., the minimum of the lengths of paths from x to y.

Definition 2.1. Let $u, v, z \in W$ with $z \in [u, v]$. We let

$$W^z_{[u,v]} := \{p \in [z, v] : \mathrm{supp}(\Gamma) \cap [z, v] = \{p\} \text{ for all paths } \Gamma \text{ from } u \text{ to } p \text{ of length } \mathrm{d}(u, p)\},$$

and

$$\widetilde{R}^z_{u,v}(q) := \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}(q).$$

We call the elements in $W^z_{[u,v]}$ the *(upper) shortcuts of* $[u, v]$ *with respect to* z. Furthermore, we say that an element z in $[u, v]$ is an *(upper) R-element for* $[u, v]$ if $\widetilde{R}^z_{u,v} = \widetilde{R}_{u,v}$.

Remark 2.2. The elements $\min([u, v] \cap W_{S\setminus\{s_{n-1}\}} v)$, $\min([u, v] \cap W_{S\setminus\{s_1\}} v)$, $\min([u, v] \cap vW_{S\setminus\{s_{n-1}\}})$, and $\min([u, v] \cap vW_{S\setminus\{s_1\}} v)$ are always upper hypercube decompositions for the interval $[u, v]$ (possibly not mutually distinct). We call these elements the *standard* upper hypercube decompositions for $[u, v]$. Note that these hypercube decompositions are called canonical in [5].

Remark 2.3. We note that the concept of an upper shortcut can be equivalently defined in the following way. Let $u, v \in W$, and z be an upper hypercube decomposition of $[u, v]$. An element p in $[z, v]$ is an upper shortcut of $[u, v]$ with respect to z if $\mathrm{d}(u, p) < \mathrm{d}(u, x)$ for all x in $[z, p]$ satisfying $\mathrm{d}(x, p) = 1$.

3. Amazing hypercube decompositions and double shortcuts

Definition 3.1. Let $u, v \in W$. We say that an upper hypercube decomposition z of $[u, v]$ is *amazing* provided that, for all $x \in [u, v]$, the intersection $[z, v] \cap [x, v]$ has a minimum

$z \wedge x$, and this minimum $z \wedge x$ is a (necessarily amazing) hypercube decomposition of $[x, v]$. Furthermore, we say that an amazing hypercube decomposition z of $[u, v]$ is an *(upper) amazing R-element for* $[u, v]$ if $z \wedge x$ is an (upper) R-element for $[x, v]$ for all $x \in [u, v]$.

Remark 3.2. Let $u, v \in W$ and z be a standard hypercube decomposition of $[u, v]$. Since $z \wedge x$ exists and is a standard hypercube decomposition of $[x, v]$ (of the same kind), we have that z is an amazing hypercube decomposition, and, by [5, Corollary 3.10], an amazing R-element.

The following conjecture is a weak version of Conjecture of [5], which has been verified up to $W = S_6$.

Conjecture 3.3. *Let W be a Coxeter group of type A, $u, v \in W$, and $z \in [u, v]$ be an amazing upper hypercube decomposition. Then z is an R-element, i.e.*

$$\widetilde{R}_{u,v} = \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}.$$

Conjecture 3.3 implies the Combinatorial Invariance Conjecture.

Definition 3.4. Let $u, v \in W$, and z and z' be two amazing upper hypercube decomposition of $[u, v]$. We say that an element b in $[z, v] \cap [z', v]$ is a (z, z')-double shortcut of $[u, v]$ if there exists $p \in W^z_{[u,v]}$ such that $b \in W^{z \wedge z'}_{[p,v]}$. We denote by $DS(z, z')$ the multiset

$$\{(\mathrm{d}(u, p) + \mathrm{d}(p, b), b) : p \in W^z_{[u,v]} \text{ and } b \in W^{z \wedge z'}_{[p,v]}\}.$$

Theorem 3.5. *Let W be a Coxeter group of type A, $u, v \in W$, and $z, z' \in [u, v]$ be two amazing upper hypercube decomposition. Suppose that*

(1) z is an amazing R-element,
(2) $z' \wedge x$ is an (amazing) R-element for all $x \in [u, v] \setminus \{u\}$,
(3) $DS(z, z') = DS(z', z)$ (as multisets).

Then, z' is an (amazing) R-element.

Proof. We have:

$$\begin{aligned}
\widetilde{R}_{u,v} &= \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v} \\
&= \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \Big(\sum_{b \in W^{z' \wedge p}_{[p,v]}} q^{\mathrm{d}(p,b)} \widetilde{R}_{b,v} \Big) \\
&= \sum_{(a,b) \in DS(z,z')} q^a \widetilde{R}_{b,v} \\
&= \sum_{(a,b) \in DS(z',z)} q^a \widetilde{R}_{b,v} \\
&= \sum_{p' \in W^{z'}_{[u,v]}} q^{\mathrm{d}(u,p')} \Big(\sum_{b \in W^{z \wedge p'}_{[p',v]}} q^{\mathrm{d}(p',b)} \widetilde{R}_{b,v} \Big) \\
&= \sum_{p' \in W^{z'}_{[u,v]}} q^{\mathrm{d}(u,p')} \widetilde{R}_{p',v},
\end{aligned}$$

where the first equation follows by (1), the second equation follows by (2), the third equation by the definition of the multiset $DS(z, z')$, the fourth by (3), the fifth by the definition of the multiset $DS(z', z)$, and the sixth equation follows by (1). □

Fix $u, v \in W$. Let us define an equivalence relation on the set of amazing hypercube decompositions of $[u, v]$ as the transitive closure of the relation $z \sim z'$ if $DS(z, z') = DS(z', z)$.

Conjecture 3.6. *Let W be a Coxeter group of type A. The above relation is trivial (i.e., it has only one equivalence class).*

The following conjecture is a weakening of Conjecture 3.6.

Conjecture 3.7. *Let W be a Coxeter group of type A. Every equivalence class of the above relation contains an amazing R-element.*

Remark 3.8. Up to S_6, we checked by computer that, in fact, a much stronger statement than Conjecture 3.7 holds: $DS(z, z') = DS(z', z)$ for all amazing hypercube decompositions of any interval.

Theorem 3.9. *Conjecture 3.7 implies Conjecture 3.3.*

Proof. Towards a contradiction, let $[u, v]$ be an interval of minimal length having an amazing hypercube decomposition z thath fails to be an R-element. By hypothesis, there exists a sequence $(z_0, \dots, z_r)$ of amazing hypercube decompositions such that $z_0 = z$, and z_r is an amazing R-element, and $DS(z_{i-1}, z_i) = DS(z_i, z_{i-1})$ for each $i \in [r]$. One gets a contradiction by iteratively applying Theorem 3.5. □

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