BANACH LIE GROUPOID OF PARTIAL ISOMETRIES OVER RESTRICTED GRASSMANNIAN

TOMASZ GOLIŃSKI, GRZEGORZ JAKIMOWICZ, ANETA SLIŻEWSKA

University of Bia lystok
Faculty of Mathematics
Cio lkowskiego 1M, 15-245 Bia lystok, Poland
email: tomaszg@math.uwb.edu.pl, g.jakimowicz@uwb.edu.pl, anetasl@uwb.edu.pl

ABSTRACT. The differential structure on the set of partial isometries over the restricted Grassmannian is constructed, which makes it a Banach Lie groupoid.

Keywords: partial isometry, restricted Grassmannian, Banach Lie groupoid, Banach manifold, Hilbert–Schmidt operators

CONTENTS

1. INTRODUCTION

The theory of Lie groupoids and algebroids in infinite-dimensional setting is still very young. First results in the Banach context were obtained in [Ana11] and [CP12]. Further results can be found in [BGJP19], where the definitions of the notions of Banach Lie groupoid and algebroid have been proposed and many analogues of finite dimensional results were proven. There was also some work done in a more general context, see e.g. [SW15, AGS18, PC19, Sch22].

An interesting class of examples of Banach Lie groupoids was presented in the paper [OS16]. Namely a structure of Banach Lie groupoid was constructed

1

on set of partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and set of partially invertible elements $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of a W^*-algebra $\mathfrak{M}$. The base of these groupoids is the lattice $\mathcal{L}(\mathfrak{M})$ of projectors in a given W^*-algebra. In follow-up papers [OJS15, OJS18] the Lie algebroid and Poisson side of the picture was investigated. A particular example can be obtained by taking as $\mathfrak{M}$ the W^*-algebra of bounded operators on a separable Hilbert space $L^\infty(\mathcal{H})$. For simplicity we will denote the groupoid of partial isometries in that case by $\mathcal{U}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$. This groupoid is also useful for the study of differential structure of orbits of normal operators, see [BL23]. Moreover in the papers [GT24a, GT24b] the hierarchy of differential equations on $\mathcal{U}(\mathcal{H})$ was discussed.

Generally one can say that there are not so many examples of Banach Lie groupoids and algebroids in the literature so far. The purpose of this paper is to investigate the groupoid of partial isometries $\mathcal{U}_{\mathrm{res}} \rightrightarrows \mathrm{Gr}_{\mathrm{res}}$ related to the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$. From an algebraic point of view it is a subgroupoid of $\mathcal{U}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$, but the topology and differential structure need to be different. Another motivation is the hope that this groupoid will have applications in the integrable systems related to the restricted Grassmannian.

The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ (known also as the Sato Grassmannian) is a strongly symplectic (or even Kähler) manifold modelled on a Hilbert space. It first appeared in the study of the KdV and KP equations, see [SS82, SW85]. It has however many other applications including quantum field theory [SV94, Wur01] or loop groups [PS86, Ser10]. It also provides non-trivial examples of Banach Lie–Poisson spaces, which are useful for studying integrable Hamilton systems, see [BRT07, GO10, GO12], and it is also connected with Banach Poisson–Lie groups [Tum20] and Siegel disc [GBRT24]. As such it is an important object in infinite dimensional differential geometry, see also [Tum07, AL08, Zel06] for wider context.

The structure of the paper is as follows. For the sake of self-consistency we start by recalling basic notions and results which will be needed later on. In Section 2 the definition of Banach Lie groupoid is presented in a particular case of Hausdorff manifolds (see [BGJP19] for general case). Next in Section 3 the construction of the groupoid of partial isometries in the Hilbert space is described (see [OS16]). The Section 4 recalls the definition and basic facts about the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$. The main results of the paper are contained in Section 5, where the set $\mathcal{U}_{\mathrm{res}}$ of partial isometries over $\mathrm{Gr}_{\mathrm{res}}$ is introduced and the structure of Banach Lie groupoid is defined on it. As said before, there are Banach Lie groupoids associated to W^*-algebras and even unital associative Banach algebras.

2. Banach Lie groupoids

Before we start let us fix the terminology that will be used later on. Note that various conventions are used in literature in general. In our paper a smooth map $f : N \to M$ between two Banach manifolds will be called a *submersion* if for each $x \in N$ the tangent map $Tf : T_x N \to T_{f(x)} M$ is a surjection and $\ker T_x f$ is a split subspace of $T_x N$.

Let us recall briefly the definition of the Banach Lie groupoid from the paper [BGJP19]. We restrict our attention here to Hausdorff case. For more detailed discussion we refer to aforementioned paper.

Definition 2.1. A Banach Lie groupoid $\mathcal{G} \rightrightarrows M$ is a pair $(\mathcal{G}, M)$ of Banach manifolds and the following maps:

- surjective submersions $s : \mathcal{G} \to M$ and $t : \mathcal{G} \to M$ called *source* and *target* maps, respectively.
- smooth map $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where

$$\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$$

  is endowed with the induced topology from the product $\mathcal{G} \times \mathcal{G}$, called a *multiplication* denoted $m(g, h) = gh$ and satisfying associativity condition in the sense that the product $(gh)k$ is defined if and only if $g(hk)$ is defined and in this case they coincide.
- continuous embedding $\epsilon : M \to \mathcal{G}$ called *identity section* such that $g\epsilon(x) = g$ for all $g \in s^{-1}(x)$, and $\epsilon(x)g = g$ for all $g \in t^{-1}(x)$.
- diffeomorphism $\iota : \mathcal{G} \to \mathcal{G}$, called *inversion*, which satisfies $g\iota(g) = \epsilon(t(g))$, $\iota(g)g = \epsilon(s(g))$ for all $g \in \mathcal{G}$.

The manifold M is called the base of the groupoid, and $\mathcal{G}$ is called the total space of the groupoid.

It was proved in [BGJP19, Proposition 3.1] (see also [Glö15]) that if those conditions hold, the following facts are true:

(1) $\mathcal{G}^{(2)}$ is a submanifold of $\mathcal{G} \times \mathcal{G}$,
(2) the map ϵ is smooth,
(3) $\epsilon(M)$ is a closed Banach submanifold of $\mathcal{G}$.

Moreover conditions on t can be relaxed as it is automatically a surjective submersion given that s is a surjective submersion and ι is a diffeomorphism as it uniquely defined by them $t = s \circ \iota$.

Standard examples of Banach Lie groupoids include of course Banach Lie groups, pair groupoid on Banach manifold, general linear Banach Lie groupoid of a vector bundle or the the action groupoid of a smooth action of a Banach Lie group on a Banach manifold.

For each Banach Lie groupoid there exists a canonically associated Banach Lie algebroid, but we will not present this construction and the necessary definitions here.

3. Banach Lie groupoid of partial isometries

In this section we recall some of the results of the paper [OS16] adapting them to the particular case of W^*-algebra $L^\infty(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. We focus only on the groupoid of partial isometries $\mathcal{U}(L^\infty(\mathcal{H})) = \mathcal{U}(\mathcal{H})$ and the other groupoid constructed in the paper (groupoid of partially invertible elements) will be subject of a separate study.

By $\mathcal{L}(\mathcal{H})$ we denote the lattice of orthogonal projectors in Hilbert space $\mathcal{H}$

$$\mathcal{L}(\mathcal{H}) = \{p \in L^\infty(\mathcal{H}) \mid p^2 = p^* = p\}.$$

For our purposes it will be useful to identify the projector with its image. In this setting $\mathcal{L}(\mathcal{H})$ can be seen as the Grassmannian of all closed subspaces of $\mathcal{H}$.

The set $\mathcal{U}(\mathcal{H})$ of partial isometries acting on $\mathcal{H}$ is defined by

$$u \in \mathcal{U}(\mathcal{H}) \iff u^*u \in \mathcal{L}(\mathcal{H}) \iff uu^* \in \mathcal{L}(\mathcal{H}).$$

Alternatively they can be described as unitaries between orthogonal complement of their kernel and their image and they fulfill the following equalities

$$u^*uu^* = u^* \qquad uu^*u = u.$$

From the definition it is easy to see that $\mathcal{L}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$.

The groupoid structure on $\mathcal{U}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$ introduced in [OS16] is given by the following natural maps:

$$s(u) = u^*u, \tag{3.1}$$

$$t(u) = uu^*, \tag{3.2}$$

$$m(u_1, u_2) = u_1u_2, \tag{3.3}$$

$$\epsilon(p) = p, \tag{3.4}$$

$$\iota(u) = u^*, \tag{3.5}$$

for $u, u_1, u_2 \in \mathcal{U}(\mathcal{H})$ such that $u_2u_2^* = u_1^*u_1$ and $p \in \mathcal{L}(\mathcal{H})$.

Note that the condition $u_2u_2^* = u_1^*u_1$ implies that the product u_1u_2 is again a partial isometry which is not true in general.

In order to obtain Banach Lie groupoid structure on $\mathcal{U}(\mathcal{H})$ one needs first to describe a differential structure on the Grassmannian $\mathcal{L}(\mathcal{H})$. In the paper [OS16] it is constructed in terms of W^*-algebras but a more direct approach from [AMR88, Section 3.1.8.G] for the case of split Grassmannian of a Banach space is applicable. We will show more straightforward formulas that can be obtained in this particular case.

The construction goes through a family of charts $\phi_W : U_W \to L^\infty(W, W^\perp)$ indexed by $W \in \mathcal{L}(\mathcal{H})$ defined by

$$\phi_W(V) = P_{W^\perp}(P_W|_V)^{-1}, \tag{3.6}$$

where

$$U_W = \{V \in \mathcal{L}(\mathcal{H}) \mid V \oplus_B W^\perp = \mathcal{H}\}, \tag{3.7}$$

P_W is orthogonal projection on W and $\oplus_B$ denotes a direct sum in the sense of Banach spaces (i.e. not necessarily orthogonal). Note that the condition $V \in U_W$ is equivalent to invertibility of $P_W|_V$.

The inverse map $\phi_W^{-1} : L^\infty(W, W^\perp) \to \mathcal{L}(\mathcal{H})$ to a chart assigns to a bounded operator its graph in $W \oplus W^\perp = \mathcal{H}$

$$\phi_W^{-1}(A) = \{(w, Aw) \in W \times W^\perp \mid w \in W\}. \tag{3.8}$$

The transition functions can be computed explicitly, see e.g. [AMR88, PS86, OS16] and are homographies.

The differential structure on $\mathcal{U}(\mathcal{H})$ compatible with the groupoid structure was described in details in [OS16]. We will not present the explicit formulas here. Note that one can adapt the expressions from the Section 5, one only needs to be careful as $\mathcal{U}(\mathcal{H})$ does not act transitively on $\mathcal{L}(\mathcal{H})$ and in consequence one needs to consider each orbit enumerated by dimension and codimension of the subspace separately.

One can also consider $\mathcal{L}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ as level sets inside $L^\infty(\mathcal{H})$ and use this point of view to define a differential structure, see e.g. [PR87, ACM05]. It is straightforward to check that it would produce the same differential structure.

4. Restricted Grassmannian as a Hilbert manifold

In this section we will recall necessary information about the restricted Grassmannian. More detailed exposition can be found in [PS86] or [Wur01].

We fix an orthogonal decomposition (called polarization) of the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{4.1}$$

onto infinite dimensional closed subspaces $\mathcal{H}_\pm$. We will denote by P_+ and P_- the orthogonal projectors onto $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively.

$L^p(\mathcal{H})$ will denote the Schatten class of operators acting in $\mathcal{H}$ equipped with the norm $\|\cdot\|_p$. The $L^p(\mathcal{H})$ spaces are ideals in associative algebra $L^\infty(\mathcal{H})$ of bounded operators in $\mathcal{H}$. In particular $L^1(\mathcal{H})$ denotes the ideal of trace-class operators and $L^2(\mathcal{H})$ is the ideal of Hilbert–Schmidt operators, which possesses additionally the structure of Hilbert space. By $L^0(\mathcal{H}) \subset L^\infty(\mathcal{H})$ one denotes the ideal of compact operators, which is operator norm $\|\cdot\|_\infty$ closure of any Schatten ideal.

In order to simplify the notation throughout the paper we will employ the indices $\pm$ or $\pm\pm$ to denote sets of operators acting in $\mathcal{H}_\pm$. For example L_+^1 would stand for $L^1(\mathcal{H}_+)$ and L_{+-}^2 would stand for $L^2(\mathcal{H}_-, \mathcal{H}_+)$ (note the order of signs in the last symbol). Similarly, one can introduce a block decomposition with respect of the polarization of an operator A acting on $\mathcal{H}$:

$$A = \left(\begin{array}{cc} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{array} \right). \tag{4.2}$$

We will also identify the operators $A_{\pm\pm} : \mathcal{H}_\pm \to \mathcal{H}_\pm$ with $P_\pm A P_\pm$ when it won't lead to any confusion.

The **restricted Grassmannian** $\mathrm{Gr}_{\mathrm{res}}$ is defined as a set of closed subspaces $W \subset \mathcal{H}$ such that:

  i) the orthogonal projection $p_+ : W \to \mathcal{H}_+$ is a Fredholm operator;
 ii) the orthogonal projection $p_- : W \to \mathcal{H}_-$ is a Hilbert–Schmidt operator.

As said before, we identify a closed subspace W with an orthogonal projector onto this subspace, which we denote P_W. Thus we identify $\mathrm{Gr}_{\mathrm{res}}$ with the set of projectors $\{ P_W \mid W \in \mathrm{Gr}_{\mathrm{res}} \} \subset \mathcal{L}(\mathcal{H})$.

From [SV94] one has the following equivalent description of $\mathrm{Gr}_{\mathrm{res}}$:

Proposition 4.1.
$$W \in \mathrm{Gr}_{\mathrm{res}} \Longleftrightarrow P_W - P_+ \in L^2$$

Special role is played by operators of various classes satisfying additionally condition $[X, P_+] \in L^2(\mathcal{H})$. It means that off-diagonal blocks X_{-+}, X_{+-} of block decomposition (4.2) are L^2 class. Note that due to the fact that $L^2(\mathcal{H})$ is an ideal, composition of operators satisfying this condition also satisfies this condition.

One introduces the unitary restricted group U_{res}:

$$U_{\mathrm{res}} := \{ u \in U \mid [u, P_+] \in L^2 \}, \tag{4.3}$$

where U is a unitary group of operators acting in $\mathcal{H}$. It possesses the structure of Banach Lie group with Banach Lie algebra

$$\mathfrak{u}_{\mathrm{res}} := \{ x \in \mathfrak{u} \mid [x, P_+] \in L^2 \}, \tag{4.4}$$

where $\mathfrak{u}$ is a Banach Lie algebra of skew symmetric operators on $\mathcal{H}$. It also possesses the structure of Banach Poisson–Lie group, see [Tum20].

The Banach Lie group U_{res} acts transitively on $\mathrm{Gr}_{\mathrm{res}}$ and the stabilizer of $\mathcal{H}_+$ for this action is $U_+ \times U_-$, which is a Banach Lie subgroup of U_{res}, see [Bel06, Definition 4.1]. In this way, $\mathrm{Gr}_{\mathrm{res}}$ can be seen as a smooth homogeneous space $U_{\mathrm{res}}/(U_+ \times U_-)$.

The differential structure on $\mathrm{Gr}_{\mathrm{res}}$ is obtained using the same charts as in the case of $\mathcal{L}(\mathcal{H})$. One notes though that in case of $\mathrm{Gr}_{\mathrm{res}}$ these charts take value in the Hilbert spaces $L^2(W, W^\perp)$ and transition functions are smooth

with respect to the L^2 topology, see also [PS86, Wur01]. For clarity we will now write down some of the formulas in more details.

Analogously to the previous situation, the inverse of the chart $\phi_W^{-1} : L^2(W, W^\perp) \to$ $\mathrm{Gr}_{\mathrm{res}}$ assigns to a Hilbert–Schmidt operator its graph in $W \oplus W^\perp = \mathcal{H}$ or equivalently the related projector. For particular case of $W = \mathcal{H}_+$ this map assumes the form

$$\phi_{\mathcal{H}_+}^{-1}(A) = \begin{pmatrix} (1 + A^*A)^{-1} & (1 + A^*A)^{-1}A^* \\ A(1 + A^*A)^{-1} & A(1 + A^*A)^{-1}A^* \end{pmatrix} \qquad (4.5)$$

for $A \in L^2_{+-}$. One easily sees using the Proposition 4.1 that indeed $\phi_{H_+}^{-1}$ takes values in $\mathrm{Gr}_{\mathrm{res}}$. Namely, upper right corner of the block decomposition of $\phi_{\mathcal{H}_+}^{-1}(A) - P_+$ reads $(1 + A^*A)^{-1} - 1 = -(1 + A^*A)^{-1}A^*A$. It is easily seen that this and all other terms are L^2.

One can rewrite the formula (4.5) as:

$$\phi_{\mathcal{H}_+}^{-1}(A) = (1_{\mathcal{H}_+} + A)(1_{\mathcal{H}_+} + A^*A)^{-1}(P_+ + A^*P_-). \qquad (4.6)$$

As a consequence we get that for an arbitrary $W \in \mathrm{Gr}_{\mathrm{res}}$ the inverse of the chart can be written as

$$\phi_W^{-1}(A) = (1_W + A)(1_W + A^*A)^{-1}(P_W + A^*P_{W^\perp}). \qquad (4.7)$$

Now, modifying the formula (3.6) we get that the chart $\phi_{\mathcal{H}_+} : \Omega_{\mathcal{H}_+} \to L^2_{+-}$ can be expressed as

$$\phi_{\mathcal{H}_+}(p) = P_- p P_+ (P_+ p P_+)^{-1} = p_{-+}(p_{++})^{-1} = p(p_{++})^{-1} - P_+, \quad (4.8)$$

where

$$\Omega_{\mathcal{H}_+} = \{p \in \mathrm{Gr}_{\mathrm{res}} \mid p_{++} \text{ is invertible in } \mathcal{H}_+\}. \qquad (4.9)$$

This map indeed takes values in L^2_{+-} since $P_- p$ is Hilbert–Schmidt operator by the definition of the restricted Grassmannian.

Thus by direct computation we get that the transition map is equal

$$\psi_{\mathcal{H}_+, W}(A) = \phi_{\mathcal{H}_+} \circ \phi_W^{-1}(A) = \qquad (4.10)$$

$$P_-(1_W + A)(1_W + A^*A)^{-1}(P_W + A^*P_{W^\perp})P_+\big(P_+(1_W + A)(1_W + A^*A)^{-1}(P_W + A^*P_{W^\perp})P_+\big)^{-1}$$

for $A \in L^2(W, W^\perp)$ such that $\phi_W^{-1}(A) \in \Omega_{\mathcal{H}_+}$. This condition on A means that both the projections P_W and P_+ restricted to graph of A are invertible. Explicitly it is equivalent to the fact that the operator $P_+(1_W + A)(1_W + A^*A)^{-1}(P_W + A^*P_{W^\perp})_{\mathcal{H}_+}$ is invertible in $\mathcal{H}_+$. Observe that this operator is a product of three terms and the middle one $(1_W + A^*A)^{-1}$ is always invertible in W. Moreover operator $(1_W + A)$ is an isomorphism from W to the graph of A. In consequence the operator $P_+(1_W + A) : W \to P_+$, under the stated assumptions on A, is also invertible. It follows now that third term is also invertible and inverting the expression term-by-term we obtain

$$\psi_{\mathcal{H}_+, W}(A) = P_-(1_W + A)\big(P_+(P_W + A)\big)^{-1}. \qquad (4.11)$$

As expected, it is a fractional map and in consequence it is smooth as a composition of the inverse map in the space $L^\infty(W, \mathcal{H}_+)$ which is smooth and multiplication $L^2(W, \mathcal{H}_-) \times L^\infty(\mathcal{H}_+, W) \to L^2(\mathcal{H}_+, \mathcal{H}_-)$ which is linear and bounded.

Note that partial inverse (or Moore–Penrose inverse) is in general not smooth and not even continuous, see the discussion in [BGJP19] and references therein.

5. Banach Lie groupoid of partial isometries over restricted Grassmannian

Since $\mathrm{Gr}_{\mathrm{res}}$ is a subset of $\mathcal{L}(\mathcal{H})$, one can define a groupoid $\mathcal{U}_{\mathrm{res}} \rightrightarrows \mathrm{Gr}_{\mathrm{res}}$ as a subgroupoid of $\mathcal{U}(\mathcal{H})$ by

$$\mathcal{U}_{\mathrm{res}} = s^{-1}(\mathrm{Gr}_{\mathrm{res}}) \cap t^{-1}(\mathrm{Gr}_{\mathrm{res}}) \tag{5.1}$$

or equivalently

$$\mathcal{U}_{\mathrm{res}} = \{u \in \mathcal{U}(\mathcal{H}) \mid u^*u, uu^* \in \mathrm{Gr}_{\mathrm{res}}\}. \tag{5.2}$$

Directly from this definition it follows that this set is a subgroupoid of $\mathcal{U}(\mathcal{H})$ in the algebraic sense. However one still needs to introduce a differential structure on it to make it into a Banach Lie groupoid in the sense of Definition 2.1.

As a consequence of definition of $\mathcal{U}_{\mathrm{res}}$ we note that:

Proposition 5.1. *For $u \in \mathcal{U}_{res}$ we have $u_{+-}, u_{-+} \in L^2$.*

Proof. First of all the conclusion holds trivially for the case $u^*u = uu^* = P_+$, i.e. $u \in U_+$. Now since U_{res} acts transitively on $\mathrm{Gr}_{\mathrm{res}}$, any $u \in \mathcal{U}_{\mathrm{res}}$ can be written as $u = g_1 \tilde{u} g_2$ for $\tilde{u} \in U_+$ and $g_1, g_2 \in U_{\mathrm{res}}$. The proposition follows since all operators on the right hands side of this expression have off-diagonal blocks of Hilbert–Schmidt class.   □

To avoid confusion let us note that unitary restricted group U_{res} is not contained in $\mathcal{U}_{\mathrm{res}}$ (for example identity operator belongs to U_{res}, but not to $\mathcal{U}_{\mathrm{res}}$). That also shows that the conditions from the Proposition 5.1 above are not sufficient for u to belong to $\mathcal{U}_{\mathrm{res}}$.

In order to define a differential structure on $\mathcal{U}_{\mathrm{res}}$ we need first the following fact:

Proposition 5.2. *For every point $W \in \mathrm{Gr}_{\mathrm{res}}$ there exists a neighbourhood $\Omega_W \subset \mathrm{Gr}_{\mathrm{res}}$ and a smooth map $\sigma_W : \Omega_W \to U_{\mathrm{res}}$ such that*

$$\forall\, W' \in \Omega_W \quad W' = \sigma_W(W')\mathcal{H}_+. \tag{5.3}$$

Proof. We treat $\mathrm{Gr}_{\mathrm{res}}$ as a homogeneous space $U_{\mathrm{res}}/(U_+ \times U_-)$. If by $\pi : U_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ we denote the projection onto the orbits, then σ_W is a local cross section of π, i.e. $\pi \circ \sigma = \mathrm{id}$.

Since $U_+ \times U_-$ is a Banach Lie subgroup of U_{res}, the theorem [Bel06, Theorem 4.19] implies the result. $\square$

Using a family of such local cross sections we can transport every partial isometry to a unitary operator $\sigma_W(uu^*)^{-1}\, u\, \sigma_{W'}(u^*u)_{|\mathcal{H}_+}$ acting in the Hilbert space $\mathcal{H}_+$. In this way we construct injective maps:

$$\mathcal{U}_{\mathrm{res}} \supset s^{-1}(W') \cap t^{-1}(W) \longrightarrow \mathrm{Gr}_{\mathrm{res}} \times \mathrm{Gr}_{\mathrm{res}} \times U_+ \qquad (5.4)$$

given by

$$u \mapsto (uu^*, u^*u, \sigma_W(uu^*)^{-1}\, u\, \sigma_{W'}(u^*u)_{|\mathcal{H}_+}), \qquad (5.5)$$

which assign to a partial isometry its final space, initial space and a unitary operator in $\mathcal{H}_+$. Naturally, knowing value of this map it is possible to reconstruct the partial isometry.

Now we couple these maps with charts on the respective manifolds. Namely let (V_α, ψ_α) denote the atlas on U_+, where $V_\alpha \subset U_+$ are open sets and $\psi_\alpha : V_\alpha \to \mathfrak{u}_+$. By $(\Omega_\beta, \sigma_\beta)$ denote a family of cross sections (5.3) covering $\mathrm{Gr}_{\mathrm{res}}$. Assume additionally that sets Ω_β are chosen open and small enough that they are contained in chart domains of $\mathrm{Gr}_{\mathrm{res}}$, i.e. there exists a map $\tilde{\psi}_\beta : \Omega_\beta \to L^2(W_\beta, W_\beta^\perp)$ for appropriately chosen $W_\beta \in \Omega_\beta \subset \mathrm{Gr}_{\mathrm{res}}$.

Denote by $\Omega_{\alpha\beta\gamma}$ the set

$$\Omega_{\alpha\beta\gamma} = \{u \in \mathcal{U}_{\mathrm{res}} \mid u^*u \in \Omega_\beta,\, uu^* \in \Omega_\gamma,\, (\sigma_\gamma(uu^*)^{-1}u\sigma_\beta(u^*u))_{|\mathcal{H}_+} \in V_\alpha\} \qquad (5.6)$$

and by $\Phi_{\alpha\beta\gamma} : \Omega_{\alpha\beta\gamma} \to L^2(W_\gamma, W_\gamma^\perp) \times L^2(W_\beta, W_\beta^\perp) \times \mathfrak{u}_+$ the map

$$\Phi_{\alpha\beta\gamma}(u) = (\tilde{\psi}_\gamma(uu^*), \tilde{\psi}_\beta(u^*u), \psi_\alpha(\sigma_\gamma(uu^*)^{-1}u\sigma_\beta(u^*u))_{|\mathcal{H}_+}). \qquad (5.7)$$

Theorem 5.3. *The family $(\Omega_{\alpha\beta\gamma}, \Phi_{\alpha\beta\gamma})$ constitutes a smooth atlas on the set of partial isometries $\mathcal{U}_{\mathrm{res}}$.*

Proof. The collection of all sets $\Omega_{\alpha\beta\gamma}$ for all values of indices covers $\mathcal{U}_{\mathrm{res}}$ and the maps $\Phi_{\alpha\beta\gamma}$ are injective. Their inverses can be written as

$$\Phi_{\alpha\beta\gamma}^{-1}(A, B, X) = \sigma_\gamma(\tilde{\psi}_\gamma^{-1}(A))\psi_\alpha^{-1}(X)\sigma_\beta(\tilde{\psi}_\beta^{-1}(B))^{-1} \qquad (5.8)$$

for $A \in L^2(W_\gamma, W_\gamma^\perp)$, $B \in L^2(W_\beta, W_\beta^\perp)$, $X \in \mathfrak{u}_+$ belonging to the image of $\Phi_{\alpha\beta\gamma}$. Note that we consider here $\psi_\alpha^{-1}(X)$ as an operator in $\mathcal{H}$ by extending it trivially from $\mathcal{H}_+$.

It remains to demonstrate that image is open and that the transition functions $\Phi_{\alpha\beta\gamma} \circ \Phi_{\alpha'\beta'\gamma'}^{-1}$ are smooth.

To prove openness of $\Phi_{\alpha\beta\gamma}(\Omega_{\alpha\beta\gamma})$ notice that the sets $\tilde{\psi}_\beta(\Omega_\beta)$, $\tilde{\psi}_\gamma(\Omega_\gamma)$, $\psi_\alpha(V_\alpha)$ are open. For any point in $\tilde{\psi}_\beta(\Omega_\beta) \times \tilde{\psi}_\gamma(\Omega_\gamma) \times \psi_\alpha(V_\alpha)$ the right hand side of formula (5.8) is well-defined and produces a partial isometry which by construction belongs to $\Omega_{\alpha\beta\gamma}$. Thus we conclude that the image $\Phi_{\alpha\beta\gamma}(\Omega_{\alpha\beta\gamma})$ coincides with the Cartesian product $\tilde{\psi}_\beta(\Omega_\beta) \times \tilde{\psi}_\gamma(\Omega_\gamma) \times \psi_\alpha(V_\alpha)$.

In order to prove smoothness of transition functions $\Phi_{\alpha\beta\gamma} \circ \Phi_{\alpha'\beta'\gamma'}^{-1}$ we will split them into three components:

$$\Phi_{\alpha\beta\gamma} \circ \Phi_{\alpha'\beta'\gamma'}^{-1} : (A, B, X) \mapsto (A', B', X'), \tag{5.9}$$

where

$$A' = \tilde{\psi}_\gamma \circ \tilde{\psi}_{\gamma'}^{-1}(A), \tag{5.10}$$

$$B' = \tilde{\psi}_\beta \circ \tilde{\psi}_{\beta'}^{-1}(B), \tag{5.11}$$

$$X' = \psi_\alpha\big(\sigma_\gamma(\tilde{\psi}_{\gamma'}^{-1}(A))^{-1}\sigma_{\gamma'}(\tilde{\psi}_{\gamma'}^{-1}(A))\psi_{\alpha'}^{-1}(X)\sigma_{\beta'}(\tilde{\psi}_{\beta'}^{-1}(B))^{-1}\sigma_\beta(\tilde{\psi}_{\beta'}(B))|_{\mathcal{H}_+}\big). \tag{5.12}$$

The expressions on A' and B' are smooth since they are just transition maps for the manifold $\mathrm{Gr}_{\mathrm{res}}$. The expression on X' is a composition of smooth maps (between manifolds), operator multiplication and inversion map in the Banach Lie group U_{res}. In consequence it is also smooth. $\square$

Now we proceed to prove that $\mathcal{U}_{\mathrm{res}}$ is a Banach Lie groupoid in the sense of definition 2.1. Note that source s and target t maps given by (3.1) and (3.2) in the introduced charts (5.7) are exactly projection onto first and second component of the Cartesian product. As such they are submersions. Inversion map ι (3.5) assumes the form

$$\Phi_{\beta\alpha'\gamma} \circ \iota \circ \Phi_{\alpha\beta\gamma}^{-1}(A, B, X) = (B, A, \psi_\alpha'((\psi_\alpha^{-1}(X))^*)) \tag{5.13}$$

and taking into account the fact that $*$ is an inversion map in the Banach Lie group U_{res}, it is smooth. Now, multiplication map can be written as

$$\Phi_{\alpha''\beta'\gamma}(m(\Phi_{\alpha\beta\gamma}^{-1}(A, B, X), \Phi_{\beta\beta'\gamma'}^{-1}(B, C, X'))) = (A, C, \psi_\alpha''(\psi_\alpha^{-1}(X)\psi_{\alpha'}^{-1}(X'))). \tag{5.14}$$

Again since U_{res} is a Banach Lie group, the multiplication is smooth.

Identity section ϵ (3.4) in the coordinates assumes the form

$$\Phi_{0\beta\beta} \circ \epsilon \circ \tilde{\psi}_\beta^{-1}(A) = (A, A, 0), \tag{5.15}$$

where ψ_0 is a particular chart on identity component of U_+ mapping unit element to $0 \in \mathfrak{u}_+$.

Thus we have proved what follows.

Theorem 5.4. *The manifold of partial isometries $\mathcal{U}_{res}$ is a Banach Lie groupoid with respect to the maps* (3.1)-(3.5)*.*

Unlike $\mathcal{U}(\mathcal{H})$, the groupoid $\mathcal{U}_{\mathrm{res}}$ is transitive and pure, i.e. the map $(s, t) : \mathcal{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}} \times \mathrm{Gr}_{\mathrm{res}}$ is surjective and both base and total space are modeled on a single (up to isomorphism) Banach space.

Note that $\mathcal{U}_{\mathrm{res}}$ is not a Banach Lie subgroupoid of the Banach Lie groupoid of all partial isometries $\mathcal{U}(\mathcal{H})$. The obstacle is evident even on the level of bases of those groupoids. Namely, the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ is not a submanifold of the Grassmannian $\mathcal{L}(\mathcal{H})$. It is only a weakly immersed

submanifold (see [BGJP19]) as the image of the tangent of the inclusion map is not closed.

One might consider a family of more general p-restricted Grassmannians $\text{Gr}_{\text{res}}^{\text{p}}$, $1 < p < \infty$ or $p = 0$ by replacing L^2 space in the definition with L^p. Note that Grassmannian Gr_{res}^0 was actually the first one to be introduced, see [SW85]. For all those Grassmannians it is possible to define a structure of Banach manifold in analogous way as in Section 4.

Proposition 5.5. *The sets $\mathcal{U}_{res}^p$ defined*

$$\mathcal{U}_{res}^p = s^{-1}(\text{Gr}_{\text{res}}^{\text{p}}) \cap t^{-1}(\text{Gr}_{\text{res}}^{\text{p}}) \tag{5.16}$$

are Banach Lie groupoids modelled on the Banach spaces $L^p(W, W^\perp) \times L^p(W', W'^\perp) \times \mathfrak{u}_+$.

The proof is completely analogous to the proofs of Theorems 5.3 and 5.4. As before they are not Banach Lie subgroupoids due to the difference in topologies.

Proposition 5.6. *The Banach Lie groupoid $\mathcal{U}_{res}^0 \rightrightarrows \text{Gr}_{\text{res}}^0$ is an immersed submanifold of $\mathcal{U}_{res} \rightrightarrows \mathcal{L}(\mathcal{H})$, but not split immersed submanifold.*

Proof. It follows from the fact that the ideal of compact operators L^0 is closed with respect to operator norm. However it doesn't admit a complementary Banach space, i.e. there is no such E that $L^0 \oplus E = L^\infty$. □

Acknowledgement

References

[ACM05] E. Andruchow, G. Corach, M. Mbekhta: On the geometry of generalized inverses. *Mathematische Nachrichten*, **278**(7-8):756–770, 2005.

[AGS18] H. Amiri, H. Glöckner, A. Schmeding: Lie groupoids of mappings taking values in a Lie groupoid. *arXiv:1811.02888*, 2018.

[AL08] E. Andruchow, G. Larotonda: Hopf–Rinow theorem in the Sato Grassmannian. *J. Funct. Anal.*, **255**(7):1692–1712, 2008. doi:https://doi.org/10.1016/j.jfa.2008.07.027.

[AMR88] R. Abraham, J. E. Marsden, T. S. Ratiu: Manifolds, Tensor Analysis, and Applications, *Applied Mathematical Sciences*, volume 75. Springer-Verlag, Berlin-Heidelberg, second edition, 1988.

[Ana11] M. Anastasiei: Banach Lie algebroids. *Analele Ştiinţifice Ale Universităţii "Alexandru Ioan Cuza" Din Iaşi – Matematică*, **57**, 2011.

[Bel06] D. Beltiţă: Smooth homogeneous structures in operator theory. CRC Press, 2006.

[BGJP19] D. Beltiţă, T. Goliński, G. Jakimowicz, F. Pelletier: Banach–Lie groupoids and generalized inversion. *J. Funct. Anal.*, **276**(5):1528–1574, 2019.

[BL23] D. Beltiţă, G. Larotonda: Unitary group orbits versus groupoid orbits of normal operators. *J. Geom. Anal.*, **33**(3):95, 2023.

[BRT07] D. Beltiţă, T. S. Ratiu, A. B. Tumpach: The restricted Grassmannian, Banach Lie–Poisson spaces, and coadjoint orbits. *J. Funct. Anal.*, **247**:138–168, 2007.

[CP12] P. Cabau, F. Pelletier: Almost Lie structures on an anchored Banach bundle. *J. Geom. Phys.*, **62**(11):2147–2169, 2012.

[GBRT24] F. Gay-Balmaz, T. S. Ratiu, A. B. Tumpach: The restricted siegel disc as coadjoint orbit. In XL Workshop on Geometric Methods in Physics (edited by P. Kielanowski, D. Beltiţă, A. Dobrogowska, T. Goliński), Trends in Mathematics. Birkhäuser, 2024.

[Glö15] H. Glöckner: Fundamentals of submersions and immersions between infinite-dimensional manifolds. *arXiv:1502.05795*, 2015.

[GO10] T. Goliński, A. Odzijewicz: Hierarchy of Hamilton equations on Banach Lie–Poisson spaces related to restricted Grassmannian. *J. Funct. Anal.*, **258**:3266–3294, 2010.

[GO12] T. Goliński, A. Odzijewicz: Hierarchy of integrable Hamiltonians describing nonlinear n-wave interaction. *J. Phys. A Math. Theor.*, **45**(4):045204, 2012.

[GT24a] T. Goliński, A. B. Tumpach: Geometry of integrable systems related to the restricted Grassmannian. *to appear*, 2024.

[GT24b] T. Goliński, A. B. Tumpach: Integrable system on partial isometries: a finite dimensional picture. In XL Workshop on Geometric Methods in Physics (edited by P. Kielanowski, D. Beltiţă, A. Dobrogowska, T. Goliński), Trends in Mathematics. Birkhäuser, 2024.

[OJS15] A. Odzijewicz, G. Jakimowicz, A. Sliżewska: Banach–Lie algebroids associated to the groupoid of partially invertible elements of a W^*-algebra. *J. Geom. Phys.*, **95**:108–126, 2015.

[OJS18] A. Odzijewicz, G. Jakimowicz, A. Sliżewska: Fiber-wise linear Poisson structures related to W^*-algebras. *J. Geom. Phys.*, **123**:385–423, 2018.

[OS16] A. Odzijewicz, A. Sliżewska: Banach–Lie groupoids associated to W^*-algebras. *J. Sympl. Geom.*, **14**:687–736, 2016.

[PC19] F. Pelletier, P. Cabau: Convenient partial Poisson manifolds. *J. Geom. Phys.*, **136**:173–194, 2019.

[PR87] H. Porta, L. Recht: Minimality of geodesics in Grassmann manifolds. *Proc. Amer. Math. Soc.*, **100**(3):464–466, 1987.

[PS86] A. Pressley, G. B. Segal: Loop Groups. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1986.

[Sch22] A. Schmeding: An introduction to infinite-dimensional differential geometry, *Cambridge Studies in Advanced Mathematics*, volume 202. Cambridge University Press, 2022.

[Ser10] A. Sergeev: Kähler geometry of loop spaces. Mathematical Society of Japan, 2010.

[SS82] M. Sato, Y. Sato: Soliton equations as dynamical systems on infinite dimensional Grassmann manifold. *Lect. Notes in Num. Appl. Anal.*, **5**:259–271, 1982.

[SV94] M. Spera, G. Valli: Plücker embedding of the Hilbert space Grassmannian and the CAR algebra. *Russ. J. Math. Phys.*, **2**:383–392, 1994.

[SW85]   G. B. Segal, G. Wilson: Loop groups and equations of KdV type. *Inst. Hautes Études Sci. Publ. Math.*, **61**:5–65, 1985.

[SW15]   A. Schmeding, C. Wockel: The Lie group of bisections of a Lie groupoid. *Annals of Global Analysis and Geometry,* **48**(1):87–123, 2015.

[Tum07]   A. B. Tumpach: Hyperkähler structures and infinite-dimensional Grassmannians. *J. Funct. Anal.*, **243**(1):158–206, 2007.

[Tum20]   A. B. Tumpach: Banach Poisson–Lie groups and Bruhat–Poisson structure of the restricted Grassmannian. *Comm. Math. Phys.*, **373**(3):795–858, 2020.

[Wur01]   T. Wurzbacher: Fermionic second quantization and the geometry of the restricted Grassmannian. In Infinite Dimensional Kähler Manifolds, DMV Seminar, volume 31 (edited by A. Huckleberry, T. Wurzbacher). Birkhäuser, Basel, 2001.

[Zel06]   M. I. Zelikin: Geometry of operator cross ratio. *Sb. Math.*, **197**(1):37–51, 2006.