# Orbital Stability of Optical Solitons in 2d 

S. Moroni


#### Abstract

We present a stability result for ground states of a Schrodinger-Poisson system in $(2+1)$ dimension, modelling the propagation of a light beam through a liquid crystal with nonlocal nonlinear response. A new estimate for perturbations of the medium configuration allows to explicitly prove strict positivity of the second derivative of the action on a ground state. In addition we prove existence of a ground state with frequency $\sigma$ for any $\sigma \in(0,1)$ by a continuity method.


## Introduction

Optical properties of nematic liquid crystals have received great attention in the last years, as they can support stationary optical waves, of large interest both in theory and in applications. Heuristically, when a light wave propagates through a nematic liquid crystal, its electric field induces a dipolar polarization in the anisotropic medium. The electromagnetic action of the dipoles cause a reorientation of the molecules in the liquid crystal, and hence a modification of the light refractive index of the material. Due to high susceptibility of nematic liquid crystals, the response is nonlocal, meaning that has effects far beyond the region occupied by the light wave, and nonlinear. This response has a self-focusing effect on the light beam, supporting waveguides that counterbalance the diffraction spreading nature of light beam, and, in optimal shapes, allows the existence of stationary waves.
The interested reader is referred to [11] or [1], for a physical overview of the topic and a presentation of the main experiments in the field, or to [9], Chapters 2 and 6, for a wider mathematical introduction.

In this paper we study the ground states, proving orbital stabilty, existence for any
frequency $0<\sigma<1$ and a decay estimate, of the Schrodinger-Poisson system

$$
\begin{align*}
& i \partial_{z} u+\frac{1}{2} \Delta u+u \sin (2 \theta)=0  \tag{1}\\
& -v \Delta \theta+q \sin (2 \theta)=2|u|^{2} \cos (2 \theta) \tag{2}
\end{align*}
$$

in dimension $(2+1)$. The axis $z$, referred to as the optical axis, is the direction of the propagation of a light beam, while $\Delta$ is the Laplacian in the transverse coordinates $(x, y)$.
The system models the propagation of a laser beam through a planar cell filled with a nematic liquid crystal, oriented by an external electric field $E$. Equation (1) represents the evolution of the light beam, with $u: \mathbb{R}^{2} \rightarrow \mathbb{C}$ the complex amplitude of the electric field, , while (2) is the nonlocal response of the medium, with $\theta$ : $\mathbb{R}^{2} \rightarrow \mathbb{R}$ the director field angle of the light-induced reorientation. The values $q, v$ are positive constants depending, respectively, on the intensity of the pre-tilting electric field and on the elastic response of the medium, that is on its property of nonlocality. In [4] a heuristic derivation of the equations is presented in the Appendix, while [11], [1] and the references therein give a deeper understanding of the system and of the related observed phenomena.

The system was rigorously studied in [4], where the authors proved global existence and regularity for the Cauchy problem, and existence of stationary waves as minimizers, over couples $(u, \theta)$ with $L^{2}$ norm of $u$ fixed, of the Hamiltonian:

$$
\begin{equation*}
E(u, \theta):=\frac{1}{4} \int_{\mathbb{R}^{2}}|\nabla u|^{2}+v|\nabla \theta|^{2}-2|u|^{2} \sin (2 \theta)+q(1-\cos (2 \theta)) d x \tag{3}
\end{equation*}
$$

The minimal configurations ( $v, \phi$ ) satisfy the equations

$$
\begin{align*}
& -\Delta v+2 \sigma v-2 v \sin (2 \phi)=0  \tag{4}\\
& -v \Delta \phi+q \sin (2 \phi)-2|v|^{2} \cos (2 \phi)=0 \tag{5}
\end{align*}
$$

where $\sigma \in \mathbb{R}$ is the Lagrange multiplier associated to the constraint. The couple ( $\left.e^{i \sigma} v(x, y), \phi(x, y)\right)$ is then a stationary wave for the system (1)-(2) as it evolves along the optic axis changing only by a phase shift of frequency $\sigma$.

We will present a first stability result for those stationary waves. This provides a strong justification of the relevance of the mathematical model to applications, as only locally stable solutions are expected to be seen in experiments and numerical simulations. The main result of the paper is

Theorem 1. Let $(v, \phi)$ be the configuration of minimal energy E over the constraint

$$
S_{a}:=\left\{(u, \theta) \in H^{1} \times H^{1} \mid\|u\|_{L^{2}}^{2}=a\right\}
$$

Then $(\nu, \phi)$ is orbitally stable.
The definition of orbital stability will be given in the first section; loosely speaking, we ask that the evolution through equations (1)-(2) of an initial datum close to ( $v, \phi$ ) remains close up to symmetries to the ground state for all times.

We recall that the previous paper [10] studied a linearization of system (1)-(2), obtained by a Taylor approximation for small values of $\theta$. Existence of stationary waves as minimizers of the Hamiltonian over the constraint $S_{a}$ was proved, similarly to the non linear setting. In that case the problem is reduced to the study of only the variable $u$, as the linear equation allows for an explicit expression of $\theta$ in terms of $u$. It is then proved, using a concentration-compactness argument, for $a$ above a threshold, that all the minimizing sequences converge to a minimizer up to translation. At this point it is straightforward to deduce stability of the ground states of the Schrodinger-Poisson equation; see Corollary [2.3] in [10], or [2].
In the non linear case, that we focus on, it is not possible to decouple the system, as equation (2) has no explicit solution, and the concentration compactness argument is harder. In [4] existence of ground states was recovered through rearrangements techniques, noticing that all the radial minimizing sequence converge to a ground state. Since compactness holds only for this family of minimizing sequence, stability of ground states is not a direct consequence of the variational method that proves existence.

In our proof, we adapt to the coupled system the arguments of [7] and [6], who obtained the result from the positivity of the second derivative of the action $S$. In our framework the action is defiened as

$$
S(u, \theta):=E(u, \theta)+\frac{\sigma}{2} \int_{\mathbb{R}^{2}}|u|^{2} d x
$$

We provide at first a refined estimate for the $H^{1}$ norm of small perturbations of the angle variable $\phi$ around a ground state, that can be controlled by the norm of the associated perturbation of the amplitude. This allows us to simplify the stabilty problem to the evolution of variable $u$ only.
Using minimality over $S_{a}$, we can bound from below the second derivative of the energy $E$ evaluated in ( $v, \phi$ ) with a small negative constant. By multiplication method for equation (4) and simple energy consideration, we prove that the value $\sigma$ for a ground state is strictly positive. Finally, convexity of the charge functional
$\|u\|_{L^{2}}^{2}$ implies strong positivity for the second derivative of $S$.
In the second result of our paper, we prove the existence of a stable stationary wave for the system for any frequency value $\sigma \in(0,1)$ :
Theorem 2. For any $0<\sigma<1$ there exists $a>0$ and a minimizer $(v, \phi)$ for the energy $E$ over the constraints $S_{a}$ that satisfies (4)-(5) with the fixed value of $\sigma$.

Theorem 2 suggests the existence of a physical stable soliton of frequency $\sigma$ for any $\sigma$ positive and sufficiently small, namely for any $\sigma \in\left(0, \sigma_{*}\right)$ with $\sigma_{*}<1$. Physical relevance of the result is lost as the values of $\sigma$ approach 1 : in the proof will be evident that the corresponding norm constraints $a$ tend to infinity, but the model (1)-(2) is a physical approximation for $\theta$, and hence $u$, small.
We prove the result starting from equation (4), which relates the value $\sigma$ with

$$
E^{-}(u, \theta):=\frac{1}{4} \int_{\mathbb{R}^{2}}|\nabla u|^{2}-2|u|^{2} \sin (2 \theta) d x
$$

that is quadratic homogeneous in $u$.
We can prove that $E^{-}$evaluated on a ground state over $S_{a}$ is a continuous monotone function of $a$. The main obstacle is non uniqueness of the minimizer for the constrained problem, which could lead to different values of $E^{-}$, and hence of $\sigma$ for a fixed norm $a$. We can overcome it proving that all the minimizers over $S_{a}$ share the same values of $E^{-}$.
As there exists a ground state for every $a$ above a certain threshold, we can conclude the proof showing that all the values $\sigma \in(0,1)$ can be reached. Getting existence for $\sigma$ up to 1 follows from simple energy asymptotic for $\|u\| \rightarrow \infty$, while for the values close to 0 the argument is more delicate. We modify the variational arguments in [4] to show a relation between the values of $E^{-}$over some radial minimizing sequence in $S_{a}$ and the existence of a ground state over $S_{a}$. By a continuity method, this obtains existence of a ground state for smaller norm constraint, as long as $E^{-}$, and hence $\sigma$, is far from 0 .

The matter of the paper is organised as follows. In the first section we prove Theorem 1, recalling the main steps of the argument of [7] and proving explicitly the positivity for the derivative of the action. In the second we prove Theorem 2, refining the ideas for the existence of minimizers in [4]. Finally, in the last section we prove a result on the regularity, and in particular the decay, of the ground states.

## 1 Stability of ground states

We start recalling the precise results from [4] that will be used in our paper, and giving the definition of orbital stability:

Theorem 3. Let $J_{a}:=\inf _{S_{a}}$ E. For a above a certain threshold $a_{0}, J_{a}<0$ and is decreasing, while $J_{a} \equiv 0$ for $a \leq a_{0}$.
For $a>a_{0}$ there exists a minimizer $(v, \phi)$ for $J_{a}$ and it satisfies the system (4)-(5) for a real $\sigma$. Moreover we have $(v, \phi) \in H_{r a d}^{1} \times H_{r a d}^{1}$ decreasing, $v \geq 0,0 \leq \phi \leq \pi / 4$. Finally, there exists a $0<\tilde{a} \leq a_{0}$ such that there is no ( $v, \phi$ ) solution to equations (4)-(5) with $\|v\|_{L^{2}}^{2} \leq \tilde{a}$.

Theorem 4. Given $u \in L^{\infty}\left(\mathbb{R}^{2}\right) \cap L^{4}\left(\mathbb{R}^{2}\right)$, there exists a unique $\theta \in H^{2}\left(\mathbb{R}^{2}\right)$ solution of

$$
\begin{equation*}
-v \Delta \theta+q \sin (2 \theta)=2|u|^{2} \cos (2 \theta) \tag{6}
\end{equation*}
$$

satisfying $0 \leq \theta \leq \pi / 4$ and $\|\theta\|_{H^{2}} \leq C\|u\|_{L^{4}}$.
Furthermore, defining $\Theta(u)$ the map associated to $u \in L^{\infty} \cap H^{1}$ in the previous result, we have the following estimate

$$
\begin{equation*}
\left\|\Theta\left(u_{1}\right)-\Theta\left(u_{2}\right)\right\|_{H_{2}} \leq C_{q, v}\left(\left\|u_{1}\right\|_{H^{1}},\left\|u_{2}\right\|_{H^{1}}\right)\left(1+\left\|u_{1}\right\|_{L^{\infty}}+\left\|u_{2}\right\|_{L^{\infty}}\right)\left\|u_{1}-u_{2}\right\|_{H^{1}} \tag{7}
\end{equation*}
$$

Theorem 5. Given $u_{0} \in H^{1}\left(\mathbb{R}^{2}\right)$, there exists a unique $(u, \theta) \in C\left(\mathbb{R}, H^{1}\left(\mathbb{R}^{2}\right)\right) \times$ $L^{\infty}\left(\mathbb{R}, H^{2}\left(\mathbb{R}^{2}\right)\right)$ solution of the evolution problem (1)-(2) with initial datum $u_{0}$, such that $0 \leq \theta \leq \pi / 4, \nabla u \in L_{\text {loc }}^{4}\left(\mathbb{R}, L^{4}\left(\mathbb{R}^{2}\right)\right)$. Moreover, the quantities

$$
E(u, \theta) ; \quad Q(u):=\frac{\int_{\mathbb{R}^{2}}|u|^{2}}{2}
$$

are preserved for all times.
Theorem 3 recaps the results from Chapter 5 of [4], Theorem 4 resumes Proposition [3.1] and [4.1], Theorem 5 comes from Theorem [4.1] and [4.2].

Definition 6. Let $(v, \phi)$ be a stationary solution. We say that the orbit of $v$ is orbitally stable if for every $\varepsilon>0$ there exists $a \delta>0$ s. $t$.

$$
\begin{equation*}
\left\|u_{0}-v\right\|_{H^{1}\left(\mathbb{R}^{2}\right)} \leq \delta \longrightarrow \sup _{t} \inf _{y \in \mathbb{R}^{2}, \alpha \in \mathbb{R}}\left\|u(\cdot, t)-e^{i \alpha} v(\cdot-y)\right\|_{H^{1}}<\varepsilon \tag{8}
\end{equation*}
$$

where $u(t)$ is the solution of (1)-(2) with initial condition $u_{0}$.
Remark 1. Even though we have an evolution both for the orientation angle $\theta$ and the amplitude $u$, in the previous definition we have considered only the latter. At the end there is no consistent difference, as in the implication of (8) we can equally write

$$
\begin{equation*}
\sup _{t} \inf _{\alpha \in \mathbb{R}, y \in \mathbb{R}^{2}} \|\left(u(\cdot, t), \Theta(u(\cdot, t))-\left(e^{i \alpha} v(\cdot-y), \Theta\left(e^{i \alpha} v(\cdot-y)\right) \|_{H^{1} \times H^{1}}<C \varepsilon\right.\right. \tag{9}
\end{equation*}
$$

In fact we will show that if the amplitude configuration $u$ is close in $H^{1}$ to $v$, amplitude of a ground state, then the corresponding angle variables are still close in $H^{1}$.

We turn now to present an improvement of the estimate (7), before proving Theorem 1 following the ideas of [6] and [7].

### 1.1 Control of the angle deviation

In Theorem 4, the authors have already shown that the $H^{2}$-norm of the difference between two angles orientations is bounded by the difference in $H^{1}$ of the respective amplitudes. This is the type of control we need, but the previous result cannot be satisfying for our purposes. The constant on the right hand side of (7) depends on the $L^{\infty}$ norm of both the amplitudes $u_{1}, u_{2}$. Looking for a stability result, we consider small perturbations in $H^{1}$ of the orbit of $(v, \phi)$; for such perturbations we have no information for the $L^{\infty}$ norm, and hence we cannot rely on (7). We can actually recover the belonging of the perturbation to $L^{\infty}$, since by Theorem 5 we have that $\nabla u(t) \in L^{4}$ for a. e. $t$, but yet we lack any uniform bound for $\|u(t)\|_{L^{\infty}}$.

We want to loose the dependence on $\left\|u_{2}\right\|_{L^{\infty}}$ on the right hand side of estimate (7). It will be enough for our aim to get a bound on $H^{1}$ norm of the difference $\theta_{1}-\theta_{2}$. We will follow closely the proof of Theorem 4 , with the due modifications: we will unbalance the inequality at the expense of $u_{1}$ to avoid the undesired dependence on $\left\|u_{2}\right\|_{L^{\infty}}$.
This asymmetry will require a technical hypothesis on $u_{1}, \theta_{1}$, that we assume in the following Lemma, and that later we will prove to be true for the minimizer $(v, \phi)$. We will comment later that this procedure makes the dependence of the constant $C$ on $u_{1}$ much worse, but this will not be a problem as in the application we will consider $u_{1}$ a fixed ground state and $u_{2}$ a small perturbation of $u_{1}$ in $H^{1}$.

Lemma 1. Let $u_{1}, u_{2} \in L^{\infty} \cap H^{1}\left(\mathbb{R}^{2}\right)$, and $\theta_{1}, \theta_{2}$ the respective angles given by Theorem 4. Assume also there exist $\varepsilon>0, \alpha<\left\|\theta_{1}\right\|_{L^{\infty}}$ such that the following implication holds:

$$
\begin{equation*}
\text { for a.e. } x \in \mathbb{R}^{2} \text {, if } \theta_{1}(x)>\alpha \text { then }\left|u_{1}(x)\right| \geq \varepsilon \tag{10}
\end{equation*}
$$

Then exist $C=C\left(\left\|u_{1}\right\|_{L^{\infty}},\left\|u_{1}\right\|_{H^{1}},\left\|u_{2}\right\|_{H^{1}}, \varepsilon, \alpha, q, v\right)$ such that

$$
\begin{equation*}
\left\|\theta_{1}-\theta_{2}\right\|_{H_{1}} \leq C\left\|u_{1}-u_{2}\right\|_{H^{1}} \tag{11}
\end{equation*}
$$

Remark 2. The hypothesis (10) requires that a control from below $\theta_{1}(x)>\alpha$ implies a control from below for the modulus of $u_{1}$, i.e. $\left|u_{1}(x)\right| \geq \varepsilon$.

Proof. From (6), the difference $\theta_{1}-\theta_{2}$ satisfies:
$-v \Delta\left(\theta_{1}-\theta_{2}\right)=-q\left(\sin \left(2 \theta_{1}\right)-\sin \left(2 \theta_{2}\right)\right)+2\left|u_{1}\right|^{2}\left(\cos \left(2 \theta_{1}\right)-\cos \left(2 \theta_{2}\right)\right)+2\left(\left|u_{1}\right|^{2}-\left|u_{2}\right|^{2}\right) \cos \left(2 \theta_{2}\right)$
We multiply by $\theta_{1}-\theta_{2}$ and integrate in $\mathbb{R}^{2}$; we can integrate by parts neglecting the boundary terms the left hand side, as $\theta_{i} \in H^{2}\left(\mathbb{R}^{2}\right)$. It follows

$$
\begin{align*}
& \int_{\mathbb{R}^{2}}\left|\nabla\left(\theta_{1}-\theta_{2}\right)\right|^{2}=-\frac{q}{v} \int_{\mathbb{R}^{2}}\left(\sin \left(2 \theta_{1}\right)-\sin \left(2 \theta_{2}\right)\left(\theta_{1}-\theta_{2}\right)+\right.  \tag{12}\\
& \quad+\frac{2}{v} \int_{\mathbb{R}^{2}} 2\left|u_{1}\right|^{2}\left(\cos \left(2 \theta_{1}\right)-\cos \left(2 \theta_{2}\right)\right)\left(\theta_{1}-\theta_{2}\right)+\frac{2}{v} \int_{\mathbb{R}^{2}}\left(\left|u_{1}\right|^{2}-\left|u_{2}\right|^{2}\right) \cos \left(2 \theta_{2}\right)\left(\theta_{1}-\theta_{2}\right)
\end{align*}
$$

As in [4], we want to use the inequality, for $0 \leq x<y \leq \pi / 4$

$$
\begin{equation*}
\sin 2 y-\sin 2 x \geq 2 \cos (2 y)(y-x) \tag{13}
\end{equation*}
$$

to reconstruct, from the first integral on the right hand side of (12), the $L^{2}$ norm of $\theta_{1}-\theta_{2}$. We have to pay attention to the areas where $\cos (2 y)$ is too close to 0 , that is when $2 y$ is close to $\pi / 2$, as in these areas the smallness of $\cos (2 y)$ is worsening the constant that will control $\left\|\theta_{1}-\theta_{2}\right\|_{H^{1}}$.
We recall that it was proved in [4] that $\theta_{1} \leq \beta$, with $\beta=\frac{1}{2} \arctan \left(2 R^{2} / q\right) \in[0, \pi / 4)$, $R=\left\|u_{1}\right\|_{L^{\infty}}$. We consider at first the set

$$
A:=\left\{\theta_{2} \leq \beta\right\} \cup\left\{\theta_{1} \leq \alpha\right\}
$$

where both the angles are bounded by a constant depending on $\left\|u_{1}\right\|_{L^{\infty}}$ but independent from $u_{2}$. The need of the request $\theta_{1} \leq \alpha$ will be clarified in the consideration of the set $A^{C}$.
We notice that if $\theta_{2} \leq \beta$, from (13) and the equality $\cos (\arctan (z))=1 / \sqrt{1+z^{2}}$ the following inequality holds:

$$
\begin{equation*}
\left|\sin \left(2 \theta_{1}\right)-\sin \left(2 \theta_{2}\right)\right| \geq \frac{2 q}{\sqrt{4 R^{4}+q^{2}}}\left|\theta_{1}-\theta_{2}\right| \tag{14}
\end{equation*}
$$

A similar inequality holds, with a different constant depending on $\alpha, R$, on the remaining part of the set $A$ : if $\pi / 4 \geq \theta_{2} \geq \beta$ and $0 \leq \theta_{1} \leq \alpha$, we have

$$
\begin{equation*}
\sin \left(2 \theta_{2}\right)-\sin \left(2 \theta_{1}\right) \geq \sin (2 \beta)-\sin (2 \alpha) \geq C(\beta-\alpha) \geq C\left(\theta_{2}-\theta_{1}\right) \tag{15}
\end{equation*}
$$

Moreover, as the function sin is increasing in $[0, \pi / 2]$, the integrand in the first addend of the right hand side of (12) is positive and hence
$-\int_{\mathbb{R}^{2}}\left(\sin \left(2 \theta_{1}\right)-\sin \left(2 \theta_{2}\right)\left(\theta_{1}-\theta_{2}\right) \leq-\int_{A}\left(\sin \left(2 \theta_{1}\right)-\sin \left(2 \theta_{2}\right)\left(\theta_{1}-\theta_{2}\right) \leq-C_{1} \int_{A}\left(\theta_{1}-\theta_{2}\right)^{2} d x\right.\right.$
$C_{1}$ a positive constant depending only on $R, \alpha, q$.
We want to prove a similar estimate for the second integral in (12) to obtain the $L^{2}$ norm of $\theta_{1}-\theta_{2}$ on the set $A^{C}$. The function cos is decreasing in $(0, \pi / 2)$, and therefore the integrand is negative. Similarly to (13) we can use

$$
\cos y-\cos x \leq \sin (x)(x-y) \leq \sin (\alpha)(x-y) \text { for } y>x \geq \alpha
$$

Since in $A^{C} \theta_{1} \geq \alpha$ and $\theta_{2} \geq \beta>\alpha$, we have from the previous inequality

$$
\left(\cos \left(2 \theta_{1}\right)-\cos \left(2 \theta_{2}\right)\right)\left(\theta_{1}-\theta_{2}\right) \leq-2 \sin (\alpha)\left(\theta_{1}-\theta_{2}\right)^{2} \quad \text { in } A^{C}
$$

Hence we can estimate the second integral

$$
\begin{align*}
& \int_{\mathbb{R}^{2}} 2\left|u_{1}\right|^{2}\left(\cos \left(2 \theta_{1}\right)-\cos \left(2 \theta_{2}\right)\right)\left(\theta_{1}-\theta_{2}\right) \leq \\
& \int_{A^{C}} 2\left|u_{1}\right|^{2}\left(\cos \left(2 \theta_{1}\right)-\cos \left(2 \theta_{2}\right)\right)\left(\theta_{1}-\theta_{2}\right) \leq-C \int_{A^{C}}\left|u_{1}\right|^{2}\left(\theta_{1}-\theta_{2}\right)^{2} \leq-C_{2} \int_{A^{C}}\left|\theta_{1}-\theta_{2}\right|^{2} \tag{17}
\end{align*}
$$

with $C_{2}$ depending on $\varepsilon, \alpha$. Notice how in the last inequality we have used the condition (10), to control from below $\left|u_{1}\right|$ on the set $A^{C}$. Combining (16), (17) and (12), we have with $C_{1}, C_{2}$ new constants depending on $\alpha, \varepsilon, R, q, v$ :

$$
\begin{aligned}
\left\|\nabla\left(\theta_{1}-\theta_{2}\right)\right\|_{L^{2}\left(\mathbb{R}^{2}\right)}^{2}+ & C_{1}\left\|\theta_{1}-\theta_{2}\right\|_{L^{2}\left(\mathbb{R}^{2}\right)}^{2} \leq \frac{2}{v} \int\left(\left|u_{1}\right|^{2}-\left|u_{2}\right|^{2}\right) \cos \left(2 \theta_{2}\right)\left(\theta_{1}-\theta_{2}\right) \\
& \leq C_{2}\left(\left\|u_{1}\right\|_{L^{4}\left(\mathbb{R}^{2}\right)}+\left\|u_{2}\right\|_{L^{4}\left(\mathbb{R}^{2}\right)}\right)^{2}\left\|u_{1}-u_{2}\right\|_{L^{4}\left(\mathbb{R}^{2}\right)}^{2}+\frac{C_{1}}{2}\left\|\theta_{1}-\theta_{2}\right\|_{L^{2}\left(\mathbb{R}^{2}\right)}^{2}
\end{aligned}
$$

We have applied Holder and weighted Cauchy Schwartz inequalities. At this point the proof follows as in [4] applying the Gagliardo-Niremberg inequality for $v \in$ $H^{1}\left(\mathbb{R}^{2}\right)$

$$
\|v\|_{L^{4}\left(\mathbb{R}^{2}\right)}^{2} \leq C\|\nabla v\|_{L^{2}\left(\mathbb{R}^{2}\right)}\|v\|_{L^{2}\left(\mathbb{R}^{2}\right)}
$$

We turn now to prove that the minimizers of the energy satisfy the condition (10), but first we make the following remark:

Remark 3. We emphasize that (10) is a very weak condition if $u, \theta$ are positive, radial decreasing and nonzero, because of the weak request on $\varepsilon$ and $\alpha$. Just notice that if $\theta$ is strictly decreasing, we can immediately conclude: for an $r_{0}>0$ with $v\left(r_{0}\right)>0$, (10) holds true for $\varepsilon=v\left(r_{0}\right), \alpha=\phi\left(r_{0}\right)$. In fact by the decreasing condition, $\phi(r)>\alpha$ if and only if $r<r_{0}$, and for such $r$ we have $v(r) \geq \varepsilon$; if $r>r_{0}$ the hypothesis in the implication (10) never happens, so the affirmation is trivially
true.
It turns out that (10) can be false for any $\alpha, \varepsilon$ with $v, \phi$ radial decreasing positive and nonzero if and only if $v$ has compact support, and $\phi \equiv\|\phi\|_{L^{\infty}}$ on all the support of $v$. The argument of the following Lemma shows at first this affirmation formally; then it proves that a minimal configuration cannot have this characteristic.

Lemma 2. Let $(v, \phi)$ be a minimizing point for $J_{a},(v, \phi) \in H_{r a d}^{1} \times H_{r a d}^{1}$. Then there exists $\alpha<\|\phi\|_{L^{\infty}}, \varepsilon>0$ depending on ( $v, \phi$ ) such that the condition (10) holds.

Proof. As $v, \phi$ are both in $H_{r a d}^{1}$, they are continuous in $\mathbb{R}^{2} \backslash\{0\}$. In the rest of the proof, we will identify the functions with their continuous representatives, and we will write $v, \phi$ both for the functions in $\mathbb{R}^{2}$ and their radial restriction. Moreover, for the result of Theorem 3, we have that they are positive and decreasing w. r. t. the radius.
By the previous Remark, it is enough to prove there exist $\alpha, \varepsilon$ as in the Lemma 1 , and a $r_{0}>0$ such that $\phi\left(r_{0}\right)=\alpha$ and $v\left(r_{0}\right) \geq \varepsilon$.
Let assume that the thesis is false, i. e.

$$
\begin{equation*}
\forall \alpha<\|\phi\|_{L^{\infty}}, \forall \varepsilon>0 \exists r=r_{\alpha, \varepsilon}>0 \text { s.t. } \phi(r)=\alpha ; \quad v(r)<\varepsilon \tag{18}
\end{equation*}
$$

We firstly send $\varepsilon \rightarrow 0$. The increasing succesion $r_{\alpha, \varepsilon}$ must converge to a finite $r_{\alpha}$, as otherwise we would have $\phi \geq \alpha$ in $\mathbb{R}^{2}$, but in this case the term $\int(1-\cos (2 \phi))$ would be infinite.
After the limit operation, thanks to the continuity of $v, \phi$ we can reformulate (18) as

$$
\begin{equation*}
\forall \alpha<\|\phi\|_{L^{\infty}} \exists r=r_{\alpha}>0 \text { s.t. } \phi(r)=\alpha ; \quad v(r)=0 \tag{19}
\end{equation*}
$$

Since $v(r)=0$, we have that $r_{\alpha} \geq \rho>0$, with $\operatorname{spt}(v)=B_{\rho}(0)$.
We repeat the operation sending $\alpha \rightarrow\|\phi\|_{L^{\infty}}$; the decreasing $r_{\alpha}$ must converge to a $R \geq \rho>0$. As a result we get that $\operatorname{spt}(v) \subseteq B_{R}(0)$ and $\phi \equiv\|\phi\|_{L^{\infty}}$ on $B_{R}(0)$.
We will reach the contradiction by showing that we are able to lower the energy by using a different angle.
Fix a $0<\tau<R$ and construct the competitor angle function, still radial, continuous and decreasing

$$
\psi(r):=\left\{\begin{array}{l}
\phi(r) \quad \text { if } r<R-\tau  \tag{20}\\
\phi(r+\tau) \quad \text { if } r \geq R-\tau
\end{array}\right.
$$

We want now to confront the energy of the minimum $(v, \phi)$ with the new configuration $(v, \psi)$.
The part of the energy with the gradient of $v$ has remained unchanged. We can look
at the difference between the gradients in polar coordinates:

$$
\begin{equation*}
\int_{\mathbb{R}^{2}}|\nabla \phi|^{2}-\int_{\mathbb{R}^{2}}|\nabla \psi|^{2}=2 \pi \int_{R}^{\infty} r\left|\phi^{\prime}(r)\right|^{2} d r-2 \pi \int_{R-\tau}^{\infty} r\left|\phi^{\prime}(r+\tau)\right|^{2} d r=2 \pi \tau \int_{R}^{\infty}\left|\phi^{\prime}(r)\right|^{2} d r \tag{21}
\end{equation*}
$$

We have a similar result looking at the integral of $1-\cos (2 \phi)$ :

$$
\begin{align*}
& \int_{\mathbb{R}^{2}}(1-\cos (2 \phi))-\int_{\mathbb{R}^{2}}(1-\cos (2 \psi)) \\
& =2 \pi\left(\int_{R-\tau}^{R} r\left(1-\cos \left(2\|\phi\|_{L^{\infty}}\right)\right)+\int_{R}^{\infty} r(1-\cos (2 \phi(r)))-\int_{R-\tau}^{\infty} r(1-\cos (2 \phi(r+\tau)))\right) \\
& =2 \pi\left(\left(\tau R-\frac{1}{2} \tau^{2}\right)\left(1-\cos \left(2\|\phi\|_{L^{\infty}}\right)+\tau \int_{R}^{\infty}(1-\cos (2 \phi(r))) d r\right)\right. \tag{22}
\end{align*}
$$

We come to the last estimate, recalling $v \equiv 0$ on $B_{R}^{C}$ :

$$
\begin{align*}
& -\int_{B_{R}}|v|^{2} \sin (2 \phi)+\int_{B_{R}}|v|^{2} \sin (2 \psi)=2 \pi\left(-\int_{R-\tau}^{R} r|v(r)|^{2} \sin \left(2\|\phi\|_{L^{\infty}}\right)+\int_{R-\tau}^{R} r|v(r)|^{2} \sin (2 \phi(r+\tau))\right) \\
& \quad \geq-2 \pi \tau R|v(R-\tau)|^{2} \tag{23}
\end{align*}
$$

Combining (21), (22), (23) we get to

$$
\begin{equation*}
E(v, \phi)-E(v, \psi) \geq C_{1} \tau-C_{2} \tau|v(R-\tau)|^{2}+o(|\tau|) \tag{24}
\end{equation*}
$$

with $C_{1}, C_{2}$ positive and depending on $R, q, v, \phi$ but not on $\tau$. Since $v$ is continuous and $v(R)=0$, we can choose $\tau$ sufficiently small so that the right hand side of (24) is strictly positive. This leads to a contradiction as $(v, \psi) \in S_{a}$.

Remark 4. Summing up the Lemmas in this section, we have for the minimizer $(\nu, \phi)$ the estimate

$$
\begin{equation*}
\|\Theta(u)-\phi\|_{H_{1}} \leq C\|u-v\|_{H^{1}} \tag{25}
\end{equation*}
$$

for any $u \in H^{1}$. The constant $C$ is now depending on $u$ only through its $H^{1}$-norm. However it has a strong dependence on $v$ : not only on $\|v\|_{H^{1}},\|v\|_{L^{\infty}}$ but on the shape of the minimizers itself through Lemma 2. The result is sufficient to deduce that, for a sequence $u_{n}$ converging to $v$ in $H^{1}$, we have

$$
\Theta\left(u_{n}\right) \rightarrow \phi \text { in } H^{1} ; \quad \lim _{n} E\left(u_{n}, \Theta\left(u_{n}\right)\right)=E(v, \phi)
$$

For both the results, the estimate (7) was not strong enough.

### 1.2 Stability proof

The stability result in [6] relies on the sign of the second derivative of the action functional. We recall the main steps of his line of reasoning, adapting them to our situation.
We begin defining the action

$$
\begin{equation*}
S(u, \theta):=E(u, \theta)+\frac{\sigma}{2} \int_{R^{2}}|u|^{2} d x=E(u, \theta)+\sigma Q(u) \tag{26}
\end{equation*}
$$

with $\sigma$ given by equation (4). Moreover in the following we will use the notation, for $u \in H^{1}$ and $y \in \mathbb{R}^{2}$

$$
u_{y}(x):=u(x+y)
$$

Proposition 1. Let $(v, \phi)$ be a minimizer for $E$ over the constraint $S_{a}$, satisfying the system (4)-(5). Suppose there exists $a \tau>0$ s. $t$.

$$
\begin{equation*}
\left\langle S^{\prime \prime}(v, \phi)(\eta, \theta),(\eta, \theta)\right\rangle \geq \tau\|\eta\|_{H^{1}} \tag{27}
\end{equation*}
$$

for any $(\eta, \theta) \in H^{1} \times H^{1}$ with $\|\eta\|_{H^{1}}$ sufficiently small and

$$
\begin{equation*}
\|\eta+v\|_{L^{2}}=\|v\|_{L^{2}} ; \quad\|\theta\|_{H^{1}} \leq C\|\eta\|_{H^{1}} \quad 0=(\eta, i v)_{H^{1}}=\left(\eta, \partial_{l} v\right)_{H^{1}} \tag{28}
\end{equation*}
$$

for a constant $C>0$. Then there exist constants $D>0, \delta>0$ depending on $\tau$ such that for any $u \in H^{1}$ and for any $\psi \in H^{1}$ with
$\|u\|_{L^{2}}=\|v\|_{L^{2}} ; \quad \inf _{\theta, y}\left\|u-e^{i \theta} v_{y}\right\|_{H^{1}}=\left\|u-e^{i \alpha} v_{x}\right\|_{H^{1}} \leq \delta \quad\left\|\psi-\phi_{x}\right\|_{H^{1}} \leq C\left\|u-e^{i \alpha} v_{x}\right\|_{H^{1}}$
it holds

$$
\begin{equation*}
E(u, \psi)-E(v, \phi) \geq D \inf _{\theta, y}\left\|u-e^{i \theta} v_{y}\right\|_{H^{1}}^{2} \tag{29}
\end{equation*}
$$

Remark 5. Heuristically, the request for the angle norm in (28) is motivated by the results of Lemmas 1,2 if we consider $\theta=\Theta(\eta+v)-\phi$. Restricting angles of this form would be intuitively correct, because while looking to the perturbed amplitude $v+\eta$ we are interested only in perturbation in the angle variable as above. However, in the proof of Theorem 1 below we will have to use a normalization procedure and this expression of the hypothesis will turn out to be more convenient.
Remark 6. The last condition in (28) is due to invariance by translations and multiplication by a complex exponential of the action. As was noted in [6], [7], the invariance implies that the inequality (27) cannot hold along those directions. For example differentiating with respect to $\alpha$ the identity $S^{\prime}\left(e^{i \alpha} \nu, \phi\right) \equiv 0$, consequence of (4)-(5) and invariance, we infer $S^{\prime \prime}(v, \phi)(i v, 0) \equiv 0$.

Remark 7. The normalization process will allow us to use the conservation of the charge hypothesis in (28), which is not true for a generic perturbation, for the stability argument. The identity of the $L^{2}$-norm plays an important role in the proof of the Proposition. Moreover, it implies $(v+\eta, \phi+\theta) \in S_{a}$ and this allows a comparison with the energy of $(v, \phi)$ which will be used later.

Proof. We follow closely the proof in [6]. As in the hypothesis, we have for $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^{2}$

$$
\begin{equation*}
\left\|e^{-i \alpha} u_{-x}-v\right\|_{H^{1}}=\inf _{\theta, y}\left\|e^{-i \theta} u_{y}-v\right\|_{H^{1}} \leq \delta \tag{31}
\end{equation*}
$$

with $\delta$ to be fixed. Setting $\eta:=e^{-i \alpha} u_{-x}-v$ and $\theta:=\psi_{-x}-\phi$ the conditions in (28) are satisfied: the first two because of hypothesis (29), while the orthogonality condition is a consequence of $x, \alpha$ minimizing the distance over the symmetry group. By a Taylor expansion we have

$$
\begin{align*}
S(u, \psi) & =S\left(e^{-i \alpha} u_{-x}, \phi+\theta\right) \\
& =S(v, \phi)+S^{\prime}(v, \phi)(\eta, \theta)+\frac{1}{2}\left\langle S^{\prime \prime}(v, \phi)(\eta, \theta),(\eta, \theta)\right\rangle+o\left(\|(\eta, \theta)\|_{H^{1} \times H^{1}}^{2}\right) \\
& \geq S(v, \phi)+\tau\|\eta\|_{H^{1}}^{2}+o\left(\|\eta\|_{H^{1}}^{2}\right) \tag{32}
\end{align*}
$$

The first order term vanishes: $(v, \phi)$ is a solution of the system (4)-(5), hence $S^{\prime}(v, \phi) \equiv 0$. Moreover, we have used the condition (28) to get $o\left(\|\eta\|_{H^{1}}^{2}\right)=o\left(\|(\eta, \theta)\|_{H^{1} \times H^{1}}^{2}\right)$. The $L^{2}$ norm of $v$ and $u$ is equal, so the conclusion follows directly from the previous inequality for $\delta$ sufficiently small.

We have to prove that the hypothesis (27) holds for $(v, \phi)$. A first estimate from below of $S^{\prime \prime}$ is a consequence of minimality of $(v, \phi)$ over $S_{a}$; we can prove strict positivity thank to the positivity of the Lagrange multiplier $\sigma$.

Proposition 2. Let $(v, \phi)$ be a minimizer of $E$ over the constraint $S_{a}$. Then there exists a $\tau>0$ such that, for any couple $(\eta, \theta)$ verifying condition (28) with $\|\eta\|_{H^{1}}$ sufficiently small, the inequality (27) in the hypothesis of Proposition 1 holds.

Proof. We decompose the perturbation $\eta$ as

$$
\eta=t v+z ; \quad t \in \mathbb{R} ; \quad z \in H^{1}\left(\mathbb{R}^{2}\right) ; \quad(z, v)_{L^{2}}=0
$$

We look hence to the Taylor expansion for the charge $Q(u):=\frac{\|u\|_{L^{2}}^{2}}{2}$

$$
Q(v+\eta)=Q(v)+Q^{\prime}(v)(\eta)+O\left(\|\eta\|_{L^{2}}^{2}\right)=Q(v)+t Q^{\prime}(v)(v)+O\left(\|\eta\|_{L^{2}}^{2}\right)
$$

Since $Q(v+\eta)=Q(v)$ by hypothesis, and $Q^{\prime}(v)(v)$ is a fixed positive quantity, independent on $\eta$, we get $t=O\left(\|\eta\|_{L^{2}}^{2}\right)$.

Hence, for such perturbations $\eta$, the contribution of the orthogonal component $z$ is the leading term for the norm of the perturbation. Both in $L^{2}$ and in $H^{1}$, we have

$$
\begin{aligned}
& \|z\|_{L^{2}}^{2}=\|\eta\|_{L^{2}}^{2}+t^{2}\|v\|_{L^{2}}^{2}-2 t(\eta, v)_{L^{2}}=\|\eta\|_{L^{2}}^{2}+o\left(\|\eta\|_{L^{2}}^{2}\right) \\
& \|z\|_{H^{1}}^{2}=\|\eta\|_{H^{1}}^{2}+t^{2}\|v\|_{H^{1}}^{2}-2 t(\eta, v)_{H^{1}}=\|\eta\|_{H^{1}}^{2}+o\left(\|\eta\|_{H^{1}}^{2}\right)
\end{aligned}
$$

In the first line we have directly used the relation on $t$ obtained above, while in the second we have exploited the relation $o\left(\|\eta\|_{L^{2}}^{2}\right) /\|\eta\|_{H^{1}}^{2} \rightarrow 0$ as $\eta \rightarrow 0$ in $H^{1}$. We can then easily compute

$$
\begin{equation*}
\lim _{\eta \rightarrow 0} \frac{\|z\|_{L^{2}}^{2}}{\|\eta\|_{L^{2}}^{2}}=1 ; \quad \lim _{\eta \rightarrow 0} \frac{\|z\|_{H^{1}}^{2}}{\|\eta\|_{H^{1}}^{2}}=1 \tag{33}
\end{equation*}
$$

where the limits are respectively in $L^{2}$ and in $H^{1}$. Therefore we have

$$
\begin{equation*}
t=O\left(\|\eta\|_{L^{2}}^{2}\right)=O\left(\|z\|_{L^{2}}^{2}\right) ; \quad o\left(\|\eta\|_{L^{2}}^{2}\right)=o\left(\|z\|_{L^{2}}^{2}\right) ; \quad o\left(\|\eta\|_{H^{1}}^{2}\right)=o\left(\|z\|_{H^{1}}^{2}\right) \tag{34}
\end{equation*}
$$

We want to use the minimality of $(v, \phi)$ to deduce a sign on $E^{\prime \prime}$, but for this we need to ensure $E^{\prime}=0$. By equations (4)-(5) we have

$$
E^{\prime}(v, \phi)(w, \theta)=-\sigma Q^{\prime}(v)(w)
$$

Defined $\alpha=\phi+\theta$, we look at the Taylor expansion with respect to the $H^{1}$-norm of the energy

$$
\begin{align*}
E(v+z, \alpha) & =E(v, \phi)+E^{\prime}(v, \phi)(z, \theta)+\left\langle E^{\prime \prime}(v, \phi)(z, \theta)(z, \theta)\right\rangle+o\left(\|\theta\|_{H^{1}}^{2}+\|z\|_{H^{1}}^{2}\right) \\
& =E(v, \phi)+\left\langle E^{\prime \prime}(v, \phi)(z, \theta)(z, \theta)\right\rangle+o\left(\|z\|_{H^{1}}^{2}\right) \tag{35}
\end{align*}
$$

We have used the hypothesis (28) to write $o\left(\|\eta\|_{H^{1}}^{2}\right)$ in place of $o\left(\|\theta\|_{H^{1}}^{2}\right)$, and then the condition (34) to absorb it in $o\left(\|z\|_{H^{1}}^{2}\right)$.
To complete the energy comparison, the left hand side must be reduced to an element in $S_{a}$. We define hence

$$
\tilde{u}:=\frac{\|v\|_{L^{2}}}{\sqrt{\|v\|_{L^{2}}^{2}+\|z\|_{L^{2}}^{2}}}(v+z)=k(v+z)
$$

For the ortogonality condition, one immediately checks $\tilde{u} \in S_{a}$. It will be useful to look at the decomposition of the energy $E:=E^{+}+E^{-}$with
$E^{-}(u, \theta):=\frac{1}{4} \int_{\mathbb{R}^{2}}|\nabla u|^{2}-2|u|^{2} \sin (2 \theta) d x ; \quad E^{+}(u, \theta):=\frac{1}{4} \int_{\mathbb{R}^{2}} v|\nabla \theta|^{2}+q(1-\cos (2 \theta) d x$

Since $E^{+}$is strictly positive for any non zero $\theta$, and the minimal energy is always negative, we have that the minimizer satisfies $E^{-}(v, \phi)<E(v, \phi) \leq 0$. It easily follows from the quadratic homogeneity of $E^{-}$
$E^{+}(v+z, \alpha)+E^{-}(v+z, \alpha)=E(\tilde{u}, \alpha)+\left(\frac{1}{k^{2}}-1\right) E^{-}(\tilde{u}, \alpha) \geq E(v, \phi)+\frac{\|z\|_{L^{2}}^{2}}{\|v\|_{L^{2}}^{2}} E^{-}(\tilde{u}, \alpha)$
where in the last inequality we have taken advantage of the minimality of $(v, \phi)$. Combining (35) and (37) we have

$$
\begin{equation*}
\left\langle E^{\prime \prime}(v, \phi)(z, \theta),(z, \theta)\right\rangle \geq \frac{E^{-}(\tilde{u}, \alpha)}{\|v\|_{L^{2}}^{2}}\|z\|_{L^{2}}^{2}+o\left(\|z\|_{H^{1}}^{2}\right) \tag{38}
\end{equation*}
$$

We can infer a positive condition on $\sigma$ : in particular multiplying (4) by $v$ and integrating we get

$$
\begin{align*}
& 2 \sigma=\frac{\int_{\mathbb{R}^{2}}-|\nabla v|^{2}+2|v|^{2} \sin (2 \theta) d x}{\|v\|_{L^{2}}^{2}}=\frac{-4 E^{-}(v, \phi)}{\|v\|_{L^{2}}^{2}}>\frac{-4 J_{a}}{\|v\|_{L^{2}}^{2}} \geq 0  \tag{39}\\
& \sigma<\frac{\int_{\mathbb{R}^{2}}|v|^{2} \sin (2 \theta) d x}{\|v\|_{L^{2}}^{2}} \leq 1
\end{align*}
$$

For $\eta$ small, as in Remark $4 E^{-}(\tilde{u}, \alpha)$ is converging to $E^{-}(v, \phi)$, which is strictly negative for the inequality above. Hence combining (38), (39) and the definition of $S$ we deduce for $\eta$ small enough

$$
\begin{aligned}
& \left\langle S^{\prime \prime}(v, \phi)(z, \theta),(z, \theta)\right\rangle=\left\langle E^{\prime \prime}(v, \phi)(z, \theta),(z, \theta)\right\rangle+\sigma\|z\|_{L^{2}}^{2} \geq \\
& \quad \geq \frac{-2 E^{-}(v, \phi)+E^{-}(\tilde{u}, \alpha)}{\|v\|_{L^{2}}^{2}}\|z\|_{L^{2}}^{2}+o\left(\|z\|_{H^{1}}^{2}\right) \geq C \frac{-E^{-}(v, \phi)}{\|v\|_{L^{2}}^{2}}\|z\|_{L^{2}}^{2}+o\left(\|z\|_{H^{1}}^{2}\right)
\end{aligned}
$$

for a positive $C . S^{\prime \prime}(v, \phi)$ is bilinear and bounded, and from (34) and the definition of the decomposition we have $\|\eta-z\|_{H^{1}}^{2}=o\left(\|y\|_{H^{1}}^{2}\right)$. We can hence infer from the previous inequality a similar one for the whole perturbation: for $\eta$ small enough and a positive $C^{\prime}$ independent on $\eta$ it holds

$$
\begin{align*}
& \left\langle S^{\prime \prime}(v, \phi)(\eta, \theta),(\eta, \theta)\right\rangle=\left\langle S^{\prime \prime}(v, \phi)(z, \theta),(z, \theta)\right\rangle+o\left(\|z\|_{H^{1}}^{2}\right) \geq C\|z\|_{L^{2}}^{2}+o\left(\|z\|_{H^{1}}^{2}\right) \\
& \quad \geq C^{\prime}\|\eta\|_{L^{2}}^{2}+o\left(\|\eta\|_{H^{1}}^{2}\right) \tag{40}
\end{align*}
$$

where in the last inequality we have used (34) and the first limit in (33).
We want to extend the previous result with the $H^{1}$ - norm on the right hand side for
some constant $\tau>0$. Let us assume this is not true: then we can pick sequences $\tau_{k} \rightarrow 0,\left(\eta_{k}, \theta_{k}\right) \in H^{1} \times H^{1}$ verifying (28) s. t.

$$
\begin{equation*}
\frac{1}{\left\|\eta_{k}\right\|_{H^{1}}^{2}}\left\langle S^{\prime \prime}(v, \phi)\left(\eta_{k}, \theta_{k}\right),\left(\eta_{k}, \theta_{k}\right)\right\rangle<\tau_{k} \tag{41}
\end{equation*}
$$

Such a sequence must verify in addition

$$
\begin{equation*}
\frac{\left\|\eta_{k}\right\|_{L^{2}}}{\left\|\eta_{k}\right\|_{H^{1}}} \rightarrow 0 \tag{42}
\end{equation*}
$$

If this were not true, up to subsequences we would have the limit $\left\|\eta_{k}\right\|_{L^{2}} /\left\|\eta_{k}\right\|_{H^{1}} \rightarrow$ $c \in(0, \infty]$. But if this is the case then definitively in $k$

$$
\tau_{k}>\frac{1}{\left\|\eta_{k}\right\|_{H^{1}}^{2}}\left\langle S^{\prime \prime}(v, \phi)\left(\eta_{k}, \theta_{k}\right),\left(\eta_{k}, \theta_{k}\right)\right\rangle \geq C^{\prime} \frac{\left\|\eta_{k}\right\|_{L^{2}}^{2}+o\left(\left\|\eta_{k}\right\|_{H^{1}}^{2}\right)}{\left\|\eta_{k}\right\|_{H^{1}}^{2}} \geq c^{\prime}>0
$$

which cannot be as $\tau_{k}$ is going to 0 .
We want to show that (41) is impossible, as the left hand side remains bigger than $1 / 2$. To this end we rewrite the second derivative of the action as

$$
\begin{align*}
& \left\langle S^{\prime \prime}(v, \phi)(\eta, \theta),(\eta, \theta)\right\rangle=\frac{1}{2} \int|\nabla \eta|^{2}-2|\eta|^{2} \sin (2 \phi)+2 \sigma|\eta|^{2}+ \\
& \frac{1}{2} \int|\nabla \theta|^{2}+2|v|^{2} \sin (2 \phi) \theta^{2}+2 q \cos (2 \phi) \theta^{2}-4 \int \operatorname{Re}(v \bar{\eta}) \cos (\phi) \theta=: I+I I+I I I \tag{43}
\end{align*}
$$

We see that $I I \geq 0$, so we can ignore it.
It is easy to show that along the sequence $\eta_{k}$ we have that $I /\left\|\eta_{k}\right\|_{H^{1}}^{2} \rightarrow 1 / 2$ : by (42) we have

$$
\frac{\int_{\mathbb{R}^{2}} 2 \sigma\left|\eta_{k}\right|^{2}-\sin (2 \phi)\left|\eta_{k}\right|^{2}}{\left\|\eta_{k}\right\|_{H^{1}}^{2}} \rightarrow 0 ; \quad \frac{\int_{\mathbb{R}^{2}}\left|\nabla \eta_{k}\right|^{2}}{\left\|\eta_{k}\right\|_{H^{1}}^{2}} \rightarrow 1
$$

Finally, we can show that $I I I /\left\|\eta_{k}\right\|_{H^{1}}^{2}$ goes to 0 :

$$
\left|\frac{\int_{\mathbb{R}^{2}} \operatorname{Re}\left(v \bar{\eta}_{k}\right) \cos (\phi) \theta_{k} d x}{\left\|\eta_{k}\right\|_{H^{1}}^{2}}\right| \leq 4\|v\|_{L^{\infty}} \frac{\left\|\eta_{k}\right\|_{L^{2}}}{\left\|\eta_{k}\right\|_{H^{1}}} \frac{\left\|\theta_{k}\right\|_{L^{2}}}{\left\|\eta_{k}\right\|_{H^{1}}} \leq C \frac{\left\|\eta_{k}\right\|_{L^{2}}}{\left\|\eta_{k}\right\|_{H^{1}}} \rightarrow 0
$$

The first passage is just an Holder inequality, for the second we have applied the condition (28) where the constant $C$ depends on ( $v, \phi$ ) but not on $\eta_{k}$, and in the last one the condition (42). So we have reached the contradiction.

We finally prove the stability result, mimicking the proof of Theorem 3.5 in [7] Proof of Theorem 1. We assume for contradiction that the thesis is false. We have, for $\delta$ given by Proposition 1, a value $0<\varepsilon<\delta$, sequence of initial datum $u_{n}^{0}$ and a sequence of times $t_{n}$ such that

$$
\begin{equation*}
\inf _{\alpha, y}\left\|u_{n}\left(t_{n}\right)-e^{i \alpha} v(\cdot-y)\right\|_{H^{1}} \geq \varepsilon ; \quad u_{n}^{0} \rightarrow v \text { in } H^{1} \tag{44}
\end{equation*}
$$

For Theorem 5, the heat flow is continuous in time; we can pick, definitely in $n$, the sequence of times $t_{n}$ such that

$$
\inf _{\alpha, y}\left\|u_{n}\left(t_{n}\right)-e^{i \alpha} v_{y}\right\|_{H^{1}}=\varepsilon
$$

Moreover, recalling Remark 4 and conservation of energy along the evolution of the flow, we have

$$
E(v, \phi)=\lim _{n} E\left(u_{n}^{0}, \Theta\left(u_{n}^{0}\right)\right)=\lim _{n} E\left(u_{n}\left(t_{n}\right), \Theta\left(u_{n}\left(t_{n}\right)\right)\right)
$$

We look at the sequences

$$
a_{n}:=\frac{\|v\|_{L^{2}}}{\left\|u_{n}^{0}\right\|_{L^{2}}} ; \quad v_{n}:=a_{n} u_{n}\left(t_{n}\right)
$$

$a_{n}$ is converging to 1 ; therefore the following holds trivially

$$
\lim _{n} E\left(v_{n}, \Theta\left(u_{n}\left(t_{n}\right)\right)\right)=\lim _{n} E\left(u_{n}\left(t_{n}\right), \Theta\left(u_{n}\left(t_{n}\right)\right)\right) ; \lim _{n} \inf _{\alpha}\left\|v_{n}-e^{i \alpha} v\right\|_{H^{1}}=\varepsilon
$$

On the grounds that the $L^{2}$-norm is preserved, $\left\|v_{n}\right\|_{L^{2}}=\|v\|_{L^{2}}$. Moreover, eventually for $n$

$$
\left\|\Theta\left(u_{n}\left(t_{n}\right)\right)-\phi\right\|_{H^{1}} \leq C\left\|u_{n}\left(t_{n}\right)-v\right\|_{H^{1}} \leq C \alpha_{n}\left\|v_{n}-v\right\|_{H^{1}}+\mid 1-\alpha_{n}\| \| v\left\|_{H^{1}} \leq C^{\prime}\right\| v_{n}-v \|_{H^{1}}
$$

Thus the couple $\left(v_{n}, \Theta\left(u_{n}\left(t_{n}\right)\right)\right.$ satisfies the hypothesis (29) of Proposition 1. We have reached the contradiction as we would have

$$
\begin{equation*}
0=\lim _{n} E\left(v_{n}, \Theta\left(u_{n}\left(t_{n}\right)\right)\right)-E(v, \phi) \geq D \liminf _{n} \inf _{\theta, y}\left\|v_{n}-e^{i \theta} v(\cdot-y)\right\|_{H^{1}}=D \varepsilon>0 \tag{45}
\end{equation*}
$$

## 2 Existence for any $0<\sigma<1$

In this section we prove Theorem 2; in particular we will prove that the the values $\sigma$ associated to the family of ground states, presented in Theorem 3 as the minimizers of the energy over the constraint $S_{a}$ for $a$ above a certain threshold, span, as the parameter $a$ is varying, all the values $\sigma \in(0,1)$.

We present a sketch of the main elements of the proof. Recalling the decomposition (36) we prove at first that $E^{-}$, evaluated over a minimizer in $S_{a}$, is a continuous and decreasing function of $a$. As the variational problem is not convex, the key step is showing that all the minimizers over $S_{a}$ share the same energy decomposition. Continuity and monotonicity are deduced for $\sigma$ using the relation (39). Finally we show that all the values $\sigma \in(0,1)$ have a corresponding ground state. The technical part is showing that, as long as $\sigma_{a}>0$, existence of a ground state can be inferred for smaller norm costraints $b<a$, and hence for smaller values of $\sigma$.

We emphasize that we will not necessarily find new solution with respect to the ones of Theorem 3, but we are rather presenting some new information and a deeper understanding of the minimal configurations. Moreover, since for every $\sigma \in(0,1)$ the solution is found as a minimizer over a certain $S_{a}$, by Theorem 1 it is orbitally stable.
Remark 8. In [4] it was proved (see Corollary [5.1]) that, for $a<b$, it holds

$$
\begin{equation*}
J_{b} \leq \frac{b}{a} J_{a} \leq J_{a} \tag{46}
\end{equation*}
$$

This was actually proved assuming $J_{a}<0$, but as the minimum energy $J_{a}$ is decreasing and $J_{a} \leq 0$ for any $a$, it is true even without the aforementioned hypothesis. Let us consider now ( $v_{a}, \phi_{a}$ ) a minimizer in $S_{a}$. Then for any $b$ it holds

$$
\begin{equation*}
J_{b} \leq E\left(\sqrt{\frac{b}{a}} v_{a}, \phi_{a}\right)=\frac{b}{a} E^{-}\left(v_{a}, \phi_{a}\right)+E^{+}\left(v_{a}, \phi_{a}\right)=J_{a}+\left(1-\frac{b}{a}\right) E^{-}\left(v_{a}, \phi_{a}\right) \tag{47}
\end{equation*}
$$

In Theorem 3 in [4], the authors proved the existence of a minimizer over $S_{a}$ assuming that the energy $J_{a}$ were negative, i. e. for $a>a_{0}$. This hypothesis played a crucial role in the proof, as it prevents a minimizing sequence from weakly converging to 0 .
In treating the case $\sigma$ close to 0 , we will need a slightly refined assertion, which express a direct relation between the values of $E^{-}$and the existence of a minimizer. As will be evident later, it allows the possibility for minimizers for certain values $a \leq a_{0}$, that is to say with total energy $J_{a}=0$. For this reason, we will use in
next Lemmas the expression: "Let $c>0$ such that for any $a \in(c, \infty)$ there exists a minimum point for $E$ in $S_{a}$ ", postponing a deeper explanation of the characteristic of $c \leq a_{0}$ to the last results of the section.
Lemma 3. Let $a>0$, and $\left(v_{n}, \phi_{n}\right) \in H_{r a d}^{1} \times H_{r a d}^{1}$ a sequence such that

$$
\lim _{n}\left\|v_{n}\right\|_{L^{2}}^{2}=a ; \quad \lim _{n} E\left(v_{n}, \phi_{n}\right) \leq J_{a} ; \quad \lim _{n} E^{-}\left(v_{n}, \phi_{n}\right)=-d \in(-\infty, 0)
$$

Then there exists a subsequence not relabeled $\left(v_{n}, \phi_{n}\right)$ weakly converging in $H^{1} \times H^{1}$ to $\left(v_{a}, \phi_{a}\right)$, minimizer of the energy in $S_{a}$ with $E^{-}\left(v_{a}, \phi_{a}\right)=-d$; the convergence is strong for the amplitude variable $v_{n}$.

Proof. As the energy and the $L^{2}$ norm are bounded, the sequence is bounded in $H^{1} \times H^{1}$. By the compact embedding $H_{r a d}^{1}\left(\mathbb{R}^{2}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{2}\right)$ for any $2<p<\infty$, we can extract a subsequence ( $v_{n}, \phi_{n}$ ) converging to $(v, \phi)$ weakly in $H^{1} \times H^{1}$, strongly in $L^{3} \times L^{3}$ and pointwise almost a.e. that satisfies (see the proof of Proposition [5.3] in [4] for details):

$$
\begin{align*}
& \|\nabla v\|_{L^{2}}^{2} \leq \liminf _{n}\left\|\nabla v_{n}\right\|_{L^{2}}^{2} ; \quad\|\nabla \phi\|_{L^{2}}^{2} \leq \liminf _{n}\left\|\nabla \phi_{n}\right\|_{L^{2}}^{2} ;  \tag{48}\\
& \int(1-\cos (2 \phi)) \leq \liminf _{n} \int\left(1-\cos \left(2 \phi_{n}\right)\right) ; \quad \int|v|^{2} \sin (2 \phi)=\lim _{n} \int\left|v_{n}\right|^{2} \sin \left(2 \phi_{n}\right)
\end{align*}
$$

Hence the following hold

$$
\begin{equation*}
\lambda:=\|v\|_{L^{2}}^{2} \leq a ; E(v, \phi) \leq \lim _{n} E\left(v_{n}, \phi_{n}\right) \leq J_{a} ; \quad E^{-}(v, \phi) \leq \liminf _{n} E^{-}\left(v_{n}, \phi_{n}\right) \tag{49}
\end{equation*}
$$

The value $\lambda$ cannot be zero, as we have by hypothesis and the last inequality in (49) that $\int|v|^{2} \sin (2 \phi) \geq d / 2>0$. It cannot either be $\lambda<a$ : we would have $(\sqrt{a / \lambda} v, \phi) \in S_{a}$ and

$$
E\left(\sqrt{\frac{a}{\lambda}} v, \phi\right)=\frac{a}{\lambda} E^{-}(v, \phi)+E^{+}(v, \phi)<E(v, \phi) \leq J_{a}
$$

We have ( $v, \phi) \in S_{a}$ and it is a minimizer. Therefore, all the inequalities in (48), and hence the last one in (49), must actually be equalities, or we would have once again $E(v, \phi)<J_{a}$. Recalling $\lambda=a$, we have convergence of the $H^{1}$ norm and hence $v_{n} \rightarrow v$ strongly in $H^{1}$.

Lemma 4. Let $c>0$ be such that for any $a \in(c, \infty)$ there exists a minimum point for the energy $E$ in $S_{a}$. Then for any $c<a<b$, for any $\left(v_{a}, \phi_{a}\right)$ and ( $v_{b}, \phi_{b}$ ) minimizer in $S_{a}, S_{b}$ respectively, the following hold

$$
\begin{equation*}
E^{-}\left(v_{a}, \phi_{a}\right) b \geq E^{-}\left(v_{b}, \phi_{b}\right) a ; \quad \sigma_{a} \leq \sigma_{b} \tag{50}
\end{equation*}
$$

Here $\sigma_{a}, \sigma_{b}$ are the values of $\sigma$ for which $\left(v_{a}, \phi_{a}\right)$ and $\left(v_{b}, \phi_{b}\right)$ solve the system (4)-(5)

Proof. Fix the couples $\left(v_{a}, \phi_{a}\right),\left(v_{b}, \phi_{b}\right)$; we write for brevity $E^{-}(b):=E^{-}\left(v_{b}, \phi_{b}\right)$. Applying relation 47 twice we get

$$
J_{a} \leq J_{b}+\left(\frac{a}{b}-1\right) E^{-}(b) \leq J_{a}+\left(\frac{a}{b}-1\right) E^{-}(b)+\left(\frac{b}{a}-1\right) E^{-}(a)
$$

Rearranging we have

$$
\frac{b-a}{b} E^{-}(b)=\left(1-\frac{a}{b}\right) E^{-}(b) \leq\left(\frac{b}{a}-1\right) E^{-}(a)=\frac{b-a}{a} E^{-}(a)
$$

and consequently the first inequality dividing by $b-a>0$. The second one follows from the first and the relation (39) between $\sigma$ and $E^{-}$for a minimizer.

Remark 9. We have that the possibly multivalued function

$$
E^{-}(b):=\left\{E^{-}(v, \phi) \mid(v, \phi) \text { is a minimizer over } S_{b}\right\}
$$

is monotone by the previous Lemma: since $E^{-}$is negative for every minimizer, for $c<a<b$ it holds

$$
\begin{equation*}
\tilde{a} \geq \tilde{b} \quad \forall \tilde{b} \in E^{-}(b) ; \tilde{a} \in E^{-}(a) \tag{51}
\end{equation*}
$$

For $b>c$, we say that $E^{-}(b)$ is well defined if the set $E^{-}(b)$ contains only one element; in this case with abuse of notation we will write $E^{-}(b)$ also for the unique value contained in the set. From the monotonicity relation, it is not difficult to see that this happens whenever $E^{-}(b)$ is continuous with respect to $b$; again because of the monotonicity this is true a.e. in $(c, \infty)$.

Corollary 1. The minimal energy $J_{a}:=\inf _{S_{a}} E$ is continuous with respect to $a$.
Proof. For $a<a_{0}, a_{0}$ as in Theorem 3, we have $J_{a} \equiv 0$; for $a>a_{0}$, for $b$ close to $a$ we have by (47)

$$
J_{b} \leq J_{a}+\left(1-\frac{b}{a}\right) E^{-}\left(v_{a}, \phi_{a}\right) ; \quad J_{a} \leq J_{b}+\left(1-\frac{a}{b}\right) E^{-}\left(v_{b}, \phi_{b}\right)
$$

Sending $b \rightarrow a$, we deduce $\lim _{b \rightarrow a} J_{b}=J_{a}$. For $a \downarrow a_{0}$, if we would not have $\lim _{a} J_{a}=0$, we could find a configuration with negative energy in $S_{a_{0}}$ repeating the proof of Lemma 3.

Lemma 5. Let $c>0$ be as in Lemma 4, and $b>c$ be such that the quantity $E^{-}(b)$ is well defined in the sense of Remark 9. Then, for every $\varepsilon>0$ and for every constant $C \in(0,1)$ exists $\delta=\delta(b, \varepsilon, C)$, such that the following implication holds for any $(u, \theta) \in S_{b} \cap H_{r a d}^{1} \times H_{r a d}^{1}$ :

$$
\begin{equation*}
\text { if } E(u, \theta)-J_{b} \leq \delta \text { and }\left|E^{-}(u, \theta)\right| \geq C\left|E^{-}(b)\right|, \quad \text { then }\left|E^{-}(u, \theta)-E^{-}(b)\right| \leq \varepsilon \tag{52}
\end{equation*}
$$

Remark 10. Notice how, as $J_{b}$ is the minimum of the energy, $E(u, \theta)-J_{b}$ is positive. The hypothesis $\left|E^{-}(u, \theta)\right| \geq C\left|E^{-}(u, \theta)\right|$ is added to rule out, for $b \in\left(c, a_{0}\right)$, the minimizing sequences weakly converging to 0 , that exist whenever $J_{b}=0$.

Proof. If it were not true we could find a minimizing sequence $\left(u_{n}, \theta_{n}\right)$ in $H_{r a d}^{1} \times$ $H_{r a d}^{1} \cap S_{b}$ such that

$$
0 \neq \lim E^{-}\left(u_{n}, \theta_{n}\right)=\alpha \neq E^{-}(b)
$$

As the sequence is miniminzing in $S_{b}$, by Lemma 3 there exists a subsequence that converges in $H^{1} \times H^{1}$ to a minimizer $(v, \phi) \in S_{b}$ with $E^{-}(v, \phi)=\alpha$, contraddicting the well-definition of $E^{-}(b)$.

For $b<c$ such that there exist no minimizer over $S_{b}$, we can prove a similar property for the almost minimizing configuration:

Lemma 6. Let $c>0$ as in Lemma 4, and $b \leq c$ be such there exists no minimizer over $S_{b}$. Then, for every $\varepsilon>0$ exists $\delta=\delta(b, \varepsilon)$,such that the following implication holds for any $(u, \theta) \in S_{b} \cap H_{r a d}^{1} \times H_{r a d}^{1}$ :

$$
\begin{equation*}
\text { if } E(u, \theta) \leq \delta \text { then }\left|E^{-}(u, \theta)\right| \leq \varepsilon \tag{53}
\end{equation*}
$$

Moreover, any minimizing sequence $\left(v_{n}, \phi_{n}\right)$ in $S_{b} \cap H_{r a d}^{1} \times H_{r a d}^{1}$ verifies $v_{n} \rightarrow 0$ weakly in $H^{1}$.

Proof. The proof of the first statement follows the steps of the previous Lemma: if it were false, we could find a $\varepsilon>0$ and a minimizing sequence $\left(u_{n}, \theta_{n}\right)$ in $H_{r a d}^{1} \times$ $H_{r a d}^{1} \cap S_{b}$ such that

$$
\lim _{n} E^{-}\left(u_{n}, \theta_{n}\right) \leq-\varepsilon<0
$$

and by Lemma 3 we would deduce the existence of a minimizer.
For the second statement, if it were not true, we would have a minimizing radial sequence ( $v_{n}, \phi_{n}$ ) weakly converging to ( $v, \phi$ ), with $v \neq 0$, satisfying as in (49) $E(v, \phi) \leq J_{b}=0$.
The latter inequalities implies in particular $E^{-}(v, \phi)<0$. It follows by the same argument of Lemma 3 that $\lambda:=\|\nu\|_{L^{2}}^{2}$ has to be equal to $b$, and hence we have reached the contraddiction as $(v, \phi)$ would be a minimizer for the energy over $S_{b}$.

The previous results can be stated uniformly along converging sequences.
Lemma 7. Let $c>0$ be as in Lemma 4, and $a>c$. Let $b_{n} \uparrow a$ such that, for any $n$ the quantity $E^{-}\left(b_{n}\right)$ is well defined in the sense of Remark 9, and consider a sequence $\left(u_{n}, \theta_{n}\right) \in S_{b_{n}} \cap H_{r a d}^{1} \times H_{r a d}^{1}$ verifying

$$
\begin{equation*}
\lim _{n \rightarrow \infty} E\left(u_{n}, \theta_{n}\right)=J_{a} ; \quad C_{1} \geq\left|E^{-}\left(u_{n}, \theta_{n}\right)\right| \geq C_{2}\left|E^{-}\left(b_{n}\right)\right| \tag{54}
\end{equation*}
$$

with $C_{1}, C_{2}$ fixed positive constants. Then it holds

$$
\begin{equation*}
\lim _{n \rightarrow \infty} E^{-}\left(u_{n}, \theta_{n}\right)=\lim _{n \rightarrow \infty} E^{-}\left(b_{n}\right) \tag{55}
\end{equation*}
$$

Proof. We define through this Lemma $E^{-}(a):=\lim _{n} E^{-}\left(b_{n}\right)$, the limit existing because of monotonicity of $E^{-}$in (51).
For every $\varepsilon>0$, we fix $\bar{n}=\bar{n}(\varepsilon)$ to be chosen later, and $\delta=\delta\left(\varepsilon, b_{\bar{n}}, C_{2} / 2\right)>0$ such that (52) holds for $b=b_{\bar{n}}$. For $n>\bar{n}$ we can estimate

$$
\begin{align*}
& \left|E^{-}\left(u_{n}, \theta_{n}\right)-E^{-}\left(b_{n}\right)\right|  \tag{5}\\
& \quad \leq\left|E^{-}\left(u_{n}, \theta_{n}\right)-E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)\right|+\left|E^{-}\left(b_{\bar{n}}\right)-E^{-}\left(b_{n}\right)\right|+\left|E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)-E^{-}\left(b_{\bar{n}}\right)\right| \\
& \quad \leq C_{1}\left(\frac{b_{n}-b_{\bar{n}}}{b_{n}}\right)+\left|E^{-}\left(b_{\bar{n}}\right)-E^{-}(a)\right|+\left|E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)-E^{-}\left(b_{\bar{n}}\right)\right| \\
& \quad \leq \varepsilon+\left|E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)-E^{-}\left(b_{\bar{n}}\right)\right|
\end{align*}
$$

The last inequality is obtained by fixing $\bar{n}$ large enough, depending on $\varepsilon$. To estimate the remaining part, we can use Lemma 5 at $b=b_{\bar{n}}$ : we have

$$
\begin{aligned}
& \left|E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)\right| \geq \frac{C_{2}}{2}\left|E^{-}\left(b_{\bar{n}}\right)\right| \\
& E\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)-J_{b_{\bar{n}}} \leq \frac{b_{\bar{n}}}{b_{n}} E^{-}\left(u_{n}, \theta_{n}\right)+E^{+}\left(u_{n}, \theta_{n}\right)-\frac{b_{\bar{n}}}{b_{n}} J_{b_{n}} \leq \frac{b_{\bar{n}}}{b_{n}}\left(E\left(u_{n}, \theta_{n}\right)-J_{b_{n}}\right)
\end{aligned}
$$

where we have used the inequality (46). If $E\left(u_{n}, \theta_{n}\right)-J_{b_{n}} \leq \delta$ we can conclude

$$
\left|E^{-}\left(u_{n}, \theta_{n}\right)-E^{-}\left(b_{n}\right)\right| \leq 2 \varepsilon
$$

Mutatis mutandis, a similar uniformity holds along a sequence $b_{n} \uparrow a$ such that $J_{a}=0$ and there exists no minimizer over $S_{b_{n}}$ :
Lemma 8. Let $c>0$ be as in Lemma 4, and $a \leq c$. Let $b_{n} \uparrow$ a such that, for any $n$ there exists no minimizer over $S_{b_{n}}$, and consider a sequence $\left(u_{n}, \theta_{n}\right) \in S_{b_{n}} \cap H_{r a d}^{1} \times$ $H_{r a d}^{1}$ verifying

$$
\begin{equation*}
\lim _{n \rightarrow \infty} E\left(u_{n}, \theta_{n}\right)=J_{a}=0 ; \quad\left|E^{-}\left(u_{n}, \theta_{n}\right)\right| \leq C \tag{57}
\end{equation*}
$$

with C fixed positive constant. Then it holds

$$
\begin{equation*}
\lim _{n \rightarrow \infty} E^{-}\left(u_{n}, \theta_{n}\right)=0 \tag{58}
\end{equation*}
$$

Moreover $u_{n}$ is weakly converging to 0 .

Proof. Instead of estimate (56), we have
$\left|E^{-}\left(u_{n}, \theta_{n}\right)\right| \leq\left|E^{-}\left(u_{n}, \theta_{n}\right)-E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)\right|+\left|E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)\right| \leq \varepsilon+\left|E^{-}\left(\sqrt{\frac{b_{\bar{n}}}{b_{n}}} u_{n}, \theta_{n}\right)\right|$
In the remaining of the proof the same steps can be followed, referring to Lemma 6 in place of Lemma 5.
For the last statement, it is enough to notice that

$$
E\left(\sqrt{\frac{b_{1}}{b_{n}}} u_{n}, \theta_{n}\right)=E\left(u_{n}, \theta_{n}\right)+\left(1-\frac{b_{1}}{b_{n}}\right) E^{-}\left(u_{n}, \theta_{n}\right) \rightarrow 0
$$

and every miniminizing sequence over $S_{b_{1}}$ weakly converges to 0 by Lemma 6 .
Proposition 3. Let $c$ be as in Lemma 4, and $a>c$. If there exist two minimizers $\left(v_{1}, \phi_{1}\right)$ and $\left(v_{2}, \phi_{2}\right)$ for $E$ in $S_{a}$, then

$$
\lim _{d \downarrow a} E^{-}(d)=E^{-}\left(v_{1}, \phi_{1}\right)=E^{-}\left(v_{2}, \phi_{2}\right)=\lim _{b \uparrow a} E^{-}(b)
$$

Proof. By the monotonicity relation (51) for $E^{-}$of the minimizers we have that the limits $\lim _{b \uparrow a} E^{-}(b)$ and $\lim _{d \downarrow a} E^{-}(a)$ exist and, for any $\left(v_{a}, \phi_{a}\right)$ minimizer in $S_{a}$ for the energy it holds

$$
\begin{equation*}
\lim _{b \uparrow a} E^{-}\left(v_{b}, \phi_{b}\right) \geq E^{-}\left(v_{a}, \phi_{a}\right) \geq \lim _{d \downarrow a} E^{-}\left(v_{d}, \phi_{d}\right) \tag{59}
\end{equation*}
$$

We consider a sequence $d_{n} \downarrow a$ with the respective minimizing configurations ( $v_{n}, \phi_{n}$ ), and a sequence $b_{n} \uparrow a$ such that for every $n$ is well defined the value $E^{-}\left(b_{n}\right)$.
We can define the sequence of almost minimizers in $S_{b_{n}}$

$$
\left(u_{n}, \theta_{n}\right):=\left(\frac{b_{n}}{d_{n}} v_{n}, \phi_{n}\right) \in S_{b_{n}}
$$

that satisfies, for some fixed $C_{1}, C_{2}>0$

$$
\lim _{n} E\left(u_{n}, \theta_{n}\right)=J_{a}=\lim _{n} J_{b_{n}} ; \quad C_{1} \geq\left|E^{-}\left(u_{n}, \theta_{n}\right)\right| \geq C_{2}\left|E^{-}\left(b_{n}\right)\right|
$$

By Lemma 7:

$$
\lim _{n \rightarrow \infty} E^{-}\left(u_{n}, \theta_{n}\right)=\lim _{n \rightarrow \infty} E^{-}\left(b_{n}\right)=\lim _{b \uparrow a} E^{-}\left(v_{b}, \phi_{b}\right)
$$

On the other hand, by explicit computation we have

$$
\lim _{n \rightarrow \infty} E^{-}\left(u_{n}, \theta_{n}\right)=\lim _{n \rightarrow \infty} E^{-}\left(v_{n}, \phi_{n}\right)=\lim _{d \downarrow a} E^{-}\left(v_{d}, \phi_{d}\right)
$$

The two limits in equation (59) coincide; that is to say the value of $E^{-}\left(v_{a}, \phi_{a}\right)$ is the same for any ( $v_{a}, \phi_{a}$ ) minimizer in $S_{a}$.

Corollary 2. Let $c>0$ as in Lemma 4. Then the map $f:(c, \infty) \rightarrow(0,1]$ defined by $f(a)=\sigma_{a}$ the value $\sigma$ for which a minimizer $\left(v_{a}, \phi_{a}\right) \in S_{a}$ satisfies the system (4)-(5) is well defined, continuous and increasing.

Proof. The function is well defined and continuous because of Proposition 3 and the relation between $\sigma$ and $E^{-}$expressed in equation (39); by Lemma 4 the monotonicity is easily inferred.

Proposition 4. Let $c>0$ as in Lemma 4; the following holds

$$
\lim _{a \rightarrow c} \sigma_{a}=0 ; \quad \lim _{a \rightarrow \infty} \sigma_{a}=1
$$

Proof. By contradiction, we assume $\sigma_{a} \rightarrow \sigma_{*}<1$ for $a \rightarrow \infty$. For any $\sigma \in\left(\sigma_{*}, 1\right)$ we can find a function $u_{\sigma} \in H_{r a d}^{1}$ with compact support $B_{R}$ such that

$$
\left\|u_{\sigma}\right\|_{L^{2}}^{2}=1 ; \quad\left\|\nabla u_{\sigma}\right\|_{L^{2}}^{2}-2\left\|u_{\sigma}\right\|_{L^{2}}^{2}=-2 \sigma
$$

Fix $\phi \in H_{r a d}^{1}$ such that $0 \leq \phi \leq \pi / 2$, and $\phi \equiv \pi / 2$ over $B_{R}$. We have then that for any $a$

$$
-2 \sigma_{*} a \leq 4 J_{a} \leq 4 E\left(\sqrt{a} u_{\sigma}, \phi\right)=-2 \sigma a+4 E^{+}\left(\sqrt{a} u_{\sigma}, \phi\right)
$$

Letting $a \rightarrow \infty$ we reach the contradiction, as the quantity $E^{+}$is independent of $a$ and finite.
For $a_{0}$ as in Theorem 3, we have $J_{a} \rightarrow 0=J_{a_{0}}$ for $a \downarrow a_{0}$. For $\left(v_{a}, \phi_{a}\right)$ minimizers, either one of the following is happening

- $\lim _{a \downarrow a_{0}} E^{-}\left(v_{a}, \phi_{a}\right)=\lim _{a \downarrow a_{0}} E^{+}\left(v_{a}, \phi_{a}\right)=0$
- $0>d=\lim _{a \downarrow a_{0}} E^{-}\left(v_{a}, \phi_{a}\right)=-\lim _{a \downarrow a_{0}} E^{+}\left(v_{a}, \phi_{a}\right)$

In the first case, we have concluded: $c=a_{0}$ and $0=\lim _{a \rightarrow c} E^{-}\left(v_{a}, \phi_{a}\right)=\lim _{a \rightarrow a_{0}}-a \sigma_{a}$. In the second case, we have by Lemma $3\left(v_{a}, \phi_{a}\right)$ converges up to subsequence to ( $v_{a_{0}}, \phi_{a_{0}}$ ) minimizer of $E$ in $S_{a_{0}}$. We can show that the set

$$
\begin{equation*}
I:=\left\{b \leq a_{0} \mid \text { exists a minimizer for the energy over } S_{b}\right\} \tag{60}
\end{equation*}
$$

is left open. If it were not, there would exist a $b \leq a_{0}$ and $\left(v_{b}, \phi_{b}\right)$ minimizer over $S_{b}$; and a sequence $b_{n} \uparrow b$ that does not allow for the existence of a minimizer over $S_{b_{n}}$. We can then look at the sequence

$$
\left(u_{n}, \theta_{n}\right):=\left(\frac{b_{n}}{b} v_{b}, \phi_{b}\right) \in S_{b_{n}} \cap H_{r a d}^{1} \times H_{r a d}^{1} ; \quad \lim _{n} E\left(u_{n}, \theta_{n}\right)=J_{b}=0
$$

By Lemma 8 we would have $u_{n} \rightarrow 0$ weakly, which is clearly impossible.
The set $I$ is also left closed upon a sequence $b_{n} \downarrow b$, as long as

$$
\lim _{n} E^{-}\left(b_{n}\right)=c \neq 0
$$

by Lemma 3. Defining $c:=\inf _{I}$ takes to the thesis: we have $c \geq \tilde{a}>0$ for $\tilde{a}$ as in Theorem 3, and the condition for the limit of $\sigma_{a}$ comes from $c$ being the infimum.

At this point Theorem 2 is a direct consequence of Corollary 2 and Proposition 4.

## 3 Decaying rate

The main result of this section will be the following
Proposition 5. A radial decreasing solution ( $v, \phi$ ) of system (4)-(5) has a polynomial decay at infinity at any rate: for any $\alpha>0$ exist $C=C(\alpha)$ and $R=R(\alpha)$ such that for any $r>R_{\alpha}$ it holds

$$
|v(r)| \leq \frac{C}{r^{\alpha}} ; \quad|\theta(r)| \leq \frac{C}{r^{\alpha}}
$$

The proof is based on the representation of $v$ and $\phi$ as a convolution of the nonlinear terms in equations (4)-(5) with a rapidly decaying kernel. We recall some basic properties for the modified Bessel function $K_{0}$, that will play the role of the kernel. It is a positive, integrable decreasing function $K_{0}: \mathbb{R}^{+} \rightarrow \mathbb{R}$, that decays at infinity as

$$
K_{0}(r)=\frac{1}{2 \pi} \sqrt{\frac{\pi}{2 r}} e^{-r}\left(1+O\left(r^{-1}\right)\right) \quad \text { for } r \rightarrow \infty
$$

It is of interest for our result because of the following (see [8])
Theorem 7. For any $f \in L^{1}\left(\mathbb{R}^{2}\right) \cap L^{2}\left(\mathbb{R}^{2}\right)$ positive, for any $\sigma>0$, there exists $a$ unique solution $u \in H^{2}\left(\mathbb{R}^{2}\right)$ of the equation

$$
-\Delta u+\sigma u=f
$$

which is given by the convolution

$$
u(x)=\int_{\mathbb{R}^{2}} K_{0}(\sqrt{\sigma}|x-y|) f(y) d y
$$

The idea of the proof for Proposition 5 is the following. We can infer a polynomial decaying estimates for $v$ and $\phi$, and hence a polynomial decay of higher order for the nonlinear terms; if we can represent the solutions as a convolution between the non linear term and the rapidly decaying $K_{0}$, we can deduce that the solution decays as the nonlinear term and iterate the reasoning. In the following Lemmas we prove those two statements.

Lemma 9. Let $f \in L^{2}\left(\mathbb{R}^{2}\right)$ be a radial decreasing function. Then there exist $C>0$ such that for any $r$

$$
|f(r)| \leq \frac{C}{r^{\frac{1}{2}}}\|f\|_{L^{2}}
$$

For the proof see [3], Lemma [1.7.3].
Lemma 10. Let $K_{0}$ be the modified Bessel functional, and $f \in L^{1}\left(\mathbb{R}^{2}\right) \cap L^{\infty}\left(\mathbb{R}^{2}\right) a$ function satisfying, for constants $C, R, \alpha>0$

$$
f(y) \leq \frac{C}{|y|^{\alpha}} \forall|y| \geq R
$$

Then the convolution

$$
u(x):=\int_{\mathbb{R}^{2}} K_{0}(|y|) f(x-y) d y
$$

has the same decaying rate as $f$, with constants $C_{1}, R_{1}$ depending on $C, R, \alpha, K_{0},\|f\|_{L^{1} \cap L^{\infty}}$
Proof. Fix $R_{K}$ such that $K_{0}(r) \leq C e^{-r}$ for any $r \geq R_{k}$; let $x \in \mathbb{R}^{2}$ with $|x| \geq$ $2 \max \left\{R, R_{K}\right\}$. We have

$$
\begin{aligned}
|u(x)| & \leq \int_{B_{\frac{|x|}{2}}} K(|y|)|f(x-y)| d y+\int_{B_{\frac{|x|}{2}}^{C}} K(|y|)|f(x-y)| d y \leq \\
& \leq \frac{C}{|x|^{\alpha}} \int_{B_{\frac{|x|}{2}}} K(|y|) d y+\frac{C| | f \|_{L^{\alpha}}}{|x|^{\alpha}} \int_{B^{\frac{|x|}{C}}}|x|^{\alpha} \frac{e^{-|y|}}{\sqrt{|y|}} d y \leq \\
& \leq \frac{C| | K \|_{L^{1}}}{|x|^{\alpha}}+\frac{C| | f \|_{L^{\infty}}}{|x|^{\alpha}} \int_{B_{\frac{|x|}{2}}^{C}}|y|^{\alpha} \frac{e^{-|y|}}{\sqrt{|y|}} d y \leq \frac{C}{|x|^{\alpha}}
\end{aligned}
$$

In the first inequality we have used the decay rate of $f$ and $K$, multiplying and dividing the second integral by $|x|^{\alpha}$. In the second we have simply controlled $|x| \leq$ $C|y|$ for $y \in B_{\frac{|x|}{2}}^{C}$, and finally we have simply used the integrability of the product of a polynomial and a negative exponential.

Proof of Proposition 5. By hypothesis, $(v, \phi)$ satisfies the equation

$$
\begin{equation*}
-\Delta v+2 \sigma v=2 v \sin (2 \phi) \tag{61}
\end{equation*}
$$

The right hand side is in $L^{1}\left(\mathbb{R}^{2}\right) \cap L^{2}\left(\mathbb{R}^{2}\right)$, as $\sin (2 \phi), v \in L^{2}\left(\mathbb{R}^{2}\right) \cap L^{4}\left(\mathbb{R}^{2}\right)$; by Theorem 7 there exists $u \in H^{2}$ solution of the equation

$$
-\Delta u+\sigma u=2 v \sin (2 \phi)
$$

written as

$$
u(x)=\int K_{0}(\sqrt{\sigma}|x-y|(2 v(y) \sin (2 \phi(y))) d y
$$

Both $v$ and $\phi$ are radial decreasing functions in $L^{2}$, and for $y \operatorname{small} \sin (y) \leq 2 y$; from Lemma 9 and Lemma $10 u$ decays as $\frac{C}{r}$. The difference $u-v$ satisfies

$$
-\Delta(u-v)+\sigma(u-v)=0
$$

Since $\sigma>0$, the weak maximum principle for $H^{1}$ functions applies to the previous equation in $B_{R}$ for any $R>0$; since both $u, v$ vanish at infinity we deduce $u \equiv v$.
We look at equality (61), with $v$ decaying as $\frac{C}{r}$. A bootstrap argument leads to the decaying claim for $v$ : using the previous result we improve the decay rate of $2 v \sin (2 \phi)$ to $\mathrm{Cr}^{-\frac{3}{2}}$, deduce the same rate for $v$, and iterate the process.
The same strategy applies to prove the result on $\phi$, which is in $H^{2}$ for Theorem 4: we can rewrite (5) as

$$
-v \Delta \phi+2 q \phi=2|v|^{2} \cos (2 \phi)+q(2 \phi-\sin (2 \phi))
$$

For $r$ large enough, by Taylor expansion $|2 \phi-\sin (2 \phi)| \leq C|\phi|^{3}$. Hence the right hand side is in $L^{1}$; moreover the result about the decay of $v$ and the previous control over $\phi$ allow for the bootstrap argument as above.

## Acknowledgments

The author is supported by the Basque Government under the IKUR program and through BCAM Severo Ochoa excellence accreditation SEV-2023-2026.

## References

[1] G. Assanto, N.F. Smyth: Self-confined light waves in nematic liquid crystals Physica D (2019) 132182
[2] T. Cazenave, P.L. Lions,: Orbital stability of standing waves for some non linear Schr"odinger equations Commun. Math. Phys. 85, 549-561 (1982).
[3] T. Cazenave: Semilinear Schr"odinger equations Courant Lecture Notes in Mathematics , vol. 10, New York University, Courant Institute of Mathematical Sciences, New York, American Mathematical Society, Providence, RI, 2003.
[4] J. P. Borgna, Panayotis Panayotaros, D. Rial, C. S. F. de la Vega 3: Optical solitons in nematic liquid crystals: model with saturation effects Nonlinearity 31, 1535-1559 (2018)
[5] Berestycki, H., Lions, P.L. :Nonlinear scalar field equations, II existence of infinitely many solutions Arch. Rational Mech. Anal. 82, 347-375 (1983).
[6] R. Fukuizumi: Stability and instability of standing waves for nonlinear Schrodinger equations Ph.D thesis, Tohoku University, Sendai, Japan, 2003
[7] M. Grillakis, J. Shatah, W. Strauss : Stability theory of solitary waves in the presence of symmetry, I Journal of Functional Analysis 74, (1987)
[8] L.C. Evans: Partial Differential Equations AMS, Providence, 1998.
[9] Y.S. Kivshar, G.P. Agrawal: Optical Solitons. From Fibers to Photonic Crystals Academic Press, San Diego (2003)
[10] P. Panayotaros, T.R. Marchant: Solitary waves in nematic liquid crystals Physics Reports 516, 147-208 (2012)
[11] M. Peccianti, G. Assanto Nematicons Physics Reports 516, 147-208 (2012)

BCAM - Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Bizkaia, Spain
Email address: smoroni@bcamath.org

