HYPERGEOMETRIC SHEAVES WITH TANNAKIAN MONODROMY GROUP G_2

BEAT ZURBUCHEN

Contents

1. Introduction

The equidistribution of Frobenius conjugacy classes has been a cornerstone of number theory at least since Dirichlet's Theorem on primes in arithmetic progressions. In a geometric context, this study was successfully initiated by Deligne in [Del80]. The central object of this theory is a compact Lie group, the so-called monodromy group, in which Frobenius conjugacy classes acting on a local system on a variety over a finite field naturally occur and then equidistribute. This provides a formalism that produces equidistribution results in large generality. Of course, the application of this formalism requires the determination of the monodromy group.

These monodromy groups were determined for a large class of exponential sums, among them the so-called hypergeometric sums (see [Kat90, Ch. 8]), by Katz. For example, consider a finite field k of odd characteristic p and a non-trivial additive character ψ of k. Let χ_2 denote the unique non-trivial multiplicative character of order 2 of k and name the quadratic Gauss sum by

$$A_{\psi,k} := -\sum_{x\in k^*} \chi_2(x)\psi(x).$$

For each $a \in k^*$ and each multiplicative character χ of k define

$$\mathrm{Hyp}(a,\chi) := A_{\psi,k}^{-7} \sum_{x_1\cdots x_7 = x_8 a} \psi(x_1+\ldots+x_7-x_8)\chi\big(x_4x_5(x_6x_7)^{-1}\big)\chi_2(x_8).$$

Define the complex (see [Kat90, p. 8.2.2] for the notation)

$$\mathcal{F}(\chi,\psi,k) := (A_{\psi,k}^{-7})^{\deg} \otimes \mathrm{Hyp}(!,\psi;1,1,1,\chi,\chi,\overline{\chi},\overline{\chi};\chi_2).$$

This complex has the property

$$\mathrm{Tr}(\mathrm{Fr}_k|\mathrm{Hyp}(\chi)_a) = \mathrm{Hyp}(a,\chi)$$

for all $a \in k^*$ and all multiplicative characters χ.

Fix a multiplicative character χ of k. We define UG_2 to be the compact form of G_2 and let $UG_2^\natural$ be the space of conjugacy classes in UG_2. In [Kat07, Thm. 9.1], Katz constructs semisimple conjugacy classes $\theta_{a,\chi} \in UG_2^\natural$ such that the trace of $\theta_{a,\chi}$ acting on the unique irreducible seven-dimensional representation of G_2 is given by

$$\mathrm{Tr}(\theta_{a,\chi}) = \mathrm{Hyp}(a,\chi).$$

Denote by k_n a finite extension of k of degree $n \geq 1$. When $p \geq 17$, Deligne's equidistribution theorem [Kat90, Thm. 7.11.1] applied to [Kat07, Thm. 9.1] implies

$$\lim_{n\to\infty} \frac{1}{|k_n^*|} \sum_{a \in k_n^*} f(\theta_{a,\chi}) = \int_{UG_2} f(g)dg$$

for any continuous function $f \in C(UG_2^\natural)$ where dg denotes the probability Haar measure on UG_2. We say that the sets $\{\theta_{a,\chi} : a \in k_n^*\}$ equidistribute in $UG_2^\natural$ for the pushforward of the probability Haar measure from UG_2.

The equidistribution result by Deligne is not able to describe the distribution of the set of conjugacy classes $\{\theta_{a,\chi} : \chi \in \widehat{k_n^*}\}$ because the multiplicative characters are not representable by a variety. In [Kat12, Ch. 4], a certain subcategory of the category of perverse sheaves on $\mathbb{G}_{m,k}$ is equipped with the structure of a Tannakian category which associates an arithmetic Tannakian monodromy group to each object in this category. This formalism does produce equidistribution theorems for such sets. The book [Kat12, Ch. 25] (see also Theorem 2.2) explains how to apply this formalism to the above sums and determines the generic Tannakian monodromy group of these sums in the family defined by the variable a.

Theorem ([Kat12, Thm. 25.1])**.** *The set of all $a \in k^*$ such that the set of conjugacy classes $\{\theta_{a,\chi} : \chi \in \widehat{k_n^*}\}$ equidistribute in $UG_2^\natural$ as $n \to \infty$ has "density" 1.*

Based on a suggestion in [Kat12, Rmk. 25.8], we improve this theorem to the following result.

Theorem (Theorem 5.2)**.** *Suppose the characteristic of k is large enough and let $a \in k^*$. The set $\{\theta_{a,\chi} : \chi \in \widehat{k_n^*}\}$ equidistributes in $UG_2^\natural$ as $n \to \infty$.*

The lower bound on the characteristic of k can not be evaluated by our method. It is, however, not clear whether the result can be proven for all finite fields. For monodromy groups of hypergeometric sums, it is a well-known phenomenon (see [Kat07, Thm 9.1] and [Kat90, Thm. 14.10]) that the monodromy groups become uniform only for large primes. For example, it is proven in [Kat07, Thm. 9.1] that for $p < 17$ there are characters χ for which the monodromy group of $\mathcal{F}(\chi,\psi,k)$ is finite. However, this precise phenomenon can not occur for irreducible perverse sheaves on $\mathbb{G}_{m,\overline{k}}$ because any finite Tannakian monodromy group of a perverse sheaf on $\mathbb{G}_{m,\overline{k}}$ is cyclic by [Kat12, Thm. 8.2].

Our method should apply to more general hypergeometric families such as the families constructed in [GL96]. To be precise, let $n > 1$ and consider a perverse !-hypergeometric sheaf $\mathscr{H}$ on $\mathbb{G}_{m,k}^n$ in the sense of [GL96, Def. 8.1.2]. In favorable cases, a group morphism $\mathbb{G}_m^n \to \mathbb{G}_m$ yields a hypergeometric family by restricting $\mathscr{H}$ to the fibers. The example studied in this paper is of this form up to negligible factors. In the future, we plan to exploit this method to determine the Tannakian monodromy groups for the members of some of these families.

Our proof relies substantially on Katz's determination of the generic Tannkian monodromy group. We study the sum $\mathrm{Hyp}(a, \chi)$ in the family defined by the variable $a \in k^*$. The transformation law in Lemma 3.3, which is a defining feature of a hypergeometric family, implies that varying the base point a is equivalent to varying the additive character ψ. The crucial step is to formulate the fourth moment of the Tannakian monodromy group as a weighted Euler-Poincare characteristic (see Lemma 3.3). Then we can prove that the Tannakian monodromy group is independent of the character ψ by appealing to the uniformity properties of the Fourier transform. This implies that the Tannakian monodromy group is independent of the base point a when the characteristic of k is large enough. In particular, the Tannakian monodromy group for any base point a has to agree with the generic Tannakian monodromy group. Then we can deduce that the Tannakian monodromy group is G_2.

Notations:

- ℓ: a fixed prime.
- All sheaves and complexes of sheaves on a separated, noetherian scheme X of finite type over $\mathbb{Z}[1/\ell]$ are objects in the category $D_c^b(X, \overline{\mathbb{Q}_\ell})$ defined in [Del80, p. 1.1.2].
- We fix an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ once and for all; we apply this isomorphism implicitly whenever needed. In particular, the notation $\lim_{n\to\infty}$ always denotes a limit of complex numbers. When we say a complex is mixed or pure, we always mean with respect to this isomorphism.
- k: a finite field, the characteristic of k is always co-prime to ℓ and *odd*.
- k_n: an extension k_n/k of degree n.
- $\mathscr{L}_\eta$: η is a character $\eta\colon G(k) \to \overline{\mathbb{Q}_\ell}^*$ for some finite field k and an algebraic group G/k, then $\mathscr{L}_\eta$ denotes the local system on G constructed from χ using the Lang torsor construction.
- χ: a multiplicative character $\chi\colon k^* \to \overline{\mathbb{Q}_\ell}^*$; we extend any such character to a character on any finite extension k_n/k by putting $\chi_n(x) := \chi(\mathrm{Nm}_{k_n/k}(x))$ for all $x \in k_n$.
- ψ: an additive character $\psi\colon k^+ \to \overline{\mathbb{Q}_\ell}^*$; we extend any such character to any finite extension k_n/k by putting $\psi_n(x) := \psi(\mathrm{Tr}_{k_n/k}(x))$ for all $x \in k_n$.
- $\mathrm{Hyp}(-,-;-,)$: a hypergeometric sheaf in the sense of [Kat90, p. 8.2.2].
- $M_{2m}(X)$: if X is an object in a Tannakian category then $M_{2m}(X)$ is defined to be the number of components in a decomposition series of the object $X^m \otimes (X^\wedge)^m$ which are isomorphic to the tensor unit (i.e. the $2m$-th moment of X).

2. Construction by Katz

We recall certain definitions and results from [Kat12]. Let k be a finite field, ψ a non-trivial additive character of k, and $a \in k^*$. Denote by $\mathscr{H}_{\psi,k,a}$ the ℓ-adic constructible complex on $\mathbb{G}_{m,k}$ denoted $N(a, k)$ in [Kat12, Ch. 27, p. 165]. Let $j\colon \mathbb{G}_m \to \mathbb{P}^1$ be the inclusion. Following [Kat12, p. 21], we call a multiplicative character χ of k_n *good* (or *not bad*) for a perverse sheaf M on $\mathbb{G}_{m,k}$ if the natural morphism

$$Rj_!(M \otimes \mathscr{L}_\chi) \to Rj_*(M \otimes \mathscr{L}_\chi)$$

is an isomorphism. When the character χ is good, the Leray spectral sequence implies that the map

$$H^\bullet_c(\mathbb{G}_{m,\overline{k}}, M \otimes \mathscr{L}_\chi) \to H^\bullet(\mathbb{G}_{m,\overline{k}}, M \otimes \mathscr{L}_\chi)$$

is an isomorphism. In this case, the cohomology groups $H^\bullet_c(\mathbb{G}_{m,\overline{k}}, M \otimes \mathscr{L}_\chi)$ are concentrated in degree zero by Artin's vanishing theorem [KW01, Thm. 6.1]. This recovers [Kat12, Lem. 2.1].

Theorem 2.1. *The complex $\mathscr{H}_{\psi,k,a}$ is pure of weight zero, perverse, irreducible, has no bad characters and there is a Frobenius-equivariant isomorphism*

$$H^\bullet_c(\mathbb{G}_{m,\overline{k}}, \mathscr{H}_{\psi,k,a} \otimes \mathscr{L}_\chi) \cong \mathcal{F}(\chi, \psi_n, k_n)_a$$

for each multiplicative character $\chi \neq \chi_2$ of a finite extension of k.

Proof. This complex is perverse and pure of weight zero by construction. It has no bad characters by [Kat12, Lem. 27.5] and is irreducible by [Kat12, Lem. 27.11]. The cohomology groups are determined in [Kat12, Lem. 27.4]. ☐

The formalism of [Kat90, Ch. 4] equips the category of perverse sheaves on $\mathbb{G}_{m,k}$, which only have good characters when pulled back to $\mathbb{G}_{m,\overline{k}}$, with the structure of a Tannakian category. Given an object M in this category then the Tannakian formalism implies that the Tannakian category generated by M is equivalent to the category of representations of a complex algebraic group G after applying the isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$. This group is called the *arithmetic Tannakian monodromy group* of M.

In loc. cit. the generic monodromy group of the hypergeometric family $\mathscr{H}_{\psi,k,a}$ is determined and the possibilities of the Tannakian monodromy group are strongly restricted. This is summarized in the following theorem.

Theorem 2.2. *Let ψ, k, a be as above.*

(1) *The geometric and the arithmetic Tannakian monodromy group of $\mathscr{H}_{\psi,k,a}$ agree. The Tannkian monodromy group of this complex is a subgroup of* GL_7.
(2) *The image of the Tannakian monodromy group of $\mathscr{H}_{\psi,k,a}$ in* GL_7 *is (conjugate to) either the image of G_2 in its unique irreducible 7-dimensional representation or* SL_2 *in* $\mathrm{Sym}^6(\mathrm{std}_2)$.
(3) *There exists $n \geq 1$ and $a \in k_n^*$ such that the monodromy group of $\mathscr{H}_{\psi_n,k_n,a}$ is G_2.*
(4) *The moment satisfies $M_4(\mathscr{H}_{\psi,k,a}) = 4$ if and only if the Tannakian monodromy group of $\mathscr{H}_{\psi,k,a}$ is G_2.*

Proof. The points (1) and (2) are proven in [Kat12, Lem. 25.2]. The point (3) follows form [Kat12, Thm. 25.1]. To prove (4), note that the moment is 4 when the Tannkian monodromy group is G_2 and the moment is 7 if the Tannakian monodromy group is SL_2. ☐

3. The fourth moment

In this chapter, we apply the equidistribution result to produce formulas for the fourth moment.

Definition. Define the polynomial

$$P(x_{i,j}, y_{i,j}) := \prod_{j\in\{1,2\}} x_{4,j}x_{5,j}y_{6,j}y_{7,j}(y_{4,j}y_{5,j}x_{6,j}x_{7,j})^{-1}$$

such that

$$P \in \mathbb{Z}[(x_{i,j})_{(i,j)\in\{1,\dots,8\}\times\{1,2\}}, (y_{i,j})_{(i,j)\in\{1,\dots,8\}\times\{1,2\}}].$$

Let k be a finite field, ψ a non-trivial additive character of k, and $a \in k^*$. Define

$$f(\psi, k, a) := |k|^{-15} \sum_{\substack{x_{1,j}\dots x_{7,j}=ax_{8,j} \\ y_{1,j}\dots y_{7,j}=ay_{8,j} \\ P(x_{i,j},y_{i,j})=1}} \left(\psi\left(\sum_{j\in\{1,2\}} \left(\sum_{i=1}^{7} (x_{i,j} - y_{i,j}) - x_{8,j} + y_{8,j} \right)\right) \right.$$
$$\left. \times \chi_2\left(\prod_{j\in\{1,2\}} x_{8,j}y_{8,j}^{-1} \right)\right).$$

Theorem 3.1. *Let k be a finite field, ψ a non-trivial additive character of k, and $a \in k^*$. We have*

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} f(\psi_n, k_n, a).$$

Proof. By Theorem 2.1 all characters are good. Thus [Kat90, Thm. 7.3] and [FFK23, Eqn. 9.2] imply

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} \frac{1}{|k_n|-1} \sum_{\chi\in\widehat{k_n^*}} |\mathrm{Tr}(\mathrm{Fr}_{k_n}|H_c^\bullet(\mathbb{G}_{m,\overline{k}}, \mathscr{H}_{\psi,k,a}\otimes\mathscr{L}_\chi))|^4.$$

The complex $\mathscr{H}_{\psi,k,a}\otimes\mathscr{L}_\chi$ is pure of weight zero. Then [Del80, Var. 6.2.3] implies that the complex $H_c^\bullet(\mathbb{G}_{m,\overline{k}}, \mathscr{H}_{\psi,k,a}\otimes\mathscr{L}_{\chi_2})$ is pure of weight zero and concentrated in degree zero because χ_2 is a good character. Thus we can write

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} \frac{1}{|k_n|-1} \sum_{\chi\neq\chi_2} |\mathrm{Tr}(\mathrm{Fr}_{k_n}|H_c^\bullet(\mathbb{G}_{m,\overline{k}}, \mathscr{H}_{\psi,k,a}\otimes\mathscr{L}_\chi))|^4.$$

By using the evaluation of the cohomology groups in Theorem 2.1 and the formulas from [Kat90, p. 8.2.7] we can rewrite this as

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} \frac{|A_{\psi,k_n}|^{-28}}{|k_n|-1} \sum_{\chi\neq\chi_2} \sum_{\substack{x_{1,j}\dots x_{7,j}=ax_{8,j} \\ y_{1,j}\dots y_{7,j}=ay_{8,j}}} \left(\chi(P(x_{i,j}, y_{i,j})) \right.$$
$$\left. \times \chi_2\left(\prod_{j\in\{1,2\}} x_{8,j}(y_{8,j})^{-1} \right) \psi\left(\sum_{j\in\{1,2\}} \left(\sum_{i=1}^{7} (x_{i,j} - y_{i,j}) - x_{8,j} + y_{8,j} \right)\right)\right)$$

It follows from the cancellation theorem [Kat90, p. 8.4.7] (see also [Kat12, p. 163]) that we can complete this sum to

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} \frac{|A_{\psi,k_n}|^{-28}}{|k_n|-1} \sum_{\chi\in\widehat{k_n^*}} \sum_{\substack{x_{1,j}\dots x_{7,j}=ax_{8,j} \\ y_{1,j}\dots y_{7,j}=ay_{8,j}}} \left(\chi(P(x_{i,j}, y_{i,j})) \right.$$
$$\left. \times \chi_2\left(\prod_{j\in\{1,2\}} x_{8,j}(y_{8,j})^{-1} \right) \psi\left(\sum_{j\in\{1,2\}} \left(\sum_{i=1}^{7} (x_{i,j} - y_{i,j}) - x_{8,j} + y_{8,j} \right)\right)\right)$$

The term $\frac{|A_{\psi,k_n}|^{-28}}{|k_n|-1}$ is asymptotically equivalent to $|k_n|^{-15}$. We change the order of summation and then the orthogonality of characters of $\widehat{k_n^*}$ implies

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} f(\psi_n, k_n, a).$$

This is the statement of the theorem. □

Corollary 3.2. *Let k be a finite field, ψ a non-trivial additive character of k, $a \in k^*$, and $n \geq 1$. Then*

$$M_4(\mathscr{H}_{\psi,k,a}) = M_4(\mathscr{H}_{\psi_n,k_n,a}).$$

Proof. Theorem 3.1 implies

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{m\to\infty} f(\psi_m, k_m, a) = \lim_{m\to\infty} f(\psi_{mn}, k_{mn}, a) = M_4(\mathscr{H}_{\psi_n,k_n,a}).$$

□

The following formula for the change of the additive character is crucial to our analysis.

Lemma 3.3. *Let k be a finite field, ψ a non-trivial additive character of k, and $a, \lambda \in k^*$. Define $\psi_\lambda(x) := \psi(\lambda x)$ for all $x \in k$. Then*

$$M_4(\mathscr{H}_{\psi_\lambda,k,a}) = M_4(\mathscr{H}_{\psi,k,\lambda^6 a}).$$

Proof. Just as in the proof of Theorem 3.1 we have

$$M_4(\mathscr{H}_{\psi_\lambda,k,a}) = \lim_{n\to\infty} \frac{1}{|k_n|-1} \sum_{\chi\neq\chi_2} |\mathrm{Tr}(\mathrm{Fr}_{k_n}|H_c^\bullet(\mathbb{G}_{m,\overline{k}}, \mathscr{H}_{\psi_\lambda,k,a} \otimes \mathscr{L}_\chi))|^4$$

We define $\psi_{\lambda,n} := (\psi_\lambda)_n$. Using Theorem 2.1, we can rewrite this as

$$M_4(\mathscr{H}_{\psi_\lambda,k,a}) = \lim_{n\to\infty} \frac{1}{|k_n|-1} \sum_{\chi\neq\chi_2} |\mathrm{Tr}\big(\mathrm{Fr}_{k_n}|\mathcal{F}(\chi, \psi_{\lambda,n}, k_n)_a\big)|^4$$

Note that $A_{\psi_\lambda,k} = \chi_2(\lambda) A_{\psi,k}$. Thus [Kat90, Lem. 8.7.2] implies

$$|\mathrm{Tr}\big(\mathrm{Fr}_{k_n}|\mathcal{F}(\chi, \psi_{\lambda,n}, k_n)_a\big)|^4 = |\mathrm{Tr}\big(\mathrm{Fr}_{k_n}|\mathcal{F}(\chi, \psi_n, k_n)_{\lambda^6 a}\big)|^4.$$

So we can apply the formula from before again to get

$$M_4(\mathscr{H}_{\psi_\lambda,k,a}) = \lim_{n\to\infty} \frac{1}{|k_n|-1} \sum_{\chi\neq\chi_2} |\mathrm{Tr}\big(\mathrm{Fr}_{k_n}|\mathcal{F}(\chi, \psi_n, k_n)_{\lambda^6 a}\big) = M_4(\mathscr{H}_{\psi,k,\lambda^6 a}).$$

□

Remark 3.4. The equality of moments could also be understood as an isomorphism of sheaves on $\mathbb{G}_{m,k}^2$. More precisely, this equality follows form a change of characters formula for !-hypergeometric sheaves on $\mathbb{G}_{m,k}^2$ as in [GL96, Def. 8.1.2]. In the one-dimensional case, this formula is given by [Kat90, Lem. 8.7.2].

4. Weighted Euler-Poincare characteristics and the moment

Roughly speaking, this section expresses the function f as the trace function of an additive Fourier transform of a sheaf on $\mathbb{A}^1$. The equality of the geometric and the arithmetic Tannakian monodromy group implies a formula for the moment in terms of the weight filtration of the stalk of the Fourier transform at 1. This expression, in turn, can be controlled by appealing to the uniformity of the Fourier transform. For this, we would like to have some notion of representability by a Fourier transform for certain "unmotivated" trace functions.

Definition. Let $g(\psi, k)$ be a complex-valued function that takes as an input a finite field k and a character ψ of k. We say that the function g *representable by a Fourier transform* if there exists a dense open $U \subset \mathrm{Spec}(\mathbb{Z})$ and a mixed complex K on $\mathbb{A}^1_U$ such that

$$g(\psi, k) = \mathrm{Tr}(\mathrm{Fr}_k | H^\bullet_c(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi))$$

for all finite fields $k/\mathbb{F}_p$ with $p \in U$. The complex K is said to be a *representing complex for the function f.*

Theorem 4.1. *The function $(\psi, k) \mapsto f(\psi, k, 1)$ is representable by a Fourier transform.*

Proof. The squaring map $[2] : \mathbb{G}_{m,\mathbb{Z}[1/2\ell]} \to \mathbb{G}_{m,\mathbb{Z}[1/2\ell]}$ defines a $\mathbb{Z}/2\mathbb{Z}$-torsor. Denote by $\mathscr{L}$ the ℓ-adic local system on $\mathbb{G}_{m,\mathbb{Z}[1/2\ell]}$ which trivializes to the unique non-trivial ℓ-adic character of $\mathbb{Z}/2\mathbb{Z}$ after pullback along the squaring map. Consider the closed subscheme $Z \subset \mathbb{G}^{32}_{m,\mathbb{Z}[1/2\ell]}$ defined by the equations

$$x_{1,j} \dots x_{7,j} = x_{8,j},\ y_{1,j} \dots y_{7,j} = y_{8,j},\ P(x_{i,j}, y_{i,j}) = 1$$

for all $j \in \{1, 2\}$. Define the map $\varphi\colon Z \to \mathbb{G}_{m,\mathbb{Z}[1/2\ell]}$ by

$$\varphi(x^i_j, y^i_j) = \sum_{j \in \{1,2\}} \left(\sum_{i=1}^{7} (x_{i,j} - y_{i,j}) - x_{8,j} + y_{8,j} \right).$$

Put $K := R\varphi_!(\mathscr{L}|_Z)$. For each finite extension $k/\mathbb{F}_p$ with p co-prime to 2ℓ and each additive character ψ, the trace formula, the proper base change theorem and the projection formula imply

$$\mathrm{Tr}(\mathrm{Fr}_k | H^\bullet_c(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)) = f(\psi, k, 1).$$

This implies that K is a representing complex for the function $(\psi, k) \mapsto f(\psi, k, 1)$. □

To get the most out of Theorem 4.1, we use the following well-known Lemma. We delay the proof of this Proposition to Section 6 because it is unrelated to the rest of the article.

Proposition 4.2. *Consider $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that the limit $\lim_{N\to\infty} \sum_{i=1}^n \alpha_i \lambda_i^N$ exists. It is given by*

$$\lim_{N\to\infty} \sum_{i=1}^{n} \alpha_i \lambda_i^N = \sum_{|\lambda_i|=1} \alpha_i.$$

Recall the definition of the weighted Euler-Poincare characteristic.

Definition. Let $w \in \mathbb{R}$ and $q \in \mathbb{N}$ a prime power. Consider a $\mathbb{C}[T]$-module V, which is finite-dimensional over $\mathbb{C}$. Define $V_w \subset V$ to be the sum of all generalized eigenspaces of T acting on V with respect to eigenvalues λ whose absolute value satisfies $|\lambda| = q^{w/2}$.

Let M be a bounded complex of $\mathbb{C}[T]$-modules, whose cohomology groups $H^i(M)$ are finite-dimensional over $\mathbb{C}$. Define the *weighted Euler-Poincare characteristic of M with weight w* to be (see [Kat80, p. 92])

$$\chi_w(M) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{C}}(H^i(M)_w).$$

Theorem 4.3. *Let k be a finite field of characteristic co-prime to 2ℓ, ψ a non-trivial additive character of k, $a \in k^*$, and K a representing complex for the function $(\psi, k) \mapsto f(\psi, k, 1)$. We have*

$$M_4(\mathscr{H}_{\psi,k,a}) = \chi_0(H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)).$$

Proof. Theorem 4.1 says

$$M_4(\mathscr{H}_{\psi,k,a}) = \lim_{n\to\infty} \mathrm{Tr}(\mathrm{Fr}_k^n | H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)).$$

Denote the eigenvalues of Frobenius acting on $H_c^i(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)$ by $\lambda_{i,j} \in \mathbb{C}$. We have

$$\mathrm{Tr}(\mathrm{Fr}_k^n | H_c^i(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)) = \sum_j \lambda_{i,j}^n,$$

so Proposition 4.2 implies

$$M_4(\mathscr{H}_{\psi,k,a}) = \sum_{|\lambda_{i,j}|=1} (-1)^i = \chi_0\big(H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)\big).$$

∎

5. Determination of the Tannakian monodromy group

We recall a theorem on the uniformity of the Fourier transform by Katz.

Theorem 5.1 ([Kat80, p. 92, Cor. 1])**.** *Let $R \subset \mathbb{C}$ be a ring that is finitely generated over $\mathbb{Z}$, K a constructible complex of $\overline{\mathbb{Q}_\ell}$-sheaves, whose cohomology sheaves are mixed, and $w \in \mathbb{Z}$ an integer. There exists $r \in R$ such that for all ring morphisms $R[1/rl] \to k$ into a finite field k and all non-trivial additive characters ψ of k, the integer*

$$\chi_w(H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi)),$$

where the restriction is taken along the ring morphism, is independent of the ring morphism and the character.

We have collected all the required results to prove the main theorem.

Theorem 5.2. *There exists a constant $C > 2$ such that for all primes $p \geq C$, all finite extensions $k/\mathbb{F}_p$, all non-trivial additive characters ψ of k, and all $a \in k^*$ the perverse sheaf $\mathscr{H}_{\psi,k,a}$ has Tannakian monodromy group G_2.*

Proof. Let K be a representing complex for the function $(\psi, k) \mapsto f(\psi, k, 1)$. Remark that Theorem 5.1 implies that there is a constant $C > 0$ such that for each prime $p > C$, each finite extension $k/\mathbb{F}_p$, and each non-trivial additive character ψ of k the number

$$\chi_0(H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_\psi))$$

does not depend on ψ nor on k.

Let $p \geq C$ be a prime, $k/\mathbb{F}_p$ a finite extension, ψ a non-trivial additive character, and $a \in k^*$. By Theorem 2.2, there exists a finite extension k_n of k and an element $b \in k_n$ such that $M_4(\mathscr{H}_{\psi_n,k_n,b}) = 4$. Define the field $k_m := k_n(a^{1/6}, b^{1/6})$ and the non-trivial additive character $\psi'(x) := \psi_m(a^{1/6}x)$ of k_m. Corollary 3.2, Lemma 3.3 and Theorem 4.3 imply

$$M_4(\mathscr{H}_{\psi,k,a}) = M_4(\mathscr{H}_{\psi',k_m,1}) = \chi_0(H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_{\psi'})).$$

Define the additive character $\psi''(x) := \psi_m(b^{1/6}x)$ of k_m. Corollary 3.2, Lemma 3.3 and Theorem 4.3 imply

$$4 = M_4(\mathscr{H}_{\psi_m,k_m,b}) = M_4(\mathscr{H}_{\psi'',k_m,1}) = \chi_0(H_c^\bullet(\mathbb{A}^1_{\bar{k}}, K \otimes \mathscr{L}_{\psi''})).$$

The weighted Euler-Poincare characteristic does not depend on the additive character, so we get $4 = M_4(\mathscr{H}_{\psi,k,a})$. Hence Theorem 2.2 implies that the Tannakian monodromy group of $\mathscr{H}_{\psi,k,a}$ is G_2.

□

6. The proposition

In this section, we prove the remaining Proposition 4.2. This Proposition and the following Lemmas are well-known but we prove them here because we were not able to find a reference. Define

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

Lemma 6.1. *Let $x \in \mathbb{T}^n$ and define A to be the smallest closed subgroup $A \subset \mathbb{T}^n$ such that $x \in A$. Any continuous function $f : A \to \mathbb{C}$, such that*

$$\lim_{n\to\infty} f(x^n)$$

exists, is constant.

This is a consequence of the Kronecker-Weyl theorem, see for example [Kow21, Thm. 6.5 (i)].

Lemma 6.2. *Define the functions $\phi_z : \mathbb{N} \to \mathbb{C}$*

$$\phi_z(n) = z^n$$

for each $z \in \mathbb{C}$. These functions are linearly independent in $\mathbb{C}^{\mathbb{N}}$.

Proof. For each finite subset $S \subset \mathbb{C}$, the Vandermonde matrix

$$(z^m)_{z\in S, 0\leq m\leq |S|-1},$$

is invertible. □

We now prove Proposition 4.2. Suppose $|\lambda_i| \leq 1$. We have

$$\lim_{N\to\infty} \sum_i \alpha_i \lambda_i^N = \lim_{N\to\infty} \sum_{|\lambda_i|=1} \alpha_i \lambda_i^N.$$

Lemma 6.1 implies

$$\lim_{N\to\infty} \sum_{|\lambda_i|=1} \alpha_i \lambda_i^N = \sum_{|\lambda_i|=1} \alpha_i \lambda_i^0 = \sum_{|\lambda_i|=1} \alpha_i.$$

Consider arbitrary $\lambda_i \in \mathbb{C}$ and define

$$\beta_z := \sum_{\lambda_i = z} \alpha_i$$

for each $z \in \mathbb{C}$. Note that we can write

$$\sum_i \alpha_i \lambda_i^N = \sum_{z \in \mathbb{C}} \beta_z z^N$$

for each $N \geq 0$. Let $M := \max\{|z| : \beta_z \neq 0\}$ and suppose $M > 1$. Then

$$0 = \lim_{N \to \infty} \sum_{z \in \mathbb{C}} \beta_z (z/M)^N.$$

The first step of the argument implies

$$0 = \lim_{N \to \infty} \sum_{|z|=M} \beta_z (z/M)^N.$$

Then Lemma 6.1 implies

$$0 = \sum_{|z|=M} \beta_z (z/M)^N$$

for all $N \geq 0$. Lemma 6.2 implies $\beta_z = 0$ for all $|z| = M$. This is a contradiction to the definition of M therefore $M = 1$. We can write

$$\lim_{N \to \infty} \sum_i \alpha_i \lambda_i^N = \lim_{N \to \infty} \sum_{z \in \mathbb{C}} \beta_z z^N = \lim_{N \to \infty} \sum_{|z| \leq 1} \beta_z z^N = \sum_{|z|=1} \beta_z = \sum_{|\alpha_i|=1} \alpha_i.$$

□

References

[Del80] Pierre Deligne. “La conjecture de Weil. II”. In: *Inst. Hautes Études Sci. Publ. Math.* 52 (1980), pp. 137–252. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1980__52__137_0.

[FFK23] Arthur Forey, Javier Fresán, and Emmanuel Kowalski. *Arithmetic Fourier transforms over finite fields: generic vanishing, convolution, and equidistribution.* 2023. arXiv: 2109.11961 [math.NT].

[GL96] Ofer Gabber and François Loeser. “Faisceaux pervers ℓ-adiques sur un tore”. In: *Duke Math. J.* 83.3 (1996), pp. 501–606. ISSN: 0012-7094,1547-7398.

[Kat07] Nicholas M. Katz. “G2 and hypergeometric sheaves”. In: *Finite Fields and Their Applications* 13.2 (2007), pp. 175–223. ISSN: 1071-5797.

[Kat12] Nicholas M. Katz. *Convolution and equidistribution.* Vol. 180. Annals of Mathematics Studies. Sato-Tate theorems for finite-field Mellin transforms. Princeton University Press, Princeton, NJ, 2012, pp. viii+203. ISBN: 978-0-691-15331-5.

[Kat80] Nicholas M. Katz. *Sommes exponentielles.* Vol. 79. Astérisque. Course taught at the University of Paris, Orsay, Fall 1979, With a preface by Luc Illusie, Notes written by Gérard Laumon, With an English summary. Société Mathématique de France, Paris, 1980, p. 209.

[Kat90] Nicholas M. Katz. *Exponential sums and differential equations.* Vol. 124. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1990, pp. xii+430. ISBN: 0-691-08598-6; 0-691-08599-4.

[Kow21] Emmanuel Kowalski. *An introduction to probabilistic number theory.* Vol. 192. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2021, pp. xiv+255. ISBN: 978-1-108-84096-5.

[KW01] Reinhardt Kiehl and Rainer Weissauer. *Weil conjectures, perverse sheaves and l'adic Fourier transform.* Vol. 42. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 2001, pp. xii+375. ISBN: 3-540-41457-6.